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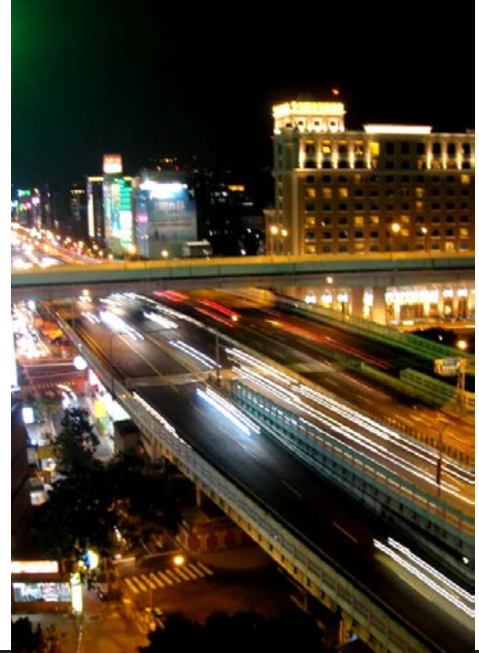

High-Frequency Electronic Circuits

# Lecture 2

# Transmission Line Theory


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# Contents

## Lecture 2: Transmission Line Theory



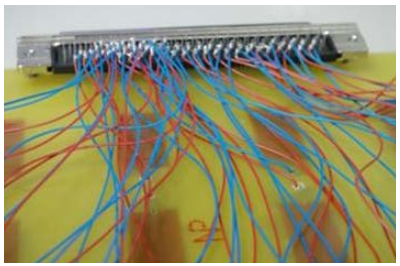
# Contents

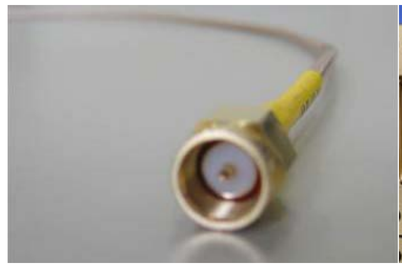
## 2.1 Why Does Frequency Matter?

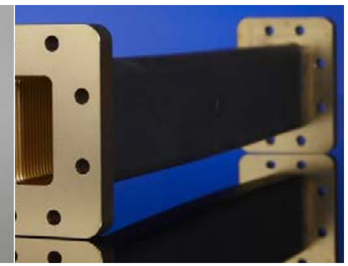
# 3D-Type Transmission Lines

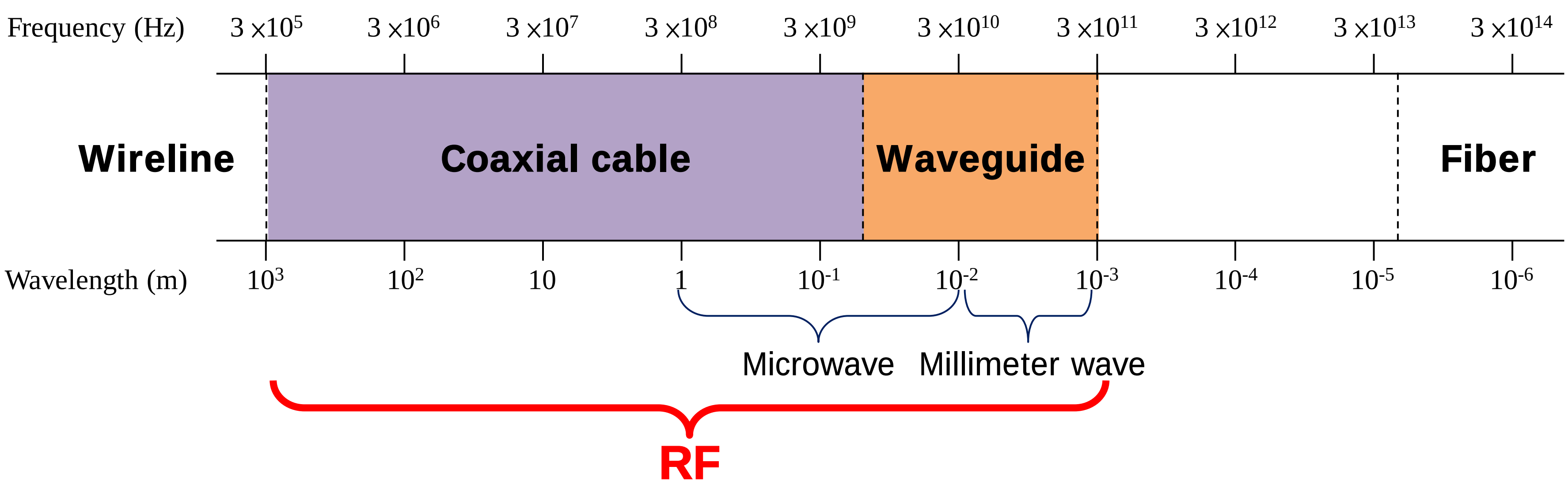
## Definition:

- Transmission line: A conductor system which delivers voltage and current signals







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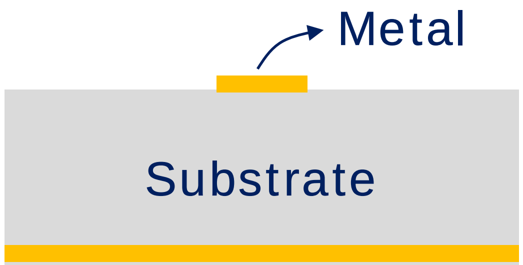
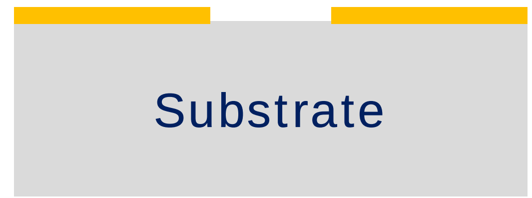
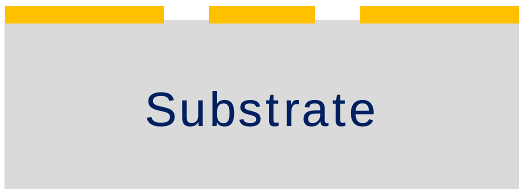
# Planar-Type Transmission Lines



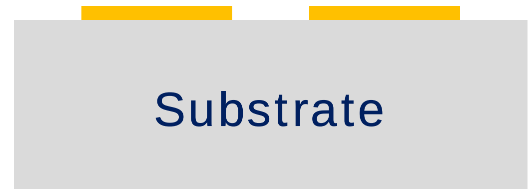

Microstrip line




Slotline




Coplanar waveguide (CPW)

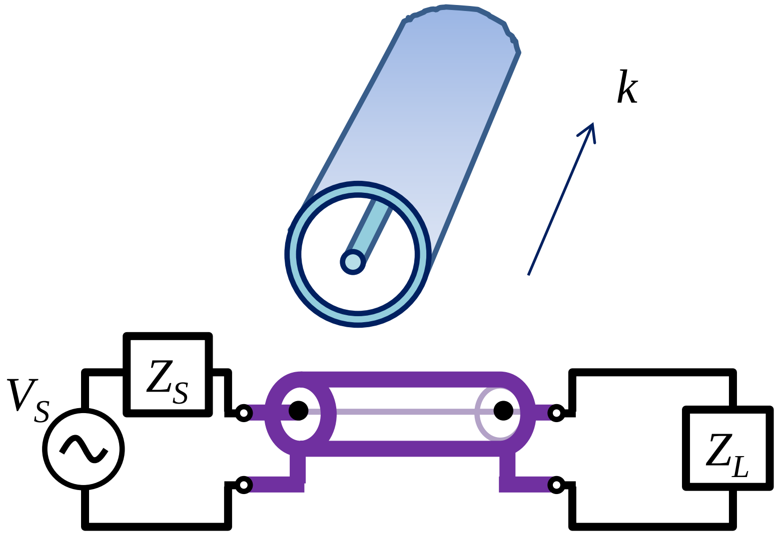



Coplanar strip (CPS)

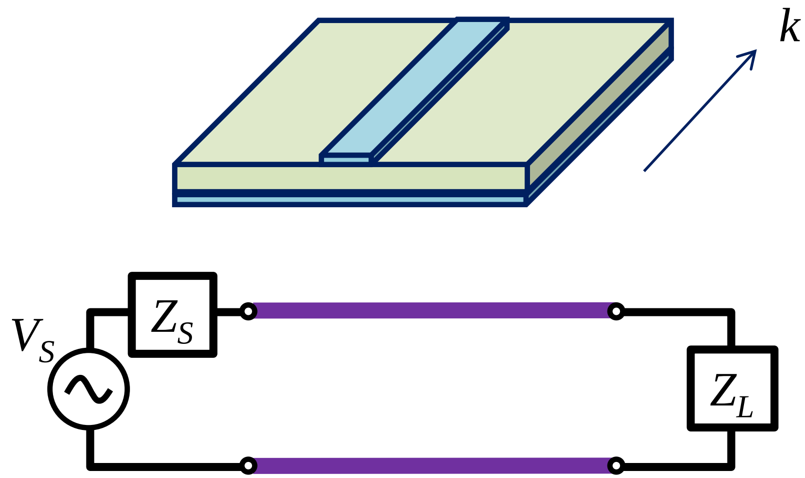
- Each transmission line serves specific scenarios
- Electrical characteristic: characteristic impedance, operational frequency, loss, electrical length, phase velocity, adjustability, etc.

# Description of Circuitry

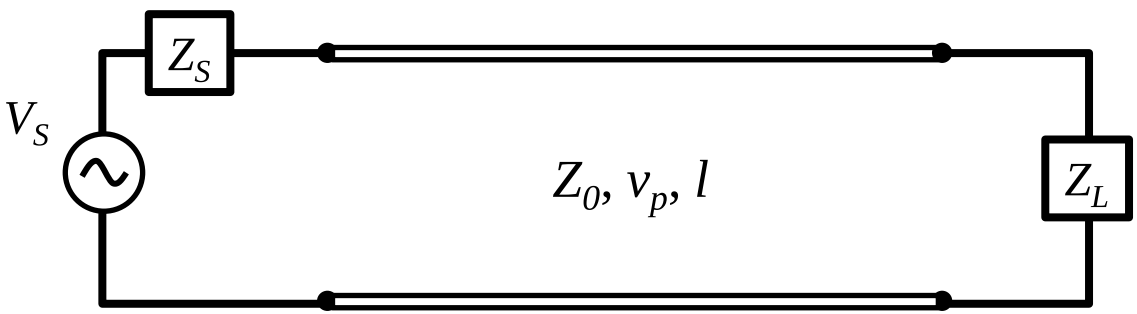
Coaxial cable



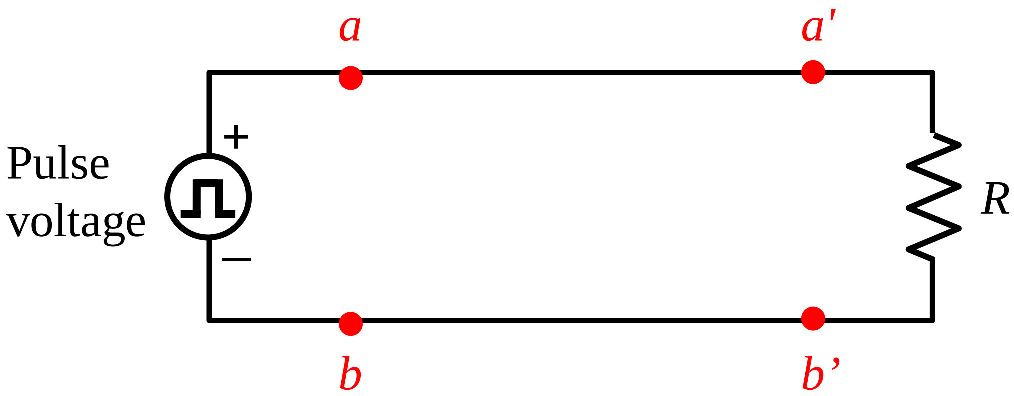

Microstrip line




In general, the transmission line is described as:



# Is the Assumption in *Circuit* Reasonable?



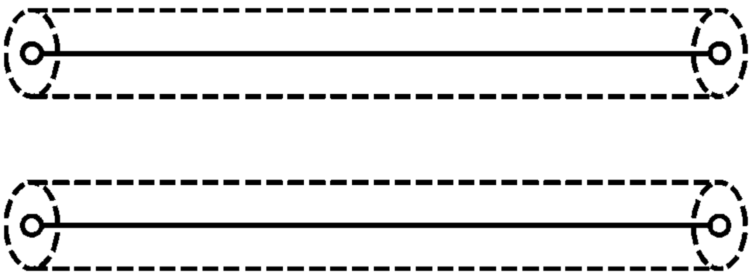

- The voltage source gives an impulse signal
- The impulse arrives at the load immediately

Is this assumption reasonable?

- Signal transmission has a number of delay
- But if the size of the circuity is ***small enough***, such a delay is omitted

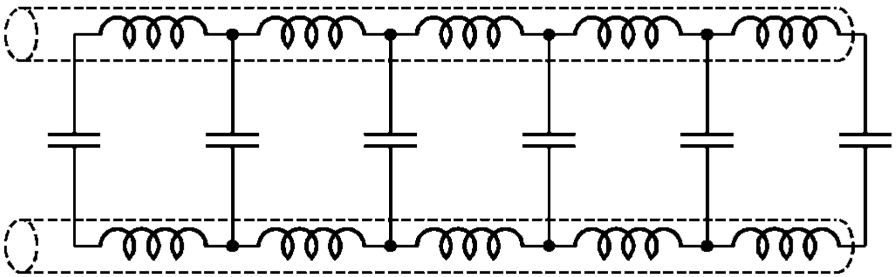
# Equivalent Circuits of the Whole Line

## Ideal: short circuit



- Voltage and current are equivalent everywhere

## Real: capacitor and inductor



- Voltage and current are different everywhere

Charges on the conductors ∝ the potential difference $C = \dfrac{Q}{v}$

Magnetic flux around the loop ∝ the current on conductor $L = \dfrac{\phi_m}{i}$

$C$ = Shunt capacitance per unit length, in F/m

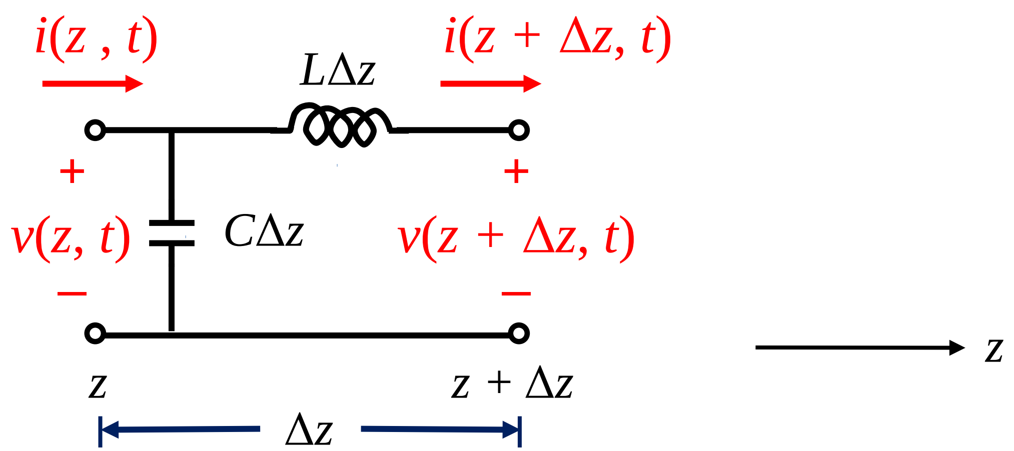
$L$ = Series inductance per unit length in H/m

} Distributed-parameter network

# Equivalent Circuits of a Short Section

Objective: Find the exact solution of $v(z, t)$ and $i(z, t)$

If we extract a ***very short*** section $\Delta z$ ($\Delta z \to 0$):




KVL: $$v(z+\Delta z,\ t)-v(z,\ t)=-(L\Delta z)\frac{\partial i(z+\Delta z,\ t)}{\partial t}$$

KCL: $$i(z+\Delta z,\ t)-i(z,\ t)=-(C\Delta z)\frac{\partial v(z+\Delta z,\ t)}{\partial t}$$

# Telegrapher's Equation

Dividing both sides by $\Delta z$ and let $\Delta z \to 0$:

$$\left.\begin{array}{ll}\dfrac{\partial v(z,\ t)}{\partial z} = -L\dfrac{\partial i(z,\ t)}{\partial t} & (1) \\[2ex] \dfrac{\partial i(z,\ t)}{\partial z} = -C\dfrac{\partial v(z,\ t)}{\partial t} & (2)\end{array}\right\}$$

Transmission line equation
or
Telegrapher's equation (1850)

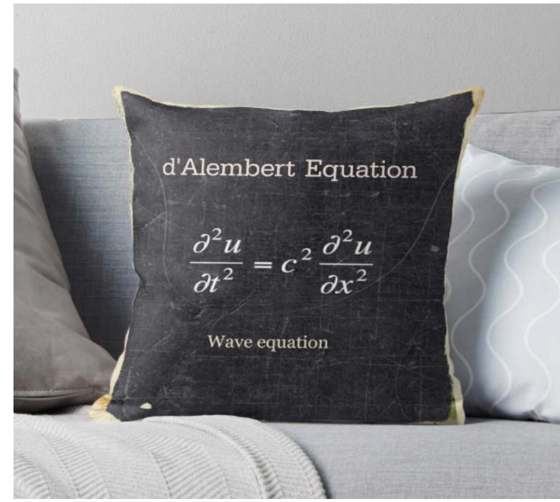
To further solve the transmission line equations:

$$\frac{\partial(1)}{\partial z}: \qquad \frac{\partial^2 v(z,\ t)}{\partial z^2} = -L\frac{\partial^2 i(z,\ t)}{\partial z \partial t} \qquad (3)$$

$$\frac{\partial(2)}{\partial t}: \qquad \frac{\partial^2 i(z,\ t)}{\partial t \partial z} = -C\frac{\partial^2 v(z,\ t)}{\partial t^2} \qquad (4)$$

# Manipulation

Combining (3) and (4):




$$\frac{\partial^2 v(z,\ t)}{\partial z^2} - LC\frac{\partial^2 v(z,\ t)}{\partial t^2} = 0 \qquad (5)$$

$$\frac{\partial^2 i(z,\ t)}{\partial z^2} - LC\frac{\partial^2 i(z,\ t)}{\partial t^2} = 0 \qquad (6)$$

(5) and (6) are identical in formation with wave equation:

$$\frac{\partial^2 f(z,\ t)}{\partial z^2} - \frac{1}{v_p^2}\frac{\partial^2 f(z,\ t)}{\partial t^2} = 0$$

where $v_p = \frac{1}{\sqrt{LC}}$ **Phase velocity**

$$f(z,\ t) = f^{i}\left(t - \frac{z}{v_p}\right) + f^{r}\left(t + \frac{z}{v_p}\right)$$

# The Solution of Telegrapher's Equation

Therefore, the voltage *wave* and current *wave* on the TL:

$$v(z,\ t) = v^i\left(t - \frac{z}{v_p}\right) + v^r\left(t + \frac{z}{v_p}\right),\ i(z,\ t) = i^i\left(t - \frac{z}{v_p}\right) + i^r\left(t + \frac{z}{v_p}\right)$$

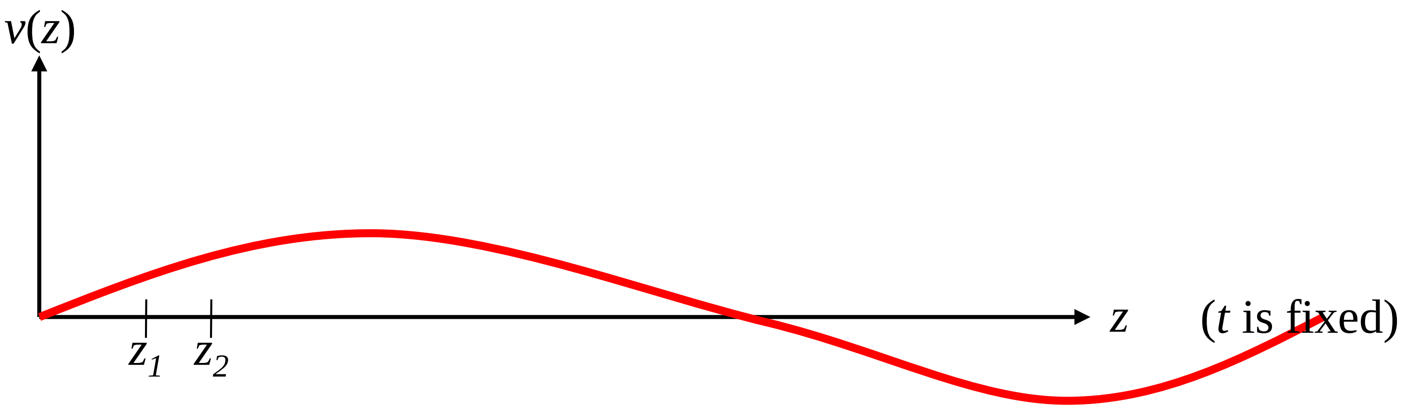
Characteristics:

1. Second-order partial differential functions → two solutions
2. Variable $t$ and $z$ cannot be separated
3. The first solution: $t\uparrow \rightarrow z$ must ↑ to keep up the argument → **Incident wave** (toward $+z$)
4. The second solution: $t\uparrow \rightarrow z$ must ↓ to keep up the argument → **Reflected wave** (toward $-z$)
5. In the circuit theorem, voltage ($v$) is calculated from any two nodes and current ($i$) is defined on any branch; $v$ and $i$ are not functions of $z$
6. In transmission line theorem, $v$ and $i$ are functions of $z$ and $t$. So we call them **voltage wave** and **current wave**

# Low Frequency Approximation

If the voltage and current change slowly with time:

- The frequency of $v$ and $i$ are low
- The variation of voltage wave and current wave **against** $z$ are slow




- Example: 60 Hz AC
- If the two positions $z_1$ and $z_2$ are close with each other, then $v(z_1) \approx v(z_2)$
- How to define the closeness? **In terms of wavelength**!
- The difference between circuit theorem and transmission line theory is electrical size

# More Explicit Expressions

Substituting $v$, $i$ to (3), (4) (telegrapher's equation)

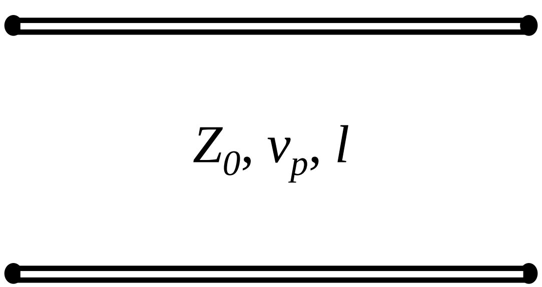
$$v(z,\ t) = v^i\left(t - \frac{z}{v_p}\right) + v^r\left(t + \frac{z}{v_p}\right)$$

According to $\dfrac{\partial v(z,\ t)}{\partial z} = -L\dfrac{\partial i(z,\ t)}{\partial t}$

$$i(z,\ t) = \frac{1}{Z_0}v^i\left(t - \frac{z}{v_p}\right) - \frac{1}{Z_0}v^r\left(t + \frac{z}{v_p}\right)$$

where $Z_0 = \sqrt{\dfrac{L}{C}}$ **Characteristic impedance**

- $Z_0$ is **NOT** an impedance of lumped components
- The resistance in *Circuit Theorem* stands for loss of energy, but TL does not cause any loss of energy if the conductor is PEC
- $Z_0$ and $v_p$ result from the distributed elements $L$ and $C$ on TL
- $Z_0$ and $v_p$ fully describe the characteristics of the TL





# Contents

## 2.2 Frequency-Domain Analysis

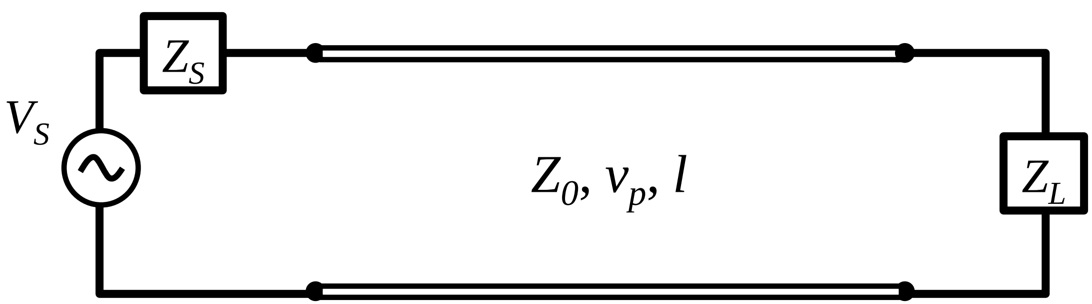
# To Completely Get the Signal Expression

Now we know the concept of voltage wave and current wave:

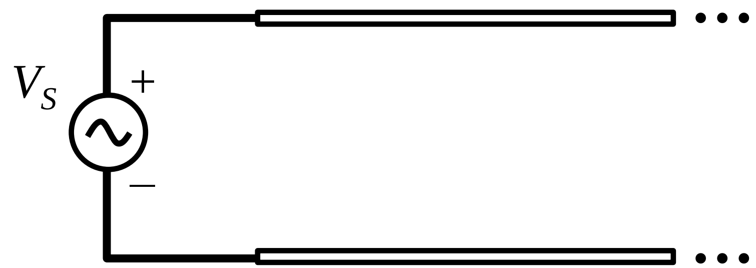
$$v(z,\ t) = v^i\left(t - \frac{z}{v_p}\right) + v^r\left(t + \frac{z}{v_p}\right) \qquad i(z,\ t) = \frac{1}{Z_0}v^i\left(t - \frac{z}{v_p}\right) - \frac{1}{Z_0}v^r\left(t + \frac{z}{v_p}\right)$$

The remaining work: To fully obtain the formulation of $v^i$ and $v^r$

- $v^i$ and $v^r$ are related the voltage source, source impedance and the load impedance
- What type of source is our concern? **Sinusoidal wave!**
- The advantage of using a sinusoidal source: phasor expression

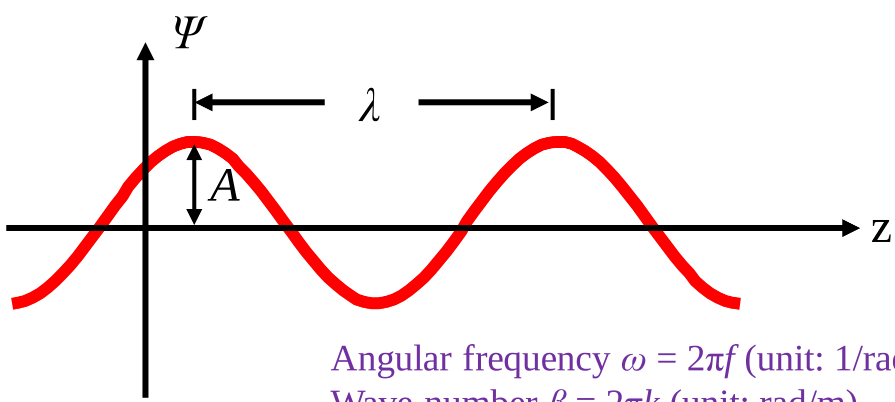


# Sinusoidal Traveling Wave




- The source in *circuit theorem* is defined as: $v_s(t) = A\cos(\omega t + \phi)$
- Now we know the traveling wave on transmission lines is:

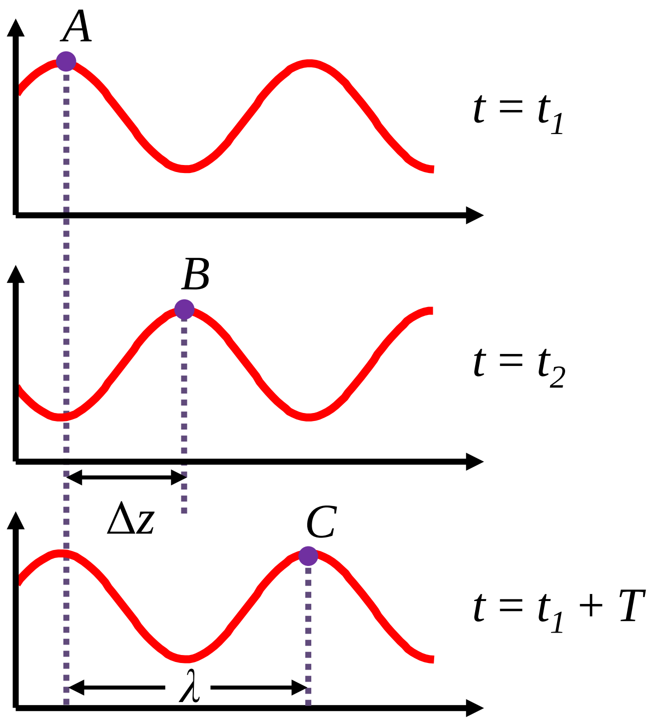
$$v_s(z,\ t) = A\cos\left[\omega\left(t - \frac{z}{v_p}\right) + \phi\right]$$




Angular frequency $\omega = 2\pi f$ (unit: 1/rad)
Wave number $\beta = 2\pi k$ (unit: rad/m)

- Frequency $f$: the number of oscillations that occur each second of time
- Amplitude $A$: the peak deviation of the function from zero
- Period $T$: the time or a complete oscillation
- Wavelength $\lambda$: the distance between two successive peaks
- Phase angle $\varphi$: specifying the relative location in a cycle
- Wave number $k$: the number of peak in one meter; $k = 1/\lambda$

# Phasor Expression




Definition of phase velocity $v_p$:

$$v_p = \frac{\Delta z}{\Delta t} = \frac{\lambda}{T}$$

According to $\beta = 2\pi k = 2\pi / \lambda$ and $\omega = 2\pi / T$:

$$v_p = \lambda f = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Therefore, the traveling wave on the TL is:

$$v(t) = A\cos\left[\omega\left(t - \frac{z}{v_p}\right) + \phi\right] = A\cos(\omega t - \beta z + \phi)$$

Now we shift our discussion to the phasor domain:

Time domain | Phasor domain

$$v(z,\ t) \xleftarrow{\mathrm{Re}\left\{V(z)e^{j\omega t}\right\}} V(z)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{\partial v(z,\ t)}{\partial t} \xleftarrow{\mathrm{Re}\left\{j\omega V(z)e^{j\omega t}\right\}} j\omega V(z)$$

where
$V(z) = |V| e^{j(-\beta z + \varphi)}$

# Frequency-Domain TL Equation (1/2)

Time-domain expression

$$\frac{\partial v(z,\ t)}{\partial z} = -L\frac{\partial i(z,\ t)}{\partial t}$$

$$\frac{\partial i(z,\ t)}{\partial z} = -C\frac{\partial v(z,\ t)}{\partial t}$$

Frequency-domain expression

$$\frac{dV(z)}{dz} = -j\omega LI(z)$$

$$\frac{dI(z)}{dz} = -j\omega CV(z)$$

$$\frac{d^2V(z)}{dz^2} + \omega^2 LCV(z) = 0$$

Observe that $\omega^2 LC = \beta^2$

$$V(z) = V^+e^{-j\beta z} + V^-e^{j\beta z}$$

where $V^+ = |V^+|e^{j\phi^+}$, $V^- = |V^-|e^{j\phi^-}$

The physical meaning of $V^+e^{-j\beta z}$ and $V^-e^{j\beta z}$ ?

# Frequency-Domain TL Equation (2/2)

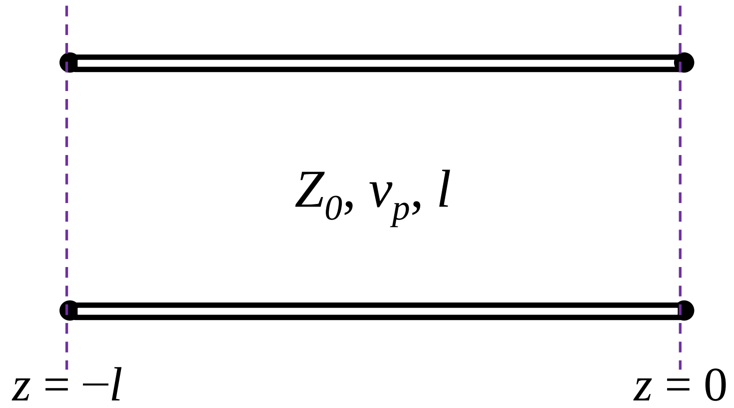
Similarly, we have $\frac{d^2 I(z)}{dz^2} + \beta^2 I(z) = 0$

$I(z) = I^+ e^{-j\beta z} + I^- e^{j\beta z}$ .Substitute $I(z)$ into $\frac{dV(z)}{dz} = -j\omega L I(z)$

$I(z) = \frac{1}{Z_0}\left[V^+ e^{-j\beta z} - V^- e^{j\beta z}\right]$ where $Z_0 = \sqrt{\frac{L}{C}}$

The complete description of $V(z)$ and $I(z)$ are only associated to $V^+$ and $V^-$, so the most imperative task is to find the expression of $V^+$ and $V^-$

# Finding Coefficients $V^+$ and $V^-$ (1/5)




Step 1: Pre-procedure definition and transformation

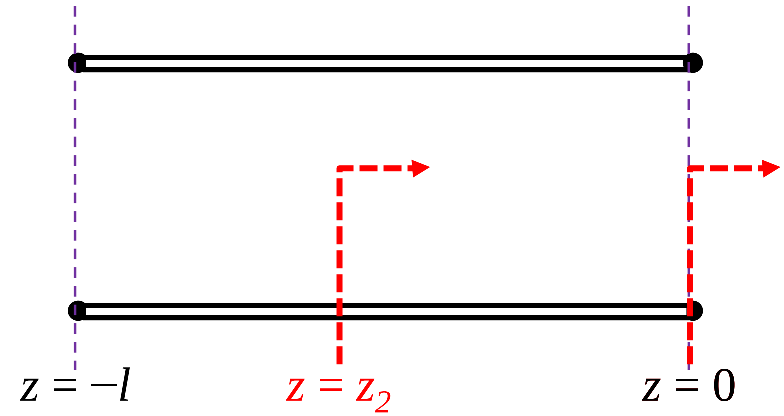
- Define "reflection coefficient $\Gamma(z)$" ***on every position*** $z$ along the TL:

$$\Gamma(z) \triangleq \frac{V^- e^{j\beta z}}{V^+ e^{-j\beta z}} = \left(\frac{V^-}{V^+}\right) e^{2j\beta z}$$

- Special case: $\Gamma(z=0)$

$$\Gamma(z=0) = \frac{V^-}{V^+} \qquad \text{(Note that } \Gamma(z) = \Gamma(0)e^{2j\beta z}\text{)}$$

# Finding Coefficients $V^+$ and $V^-$ (2/5)




## Step 2: Rearrange $V(z)$ and $I(z)$ in terms of $V^+$ and $\Gamma(z)$

$$V(z)=V^+e^{-j\beta z}+V^-e^{j\beta z}=V^+e^{-j\beta z}\left[1+\frac{V^-e^{j\beta z}}{V^+e^{-j\beta z}}\right]=\boxed{V^+}e^{-j\beta z}[1+\boxed{\Gamma(z)}]$$

and $$I(z)=\frac{1}{Z_0}\left[V^+e^{-j\beta z}-V^-e^{j\beta z}\right]=\frac{\boxed{V^+}e^{-j\beta z}}{Z_0}[1-\boxed{\Gamma(z)}]$$

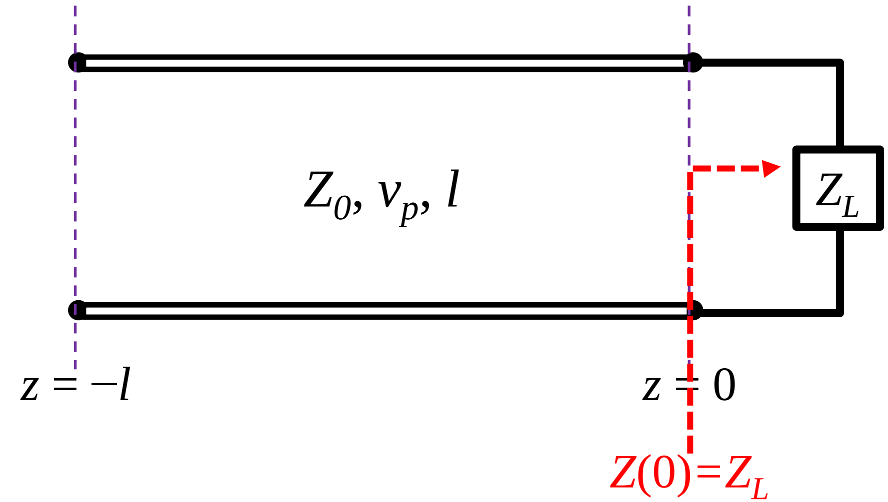
By the concept of *circuit theorem*,
the input impedance on location $z$:

$$Z(z)\equiv\frac{V(z)}{I(z)}=Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

Rearrange it:

$$\Gamma(z)=\frac{Z(z)-Z_0}{Z(z)+Z_0}$$

# Finding Coefficients $V^+$ and $V^-$ (3/5)




Step 3: Express $\Gamma(z)$ as the function of $\Gamma(0)$ and $z$

Observe that $\Gamma(z) = \Gamma(0)e^{2j\beta z}$. If we can express $\Gamma(0)$, then we know $\Gamma(z)$ on any location $z$

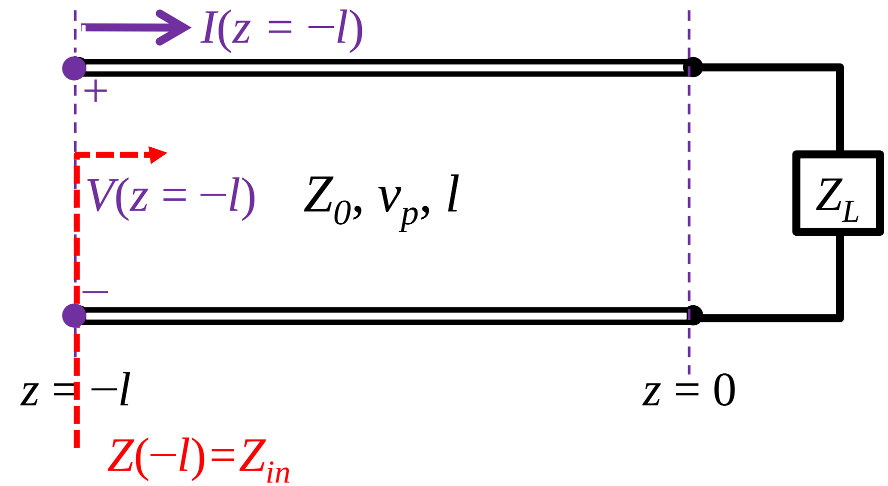
Place a lumped element $Z_L$ at $z = 0$:

$$\Gamma(z=0) = \frac{Z(z=0) - Z_0}{Z(z=0) + Z_0} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma(z) = \Gamma(0)e^{2j\beta z}$$ Now the only unknown is $V^+$

$\Gamma(0)$ is also denoted as $\Gamma_L$

# Finding Coefficients $V^+$ and $V^-$ (4/5)



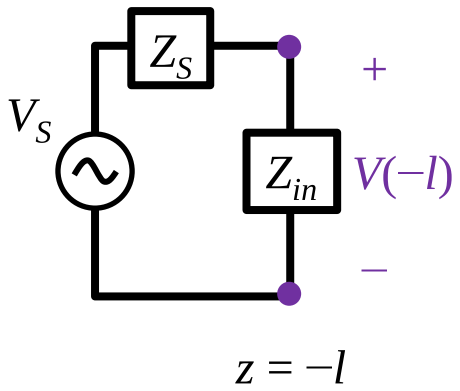

Step 4: Connect a Thévenin equivalent circuit on the source terminal

- The impedance at $Z(z=-l)$ is called an input impedance, denoted as $Z_{in}$
- How to calculate $Z_{in}$? By $Z_{in} = Z(z=-l) = Z_0 \dfrac{1+\Gamma(z=-l)}{1-\Gamma(z=-l)}$
- How to find $V^+$? Recall

$$V(z) = V^+ e^{-j\beta z}[1+\Gamma(z)]$$

- Now we can express $\Gamma(z)$; if we know $V(z)$, then we find $V^+$

# Finding Coefficients $V^+$ and $V^-$ (5/5)




- At the location $z = -l$, $V(z = -l)$ can be found by voltage division:

$$V(-l) = \frac{Z_{in}}{Z_{in} + Z_S} V_S$$

- So the only remaining unknown is:

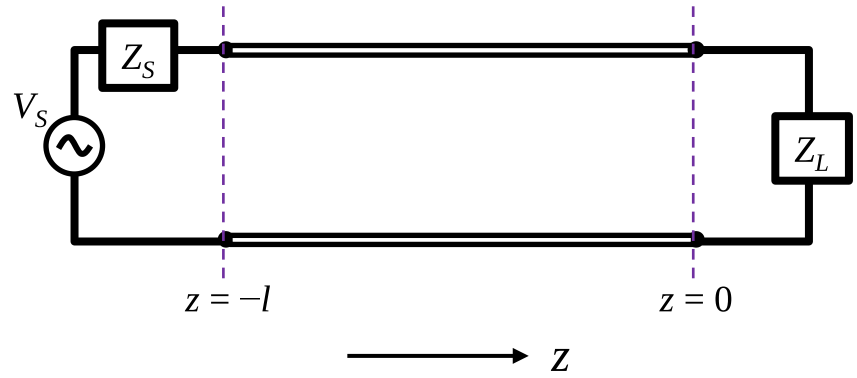
$$V^+ = V_S \frac{1}{1 + \Gamma(-l)} \frac{Z_{in}}{Z_{in} + Z_S} e^{j\beta(-l)}$$

# Impedance Transformation

## 1. Transform of reflection coefficient:

- We have found that $\Gamma(z) = \Gamma(0)e^{2j\beta z}$
- The relation can be extended to:

$$\Gamma(z_2) = \Gamma(z_1)e^{2j\beta(z_2 - z_1)}$$




## 2. Transform of impedance:

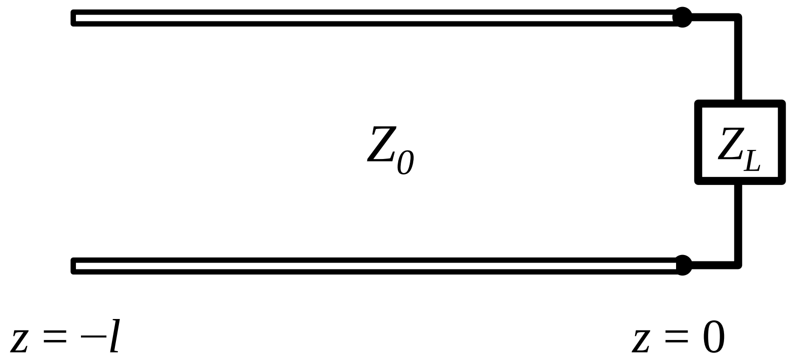
- We have found that $Z(z) = Z_0 \dfrac{1+\Gamma(z)}{1-\Gamma(z)}$
- Substituting $\Gamma(z) = \Gamma(0)e^{2j\beta z} = \dfrac{Z_L - Z_0}{Z_L + Z_0}e^{2j\beta z}$

$$Z(z) = Z_0\left(\frac{Z_L + jZ_0 \tan(-\beta z)}{Z_0 + jZ_L \tan(-\beta z)}\right)$$

$$Z(z_2) = Z_0\left(\frac{Z(z_1) + jZ_0 \tan\left(-\beta\left(z_2 - z_1\right)\right)}{Z_0 + jZ(z_1)\tan\left(-\beta\left(z_2 - z_1\right)\right)}\right)$$

Special case: $Z_{in} = Z_0\left(\dfrac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right)$

# Average Power along a Lossless TL (1/2)




- The time-average power flow along the line at the point $z$:

$$P_{av}=\frac{1}{2}\mathrm{Re}\left[V(z)I(z)^{*}\right]$$

- Substituting $V(z)=V^{+}e^{-j\beta z}[1+\Gamma(z)]$, $I(z)=\dfrac{V^{+}e^{-j\beta z}}{Z_0}\left[1-\Gamma(z)\right]$ into $P_{av}$:

$$\Rightarrow \quad P_{av}(z)=\frac{\left|V^{+}\right|^{2}}{2Z_0}\left(1-\left|\Gamma(z)\right|^{2}\right)$$

# Average Power along a Lossless TL (2/2)

$$P_{av}(z) = \frac{|V^+|^2}{2Z_0}\left(1 - |\Gamma(z)|^2\right) = \underbrace{\frac{|V^+|^2}{2Z_0}}_{\text{Incident power } (P^i)} - \underbrace{\frac{|V^+|^2}{2Z_0}|\Gamma(z)|^2}_{\text{Reflected power } (P^r)}$$

- Note that $\Gamma(z) = \Gamma_L e^{2j\beta z}$, so $|\Gamma(z)|$ is independent of $z$
- Therefore, $P_{av}(z)$ is also independent of $z$. WHY?

- The smaller the $|\Gamma(z)|$, the larger the $P_{av}$ is
- How to estimate the ratio between $P^i$ and $P^r$?

# Return Loss and Insertion Loss

## 1. Return loss (RL)

$$RL = -10\log\left(\frac{P^r}{P^i}\right) \qquad \text{Unit: dB}$$

- Return loss can also be expressed as:

$$RL = -10\log\left(\frac{V^-}{V^+}\right)^2 = -20\log\left|\Gamma(z)\right| = -20\log\left|\Gamma_L\right|$$

- Matched load: $RL = \infty$ (dB)
- Short-circuited load or open-circuited load: $RL = 0$ (dB)

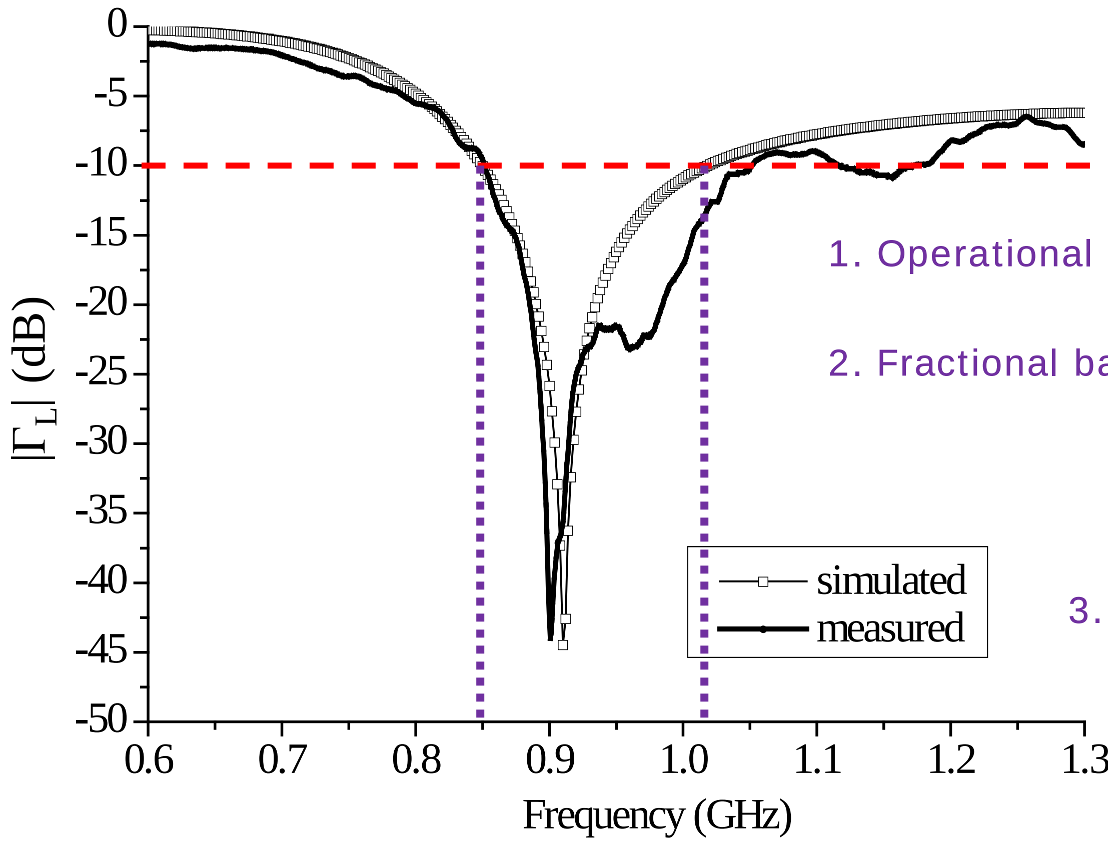
## 2. Insertion loss (IL)

$$IL = -10\log\left(\frac{P^t}{P^i}\right) = -10\log\left(\frac{P^i - P^r}{P^i}\right) = -10\log\left(1-\left|\Gamma_L\right|^2\right)$$

# Operational Bandwidth

Based on the definition of "power reflection coefficient":

- Usually we choose the frequency range which includes $RL$ > 10 dB as the operational bandwidth




1. Operational bandwidth = 170 MHz

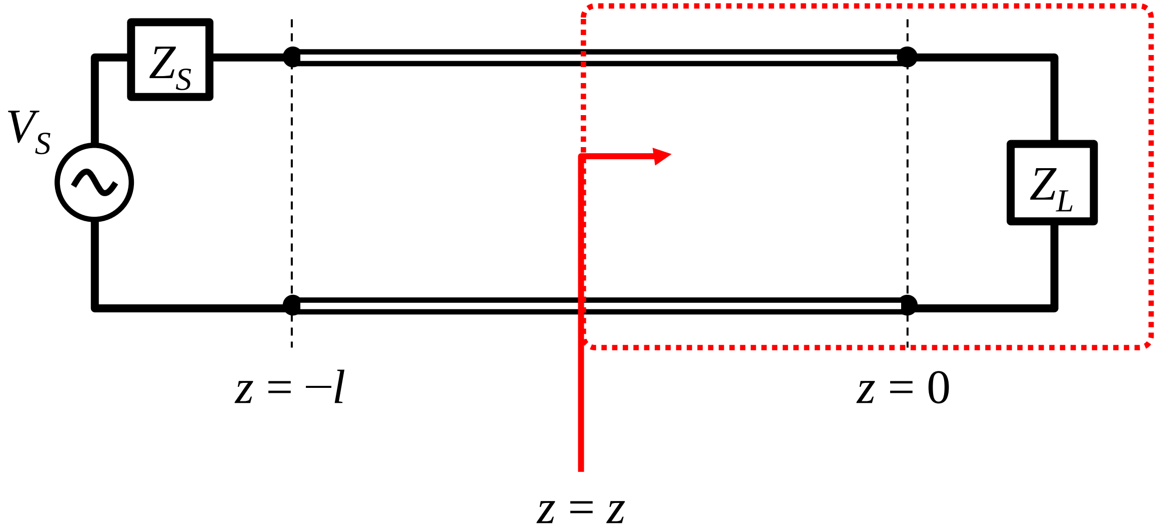
2. Fractional bandwidth $= \dfrac{f_{upper} - f_{lower}}{f_{center}}$

$$= \frac{1.02 - 0.85}{0.935} = 18.18\%$$

3. Bandwidth $= f_{upper}/f_{lower}$

$= 1.02/0.85$

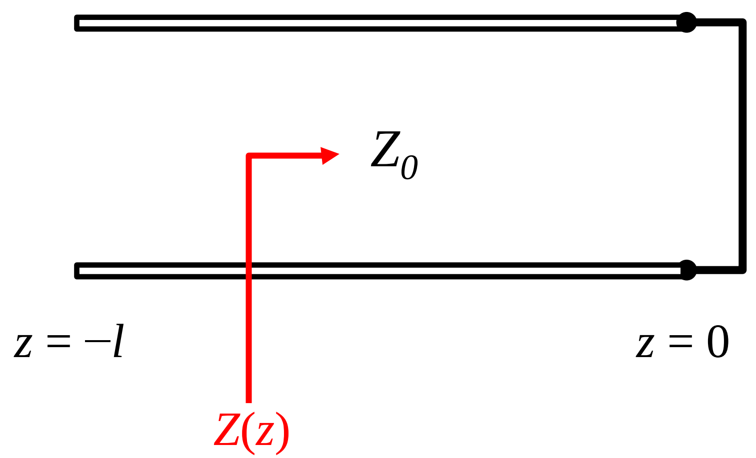
# Special Cases of Input Impedance

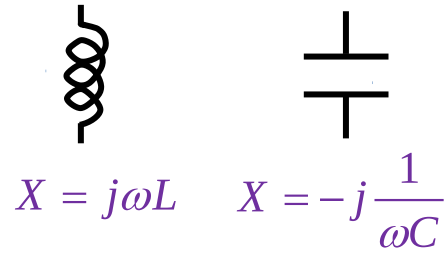



Just like the above example, most of the time we are interested in only the input impedance at the location $z$ or $-l$. The following cases are particularly useful:

1. $Z_L = 0$ (short circuit)
2. $Z_L = \infty$ (open circuit)
3. $Z_L = Z_0$ (match)
4. $z = \lambda / 2$
5. $z = \lambda / 4$

# Special Case 1: $Z_L = 0$

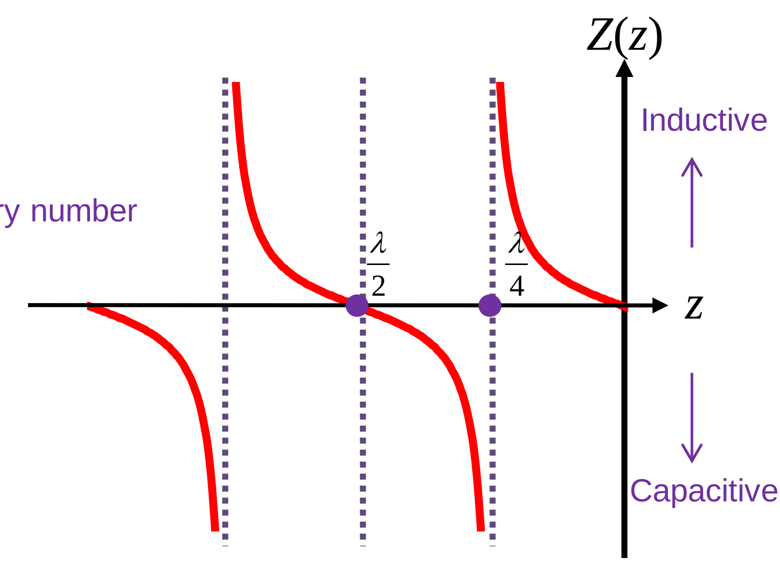



$$\Gamma_L = \Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$$

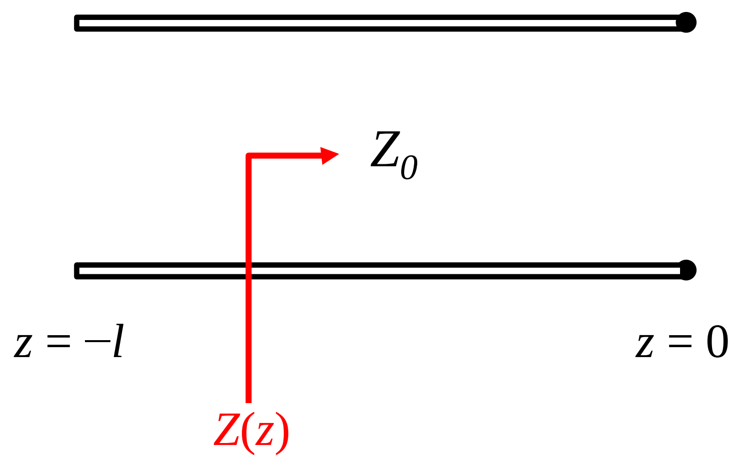
Pure imaginary number

$$Z(z) = Z_0\left(\frac{Z_L + jZ_0\tan(-\beta z)}{Z_0 + jZ_L\tan(-\beta z)}\right) = jZ_0\tan(-\beta z)$$




- Most of the locations cannot have $Z(z) = 0$

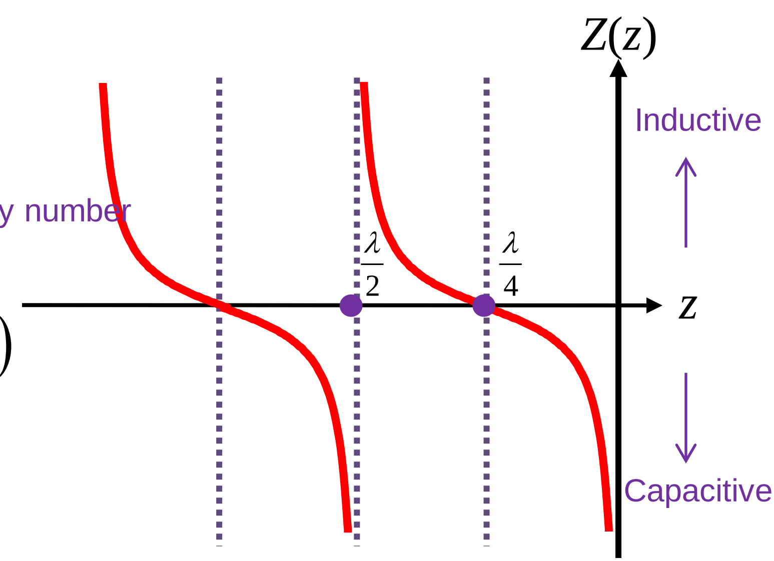
# Special Case 2: $Z_L = \infty$




$$\Gamma_L = \Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

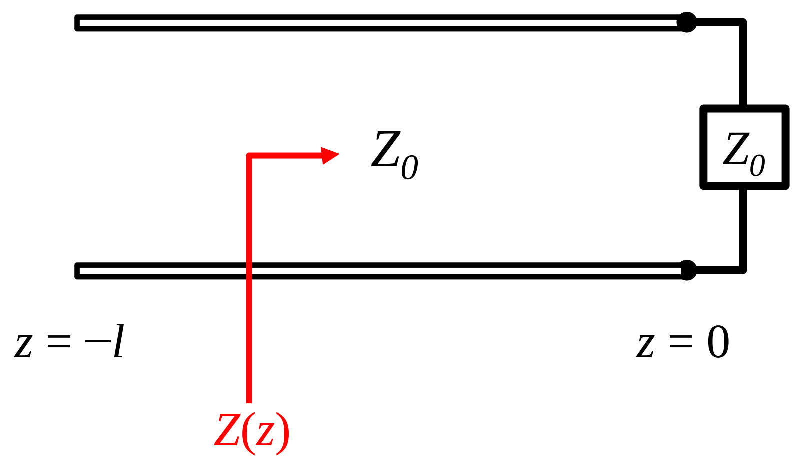
$$Z(z) = Z_0 \left( \frac{Z_L + jZ_0 \tan(-\beta z)}{Z_0 + jZ_L \tan(-\beta z)} \right) = -jZ_0 \cot(-\beta z)$$

Pure imaginary number



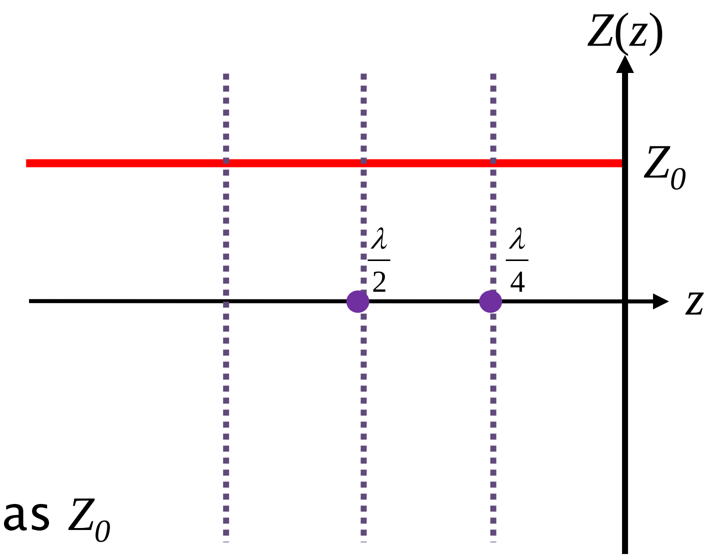

- Most of the locations cannot give $Z(z) = \infty$

# Special Case 3: $Z_L = Z_0$



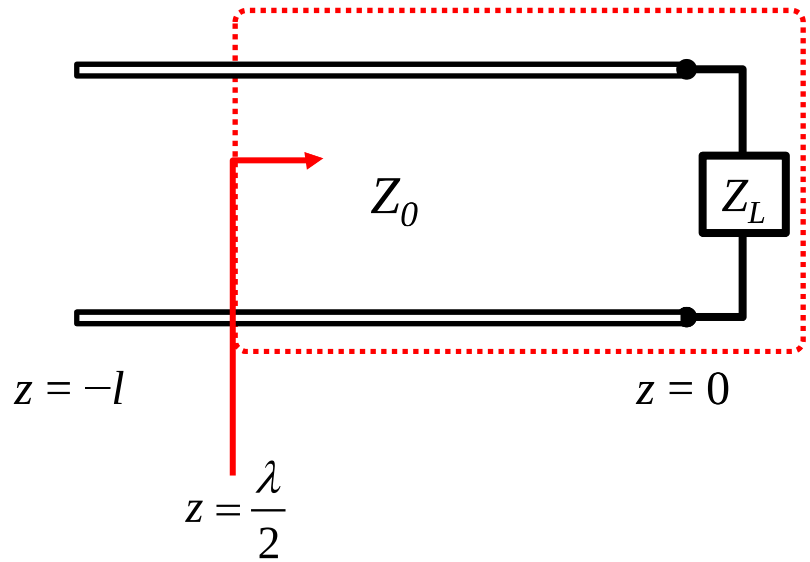

$$\Gamma_L = \Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$

$$Z(z) = Z_0 \left( \frac{Z_L + jZ_0 \tan(-\beta z)}{Z_0 + jZ_L \tan(-\beta z)} \right) = Z_0$$




- The input impedances are always identical as $Z_0$

# Special Case 4: $z = \lambda / 2$

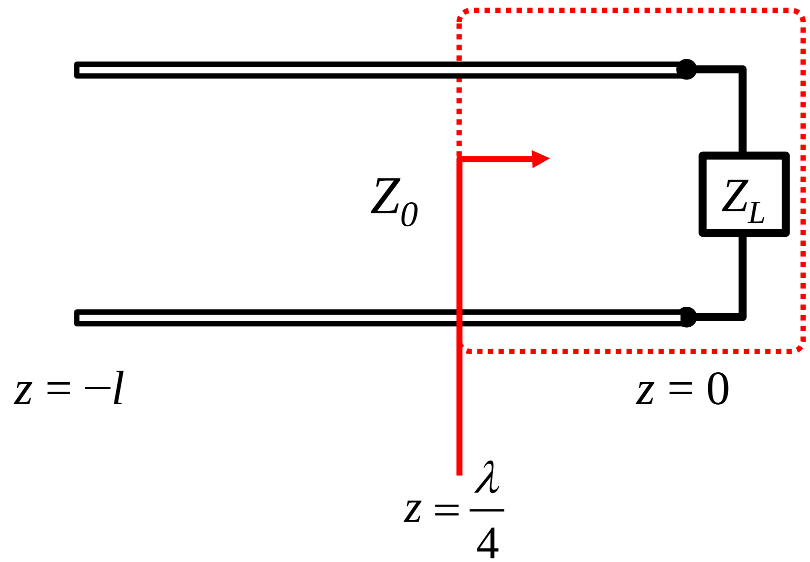



- According to $\Gamma(z) = \Gamma_L e^{2j\beta z}$, at $z = \lambda / 2 \rightarrow \beta \times z = \pi$

$\Rightarrow$ $\Gamma\left(\frac{\lambda}{2}\right) = \Gamma_L e^{j2\pi} = \Gamma_L$ The reflection coefficient is the same as $\Gamma_L$

- $$Z\left(z = \frac{\lambda}{2}\right) = Z_0 \left( \frac{Z_L + jZ_0 \tan\left(-\beta \frac{\lambda}{2}\right)}{Z_0 + jZ_L \tan\left(-\beta \frac{\lambda}{2}\right)} \right) = Z_0 \left( \frac{Z_L}{Z_0} \right) = Z_L$$

# Special Case 5: $z = \lambda / 4$

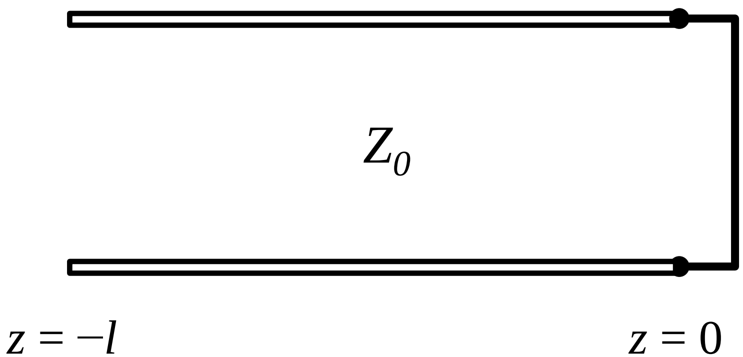



- If $z$ is a quarter-wavelength long or $z = \lambda / 4 + n \times \lambda / 2,\ n = 1, 2, 3, \ldots$

$$Z\left(z = \frac{\lambda}{4}\right) = Z_0 \left( \frac{Z_L + jZ_0 \tan\left(-\beta \frac{\lambda}{4}\right)}{Z_0 + jZ_L \tan\left(-\beta \frac{\lambda}{4}\right)} \right) = Z_0 \left( \frac{Z_0}{Z_L} \right) = \frac{Z_0^2}{Z_L}$$

Called "**quarter-wavelength transformer**"

# Voltage Standing Wave Phenomenon




$Z_L = 0$ (short circuit)

- $\Gamma_L = -1$
- $V^- = -V^+ \qquad V(0) = 0$

Frequency-domain signals:

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} = -2jV^+ \sin\beta z$$

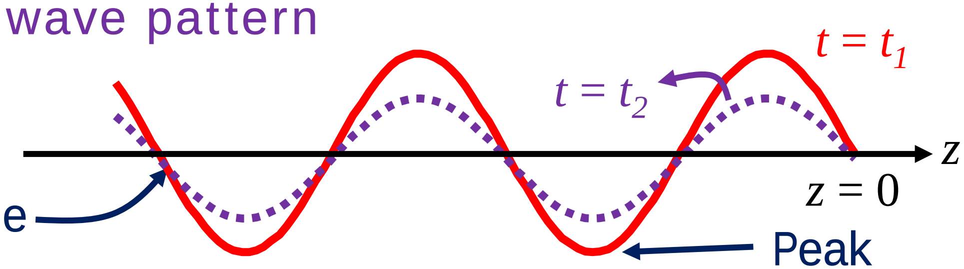
$$I(z) = \frac{1}{Z_0}\left[V^+ e^{-j\beta z} - V^- e^{j\beta z}\right] = \frac{2V^+}{Z_0}\cos\beta z$$

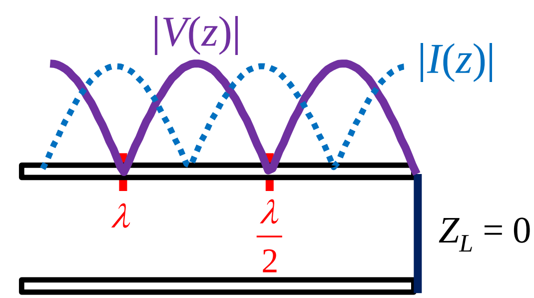
Associated time-domain waveform:

$$v(z,\, t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\} = 2\left|V^+\right| \boxed{\sin\beta z} \sin\omega t$$

$$i(z,\, t) = \mathrm{Re}\left\{I(z)e^{j\omega t}\right\} = 2\frac{\left|V^+\right|}{Z_0} \boxed{\cos\beta z} \cos\omega t$$

Standing wave pattern

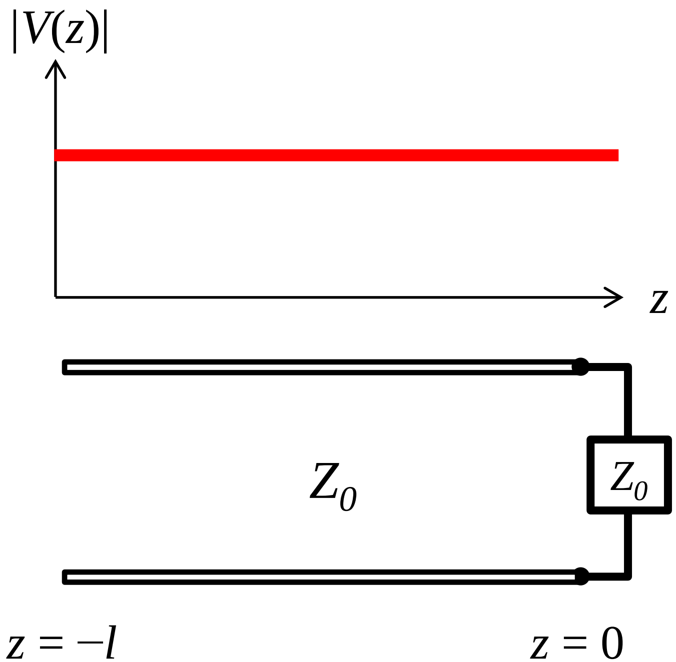


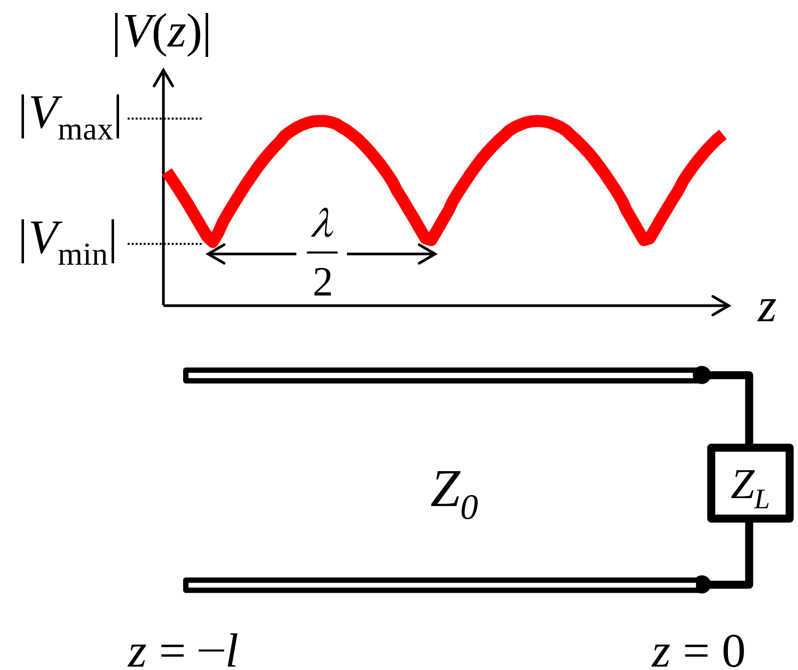
# Generalized Standing Wave

- A perfect standing wave only comes from $Z_L = 0$ or $Z_L = \infty$
- However, for a general load impedance $Z_L \neq Z_0$, the voltage $V(z)$ against $z$ also has similar phenomenon; it is called a generalized standing wave

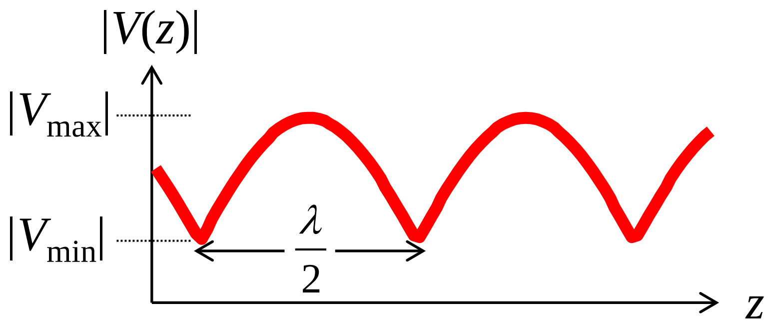
1. $Z_L = Z_0$




2. $Z_L \neq Z_0$



# What are the Extreme Values of $|V(z)|$?




## 1. Why does a standing wave exist?

- Recall $V(z) = V^{+}e^{-j\beta z}[1 + \Gamma(z)]$ and $\Gamma(z) = \Gamma_L e^{2j\beta z}$ where $\Gamma_L = \Gamma(0) = |\Gamma_L| e^{j\varphi_L}$
- The amplitude of $V(z)$:

$$|V(z)| = \sqrt{V(z)V^{*}(z)} = |V^{+}|\sqrt{[1+\Gamma(z)][1+\Gamma^{*}(z)]}$$

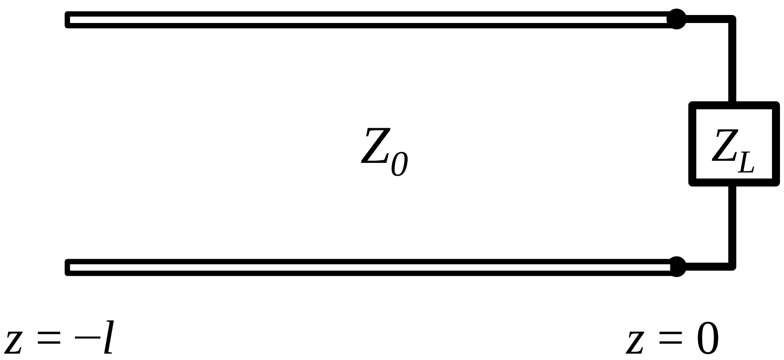
$$= |V^{+}|\sqrt{1+|\Gamma_L|^2 + 2|\Gamma_L|\cos(2\beta z + \varphi_L)}$$

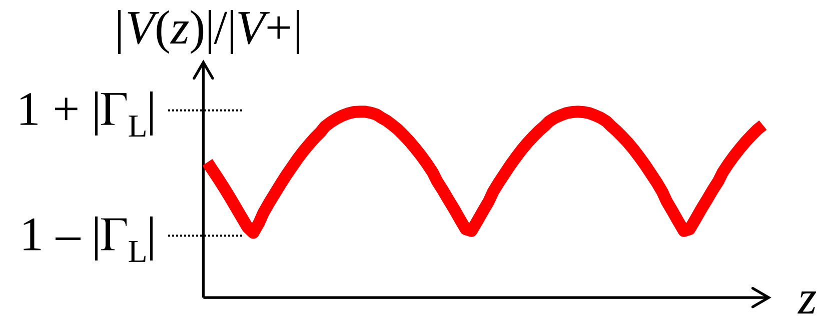
## 2. What are the extreme values of $|V(z)|$?

- $|V_{max}| = |V^{+}|\,(1 + |\Gamma_L|)$
- $|V_{min}| = |V^{+}|\,(1 - |\Gamma_L|)$

(Another perspective will be provided in §2.3)

# Voltage Standing Wave Ratio (VSWR)



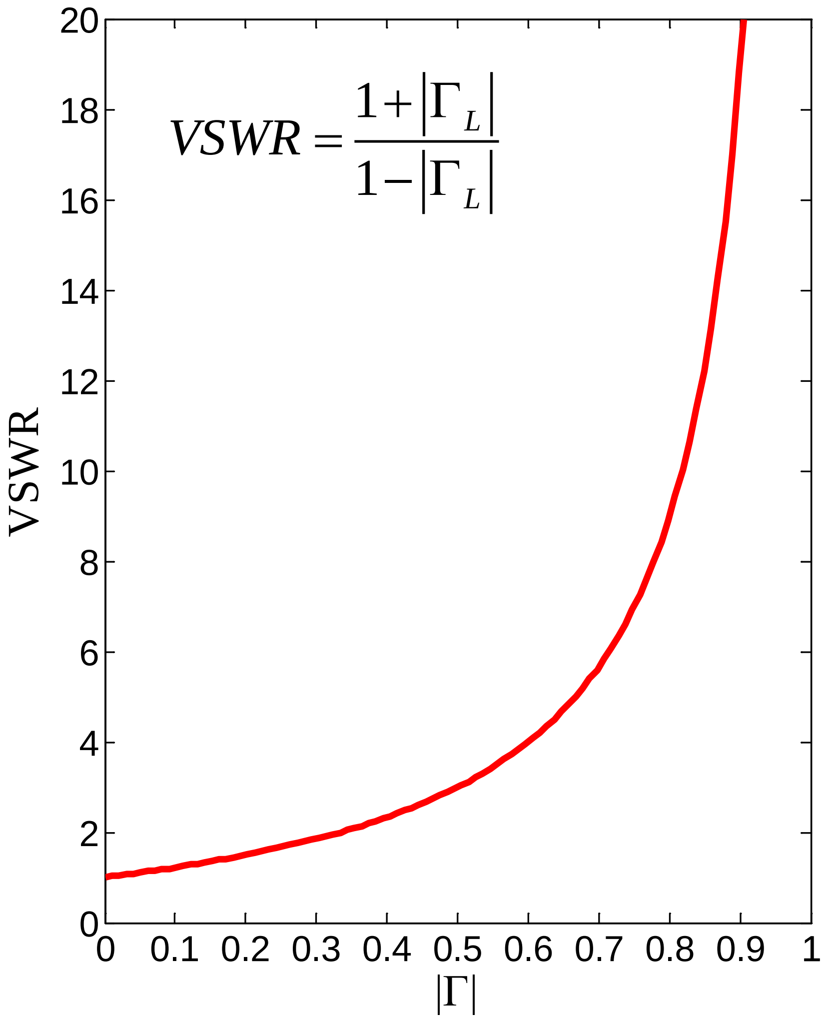

- To define a measure of the mismatch of the line, we use the **voltage standing wave ratio** defined as:

$$VSWR \triangleq \frac{\left|V(z)\right|_{\max}}{\left|V(z)\right|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

- VSWR is a real number ($1 \leq \text{VSWR} < \infty$)
  - Matched load: VSWR = 1
  - Short-circuited load or open-circuited load: VSWR $\rightarrow \infty$

# VSWR vs. Return Loss

## Return loss vs. VSWR:




- 4 standards are usually used in industrial applications:

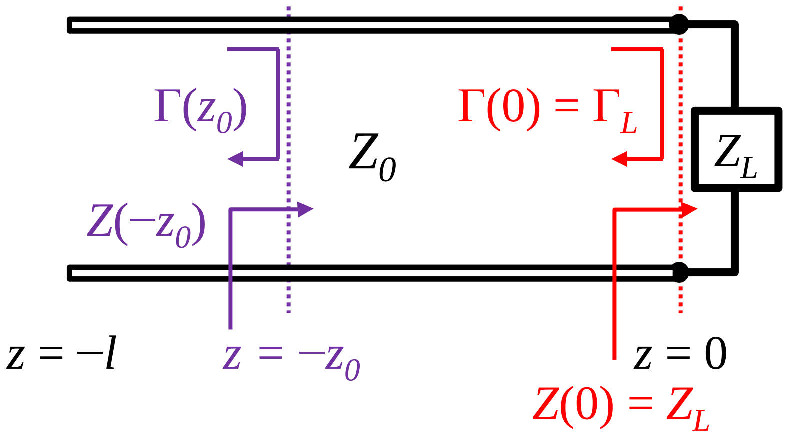
1. VSWR < 1.5:
   $|\Gamma_L| = 0.2$, RL = 14 dB
2. VSWR < 2:
   $|\Gamma_L| = 0.333$, RL = 10 dB (RL = 9.54 dB)
3. VSWR < 2.5:
   $|\Gamma_L| = 0.428$, RL = 7.4 dB
4. VSWR < 3:
   $|\Gamma_L| = 0.5$, RL = 6 dB



# Contents

## 2.3 Smith Chart

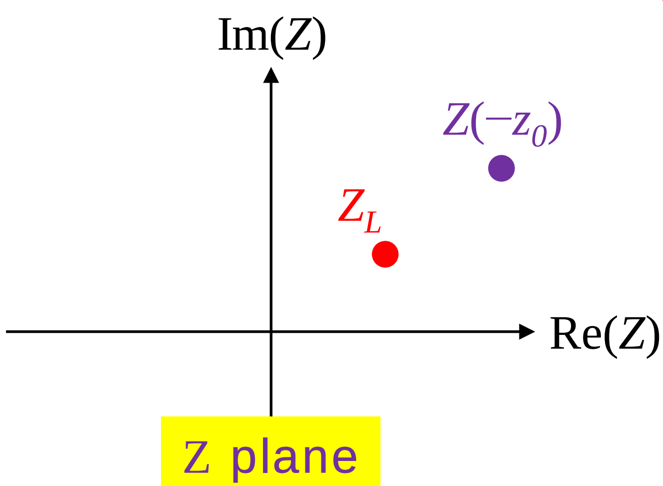
# $Z$ Plane vs. $\Gamma$ Plane

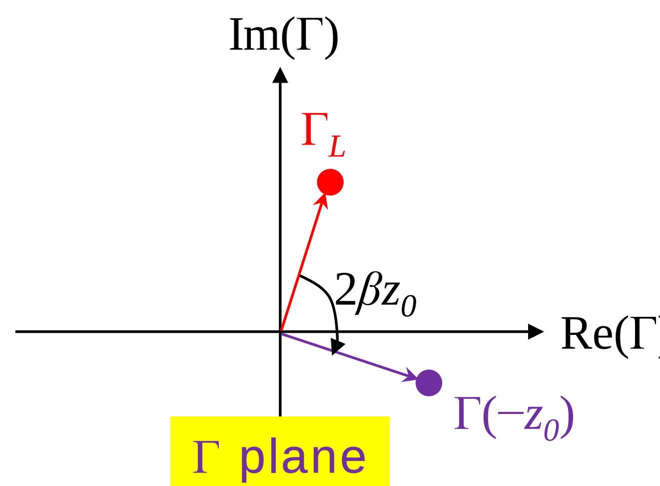



Recall that $\Gamma(z) \equiv \dfrac{V^{-}e^{j\beta z}}{V^{+}e^{-j\beta z}}$

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V^{+}e^{-j\beta z}[1+\Gamma(z)]}{\frac{V^{+}e^{-j\beta z}}{Z_0}[1-\Gamma(z)]} = Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

$$\therefore \Gamma(z) = \frac{Z(z)-Z_0}{Z(z)+Z_0}$$

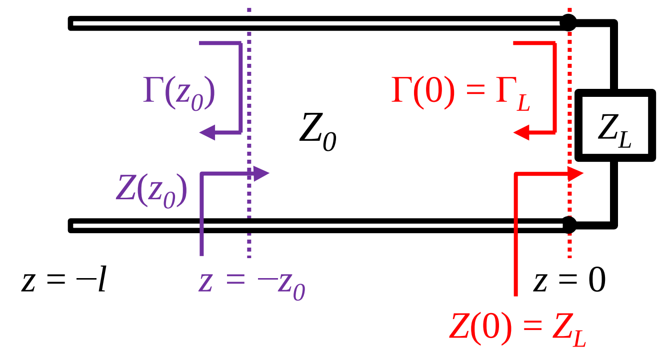


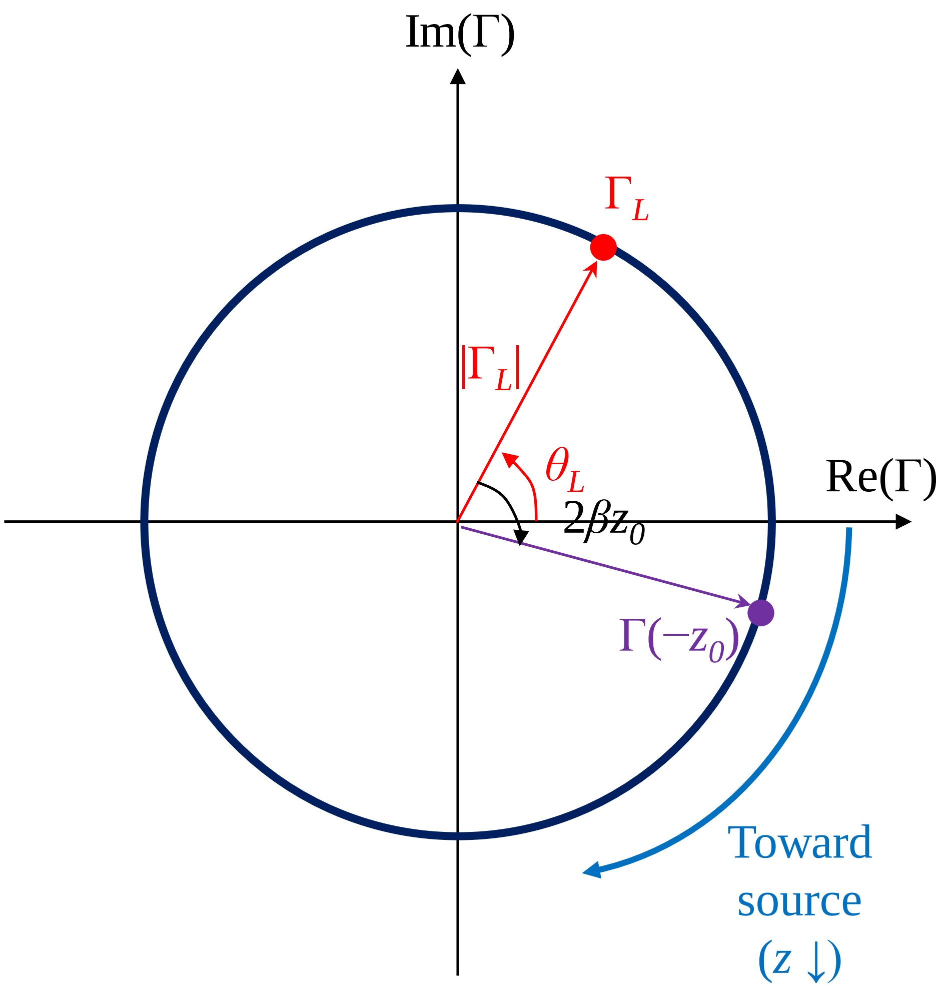




$$Z(z) = Z_0\left(\frac{Z_L + jZ_0\tan(-\beta z)}{Z_0 + jZ_L\tan(-\beta z)}\right)$$

$$\Gamma(z) = \Gamma_L e^{2j\beta z}$$

# Properties of Γ Plane




$$\Gamma(z) = \Gamma_L e^{2j\beta z}$$

Given arbitrary $Z_L$:

- We can compute $\Gamma_L$ by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta_L}$$

- The reflection coefficient at $z = -z_0$ is calculated by rotating the point ***clockwise*** an amount $2\beta z_0$ around the center
- When $z_0 = \lambda/2$, the rotation angle is $2\beta z_0 = 2\pi$, bringing the point back to its original position

# EX 2.1 Marking Impedances on $\Gamma$ Plane (1/2)

- A 50-ohm line is connected to the following load impedance respectively

  A. $Z_L = 50\ \Omega$

  B. $Z_L = 0\ \Omega$ (short circuit)

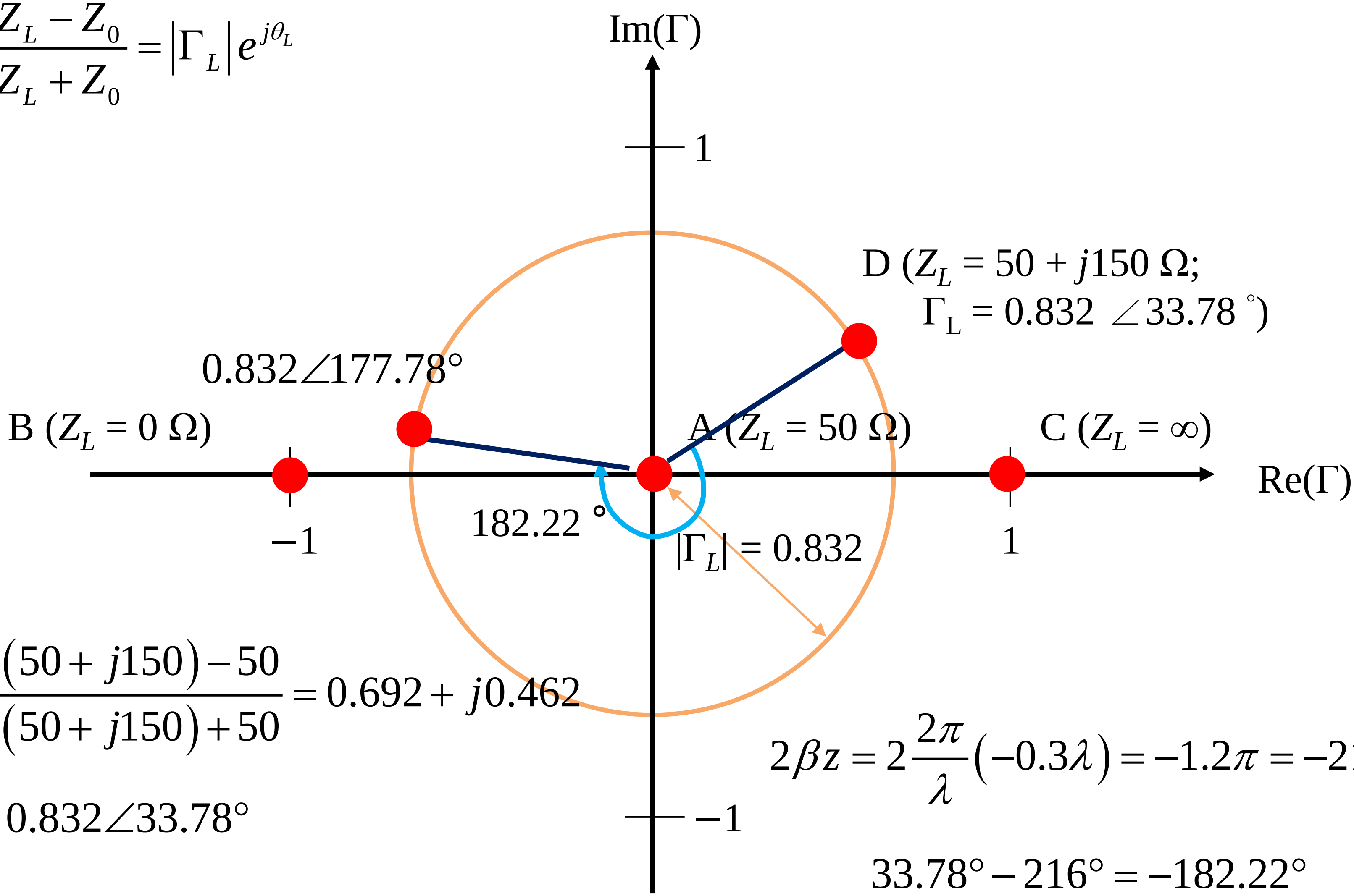
  C. $Z_L = \infty\ \Omega$ (open circuit)

  D. $Z_L = 50 + j150\ \Omega$

1. Calculate their reflection coefficients and mark them on $\Gamma$ plane
2. Calculate $\Gamma(z = -0.3\,\lambda)$ for case D

# Marking Impedances on Γ Plane (2/2)

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta_L}$$



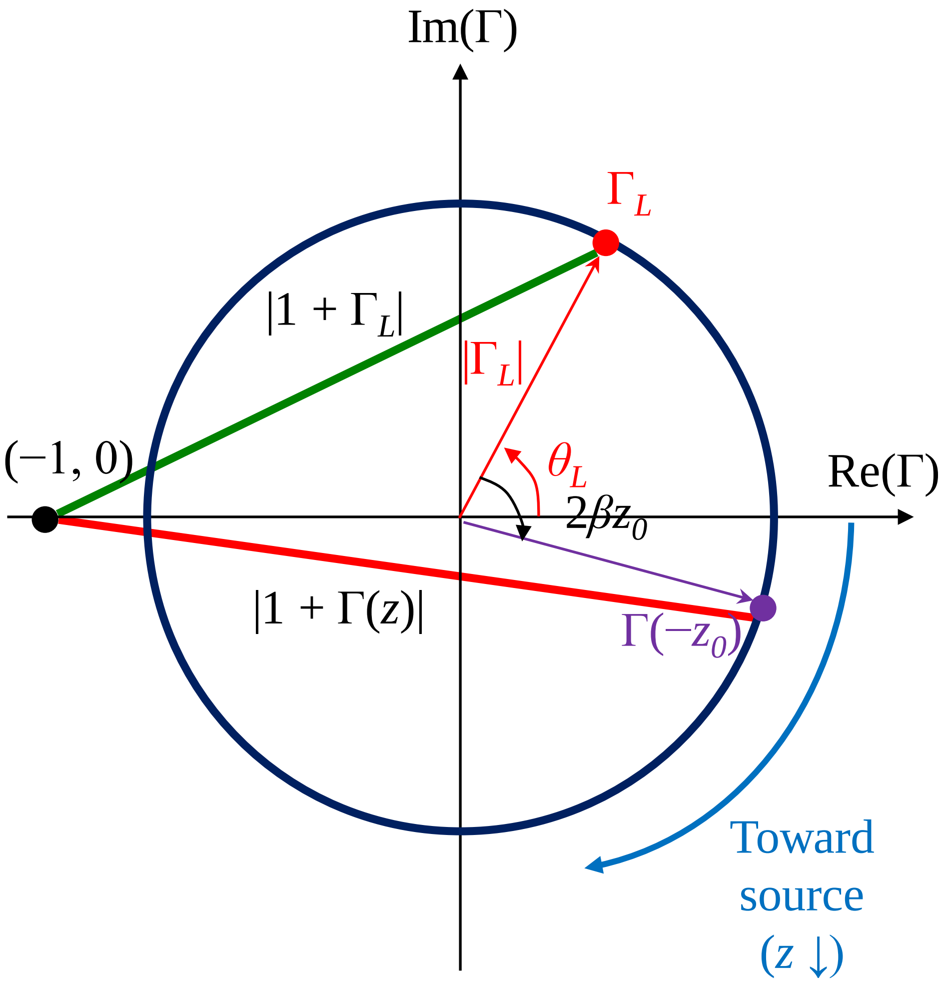

$$D : \Gamma_L = \frac{(50 + j150) - 50}{(50 + j150) + 50} = 0.692 + j0.462$$

$$\Gamma_L = 0.832\angle 33.78°$$

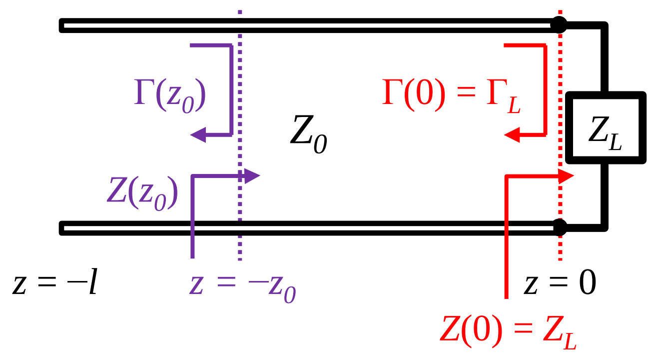
$$2\beta z = 2\frac{2\pi}{\lambda}(-0.3\lambda) = -1.2\pi = -216°$$

$$33.78° - 216° = -182.22°$$

# Amplitude of Voltage Wave




$$\Gamma(z) = \Gamma_L e^{2j\beta z}$$




$$\Gamma_L = |\Gamma_L| e^{j\theta_L}$$

The amplitude of voltage wave:

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} = V^+ e^{-j\beta z}[1 + \Gamma(z)]$$

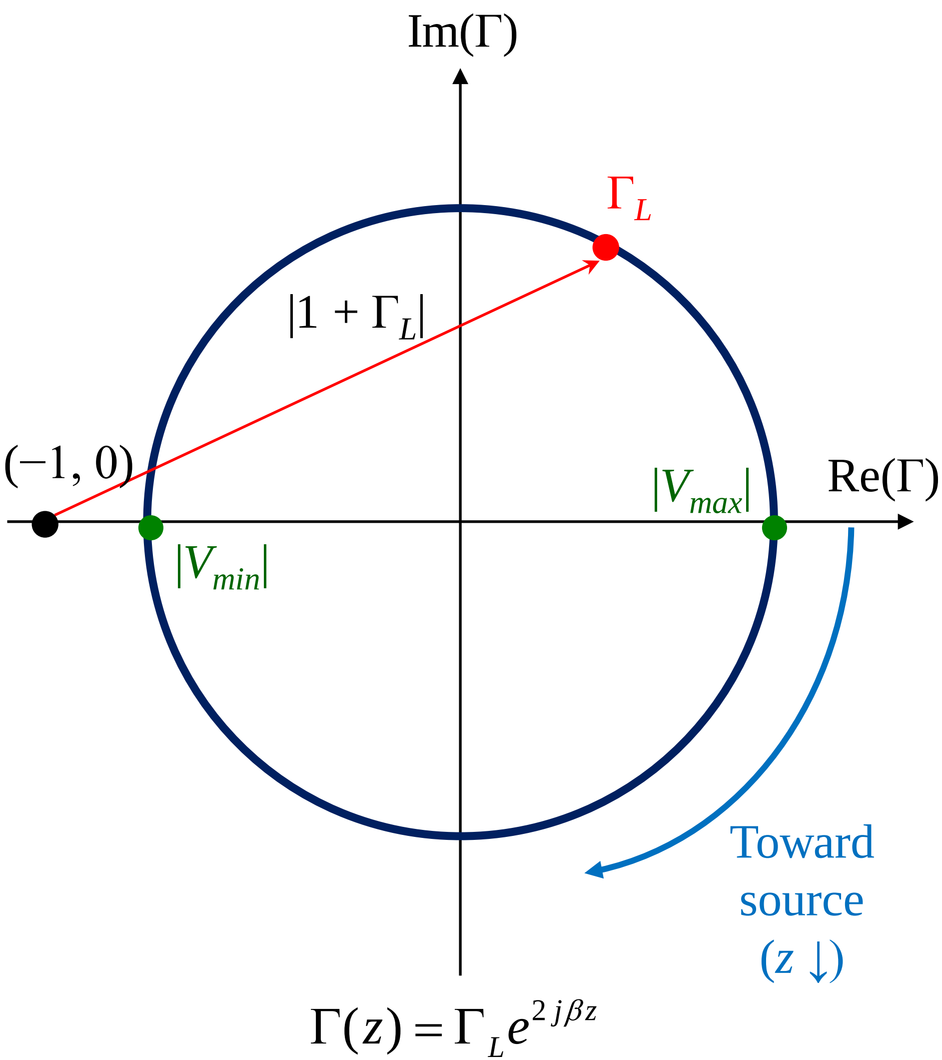
$$\Rightarrow \; |V(z)| = |V^+||1 + \Gamma(z)|$$

The amplitude of current wave:

$$I(z) = V^+ e^{-j\beta z}[1 - \Gamma(z)]/Z_0$$

$$\Rightarrow \; |I(z)| = |V^+/Z_0||1 - \Gamma(z)|$$

# Maximum Voltage and Minimum Voltage




- The maximum amplitude of voltage:

  $$|V_{\max}| = |V^+|\left(1+|\Gamma_L|\right)$$

  And now the current has its minimum value

  $$|I_{\min}| = \frac{|V^+|}{Z_0}\left(1-|\Gamma_L|\right)$$

- The minimum amplitude of voltage:

  $$|V_{\min}| = |V^+|\left(1-|\Gamma_L|\right)$$

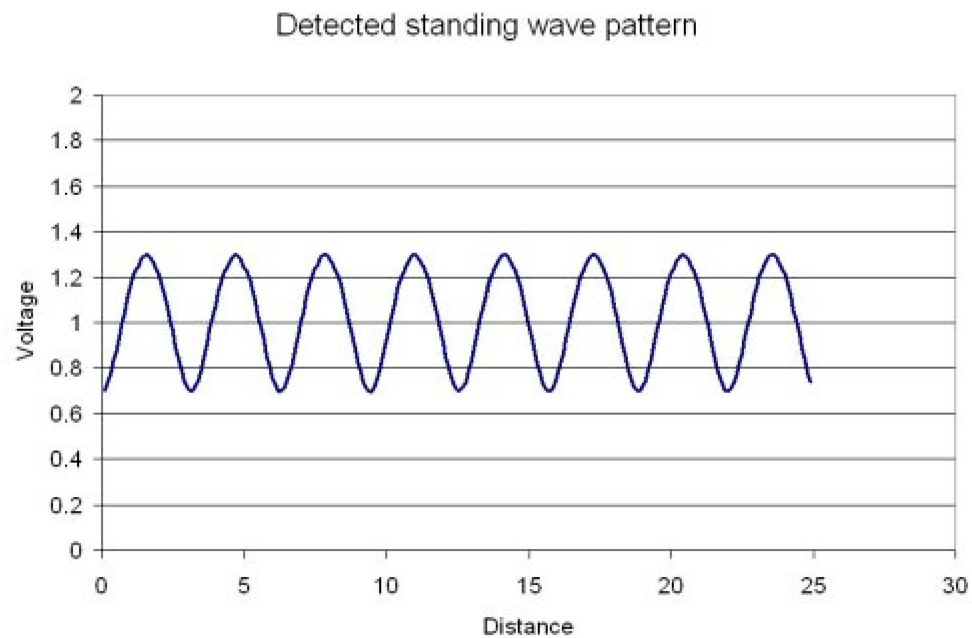
  And now the current has its maximum value

  $$|I_{\max}| = \frac{|V^+|}{Z_0}\left(1+|\Gamma_L|\right)$$

- $$VSWR = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

# Slotted Line Measurements

(C17452 General Radio 900-LB)




(Hewlett Packard C17540 HP 816A)

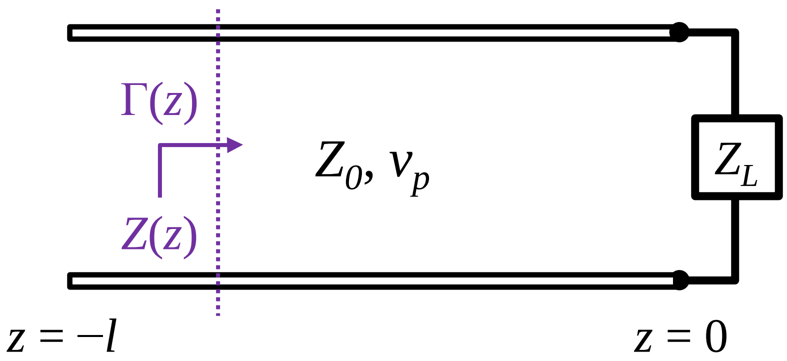
(HP 805A)

(Alford 3300)

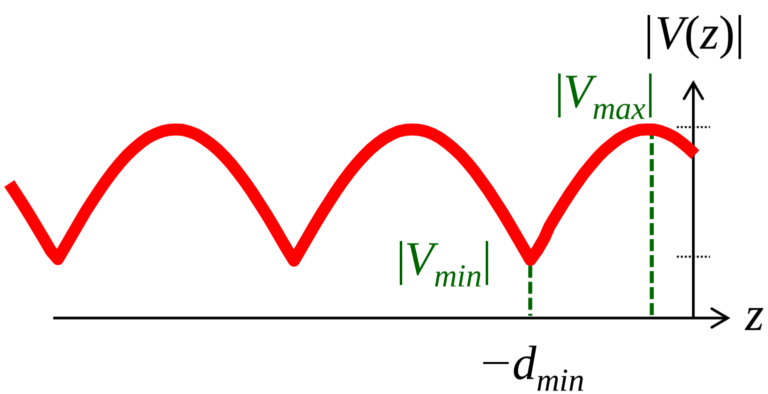


Slotted line measurements. [Online] Available: http://www.microwaves101.com/encyclopedias/545-slotted-line-measurements

## Determining an unknown load impedance by a slotted line:

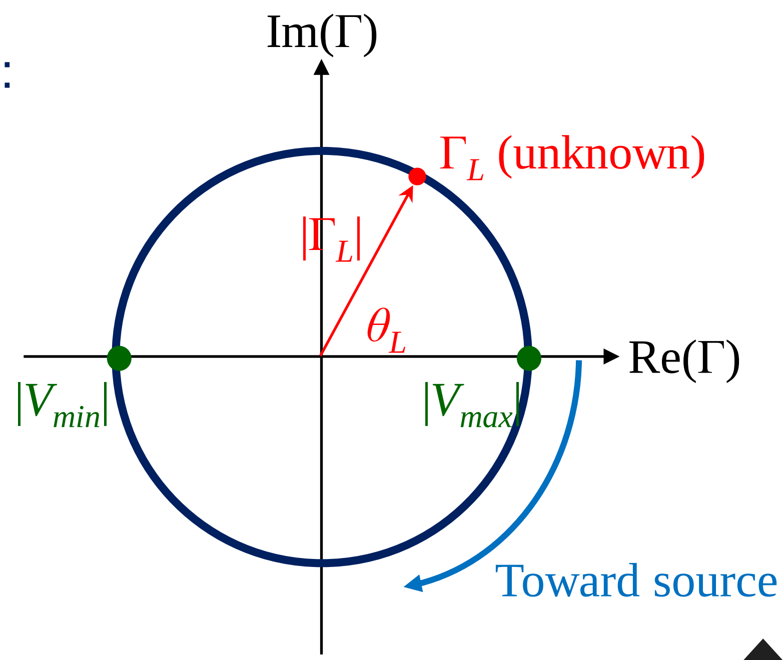



- A transmission line is connected to an unknown impedance
- Let's find this unknown $Z_L$ by measuring the voltage along the line

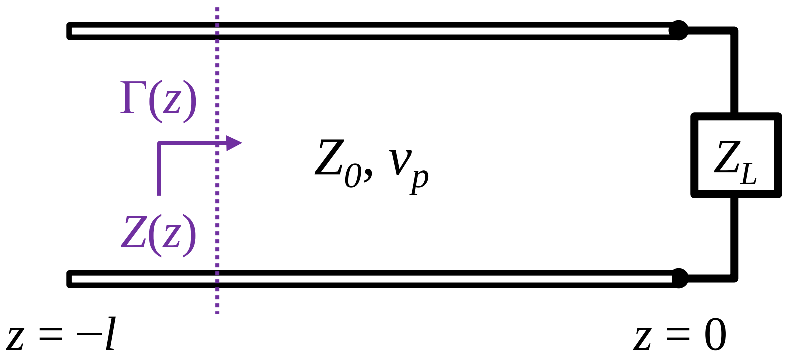


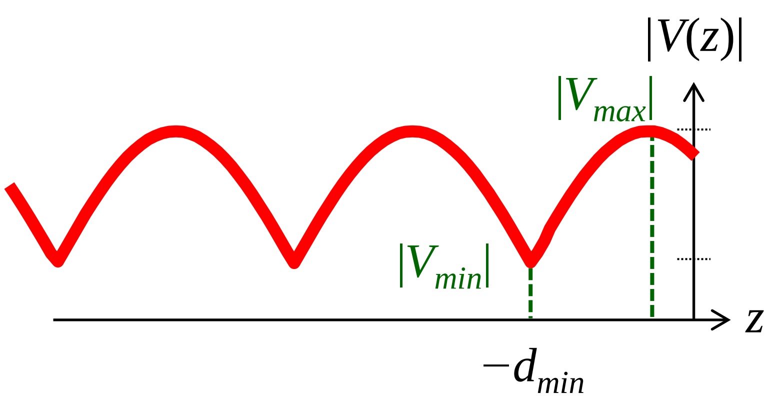

First $|V_{min}|$ occurs at $z = -d_{min}$

Step 1:



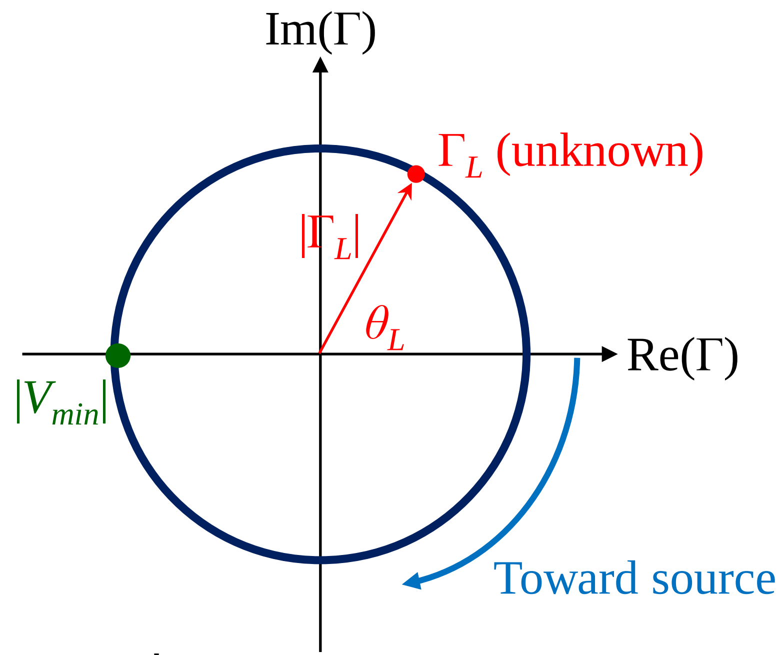
# Applications of Γ-Plane Properties (2/3)




First $|V_{min}|$ occurs at $z = -d_{min}$

Step 2:



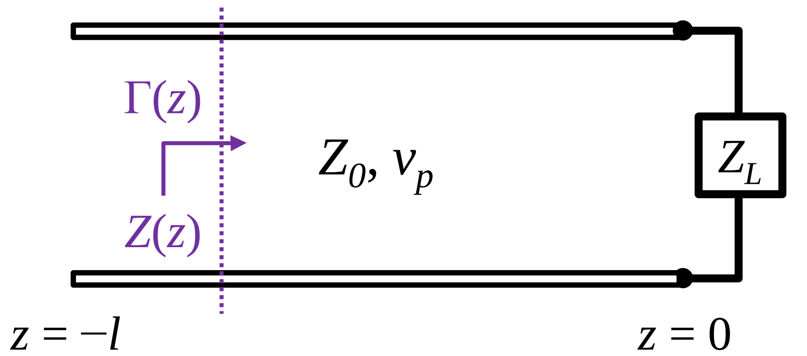

- Observe that

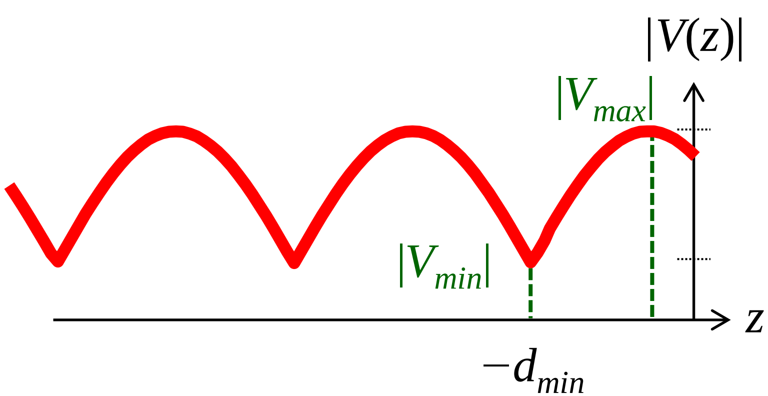
$$\theta_L + 2\beta\left(-d_{\min}\right) = -\pi \quad \left(\text{If } -\pi \le \theta_L \le \pi\right)$$

$$\Rightarrow \theta_L = \frac{4\pi d_{\min}}{\lambda} - \pi$$

- So we have the expression of $\theta_L$

# Applications of Γ-Plane Properties (3/3)




First $|V_{min}|$ occurs at $z = -d_{min}$

**Step 3:**

- How about $|\Gamma_L|$?
- From the measured $|V(z)|$:

$$VSWR = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

$$\Rightarrow \quad |\Gamma_L| = \frac{VSWR-1}{VSWR+1}$$

- Now we have $\Gamma_L = |\Gamma_L| e^{j\theta_L}$

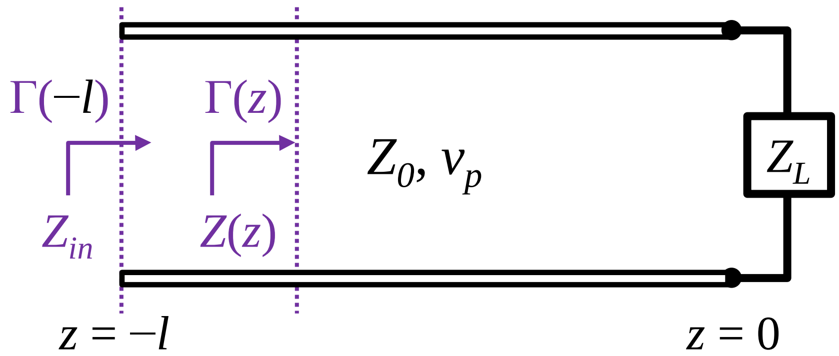
**Step 4:**

- Finally, the unknown impedance is:

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}$$

# Why Don't We Do the Entire System on $\Gamma$ Plane?

$$Z_{in} = Z_0 \left( \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right)$$




To avoid the explicit formula of $Z_{in}$, how do we find it on $\Gamma$ plane?

$$Z_L \to \Gamma_L \to \Gamma_{in} \to Z_{in}$$

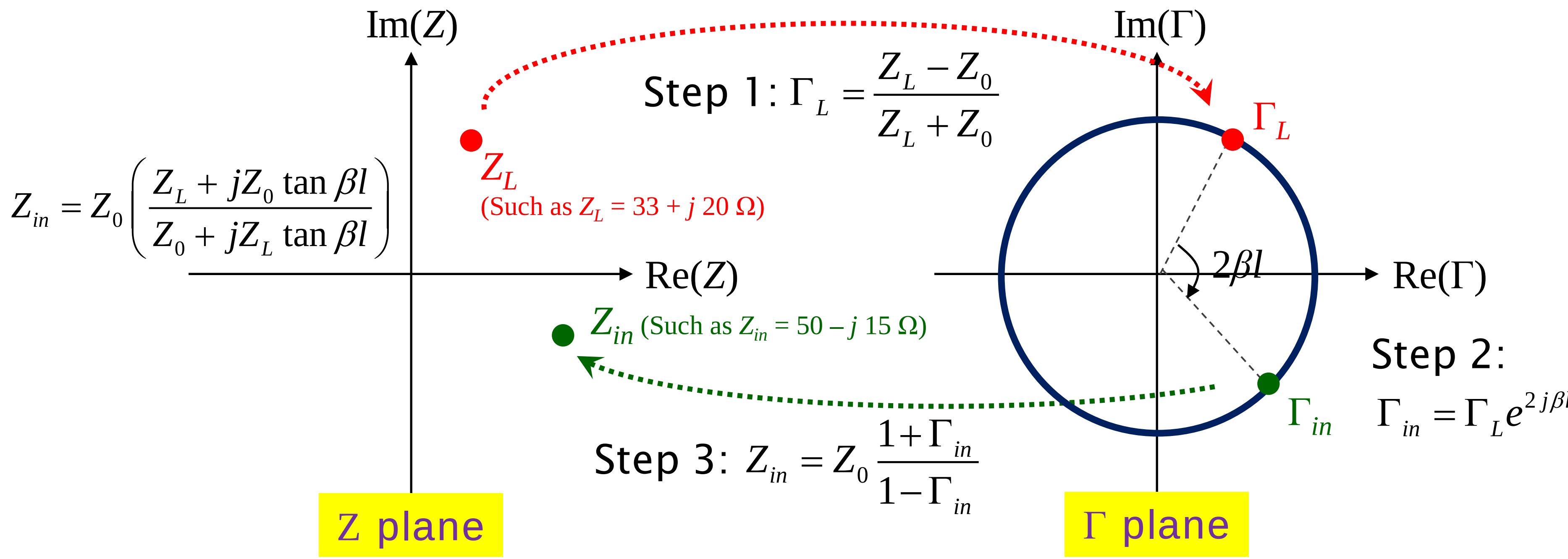
Step 1: $\Gamma_L$ results from a predetermined impedance value $Z_L$

Step 2: $\Gamma_{in}$ at $z = -l$ is found by *clockwise* rotating $\Gamma_L$ an angle $2\beta l$ around the center of $\Gamma$ plane

Step 3: Then we transform the value of $\Gamma_{in}$ to $Z_{in}$

Can we drop the procedure 1 and 3, directly finding impedance values on $\Gamma$ Plane?

# Concept of Smith Chart




Mapping Z plane to Γ plane → That's where Smith chart comes from!

- Removing the Z-plane part; directly working on Γ plane
- Smith chart marks ***every possible impedance value*** on Γ plane

Γ plane with Z values already attached on → Smith chart!

# Why Is Smith Chart So Smart?

- From the formulas of $Z$-$\Gamma$ transformation:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}, \; Z_{in} = Z_0 \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$$

- Different $Z_0$ result in different $\Gamma$ even if the same $Z_L$ is connected

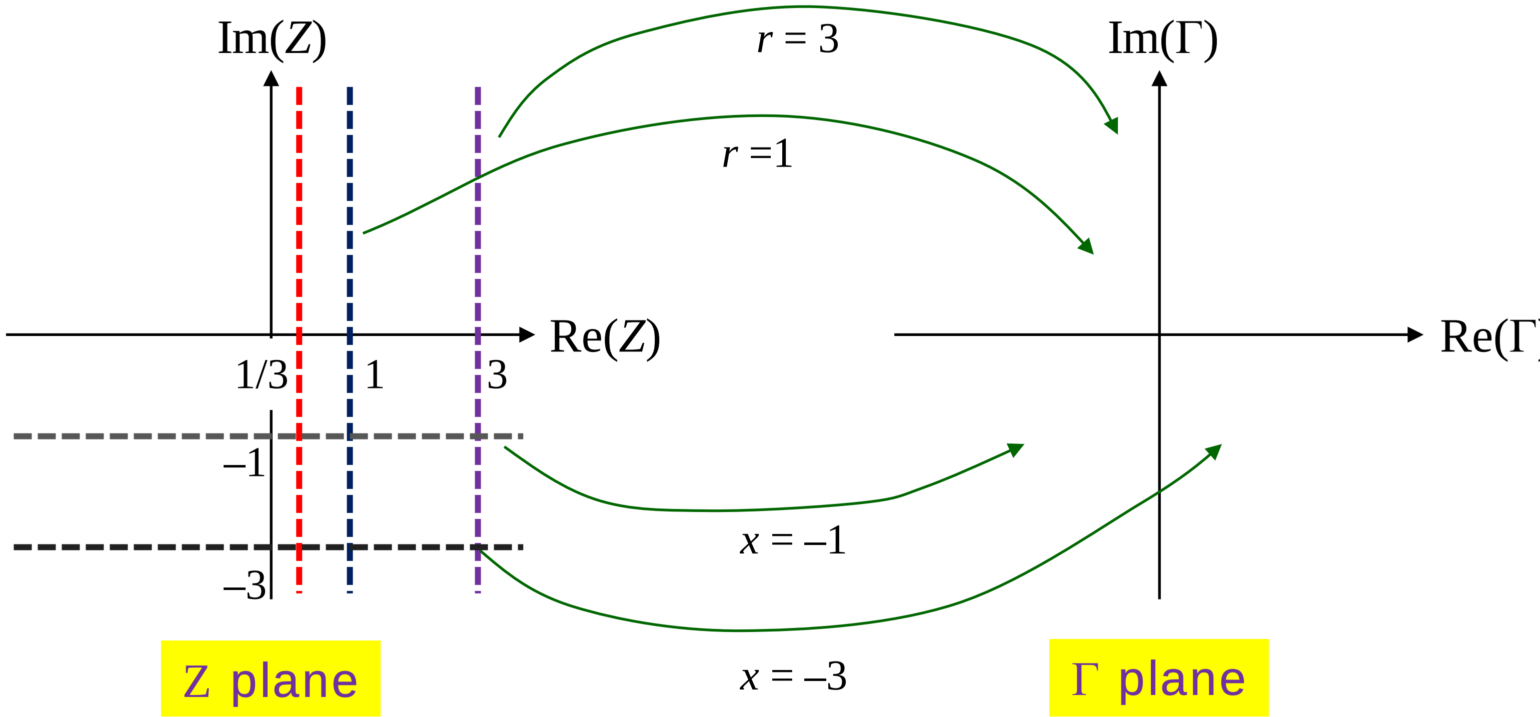
  - Normalizing the load impedance $Z_L$ to the characteristic impedance $Z_0$ of transmission line:

$$z_n = \frac{Z_L}{Z_0}$$

  - So, any point on $\Gamma$ plane has a unique mapping of $z_n$ (called normalized impedance)

$$\Gamma = \frac{z_n - 1}{z_n + 1}, \; z_n = \frac{1+\Gamma}{1-\Gamma}$$

# Normalized Impedances on Γ Plane (1/4)




- Let $z_n = r + jx$
- Idea: Transforming $r = r_0$ and $x = x_0$ to Γ plane and observing their new expression

$$\Gamma = \frac{z_n - 1}{z_n + 1}, \ z_n = \frac{1+\Gamma}{1-\Gamma}$$

# Normalized Impedances on Γ Plane (2/4)

How to mark normalized impedances on Γ plane?

- Let $z_n = r + jx$
- The real part and imaginary part of Γ plane: $\Gamma = u + jv$

$$\text{From } z_n = r + jx = \frac{1+\Gamma}{1-\Gamma} = \frac{1+(u+jv)}{1-(u+jv)}$$

$$\Rightarrow \quad r + jx = \frac{1+u+jv\,[1-u+jv]}{1-u-jv\,[1-u+jv]} = \frac{\left(1-u^2-v^2\right)+j(2v)}{(1-u)^2+v^2}$$

- Comparing the real part and the imaginary part:

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}, \quad x = \frac{2v}{(1-u)^2+v^2}$$

- Then we can rearrange them into:

$$\Rightarrow \quad \left(u-\frac{r}{r+1}\right)^2 + v^2 = \left(\frac{1}{r+1}\right)^2$$

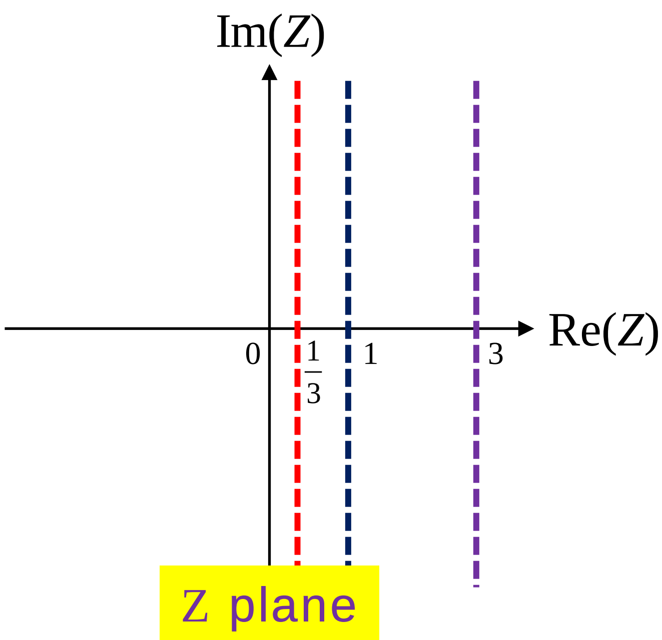
$$(u-1)^2 + \left(v-\frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

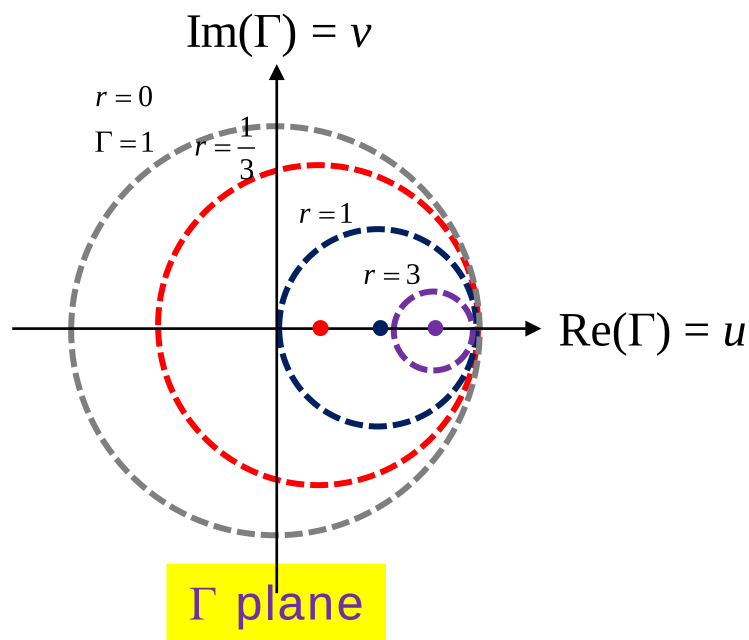
# Normalized Impedances on Γ Plane (3/4)

$$\left(u-\frac{r}{r+1}\right)^2+v^2=\left(\frac{1}{r+1}\right)^2$$

- For any given $x$, $r = r_0$ on $Z$ plane, it generates a new expression on $\Gamma$ plane
- The new expression is a circle with

$$\text{Center: } \left(\frac{r}{r+1},0\right),\ \text{Radius: } \frac{1}{r+1}$$



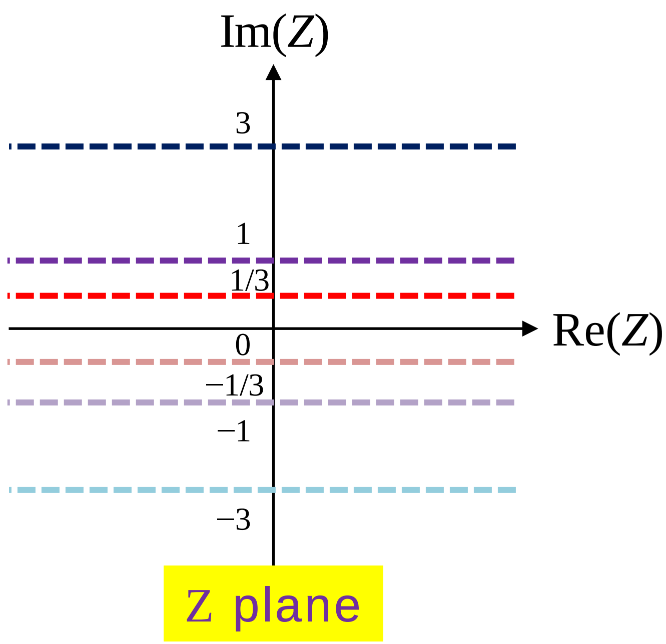

Called "constant $r$ circle" or "constant resistance circle"

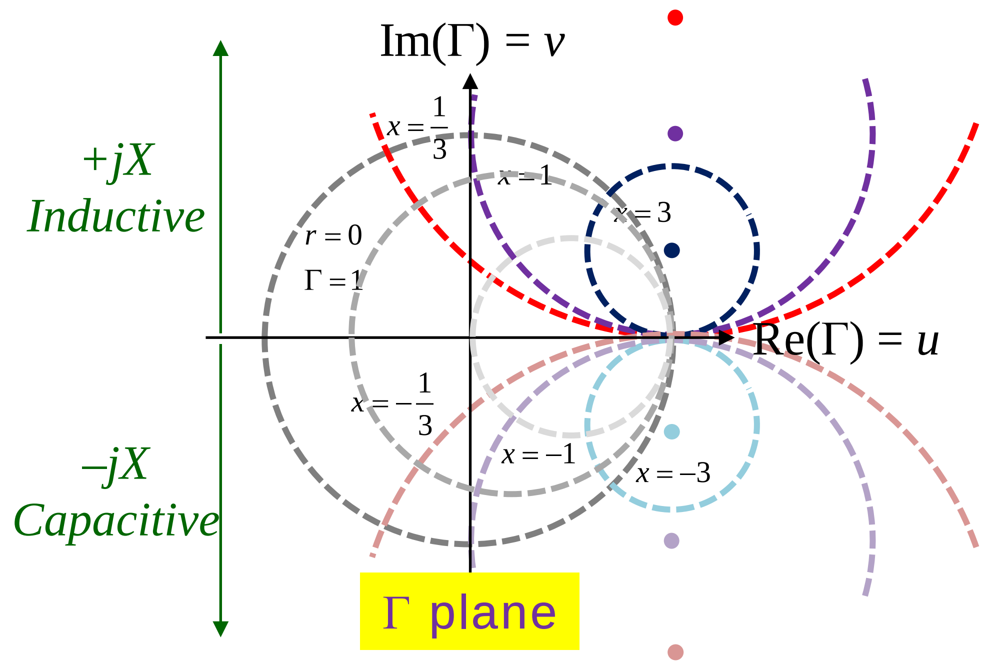
# Normalized Impedances on Γ Plane (4/4)

$$(u-1)^2+\left(v-\frac{1}{x}\right)^2=\left(\frac{1}{x}\right)^2$$

- For any given $r$, $x = x_0$ on $Z$ plane, it generates a new expression on Γ plane
- The new expression is a circle with

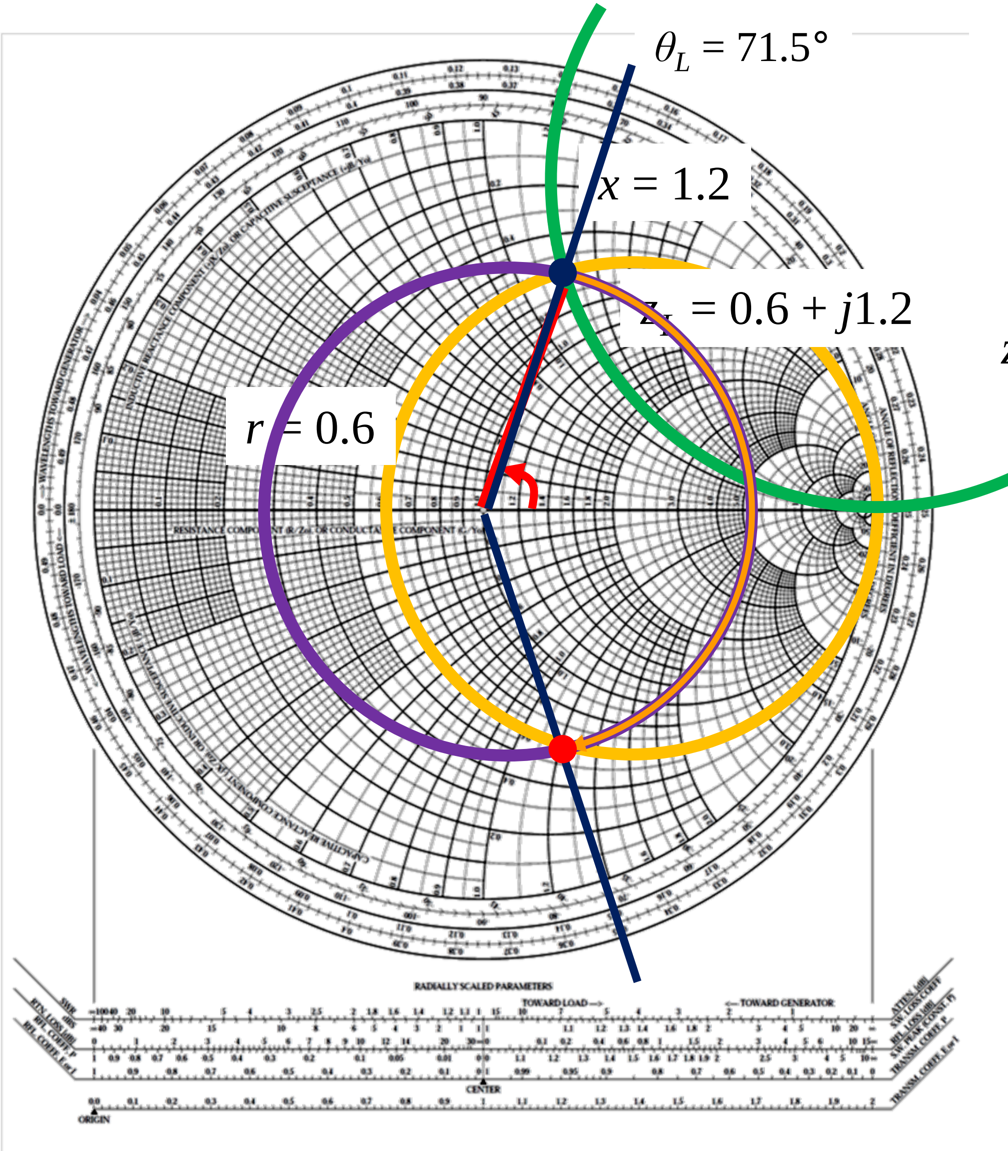
$$\text{Center: } \left(1,\frac{1}{x}\right),\ \text{Radius: } \frac{1}{|x|}$$

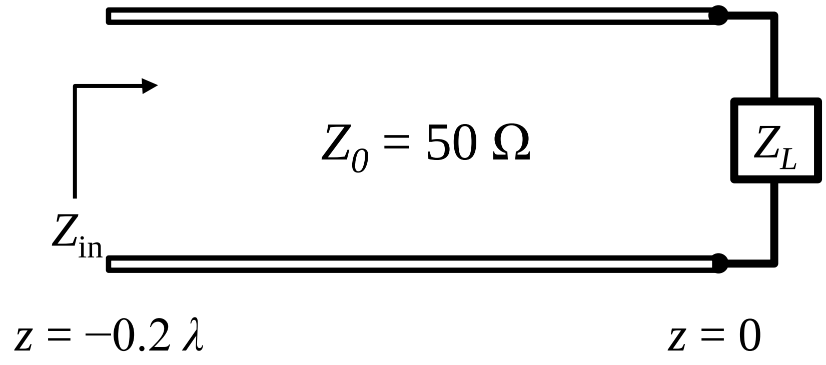



Called "constant $x$ circle" or "constant reactance circle"

# How to Read Impedances From Smith Chart?

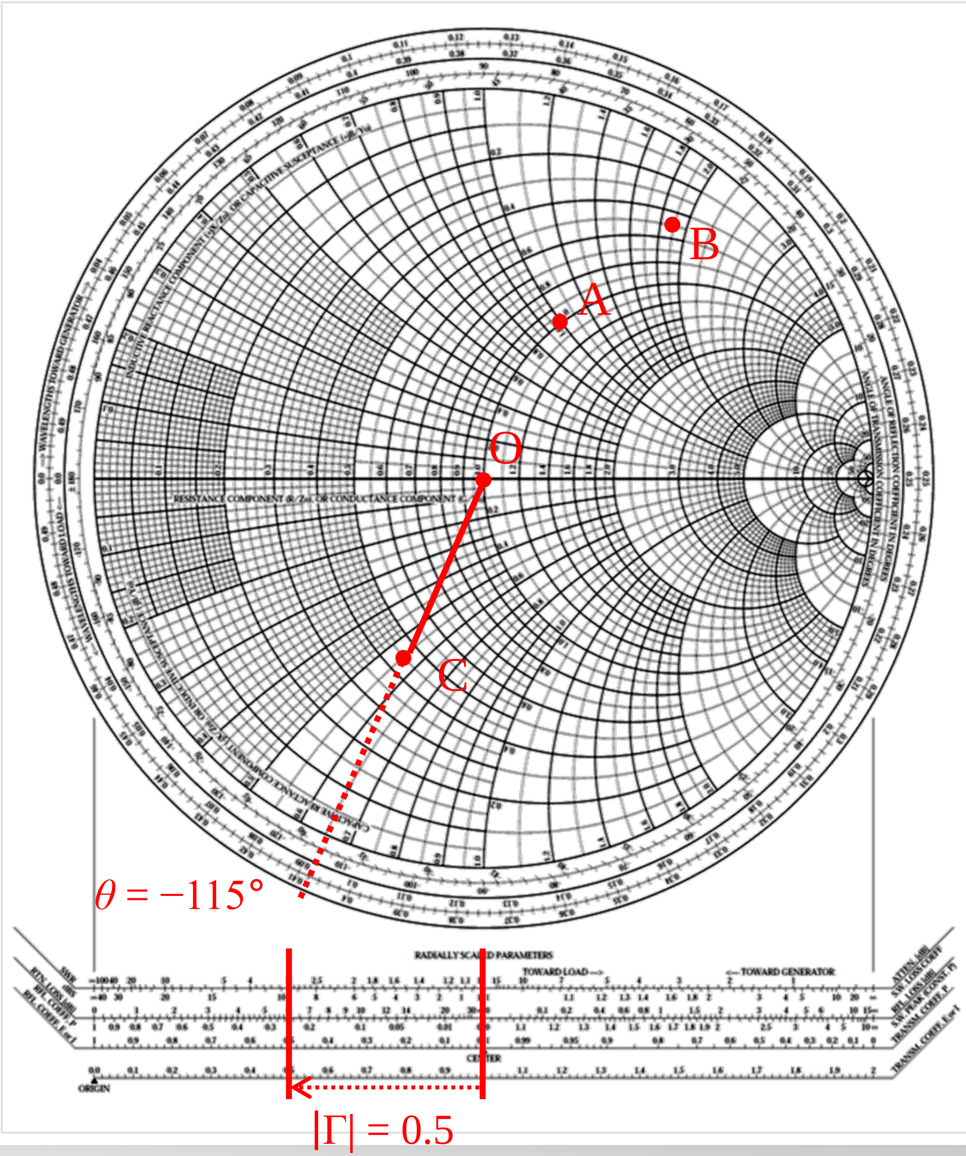


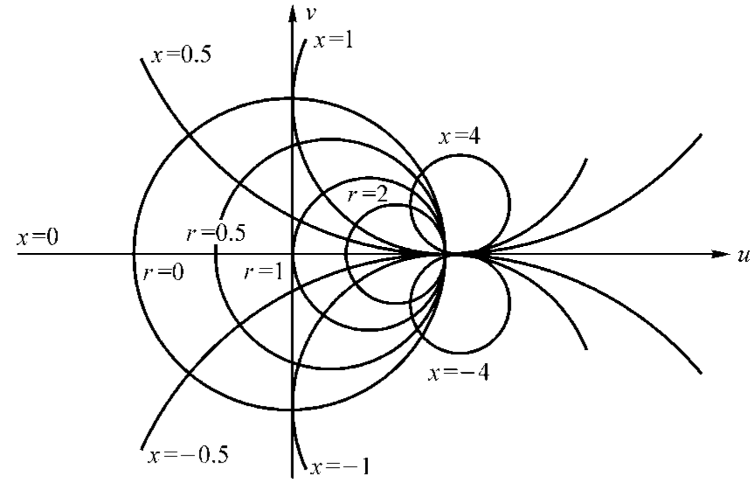




Example: $Z_L = 30 + j60\ \Omega$ ($Z_0 = 50\ \Omega$ )

1. Normalized $Z_L$ to $Z_0$: $z_L = 0.6 + j1.2$
2. $r = 0.6$
3. $x = 1.2$
4. The intersection is $z_L$
5. $|\Gamma|$: The distance between $z_L$ and the origin
6. $\theta_L$: The angle between $z_L$ and the Re($\Gamma$)
7. $\Gamma = |\Gamma|(\cos\theta_L + j\sin\theta_L)$
8. The coefficients along the transmission line: constant $|\Gamma|$ circle
9. $\Gamma(z = -0.2\ \lambda)$: $2\beta l = 0.8\ \pi = 144°$
10. $z_{in} = 0.6 - j1.2$ $\therefore$ $Z_{in} = Z_0 z_{in} = 30 - j60\ \Omega$

# Smith Chart (1/2)



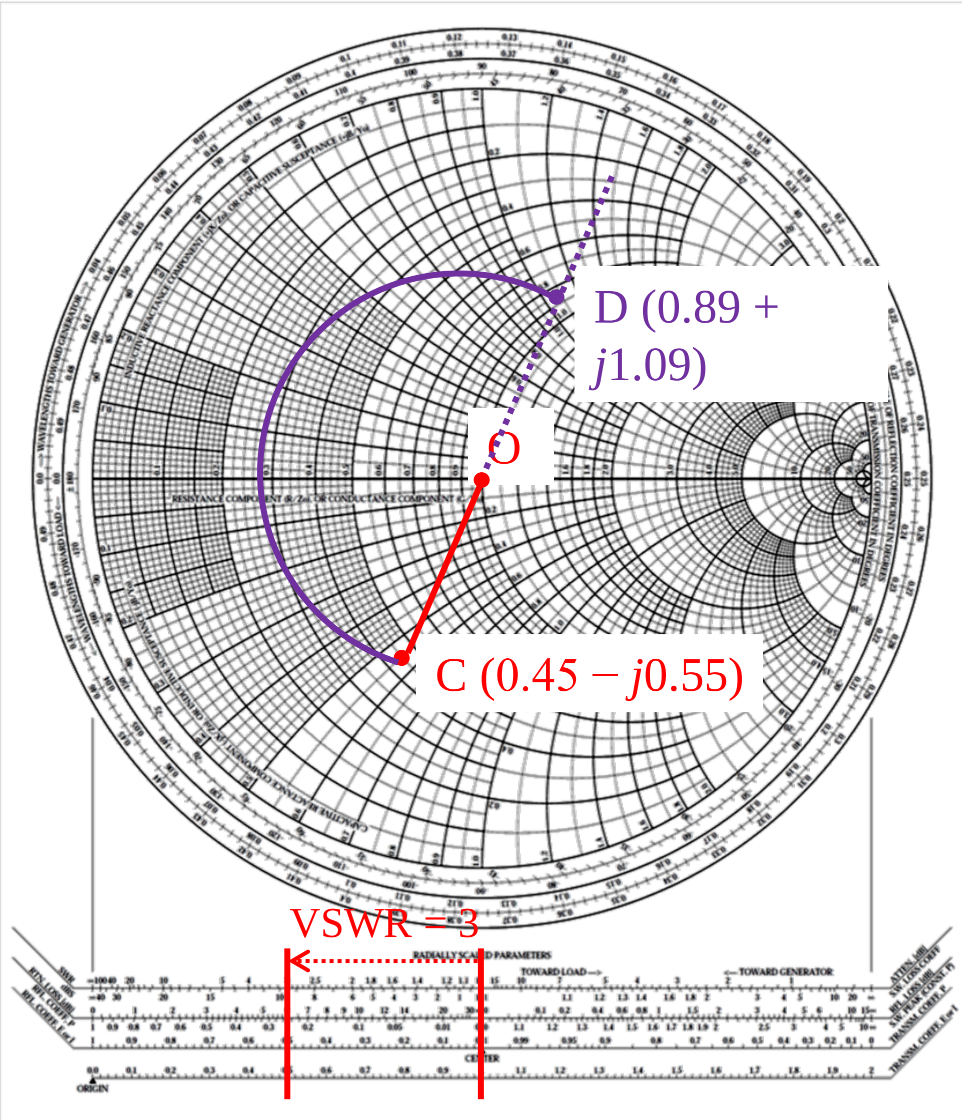




Only the curves inside $|\Gamma| \leq 1$ are needed

1. Normalized impedance $z_n = r_0 + jx_0$ on $\Gamma$ plane: the intersection of constant $r = r_0$ circle and constant $x = x_0$ circle
   - Such as O $(1 + j0)$, A $(1 + j1)$, B $(0.5 + j1.9)$, and C $(0.45 - j0.55)$
2. Reflection coefficient $\Gamma$: $|\Gamma|$ is the distance from $z_n$ to the origin O; the associated phase is read from the angular indication on the circumference of the chart
   - Such as C $(0.45 - j0.55)$: $\Gamma = 0.5\angle -115°$

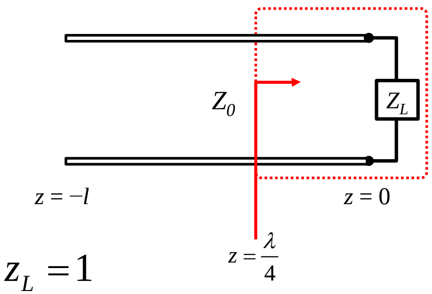
[Online]. Available: http://www.sss-mag.com/smith.html

# Smith Chart (2/2)




3. VSWR: VSWR results from the mapping of $|\Gamma|$; therefore, the VSWR value can be read from the SWR scale
   - Such as C $(0.45 - j0.55)$: VSWR $= 3$
4. Normalized admittance on $\Gamma$ plane: Recall that when the length of the TL is $\lambda/4$:

$$Z\left(z = \frac{\lambda}{4}\right) = \frac{Z_0^2}{Z_L}$$



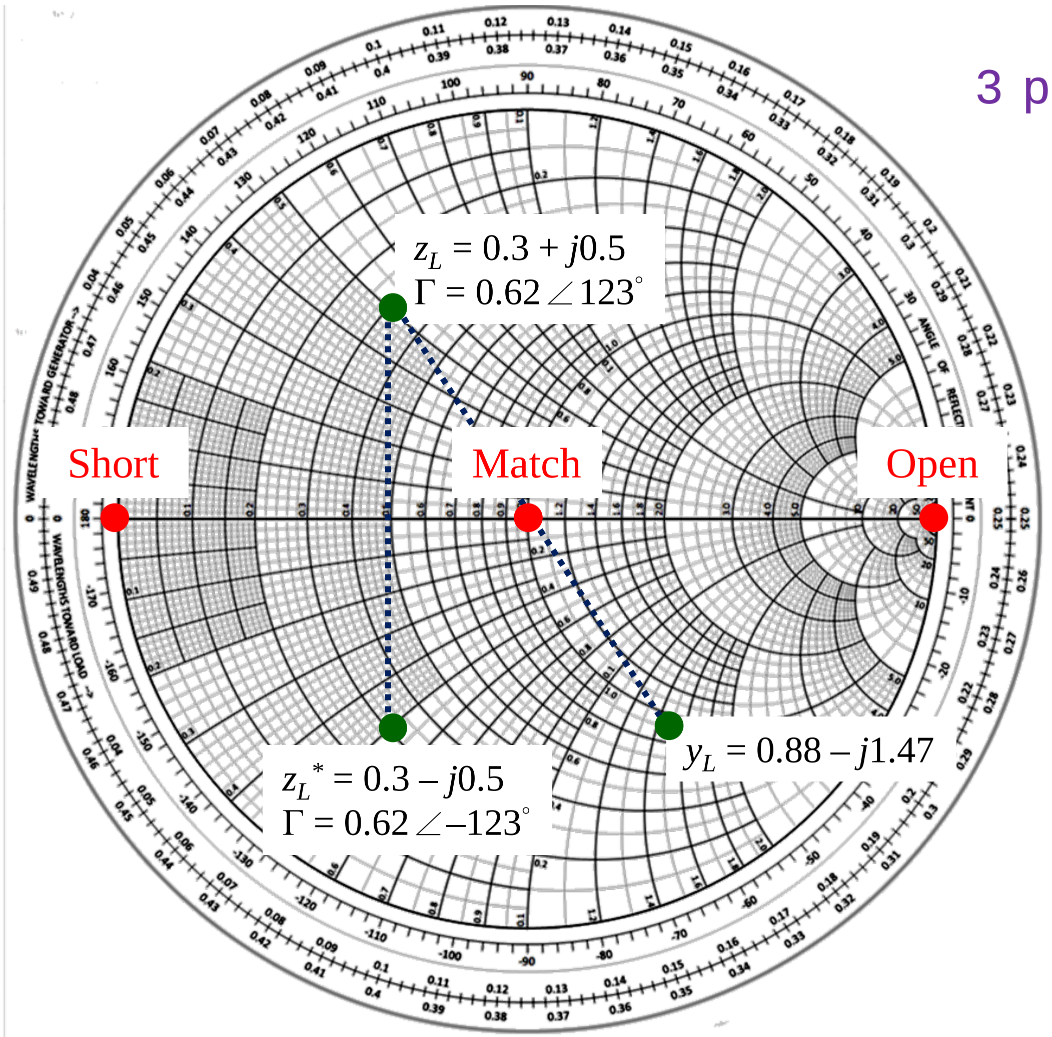

$$\Rightarrow z_n\left(z = \frac{\lambda}{4}\right) \times z_L = 1$$

$$\Rightarrow y_L = z_n\left(z = \frac{\lambda}{4}\right)$$

The normalized admittance is the same as the normalized impedance at a distance $\lambda/4$ from it

- Such as the normalized admittance of C: D $(0.89 + j1.09)$
- Admittance: $Y_0 \times y_L$ (where $Y_0 = 1/Z_0$ )

# Landmark Points

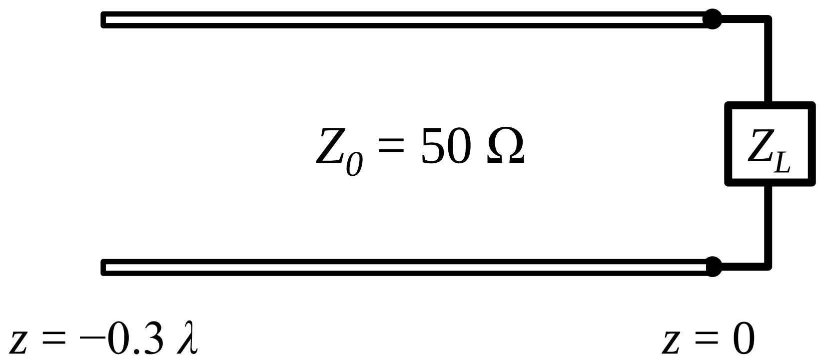



3 points you must keep in mind:

- Open circuit point $Z_L = \infty$
- Short circuit point $Z_L = 0$
- Match point $Z_L = Z_0$ $(z_L = 1)$

- Given $z_L$, $y_L$ is found by the $180^\circ$ rotation from $z_L$
- The conjugate value of $z_L$, namely, $z_L^*$, is found by the reflection in the real axis ($u$)

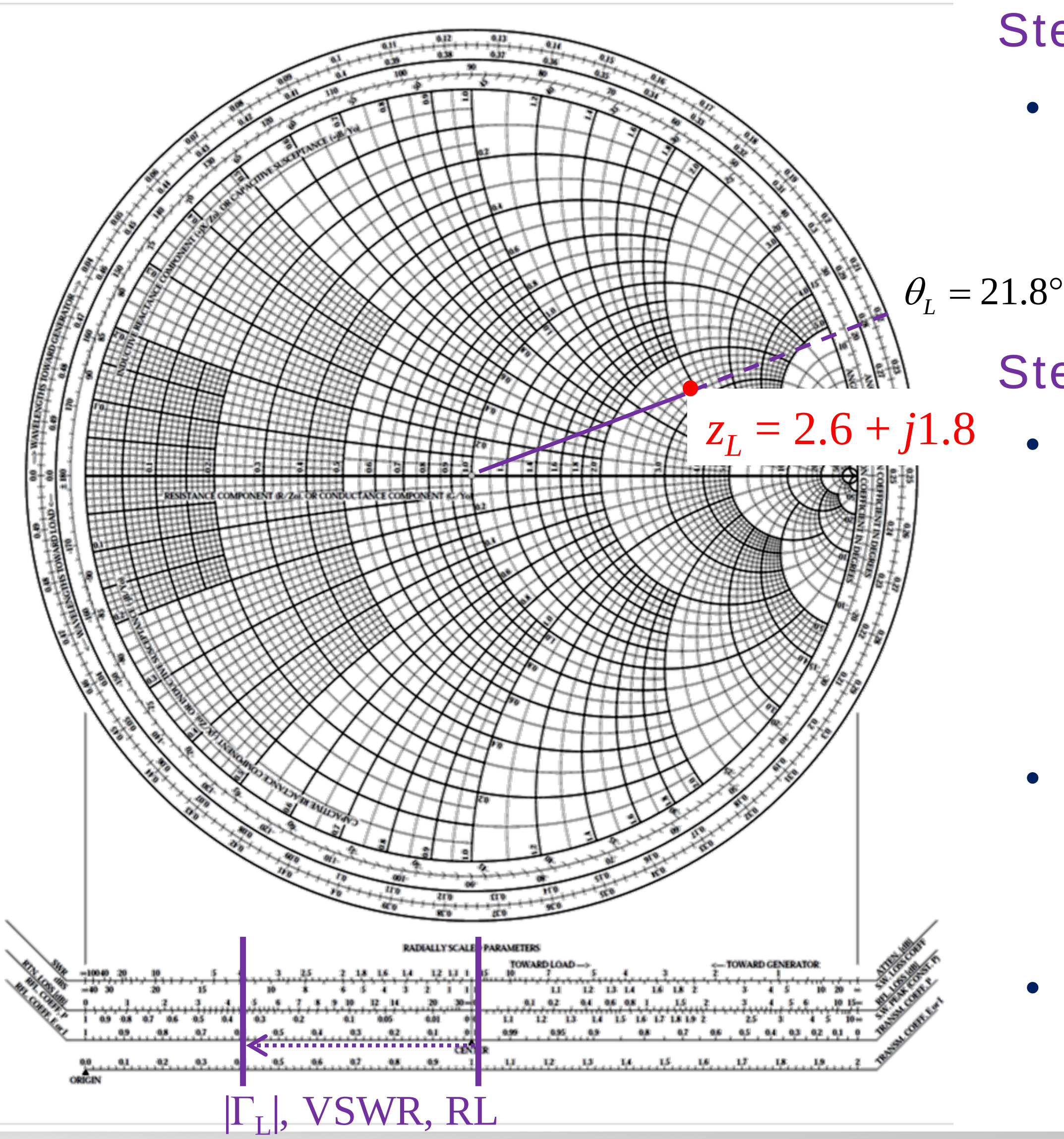
# Basic Procedures of Smith Chart (1/4)




- A load impedance of $130 + j90\ \Omega$ terminates a $50\ \Omega$ transmission line that is $0.3\ \lambda$ long

1. Find the reflection coefficient at the load
2. Find the reflection coefficient at the input to the line
3. Find the VSWR on the line
4. Find the return loss
5. Find the impedance seen at the input to the line
6. Find the admittance seen at the input to the line

# Basic Procedures of Smith Chart (2/4)




## Step 1:

- Firstly, calculate the normalized load impedance $z_L$:

$$z_L = \frac{Z_L}{Z_0} = \frac{130 + j90}{50} = 2.6 + j1.8$$

## Step 2:

- The parameters on the load can be easily read from the chart

$$|\Gamma_L| = 0.6$$
$$VSWR = 3.98$$
$$RL = 4.4 \text{ dB}$$

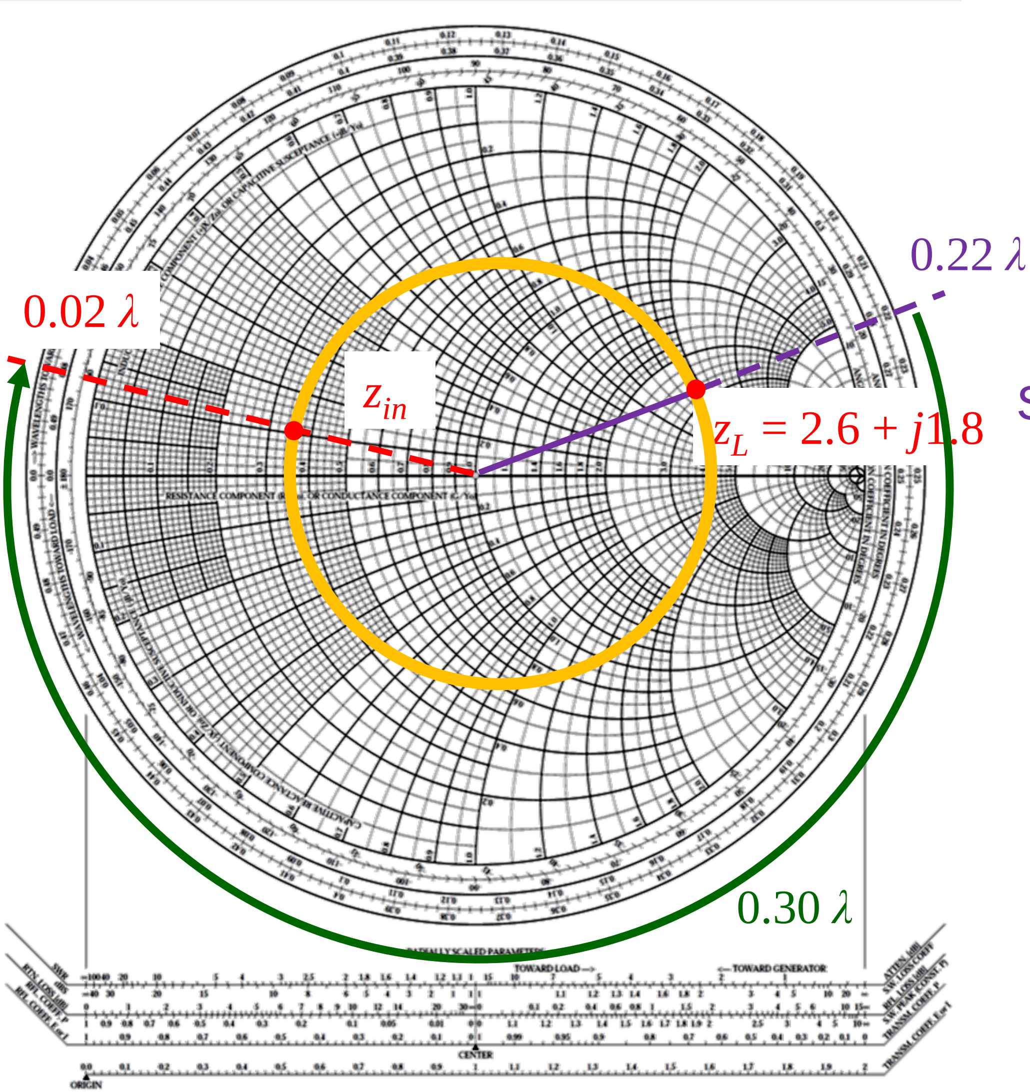
- The phase angle of $\Gamma_L$ can be read from the outer scale of the chart

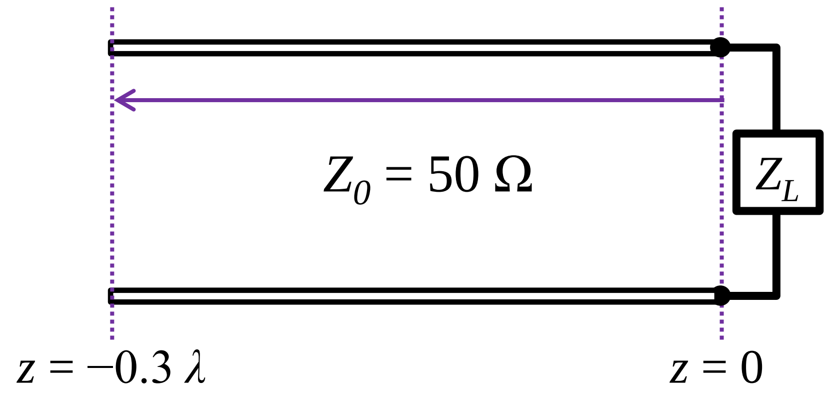
$$\theta_L = 21.8°$$

- So, the reflection coefficient:

$$\Gamma_L = 0.6\angle 21.8°$$

# EX 2.2 Basic Procedures of Smith Chart (3/4)




## Step 3:

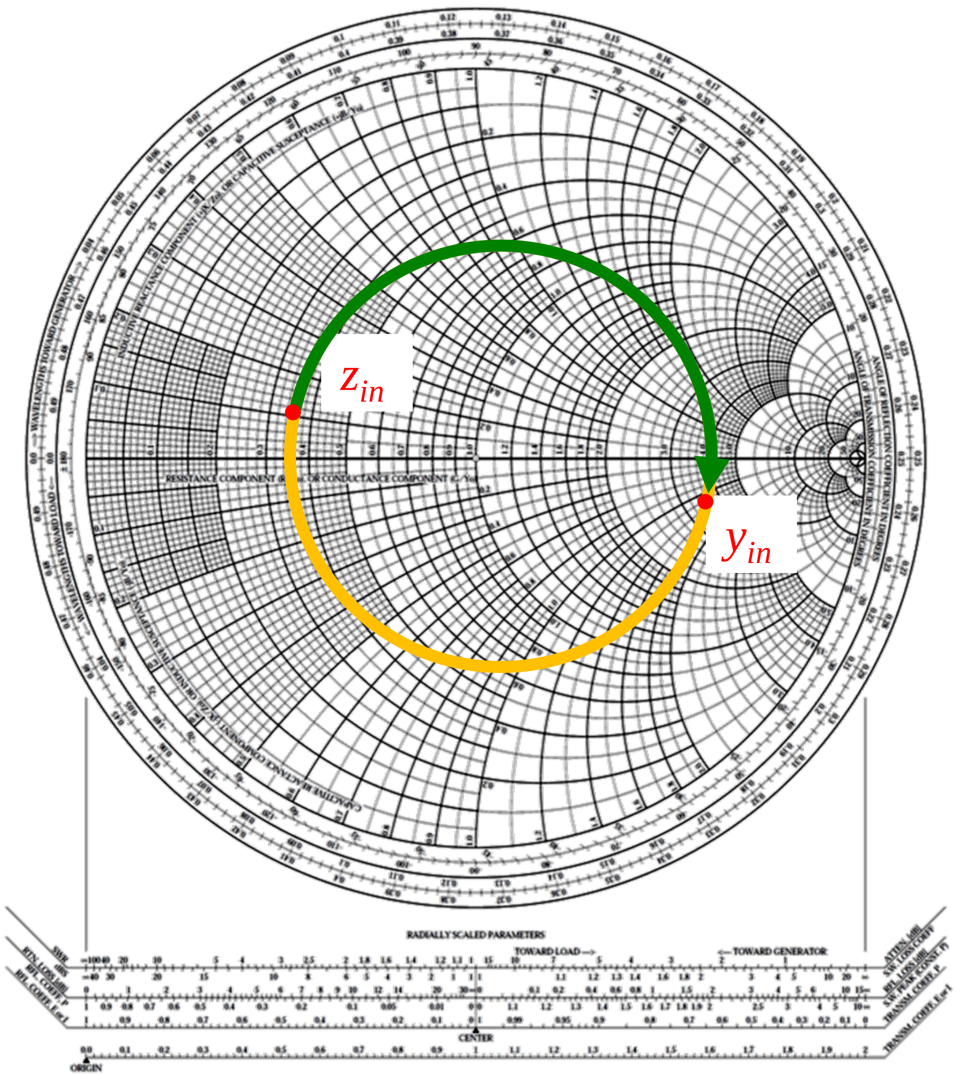
- The input impedance is found by rotating the point
- Draw a constant $|\Gamma|$ circle. It represents the locus of all possible values of $\Gamma$ that the load can present along the line
- Moving along the line a distance of $0.3\lambda$ *clockwise*

$$0.22\lambda + 0.3\lambda = 0.52\lambda \Rightarrow 0.02\lambda$$

- The intersection of the radial line and the $|\Gamma|$ circle gives

$$z_{in} = 0.255 + j0.117$$

# Basic Procedures of Smith Chart (4/4)




## Step 4:

- The input impedance at the input to the line:

$$Z_{in} = Z_0 z_{in} = 50(0.255 + j0.117) = 12.7 + j5.8\ \Omega$$

- The corresponding parameters:

$$|\Gamma_{in}| = 0.6,\ \theta_{in} = 165.8° \rightarrow \Gamma_{in} = 0.6\angle 165.8°$$

$$VSWR = 3.98$$

$$RL = 4.4\text{ dB}$$

## Step 5:

- The normalized admittance is computed by moving $z_{in}$ a distance $\lambda/4$ along the constant $|\Gamma|$ circle:

$$y_{in} = 3.24 - j1.49$$

- And the input admittance:

$$Y_{in} = Y_0 y_{in} = \frac{1}{50}(3.24 - j1.49) = 0.065 - j0.03\text{ S}$$

# Characteristics of Smith Chart

Smith chart: a design and analysis tool invented by P. H. Smith in 1939

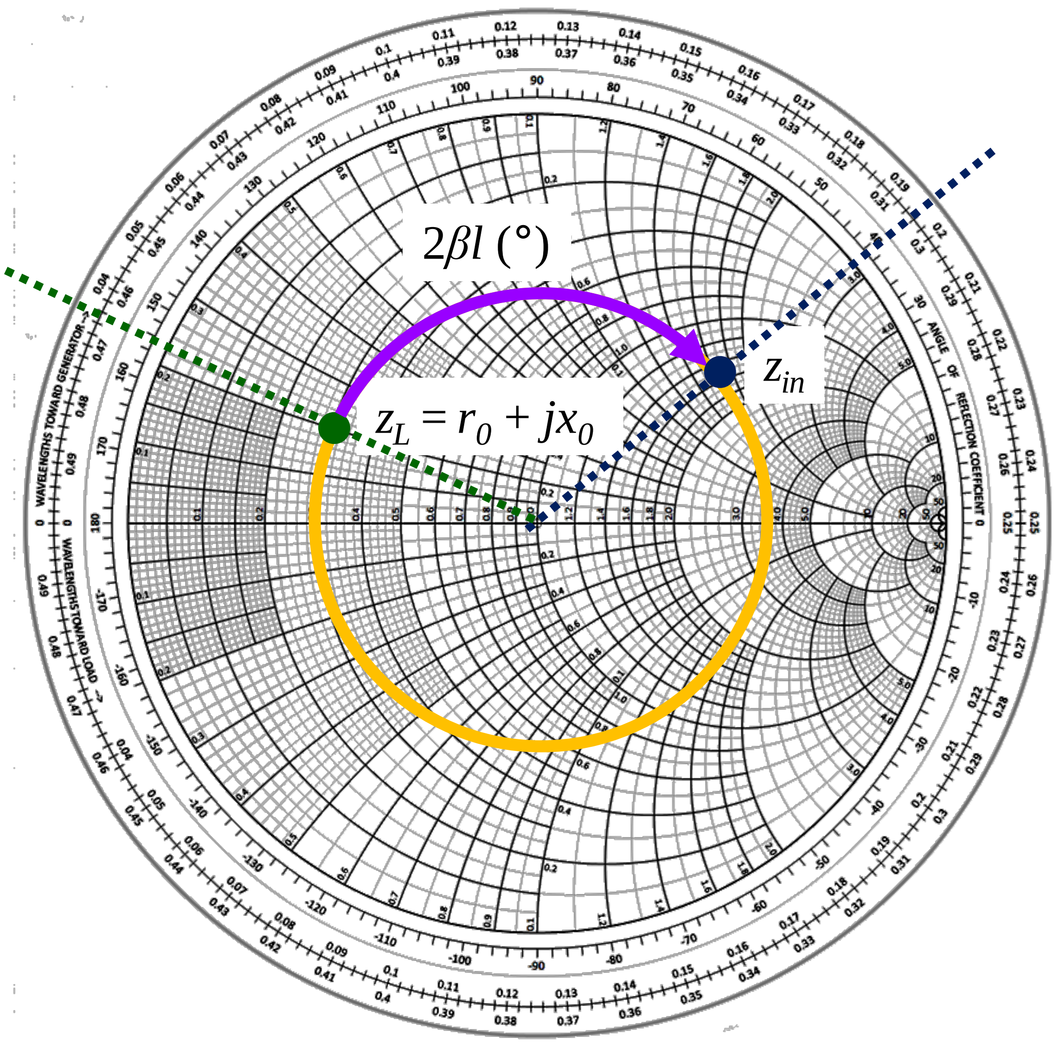
- It provides an extremely useful way of visualizing transmission-line phenomenon
- CAD tools and VNAs are equipped with this display format
- A number of studies use Smith chart to illustrate their research about transmission line and microwave components

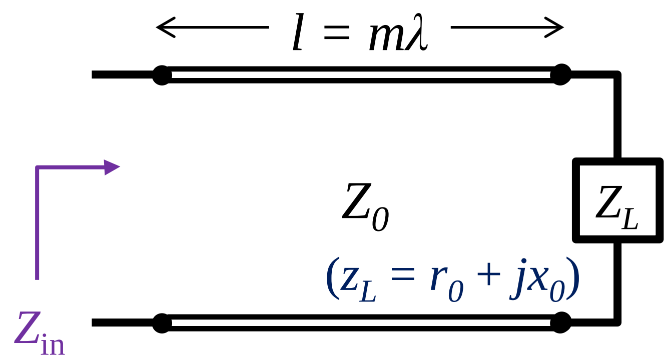
It’s powerful in:

- Evaluating effects of shunt and series impedances on the impedance of a transmission line
- Displaying complex impedances vs. frequencies
- Designing impedance matching networks

Wikipedia. [Online] Available: http://en.wikipedia.org/wiki/Network_analyzer_(electrical)

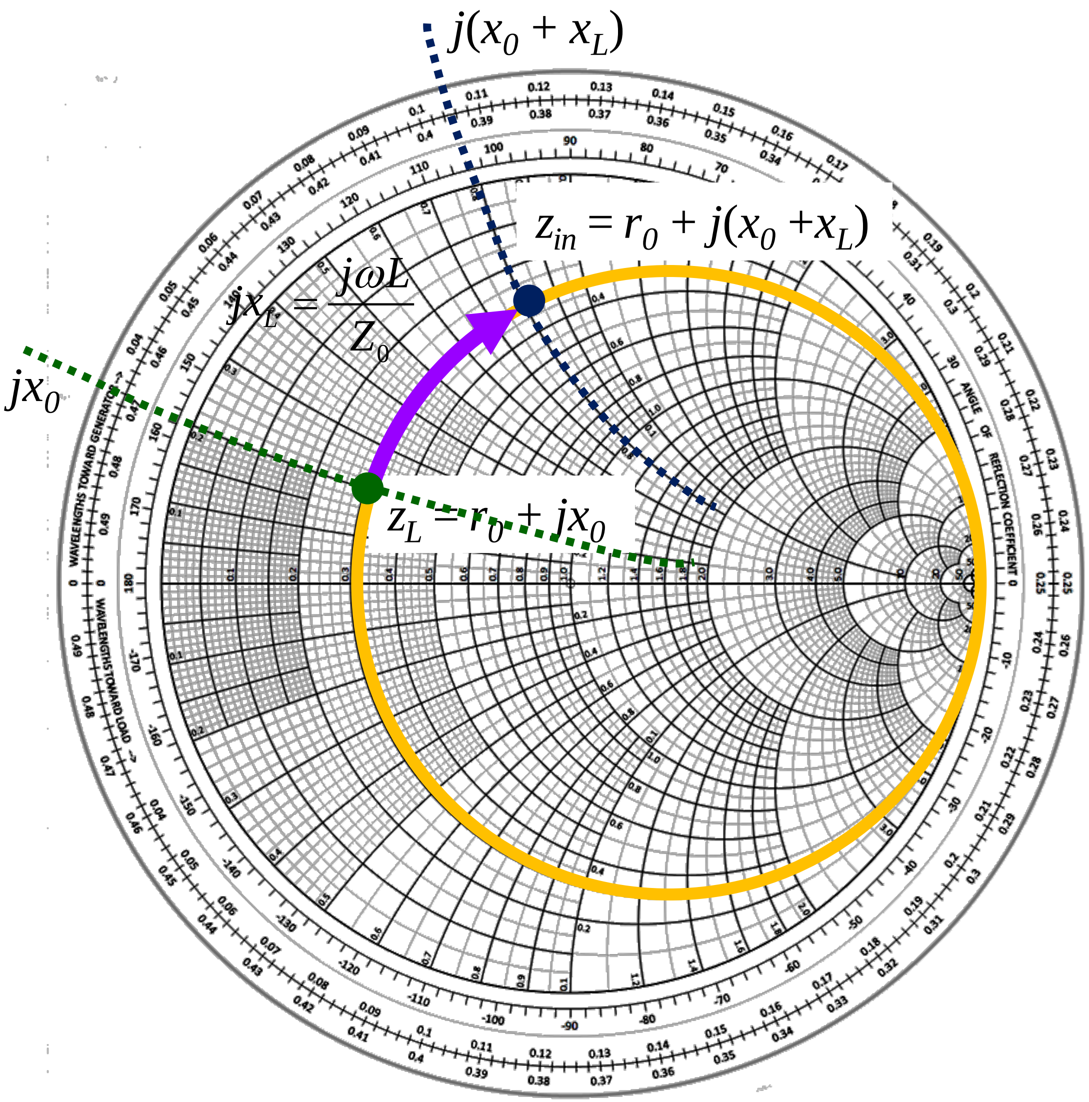
# Connecting $Z_L$ with a TL in Series

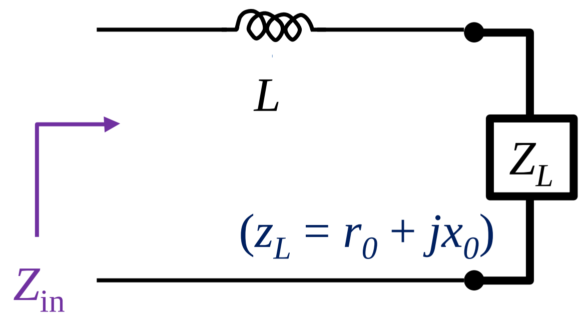



- If $Z_L$ is connected in series with a transmission line, it moves along the constant $|\Gamma|$ circle clockwise

# Connecting $Z_L$ with an Inductor in Series




- For an inductor with its value $L$, its impedance (or reactance) is:

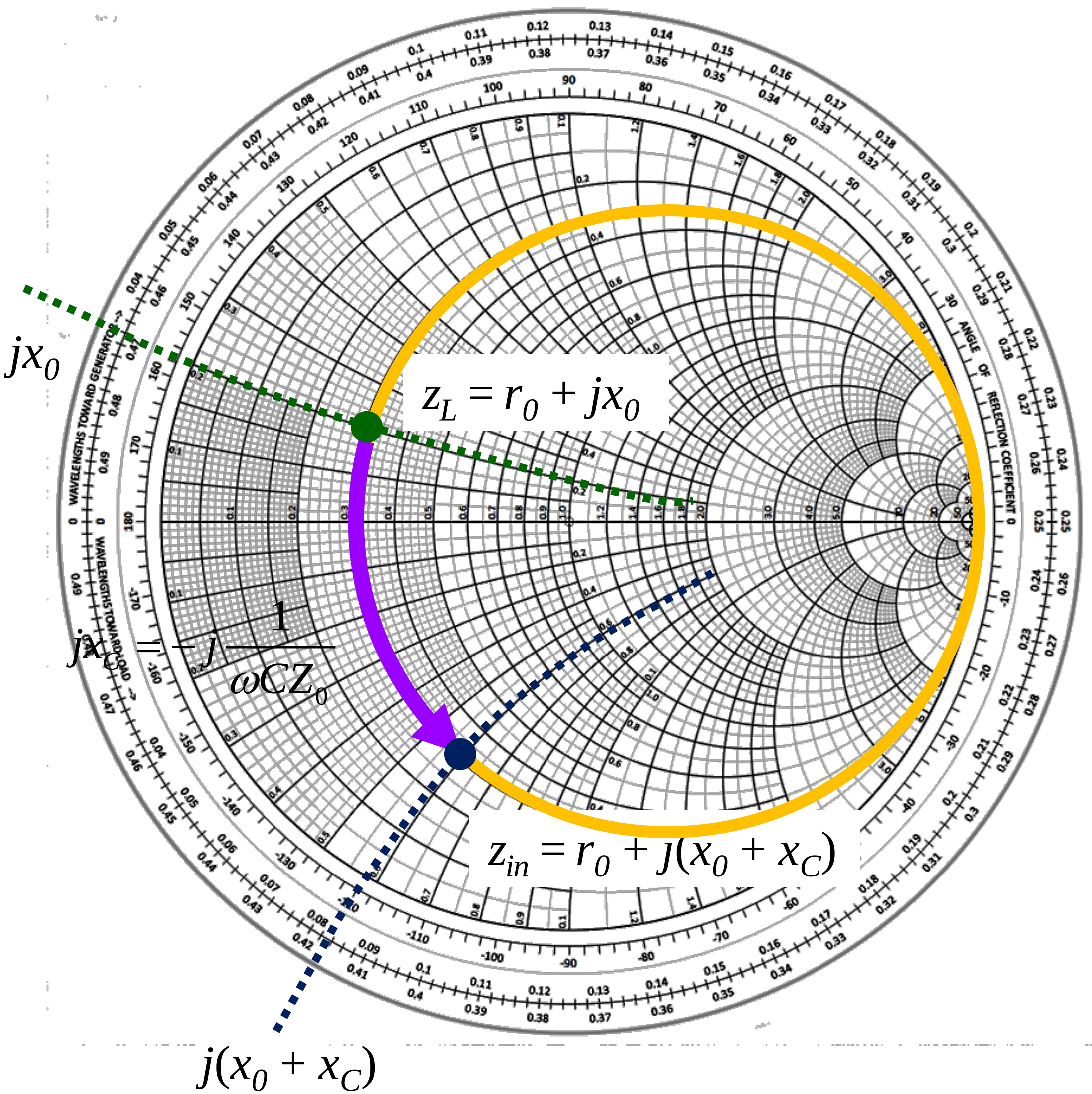
  $X_L = j\omega L$

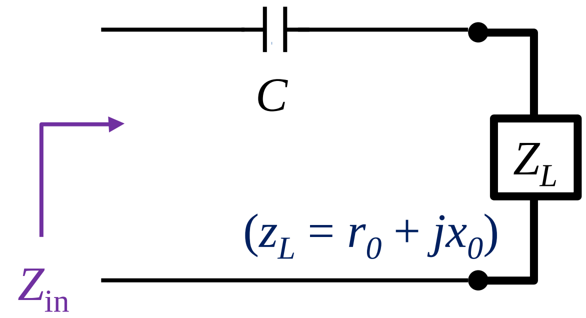
  $\omega$: operational angular frequency

- If $Z_L$ is connected in series with an inductor $L$, it moves along the constant $r$ circle clockwise

$$z_{in} = r_0 + j\left(x_0 + \frac{\omega L}{Z_0}\right)$$

# Connecting $Z_L$ with a Capacitor in Series




- For a capacitor with its value $C$, its impedance (or reactance) is:

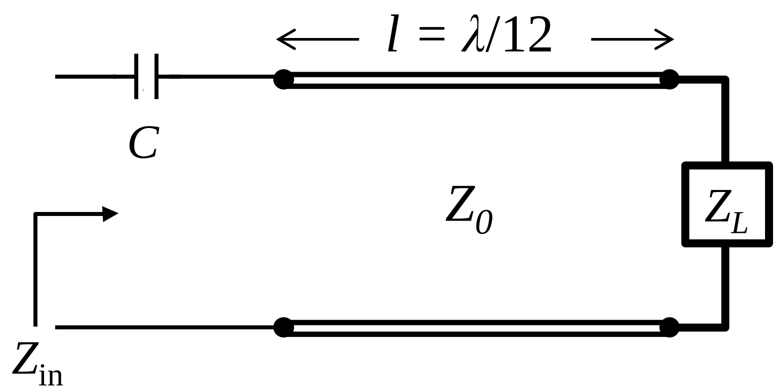
  $X_C = 1/j\omega C = -j/\omega C$

  $\omega$: operational angular frequency

- If $Z_L$ is connected in series with a capacitor $C$, it moves along the constant $r$ circle counterclockwise

$$z_{in} = r_0 + j\left(x_0 - \frac{1}{\omega C Z_0}\right)$$

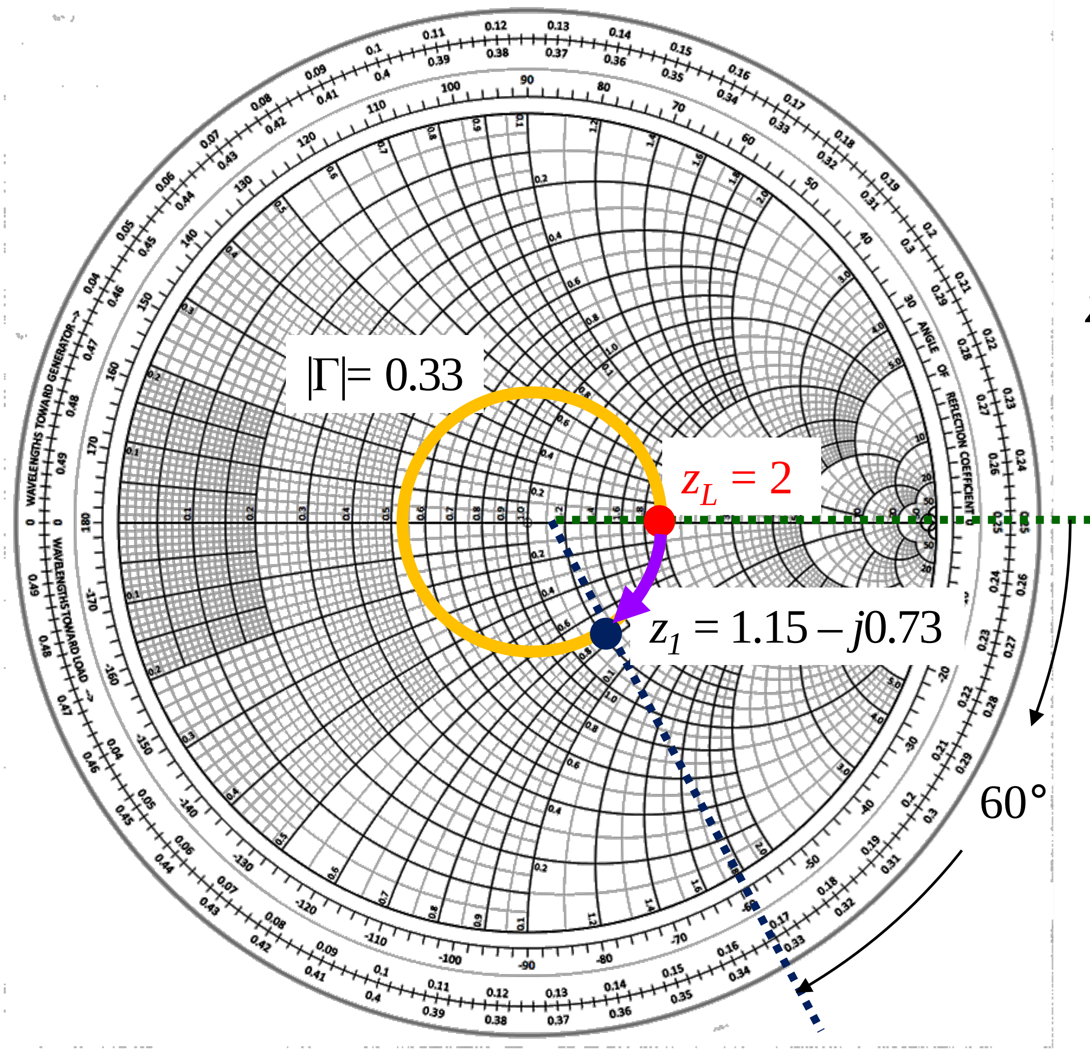
# EX 2.3 Connecting $Z_L$ with Various Components in Series

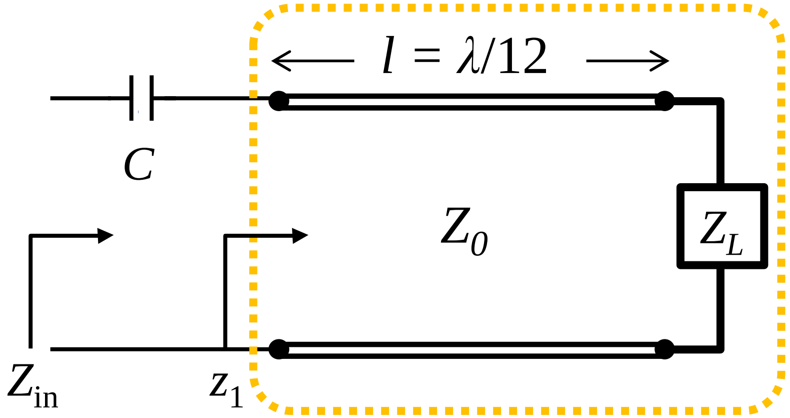



- $Z_L = 100\ \Omega$, C = 4.8 pF, $f = 2$ GHz, $Z_0 = 50\ \Omega$

➡ Find the input impedance $Z_{\text{in}}$ by Smith chart

# EX 2.3 Connecting $Z_L$ with Various Components in Series (1/2)




**Step 1:**

- The normalized load impedance $z_L$:

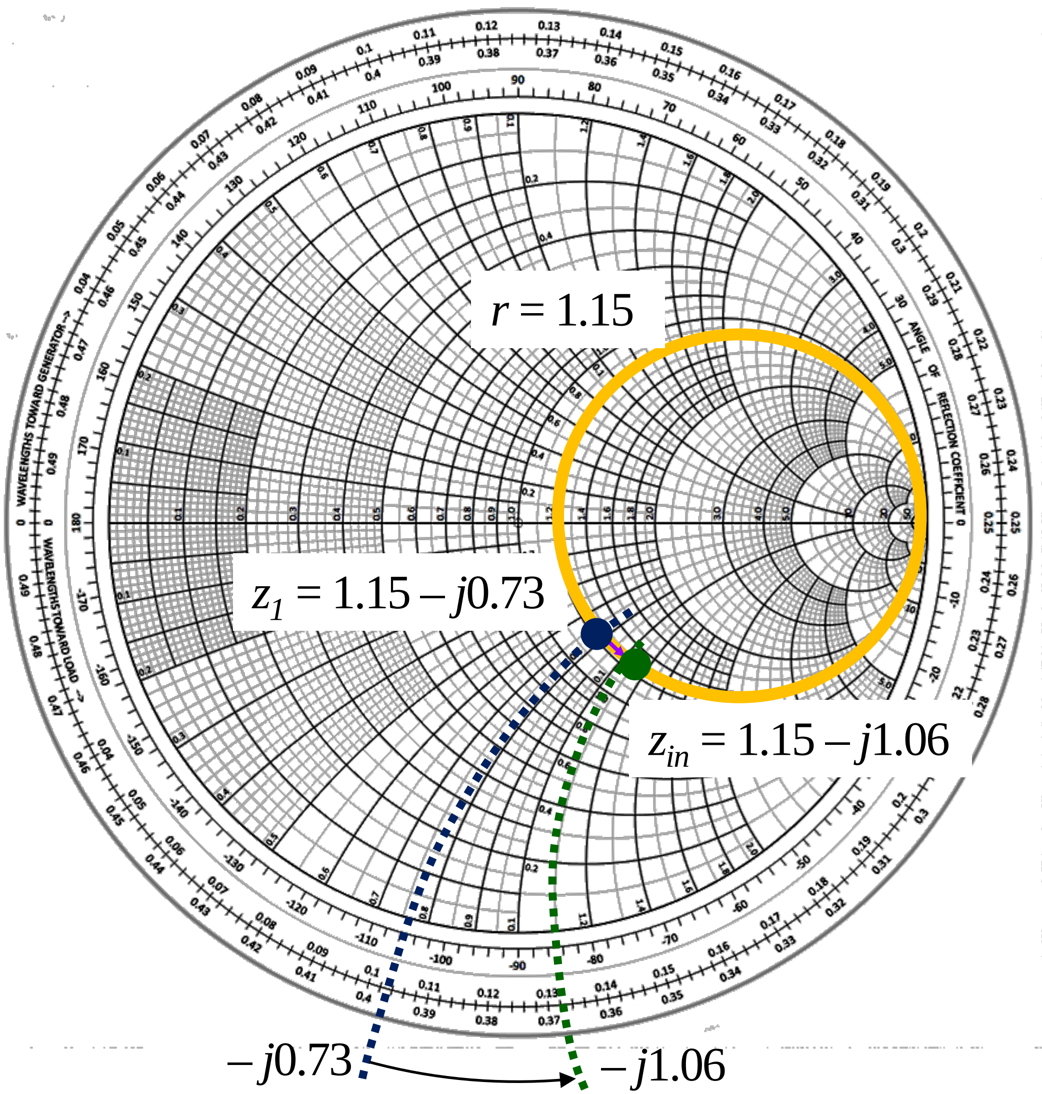
$$z_L = \frac{Z_L}{Z_0} = \frac{100}{50} = 2$$

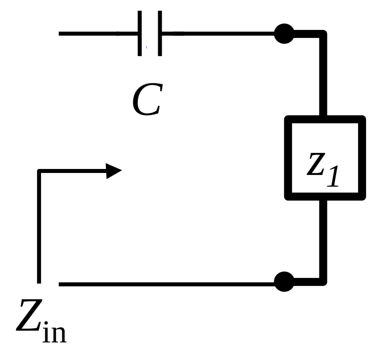
**Step 2:**

- Rotating $z_L$ clockwise an amount $2\beta l = 60°$ along the constant $|\Gamma|$ circle to $z_1$:

$$z_1 = 1.15 - j0.73$$

# EX 2.3 Connecting $Z_L$ with Various Components in Series (2/2)




## Step 3:

- The **normalized** impedance of the capacitor:

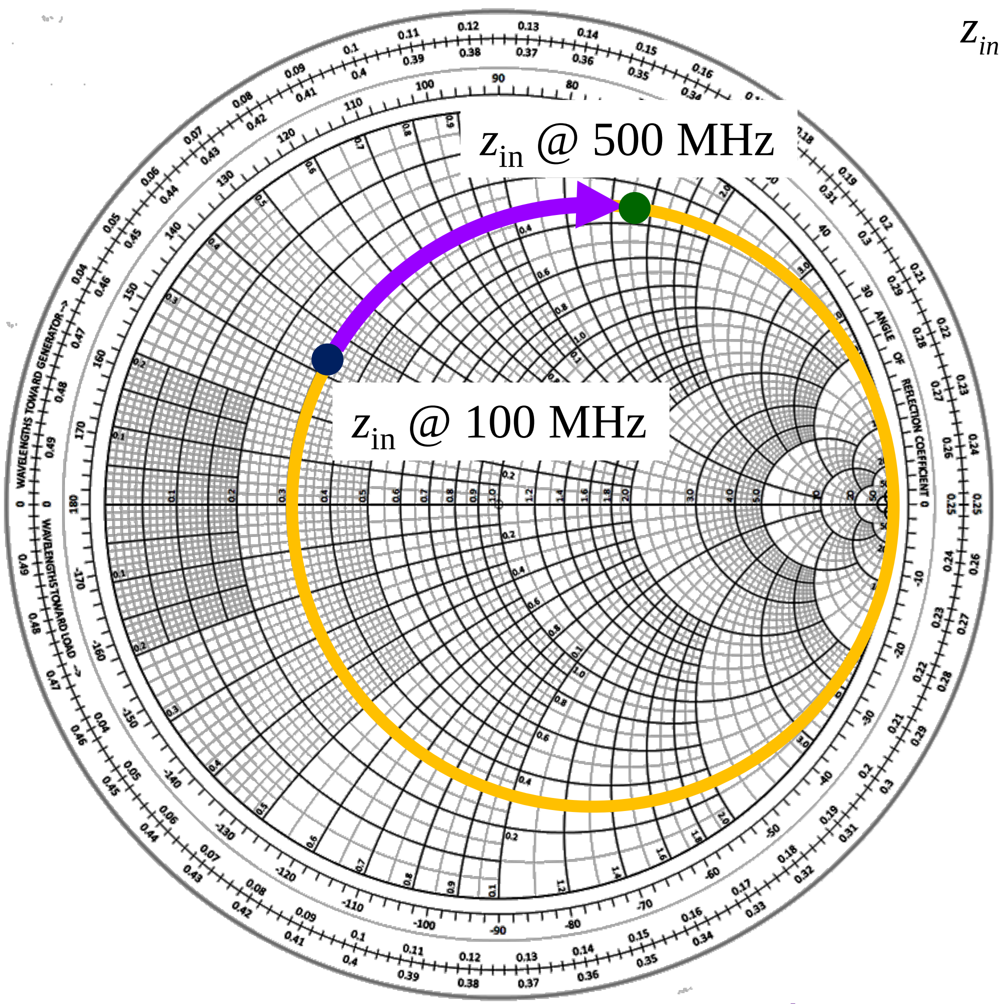
$$x_c = -j\frac{1}{2\pi \times 2\times 10^9 \times 4.8\times 10^{-12} \times 50} = -j0.33$$

- Rotating $z_1$ counterclockwise along the constant $r$ circle from $x = -j0.73$ to $-j(0.73 + 0.33) = -j1.06$

$$z_{in} = 1.15 - j(0.73 + 0.33) = 1.15 - j1.06$$

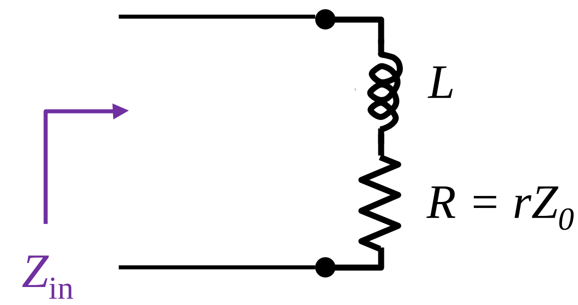
$$\Rightarrow Z_{in} = 50 \times (1.15 - j1.06) = 57.5 - j53\ \Omega$$

# Complex Impedance vs. Frequency (1/2)




How about $\omega\downarrow$?

$$z_{in} = r + j\frac{\omega L}{Z_0}$$




Frequency response of network:

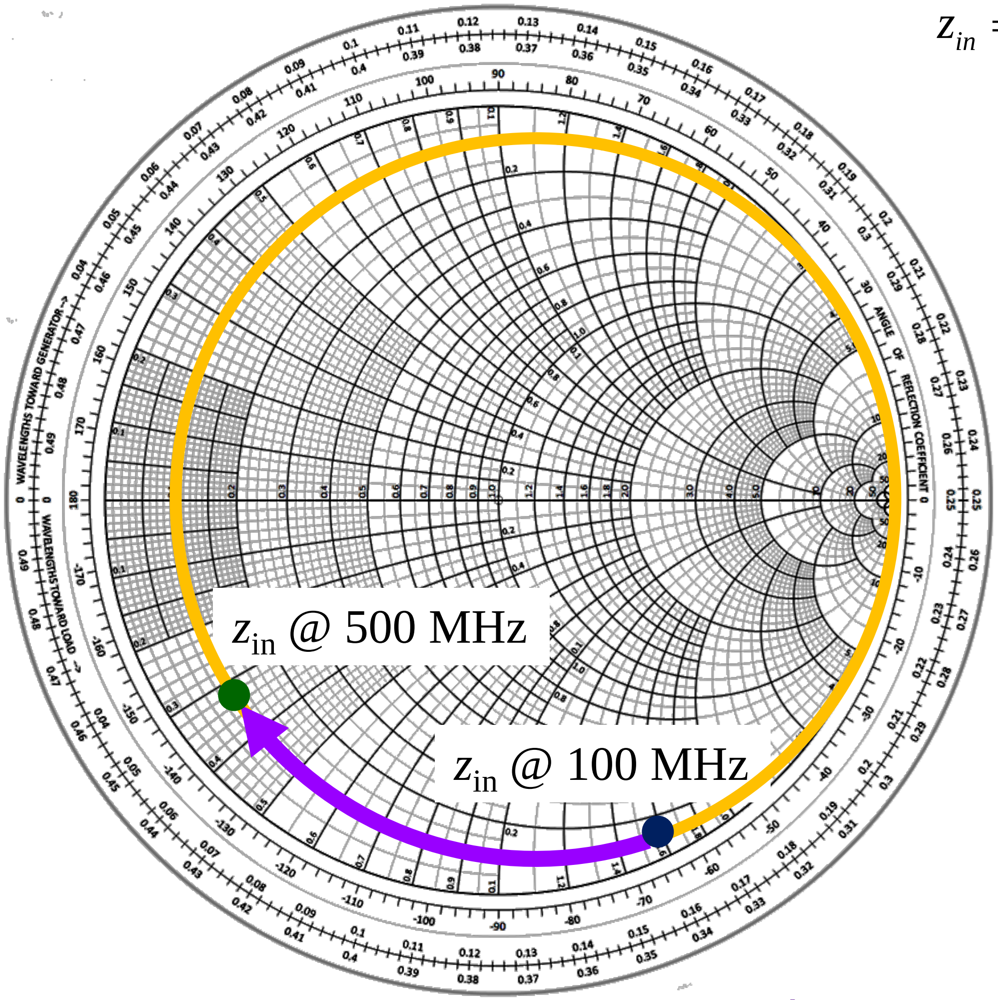
- The input reactance changes with the angular frequency $\omega$
- $\omega\uparrow$, $x_{in}\uparrow$
- $z_{in}$ moves along the constant $r$ circle
- Example: $R = 15\ \Omega$, $L = 25\ \text{nH}$, $Z_0 = 50\ \Omega$, and $f = 100\ \text{MHz} - 500\ \text{MHz}$

$$z_{in} = 0.3 + j(\omega L/Z_0)$$

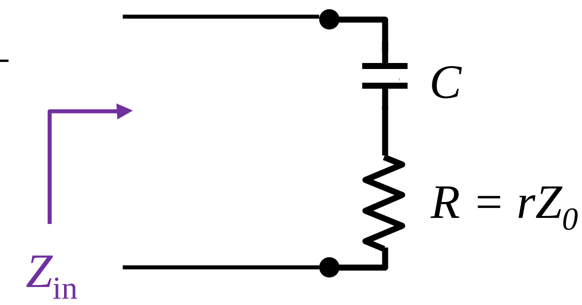
100 MHz: $z_{in} = 0.3 + j0.314$

500 MHz: $z_{in} = 0.3 + j1.57$

# Complex Impedance vs. Frequency (2/2)




$$z_{in} = r - j\frac{1}{\omega C Z_0}$$




Frequency response of network:

- The input reactance changes with the angular frequency $\omega$
- $\omega\uparrow$, $|x_{in}|\downarrow$, $x_{in}\uparrow$
- $z_{in}$ moves along the constant $r$ circle
- Example: $R = 5\ \Omega$, $C = 20$ pF, $Z_0 = 50\ \Omega$, and $f = 100$ MHz – 500 MHz

$$z_{in} = 0.1 - j(1/\omega C Z_0)$$

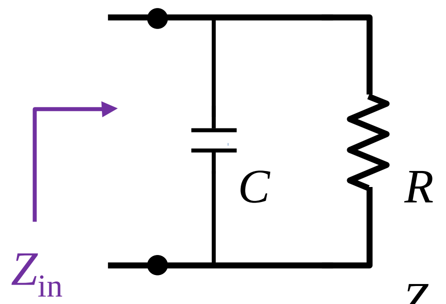
100 MHz: $z_{in} = 0.1 - j1.59$

500 MHz: $z_{in} = 0.1 - j0.32$

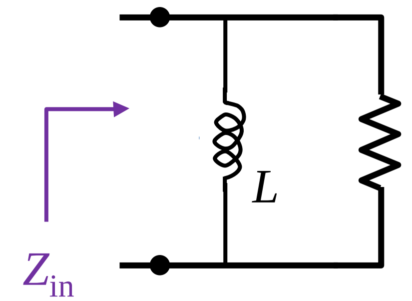
How about $\omega\downarrow$?

# If We Deal With Connection in Parallel...

If the load impedances are not connected in series but in parallel:




$$Z_C = \frac{1}{j\omega C}$$

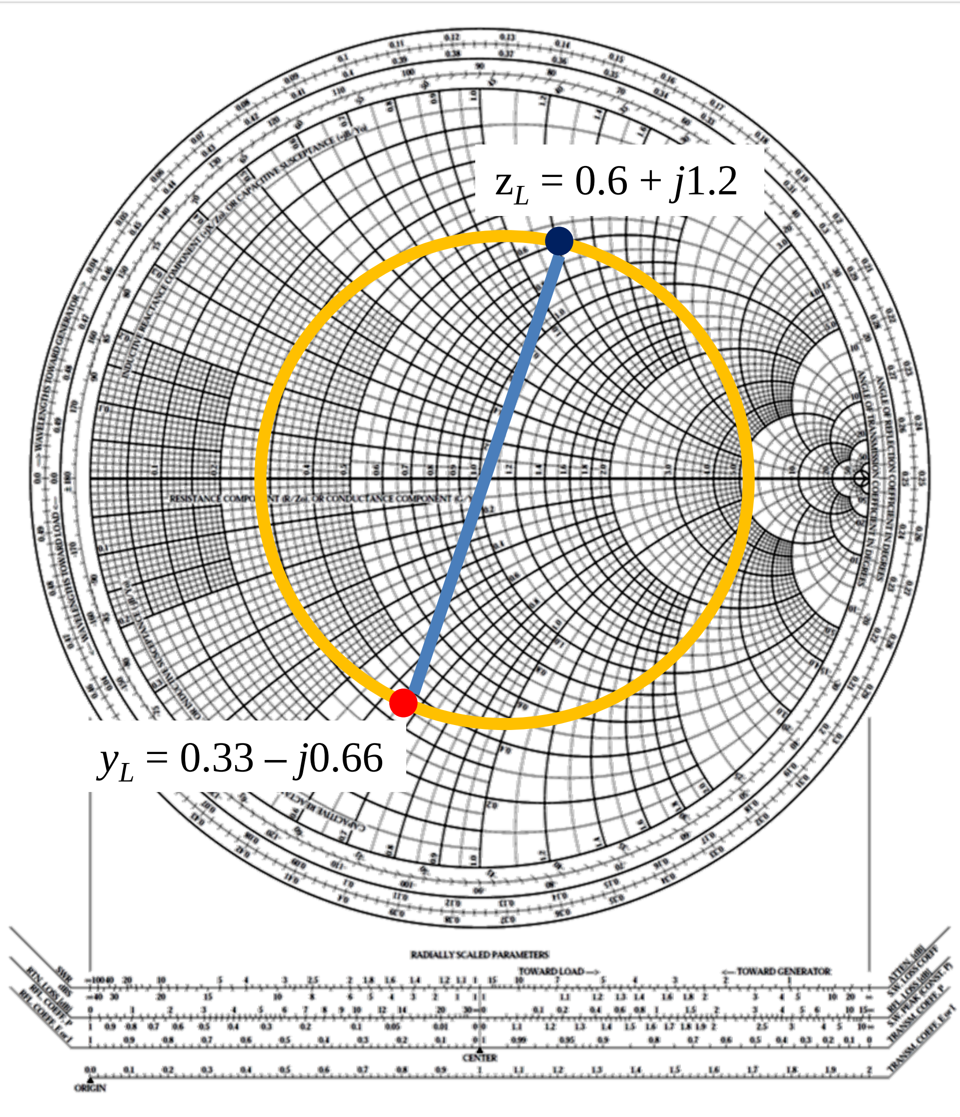



$$Z_L = j\omega L$$

- How to find $Z_{\text{in}}$?
- $Z_{\text{in}}$ for parallel R-C: $Z_{in} = \dfrac{1}{j\omega C} \parallel R = \dfrac{\dfrac{1}{j\omega C} \times R}{\dfrac{1}{j\omega C} + R} = \dfrac{R}{1 + j\omega CR} = \dfrac{R}{1-\omega^2 C^2 R^2} - j\dfrac{\omega CR^2}{1-\omega^2 C^2 R^2}$
- It's too complicated to visualize $Z_{\text{in}}$ in Smith chart!
- But, if we are interested in admittance rather than impedance, the overall admittance $Y_{\text{in}}$ is:

$$Y_{in} = Y_C + \frac{1}{R} = j\omega C + \frac{1}{R}$$

# How to Find the Admittance on Smith Chart?



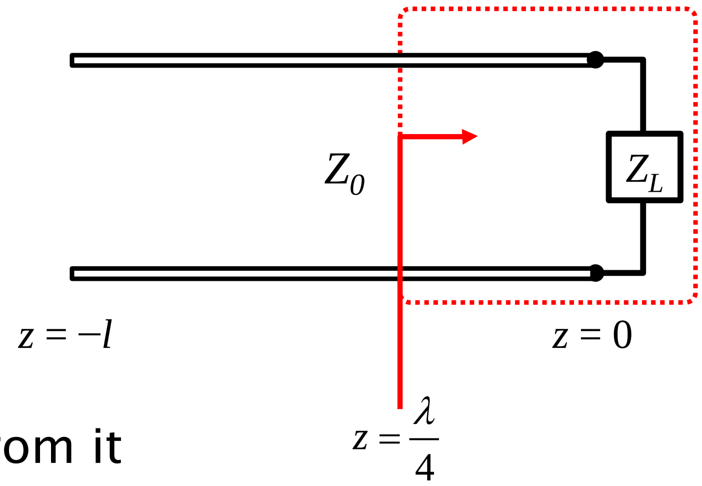

Find the admittance of $Z_L = 30 + j60\ \Omega$ ($Z_0 = 50\ \Omega$):

1. Normalized $Z_L$ to $Z_0$: $z_L = 0.6 + j1.2$
2. $r = 0.6$
3. $x = 1.2$
4. $y_L$ can be found by:

$$y_L = z_n\left(z = \frac{\lambda}{4}\right)$$

$y_L = 0.33 - j0.66$

# Rotating the Axis of Smith Chart




- We already know that the normalized admittance of $Z_L$ is

$$y_L = z_n\left(z=\frac{\lambda}{4}\right)$$

- That is, the normalized admittance is the normalized impedance at a distance $\lambda/4$ from it

In order to avoid the continual rotation on Smith chart:

- This is equivalent to rotating the $r$, $x$ scales of Smith chart by $180^\circ$ around the center
- Thus, we can read the admittance directly on this new scale
- The new scale is called "admittance Smith chart"

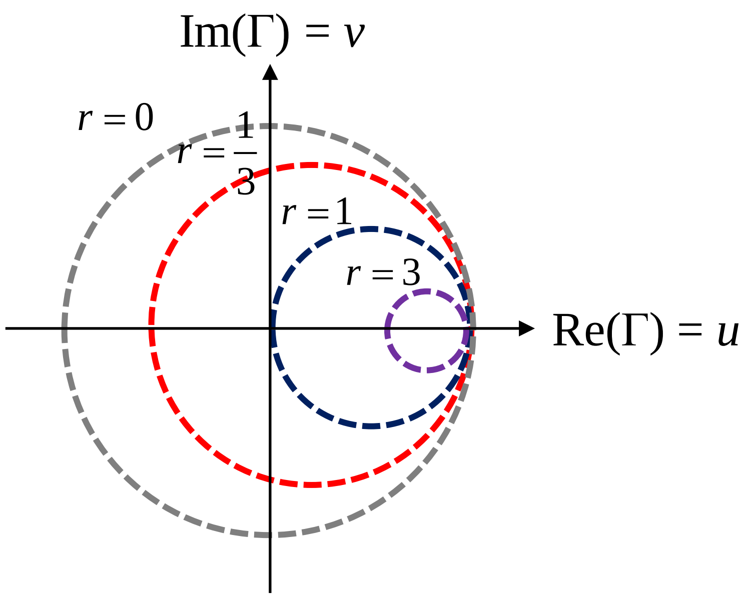
# From Z Chart to Y Chart (1/2)

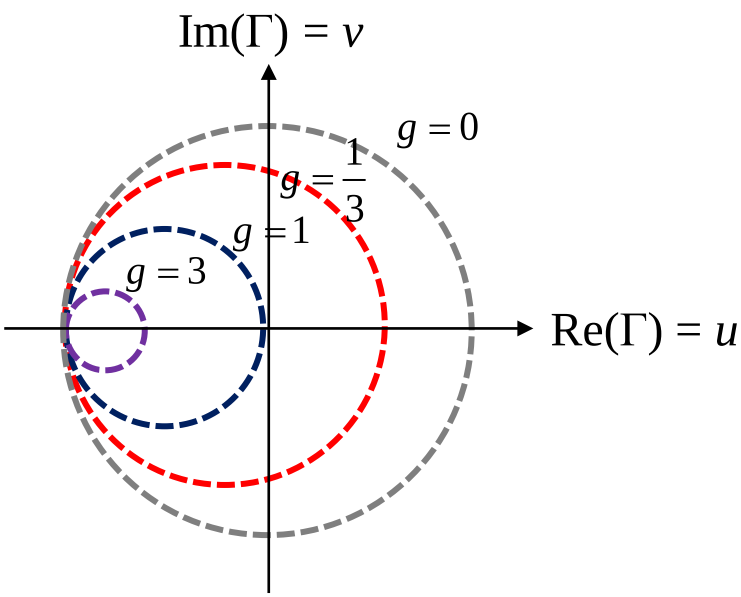
$z = r + jx$ $\longrightarrow$ $y = g + jb$

$$y = \frac{1}{z}$$

Z chart

Y chart

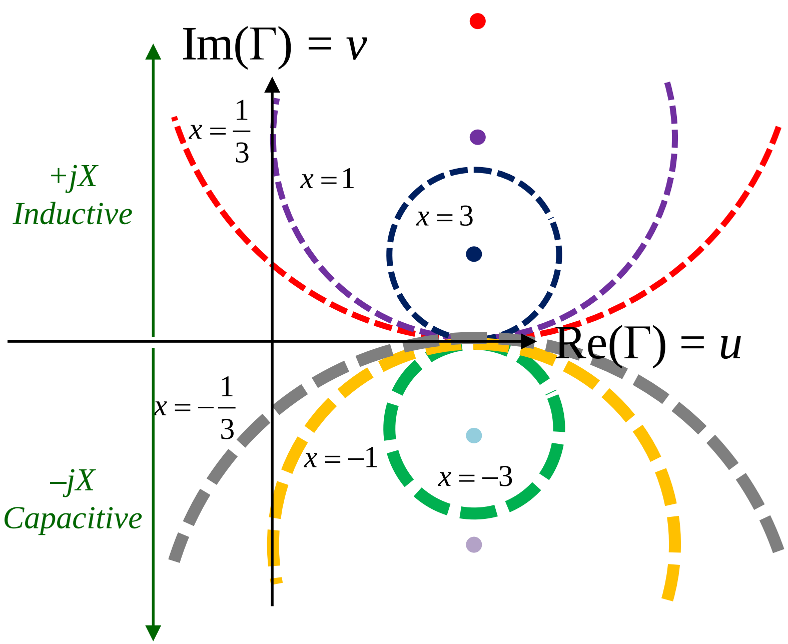


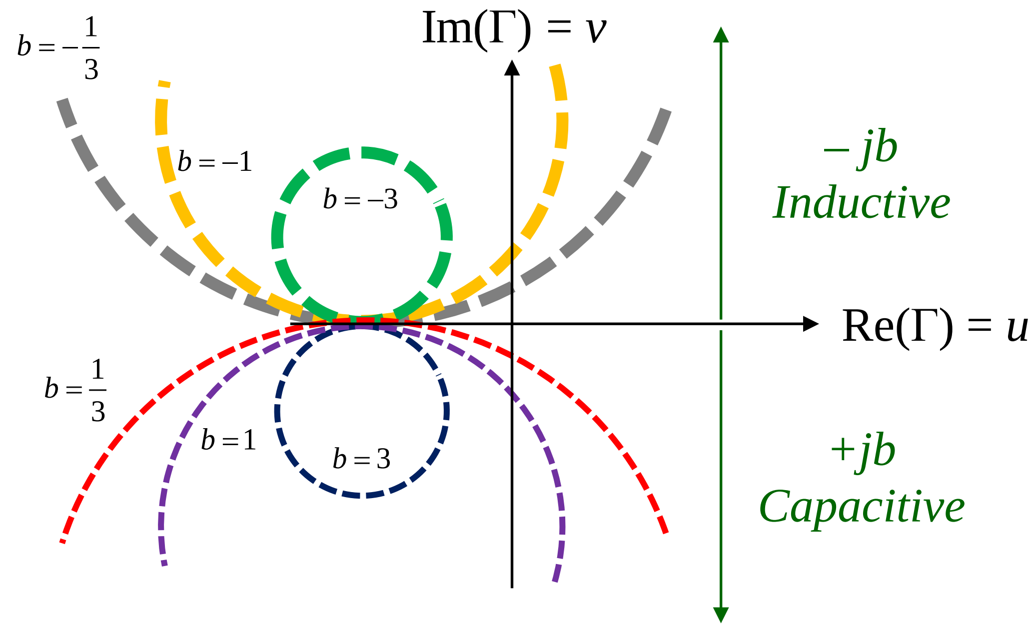
# From Z Chart to Y Chart (2/2)

$$z = r + jx \xrightarrow{\quad y = \frac{1}{z} \quad} y = g + jb$$

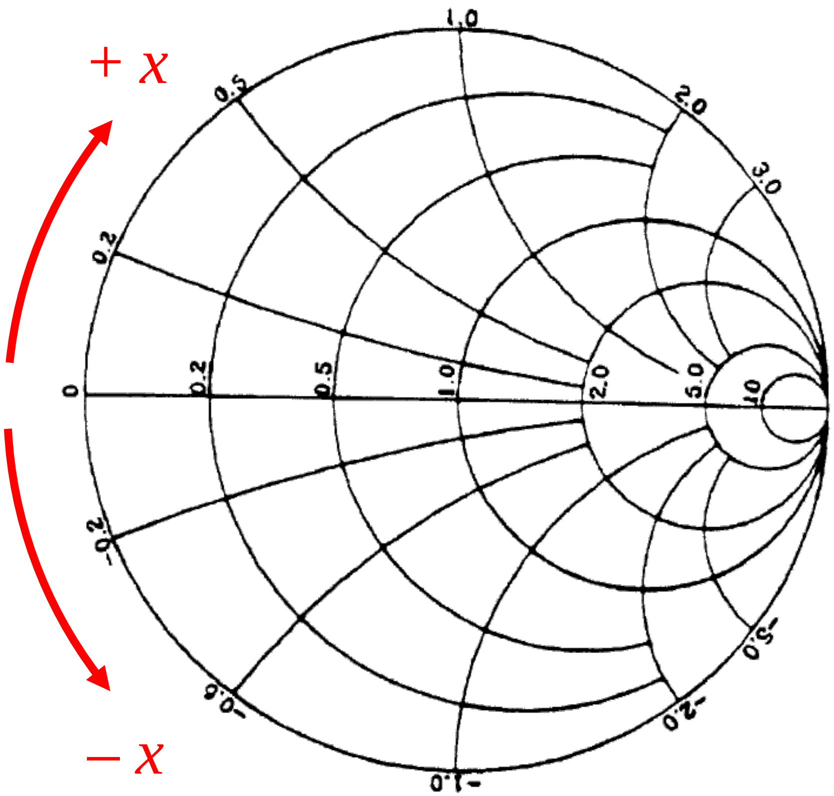
Z chart




Y chart



# Admittance Smith Chart



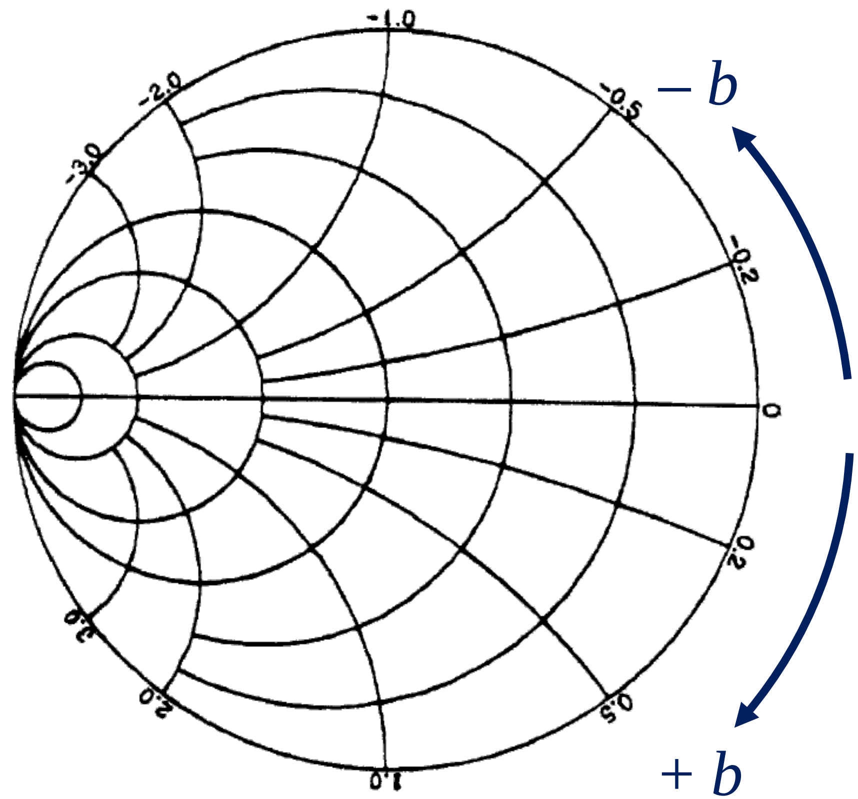

Standard Smith chart

Impedance Smith chart

Z chart ($z = r + jx$)

$r$: resistance

$x$: reactance



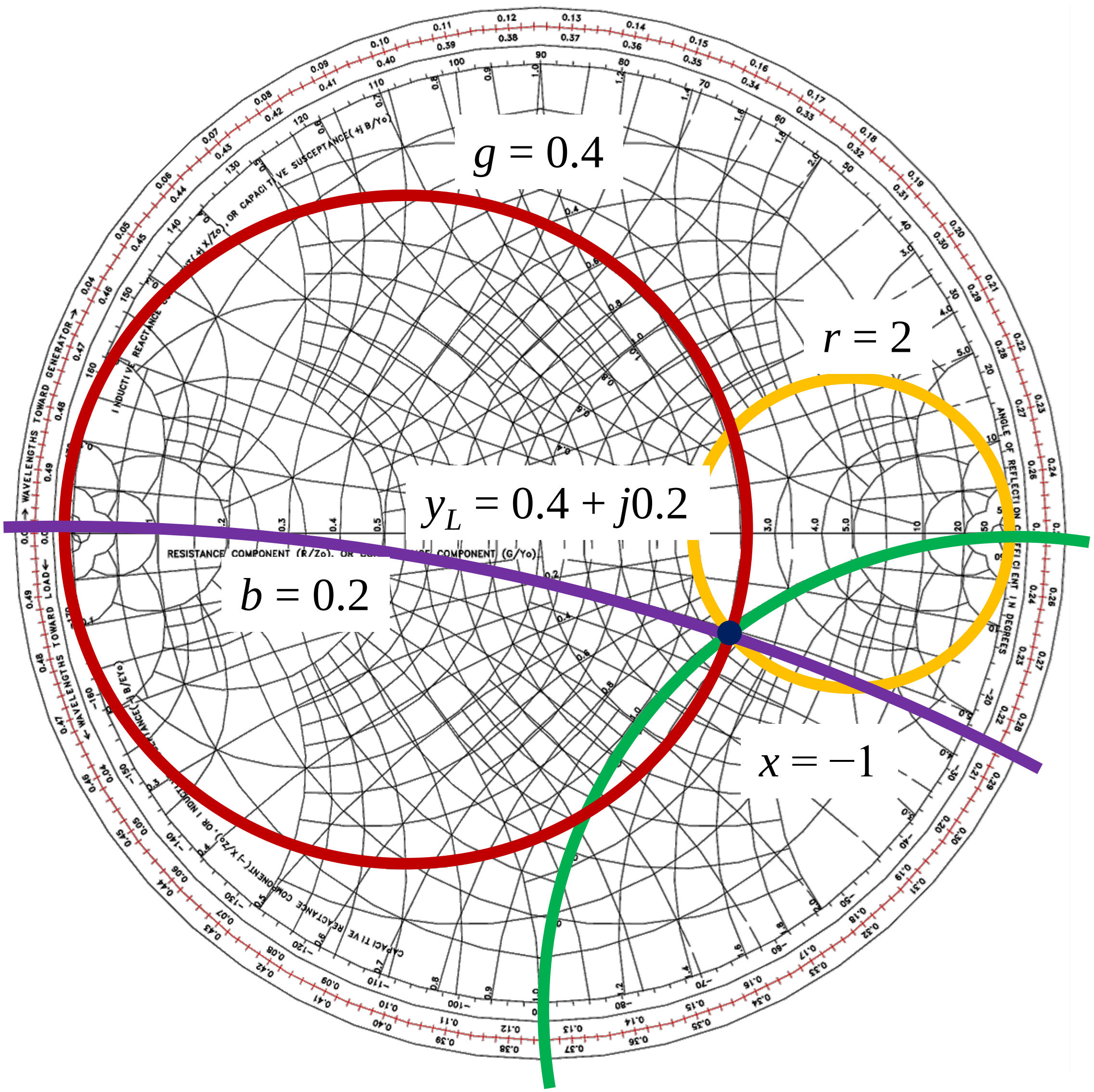

Admittance Smith chart

Y chart ($y = g + jb$)

$g$: conductance

$b$: susceptance

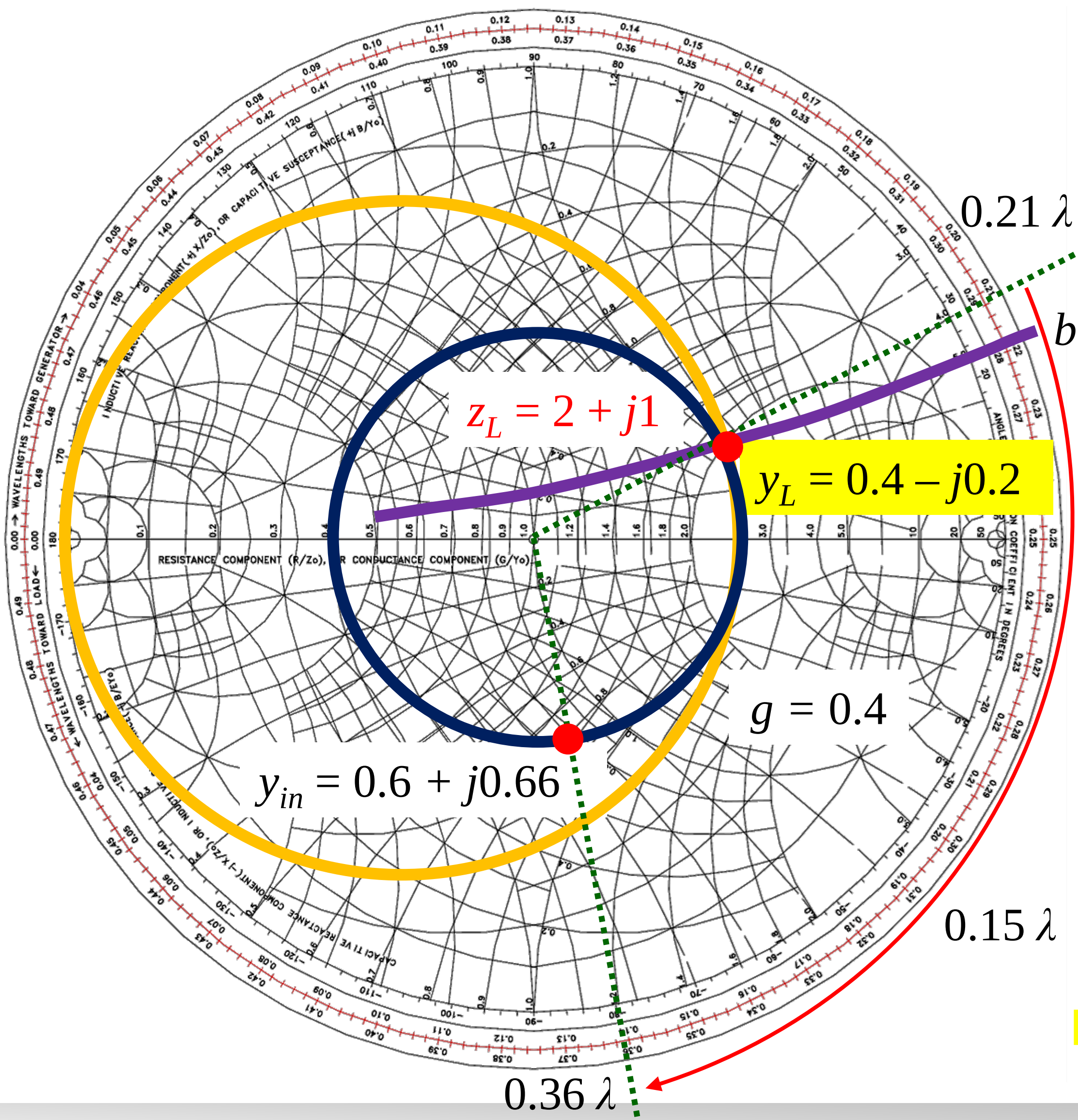
# Find the Admittance of $z_L = 2 - j1$




Find the admittance of $z_L = 2 - j1$ on Y chart:

1. $z_L = 2 - j1$
2. $g = 0.4$
3. $b = 0.2$
4. $y_L = 0.4 + j0.2$

# Impedance and Admittance Smith Chart




## Z-Y chart

- It has two scales in one chart
- No more rotation to find the associated admittance of a given $z_L$

## Example:

- $Z_L = 100 + j50\ \Omega$ and $Z_0 = 50\ \Omega$
- Find the input admittance at $z = 0.15\ \lambda$

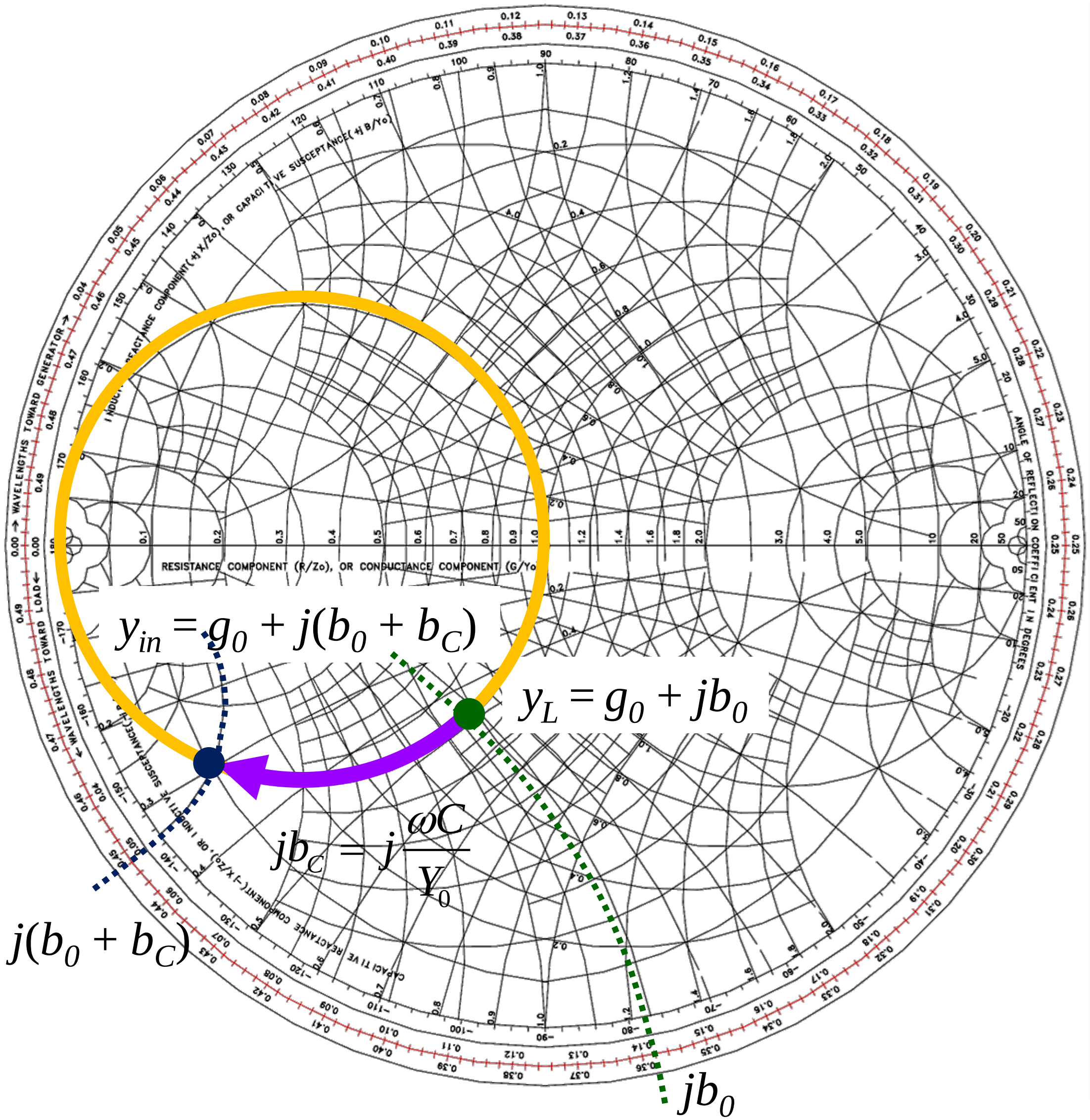
## Solution:

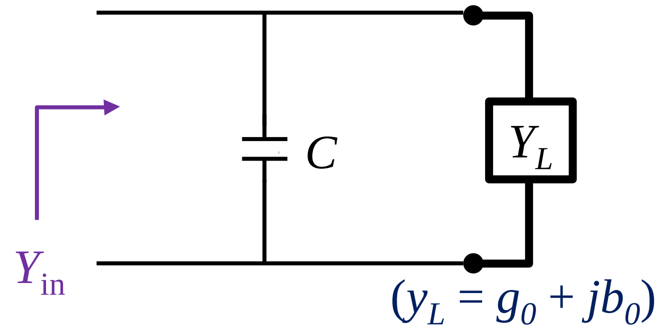
- $z_L = 2 + j1$. We can read the admittance at the same point: $y_L = 0.4 - j0.2$
- The input admittance at $z = 0.15\ \lambda$ is found by tracing the constant $|\Gamma|$ circle

$$y_{in} = 0.6 + j0.66$$

$$Y_{in} = Y_0 y_{in} = 0.012 + j0.013\ \text{S}$$

# Connecting $Z_L$ with a Capacitor in Parallel




- For a capacitor with its value $C$, its admittance is:

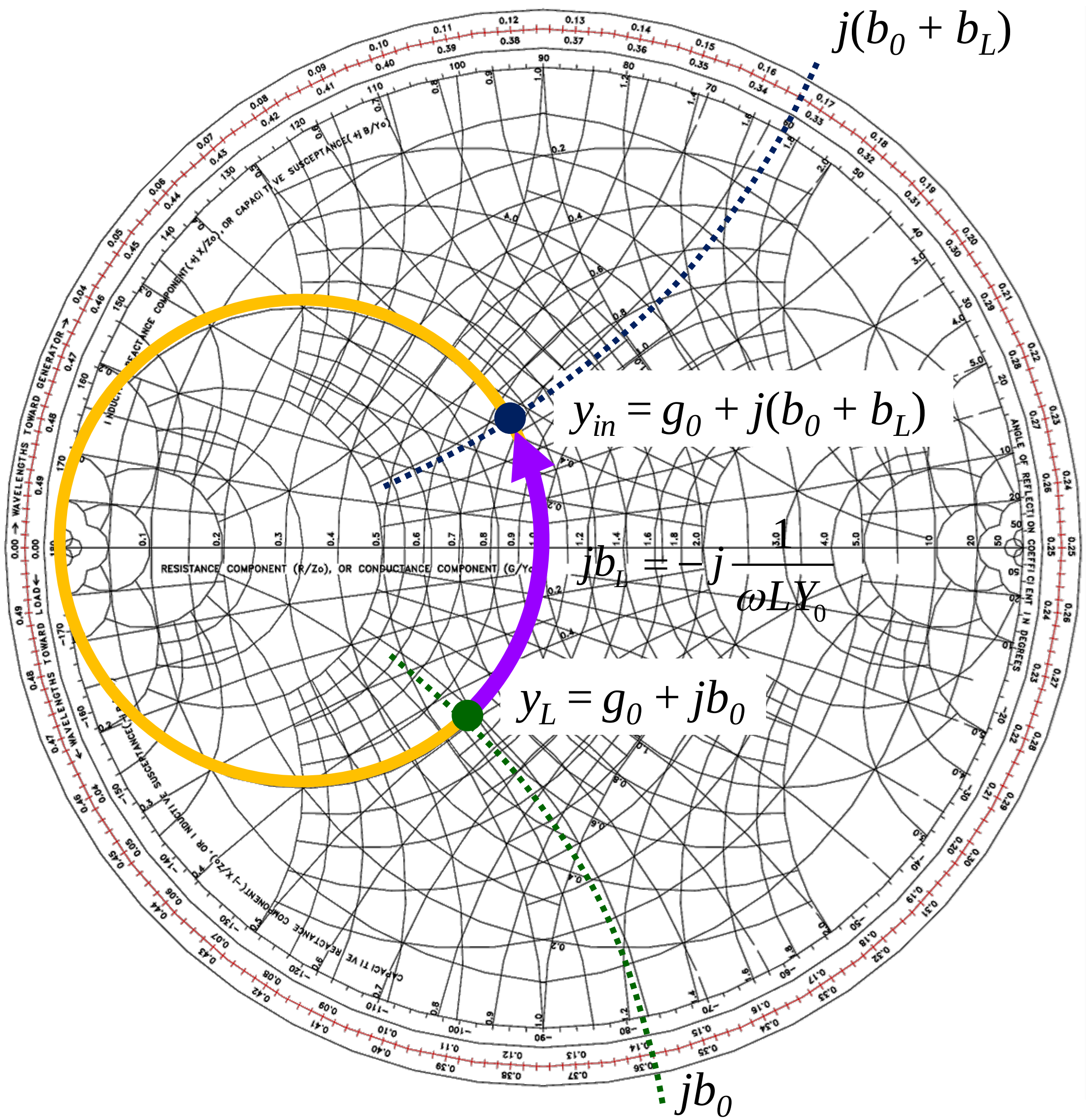
$$B_C = j\omega C$$

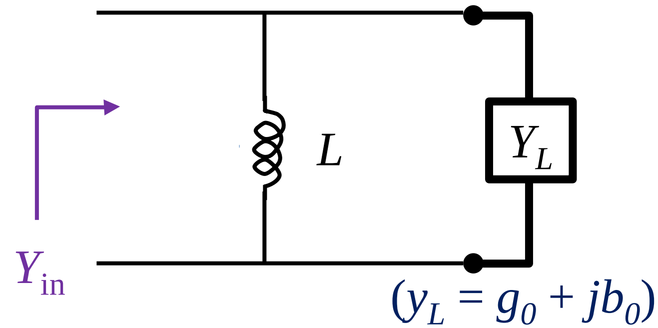
$\omega$: operational angular frequency

- If $Y_L$ is connected in parallel with a capacitor $C$, it moves along the constant $g$ circle clockwise

$$y_{in} = g_0 + j\left(b_0 + \frac{\omega C}{Y_0}\right)$$

# Connecting $Z_L$ with an Inductor in Parallel




- For an inductor with its value $L$, its admittance is:

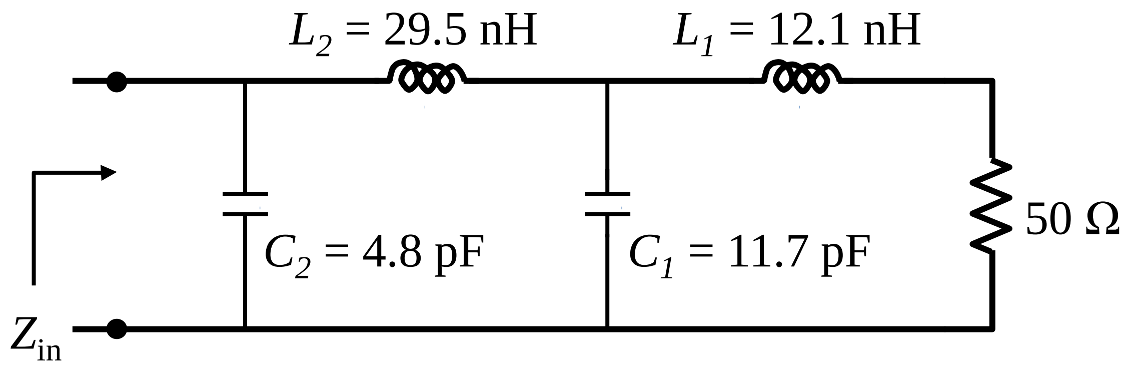
$$B_L = 1/j\omega L = -j/\omega L$$

$\omega$: operational angular frequency

- If $Y_L$ is connected in parallel with an inductor $L$, it moves along the constant $g$ circle counterclockwise

$$y_{in} = g_0 + j\left(b_0 - \frac{1}{\omega L Y_0}\right)$$

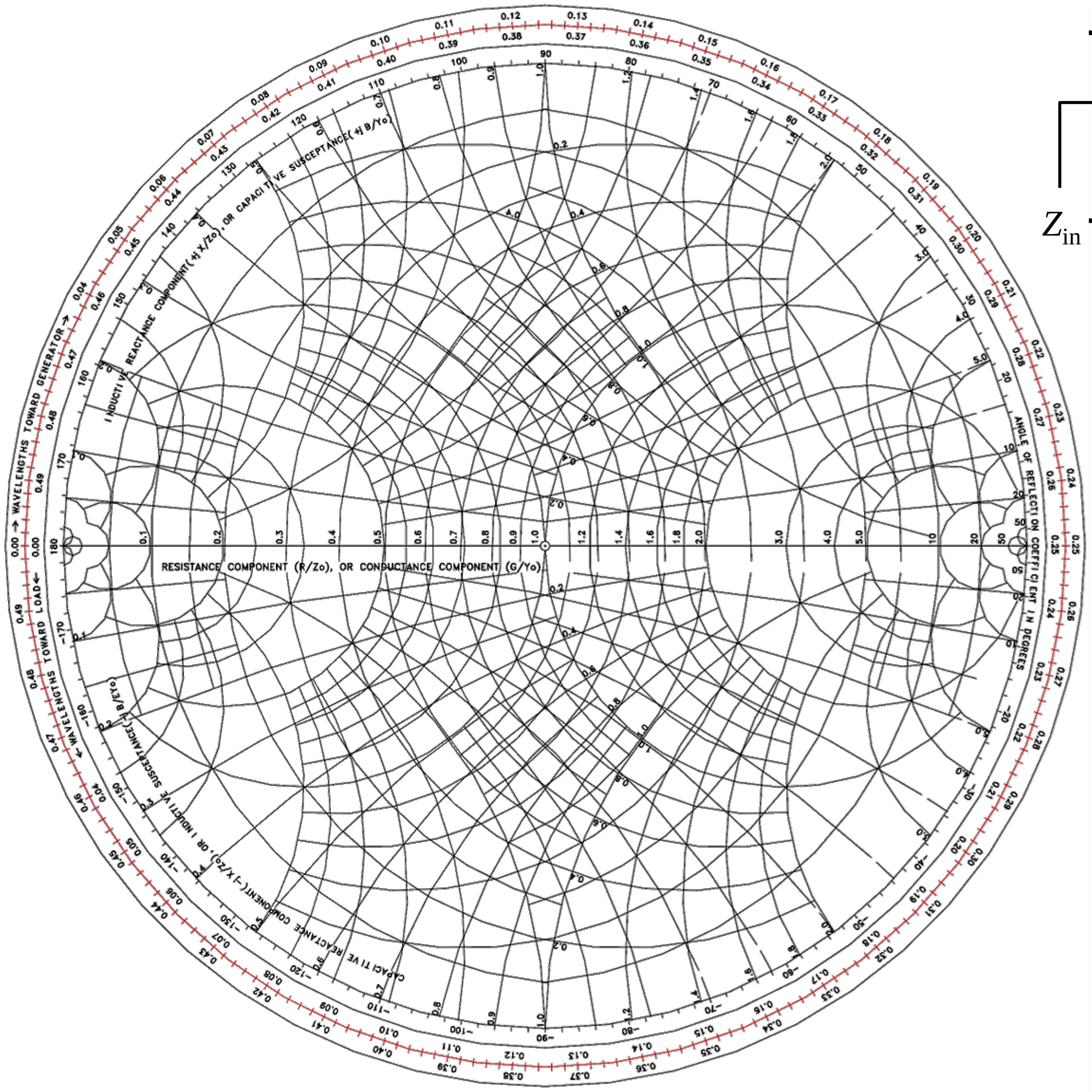
# EX 2.4 Shunt and Series Impedances (1/6)

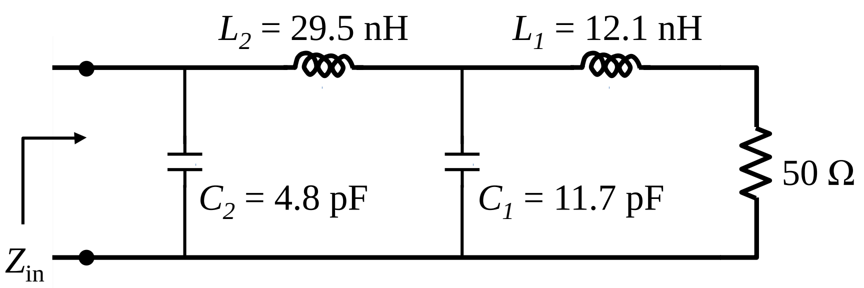



- $f$ = 1 GHz and $Z_0$ = 50 Ω

Find the input impedance $Z_{in}$ by Smith chart

# EX 2.4 Shunt and Series Impedances (2/6)






## Step 1:

- Calculate the normalized impedances and admittances of all components:

$$r = 1$$

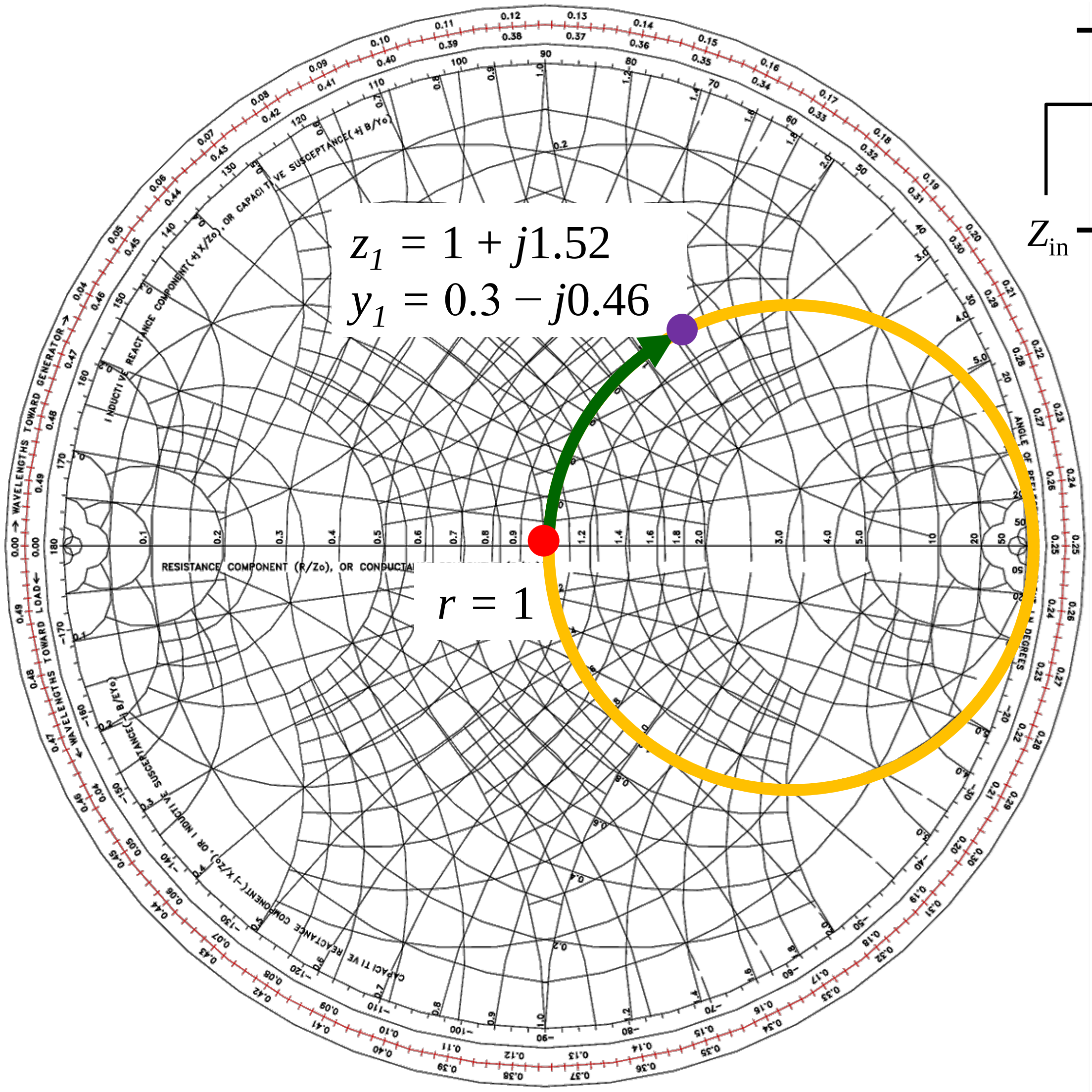
$$jx_{L1} = j\frac{\omega L}{Z_0} = j\frac{2\pi \times 10^9 \times 12.1 \times 10^{-9}}{50} = j1.52$$

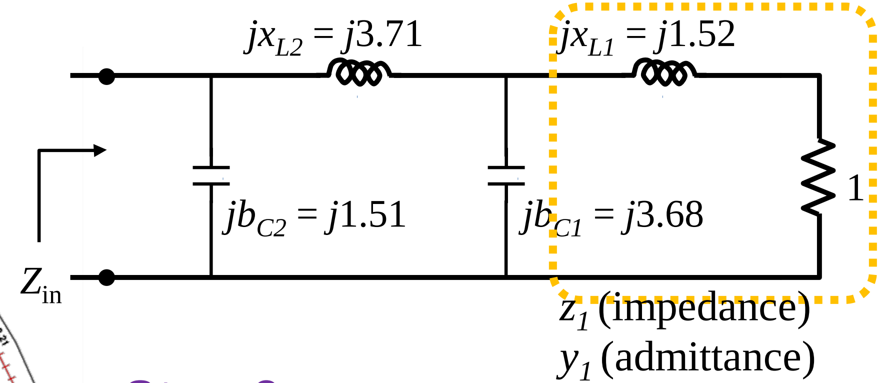
$$jx_{L2} = j\frac{2\pi \times 10^9 \times 29.5 \times 10^{-9}}{50} = j3.71$$

$$jb_{C1} = j\frac{\omega C}{Y_0} = j\frac{2\pi \times 10^9 \times 11.7 \times 10^{-12}}{0.02} = j3.68$$

$$jb_{C2} = j\frac{2\pi \times 10^9 \times 4.8 \times 10^{-12}}{0.02} = j1.51$$

# Shunt and Series Impedances (3/6)




## Step 2:

- Begin by the right side of the circuit
- Mark $r = 1$ on the $Z$ chart

## Step 3:

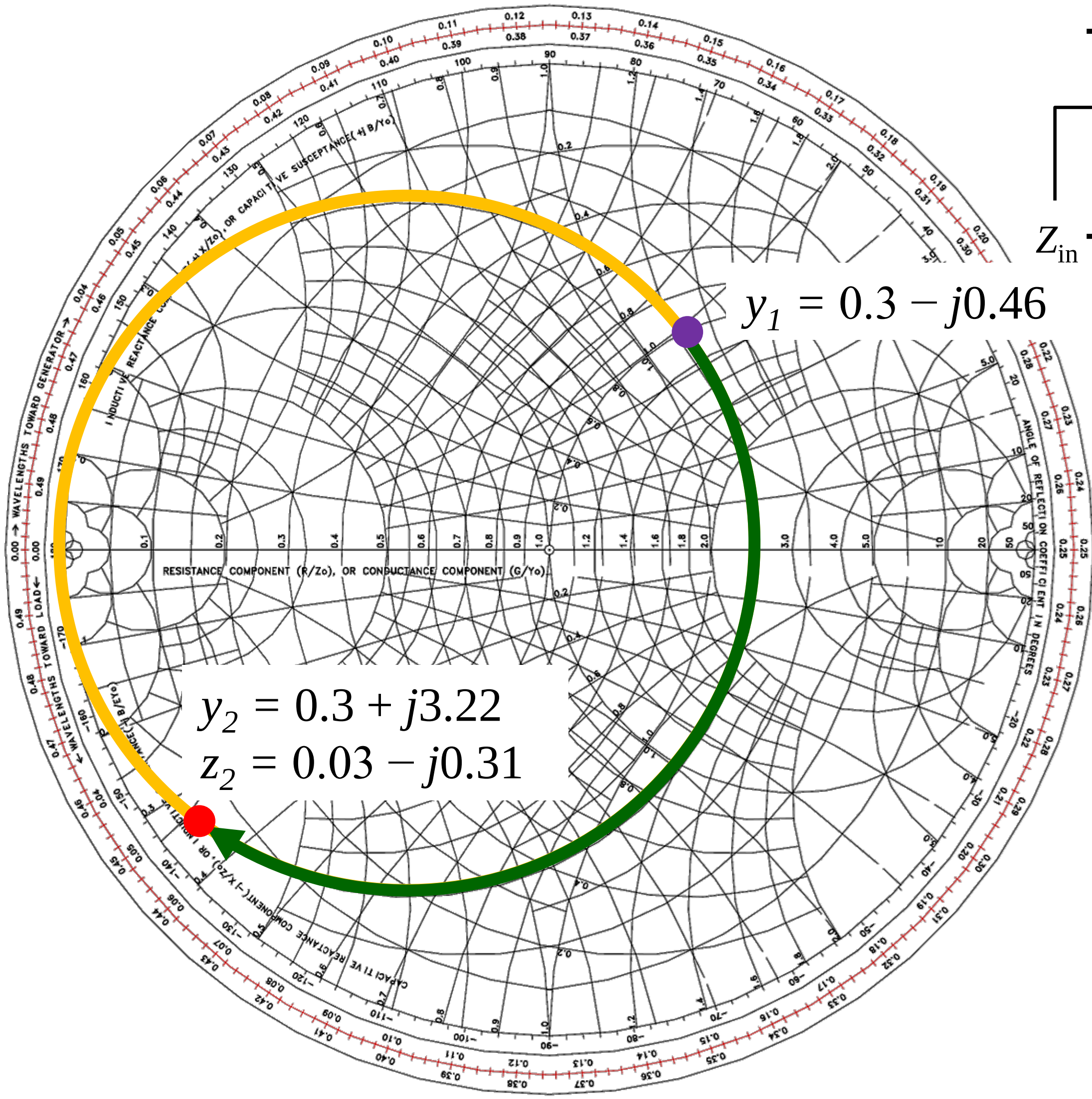
- $r = 1$ is connected with $jx_{L1}$ in series, so let's find the overall impedance $z_1$ on $Z$ chart

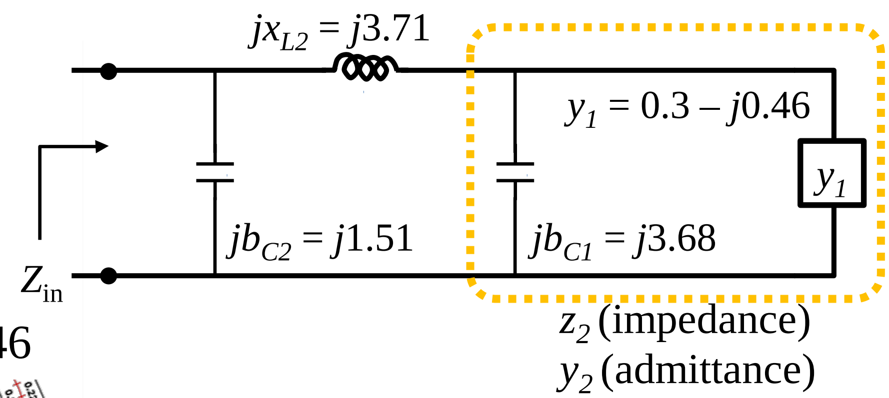
$$z_1 = 1 + j1.52$$

- Since we are going to connect $z_1$ to $jb_{C1}$ in parallel, read its admittance on $Y$ chart:

$$y_1 = 0.3 - j0.46$$

# Shunt and Series Impedances (4/6)




## Step 4:

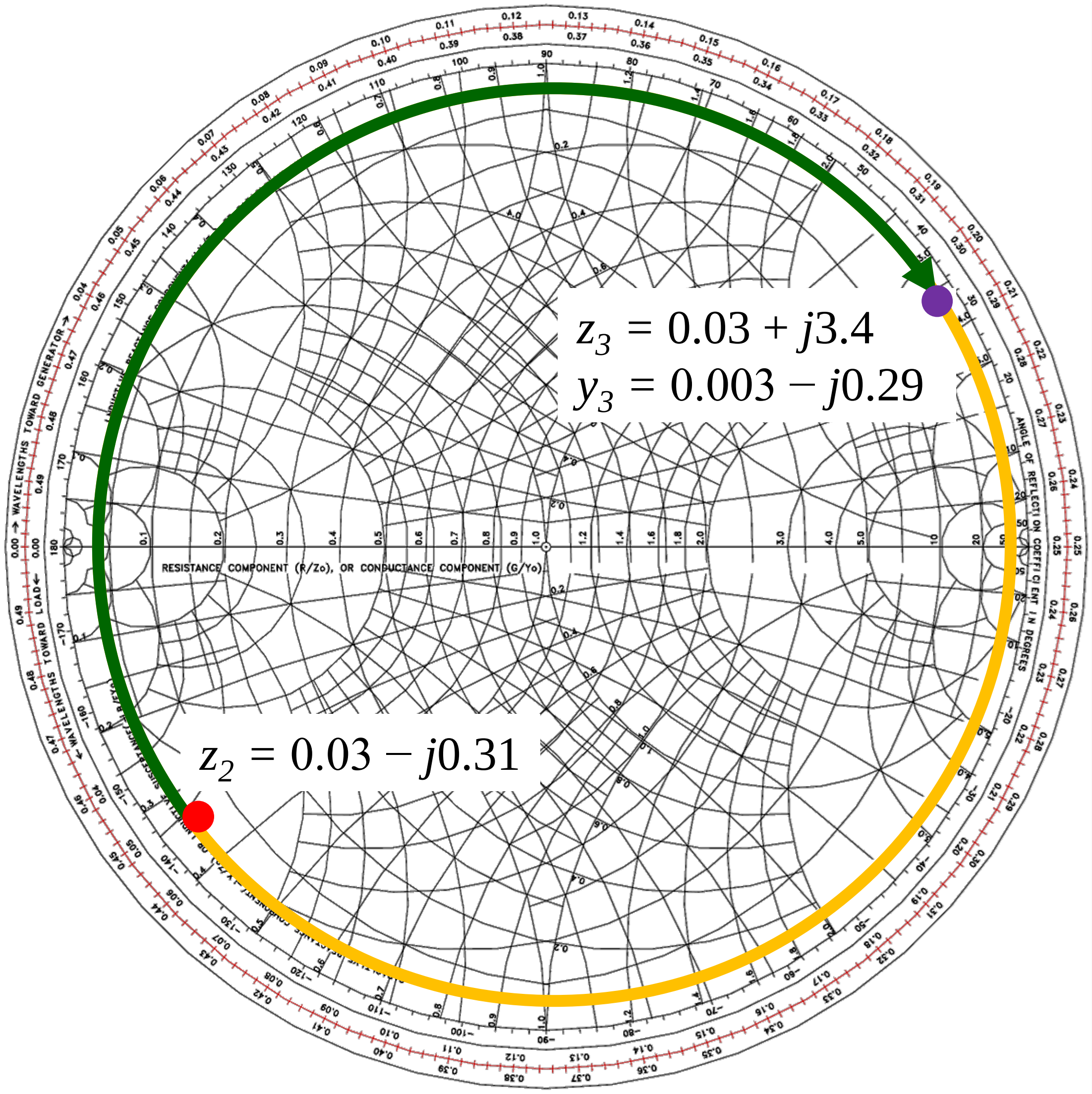
- The shunt admittance of $y_1$ and $jb_{C1}$ can be determined by the rotation on constant $g$ circle

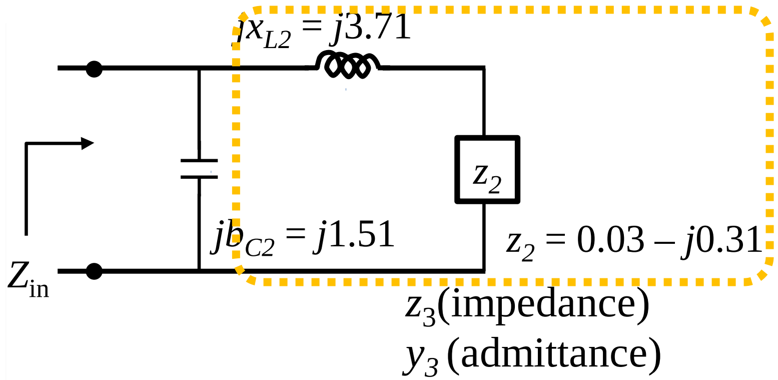
$$y_2 = (0.3 - j0.46) + j3.68 = 0.3 + j3.22$$

- Since we are going to connect $y_2$ to $jx_{L2}$ in series, read its impedance on $Z$ chart:

$$z_2 = 0.03 - j0.31$$

# Shunt and Series Impedances (5/6)




## Step 5:

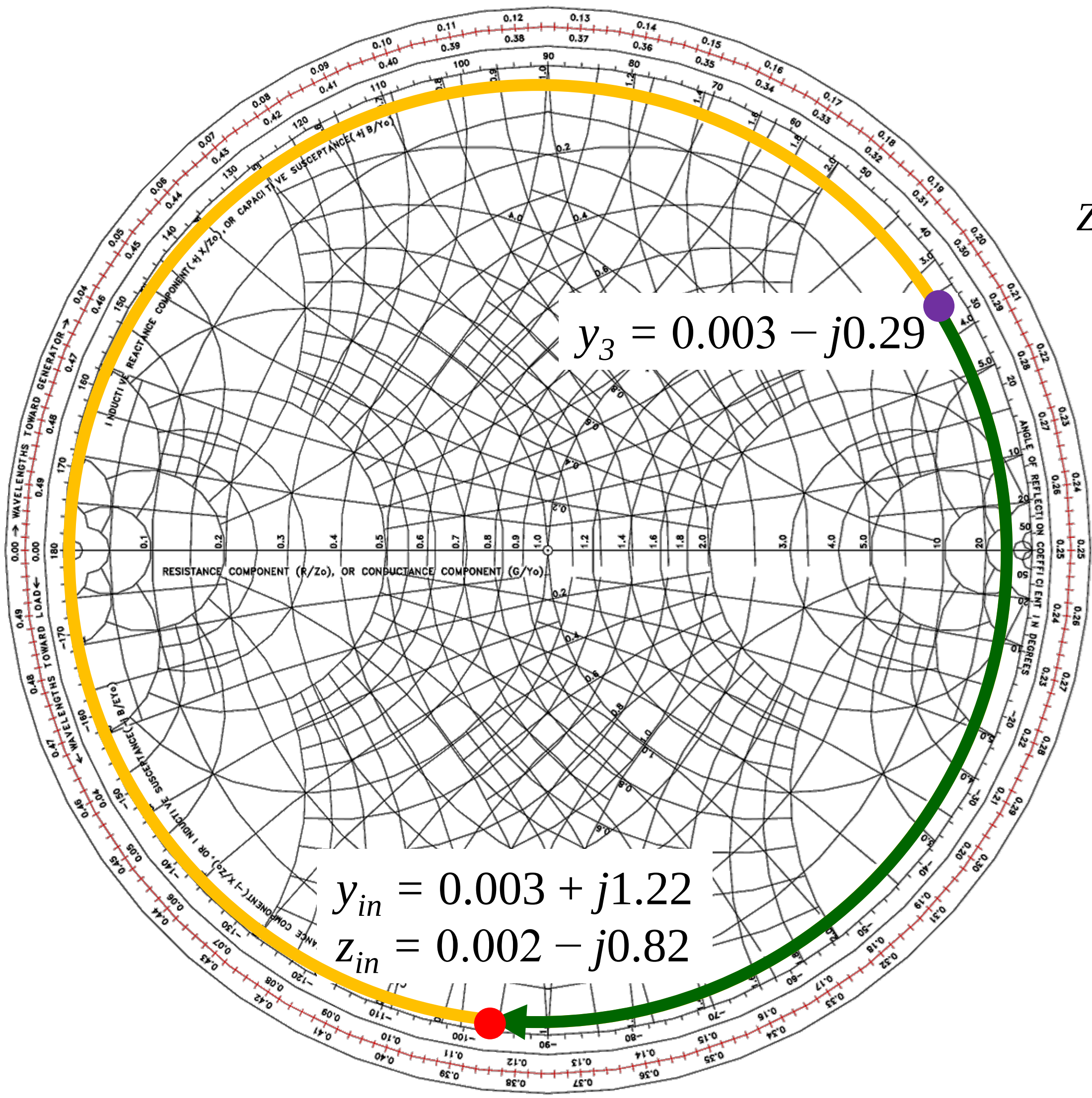
- The series impedance of $z_2$ and $jx_{L2}$ can be determined by the rotation on constant $r$ circle

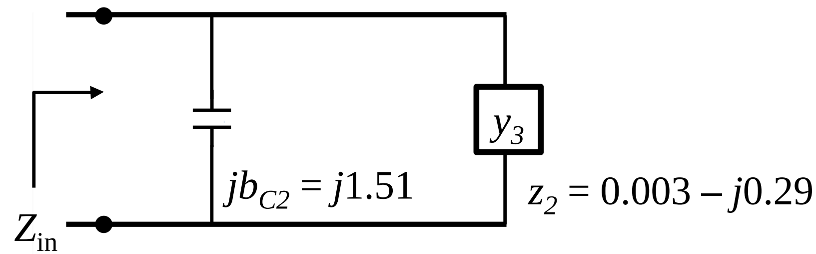
$$z_3 = (0.03 - j0.31) + j3.71 = 0.03 + j3.4$$

- Since we are going to connect $z_3$ to $jx_{L2}$ in series, read its impedance on $Z$ chart:

$$y_3 = 0.003 - j0.29$$

# Shunt and Series Impedances (6/6)




## Step 6:

- The shunt impedance of $y_3$ and $jb_{C2}$ can be determined by the rotation on constant $g$ circle

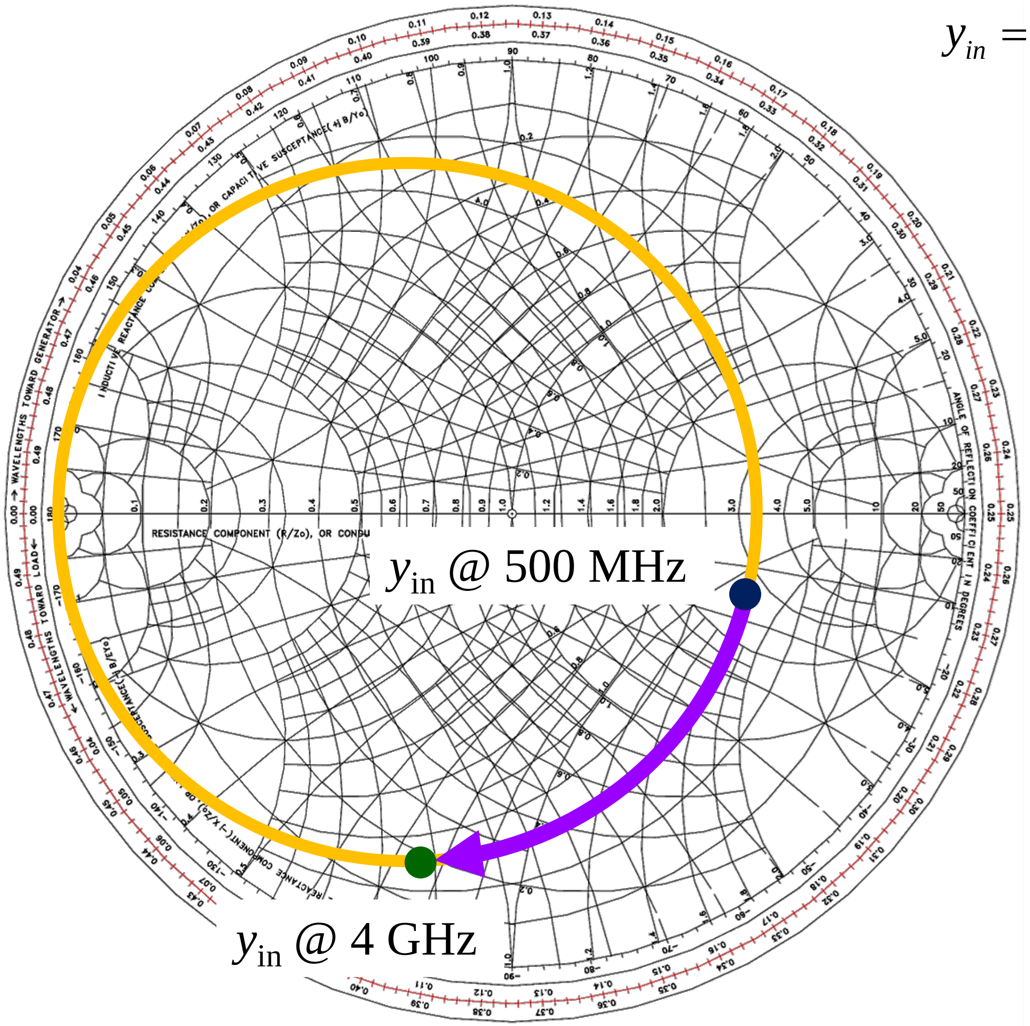
$$y_{in} = (0.003 - j0.29) + j1.51 = 0.003 + j1.22$$

$$\Rightarrow z_{in} = 0.002 - j0.82$$

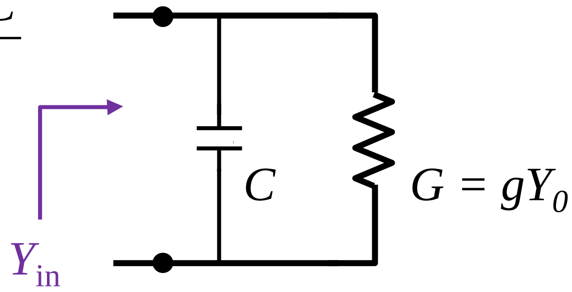
- The input impedance:

$$Z_{in} = Z_0 z_{in} = 0.1 - j40.98\ \Omega$$

# Complex Admittance vs. Frequency (1/2)




$$y_{in} = g + j\frac{\omega C}{Y_0}$$




**Frequency response of network:**

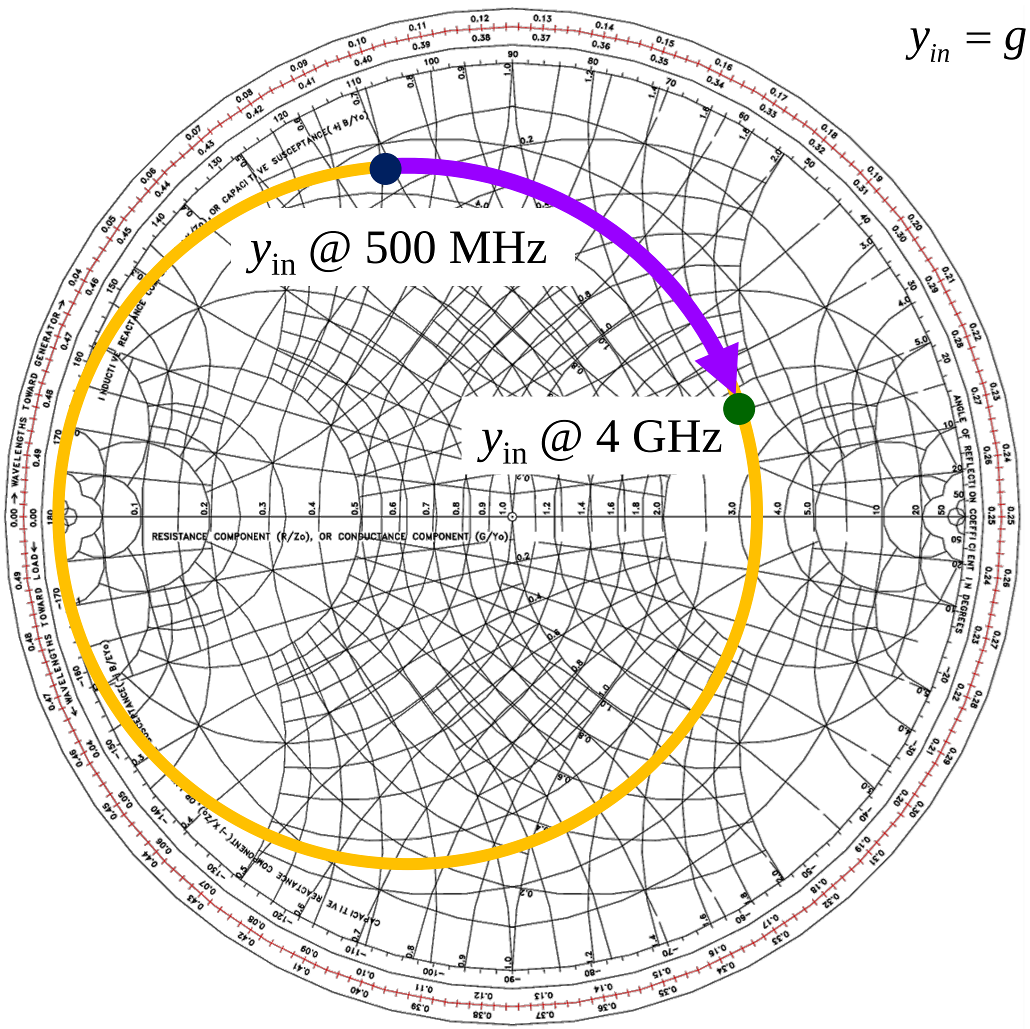
- The input admittance changes with the angular frequency $\omega$
- $\omega \uparrow$, $b_{in} \uparrow$
- $y_{in}$ moves along the constant $g$ circle
- Example: $g = 0.3$, $C = 1$ pF, $Z_0 = 50\ \Omega$, and $f = 500$ MHz – 4 GHz

$$y_{in} = 0.3 + j(\omega C/Y_0)$$

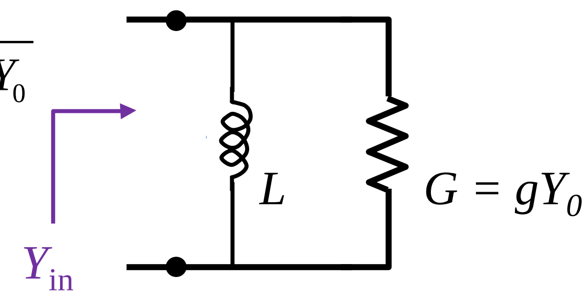
500 MHz: $z_{in} = 0.3 + j0.16$

4 GHz: $y_{in} = 0.3 + j1.26$

# Complex Admittance vs. Frequency (2/2)




$$y_{in} = g - j\frac{1}{\omega L Y_0}$$




Frequency response of network:

- The input admittance changes with the angular frequency $\omega$
- $\omega \uparrow$, $|b_{in}| \downarrow$, $b_{in} \uparrow$
- $y_{in}$ moves along the constant $g$ circle
- Example: $g = 0.3$, $L = 10$ nH, $Z_0 = 50\ \Omega$, and $f = 500$ MHz – 4 GHz

$$y_{in} = 0.3 - j(1/\omega L Y_0)$$

500 MHz: $z_{in} = 0.3 - j1.59$

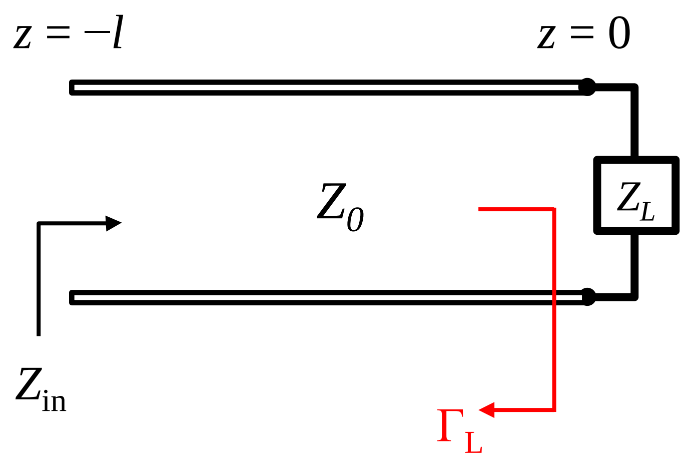
4 GHz: $y_{in} = 0.3 - j0.2$



# Contents

## 2.4 Lossy Transmission Lines

# Two Significant Results of Lossless TL




1. Input impedance

$$Z_{in} = Z_0\left(\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right)$$

2. Power transmission

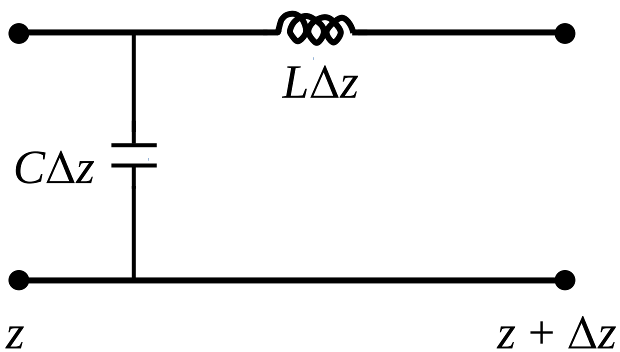
$$P_{av}(z) = \frac{\left|V^+\right|^2}{2Z_0}\left(1 - \left|\Gamma(z)\right|^2\right)$$

$$\Gamma(z=0) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

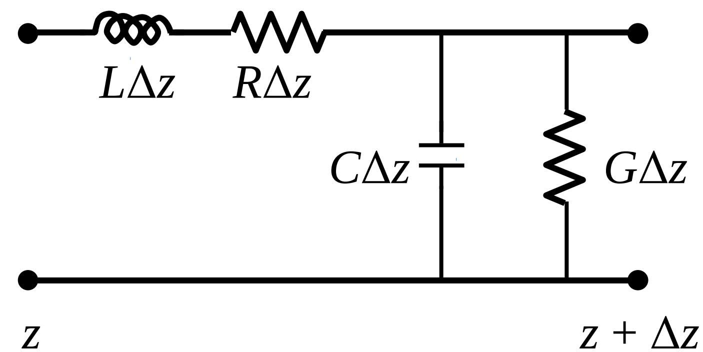
$$RL = -20\log\left|\Gamma(z)\right|$$

# Presumption: From Lossless to Lossy

Lossless transmission lines (§2.1)




Lossy transmission line (§2.4)

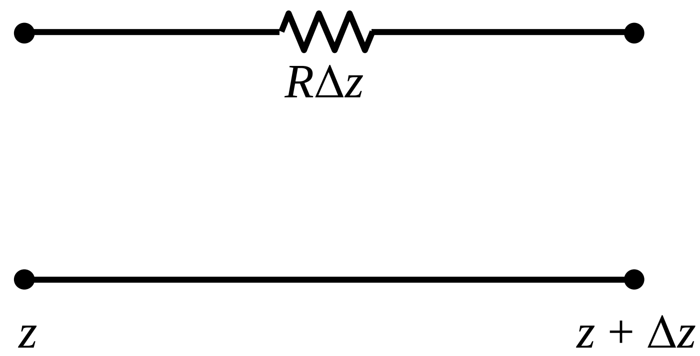



## Why does loss ($R$, $G$) exist?

- $R$: Imperfect conductors
- $G$: Imperfect dielectric
- $R$ and $G$ are also distributed element (unit: $\Omega$/m)

# Transmission Lines at Very Low Frequency

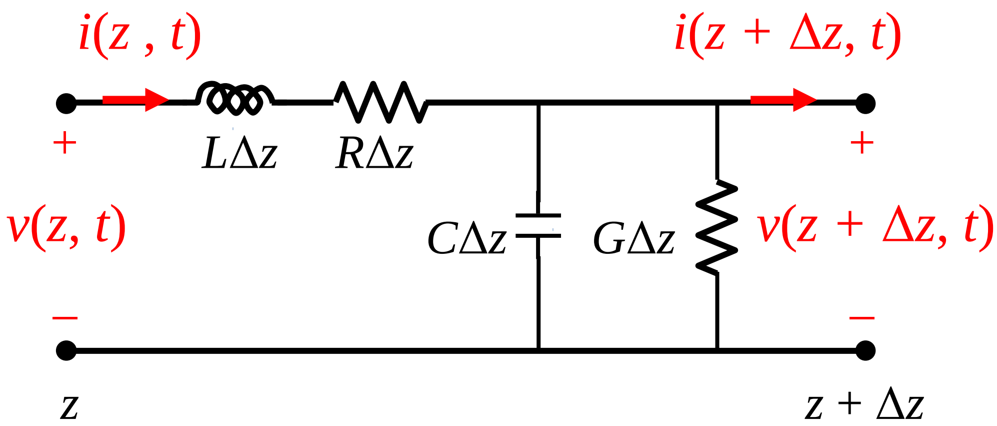
Lossy transmission line at D.C.




- Since $Z_L = j\omega L$, $Z_L \to 0$ when $\omega \to 0$
- Since $Y_C = j\omega C$, $Y_C \to 0$ when $\omega \to 0$
- $G(\omega)$ is proportional to the dielectric conductivity $\sigma(\omega)$; and, $\sigma(\omega)$ is proportional to $\omega$. So $G(\omega) \to 0$

⇒ This situation is identical to that of DC circuit analysis.
But, at high frequencies, the effects of $L$, $C$, $G$ should be evaluated

# Transmission Line Equations (1/2)



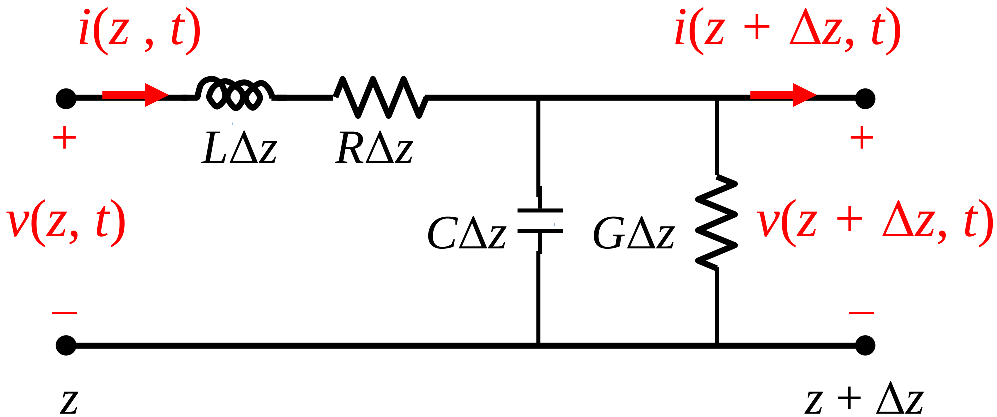

By KVL:

$$-v(z,\ t)+(L\Delta z)\frac{\partial i(z,\ t)}{\partial t}+(R\Delta z)i(z,\ t)+v(z+\Delta z,\ t)=0$$

$$\frac{\partial v(z,\ t)}{\partial z}=-Ri(z,\ t)-L\frac{\partial i(z,\ t)}{\partial t}$$

Phasor expression: $\dfrac{dV}{dz}=-(R+j\omega L)I \qquad (1)$

# Transmission Line Equations (2/2)




By KCL:

Approximated by $v(z, t)$

$$-i(z,\ t)+(C\Delta z)\frac{\partial v(z+\Delta z,\ t)}{\partial t}+(G\Delta z)v(z+\Delta z,\ t)+i(z+\Delta z,\ t)=0$$

$$\Rightarrow \frac{\partial i(z,\ t)}{\partial z}=-Gv(z,\ t)-C\frac{\partial v(z,\ t)}{\partial t}$$

$\Rightarrow$ Phasor expression: $\dfrac{dI}{dz}=-(G+j\omega C)V \qquad (2)$

# Solution of Telegrapher's Equations

$$\frac{d(1)}{dz}:\quad \frac{d^2V}{dz^2} = -\left(R + j\omega L\right)\frac{dI}{dz} \underset{By\ (2)}{=} \left(R + j\omega L\right)\left(G + j\omega C\right)V$$

Define $\gamma^2 = \left(R + j\omega L\right)\left(G + j\omega C\right)$

$$\Rightarrow \quad \frac{d^2V}{dz^2} - \gamma^2 V = 0$$

So the solution can be written as:

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

where $\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)} = \alpha + j\beta$

# Only Two Cases That We Are Interested In (1/2)

## Case 1: Lossless ($R = 0$ and $G = 0$)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC}$$

In other words, $\alpha = 0$ and $\beta = \omega\sqrt{LC}$

The solution of $V(z)$ becomes $V(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}$

Identical to the formulation in §2.2

## Case 2: Low loss and high frequency ($R \ll \omega L$ and $G \ll \omega C$)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{\left(\frac{R}{j\omega L}+1\right)\left(\frac{G}{j\omega C}+1\right)}\left(\sqrt{j\omega L \times j\omega C}\right)$$

$$\approx j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}+\frac{G}{j\omega C}} \approx j\omega\sqrt{LC}\left[1+\frac{1}{2}\left(\frac{R}{j\omega L}+\frac{G}{j\omega C}\right)\right]$$

# Only Two Cases That We Are Interested In (2/2)

Comparing the real part and imaginary part respectively:

$$\alpha = \underbrace{\frac{1}{2}R\sqrt{\frac{C}{L}}}_{\substack{\text{Conductor}\\ \text{loss } (\alpha_c)}} + \underbrace{\frac{1}{2}G\sqrt{\frac{L}{C}}}_{\substack{\text{Dielectric}\\ \text{loss } (\alpha_d)}} \ (\text{unit: Neper/m}),\ \underbrace{\beta = \omega\sqrt{LC}}_{\substack{\text{The same as the}\\ \text{expression of lossless line}}}$$

The solution of $I(z)$:

Substituting $V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$ to $\dfrac{dV}{dz} = -(R + j\omega L)I$

$$\Rightarrow I(z) = \frac{-1}{R + j\omega L}\frac{dV}{dz} = \boxed{\frac{\gamma}{R + j\omega L}}\left(V^+ e^{-\gamma z} - V^- e^{\gamma z}\right)$$

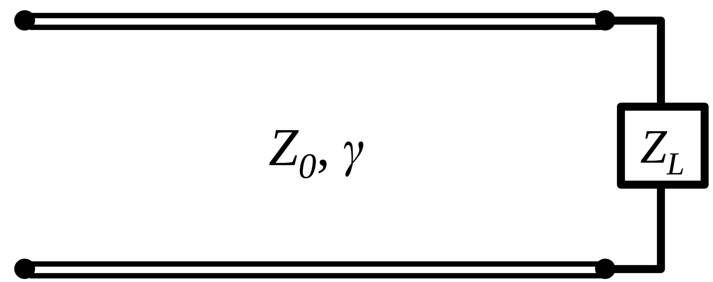
assigned as $1/Z_0$

$$\Rightarrow Z_0 = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{L}{C}} \qquad (\text{unit: } \Omega)$$

# Physical Meaning of the $V(z)$ Expression

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left(V^+ e^{-\gamma z} - V^- e^{\gamma z}\right)$$

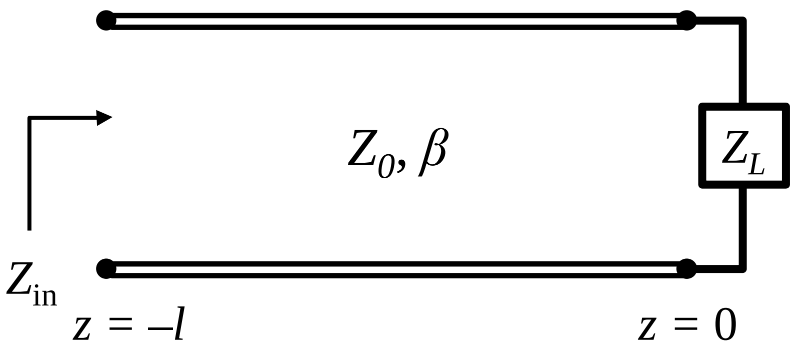


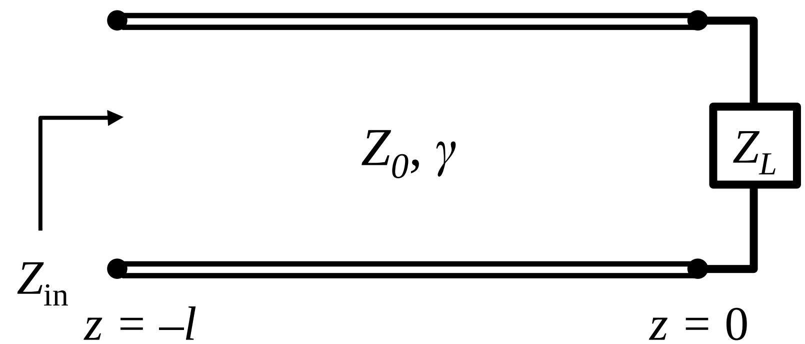

The time-domain expression:

$$v(z,t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\} = \underbrace{\left|V^+\right| e^{-\alpha z}\cos\left(\omega t - \beta z + \phi^+\right)}_{\text{Attenuation wave toward } +z} + \underbrace{\left|V^-\right| e^{\alpha z}\cos\left(\omega t + \beta z + \phi^-\right)}_{\text{Attenuation wave toward } -z}$$

- $\gamma = \alpha + j\beta$: Complex propagation constant
- $\alpha$: Attenuation constant; $\alpha = \alpha_c + \alpha_d$
- $\alpha_c = R/2Z_0$: Attenuation due to conductor loss (ohmic loss)
- $\alpha_d = GZ_0/2$: Attenuation due to dielectric loss
- $\beta$: Phase constant

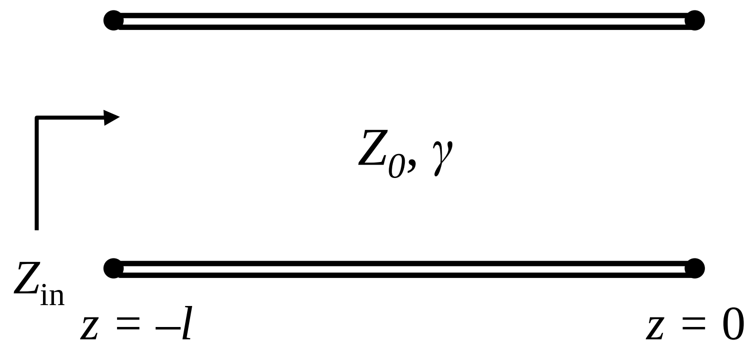
# Lossless Lines vs. Lossy Lines




| Lossless lines | | Lossy lines |
|---|---|---|
| $\beta = \omega\sqrt{LC}$ | Propagation constant | $\gamma = \alpha + j\beta$<br>$\alpha:$ attenuation, $\beta = \omega\sqrt{LC}$ |
| $Z_{in} = Z_0\left(\dfrac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right)$ | Input impedance | $Z_{in} = Z_0\left(\dfrac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l}\right)$ |
| $P_{av}(z) = \dfrac{\lvert V^+\rvert^2}{2Z_0}\left(1 - \lvert\Gamma(z)\rvert^2\right)$ | Time-average power flow along the line | $P_{av}(z) = \dfrac{\lvert V^+\rvert^2}{2Z_0}\left(1 - \lvert\Gamma(z)\rvert^2\right)e^{-2\alpha z}$ |

# Input Impedance of Lossy Lines (1/3)



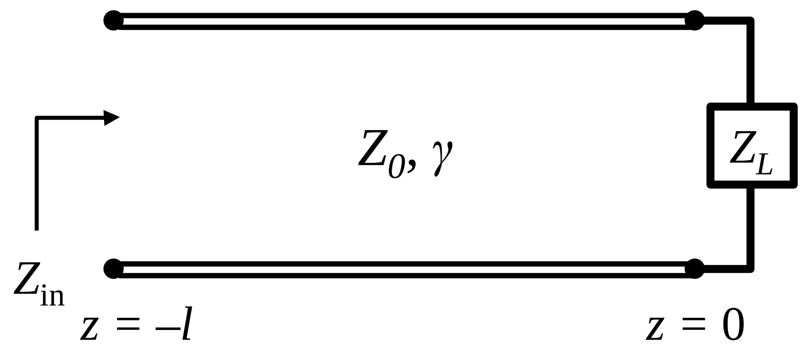

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left(V^+ e^{-\gamma z} - V^- e^{\gamma z}\right)$$

Step 1: Define reflection coefficient $\Gamma(z)$ and formulate the impedance on TL

$$\Gamma(z) \triangleq \frac{V^- e^{\gamma z}}{V^+ e^{-\gamma z}} = \left(\frac{V^-}{V^+}\right) e^{2\gamma z} \quad \text{and} \quad Z(z) = \frac{V(z)}{I(z)} = \frac{V^+ e^{-\gamma z}\left[1+\Gamma(z)\right]}{\frac{1}{Z_0} V^+ e^{-\gamma z}\left[1-\Gamma(z)\right]} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

# Input Impedance of Lossy Lines (2/3)




NOW $Z(z = 0) = Z_L$

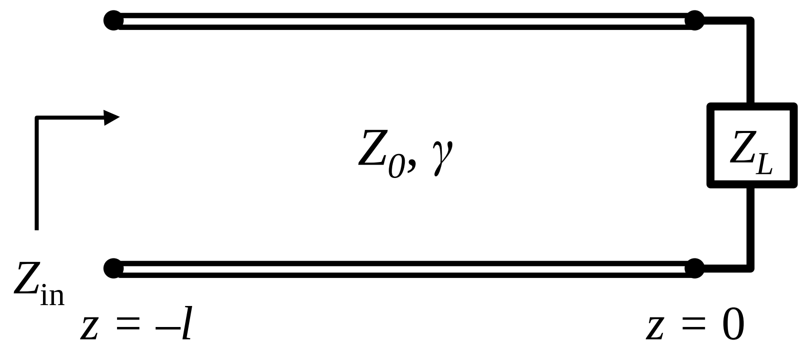
Step 2: Find the reflection coefficient at $z = 0$

$$\Gamma_L = \Gamma(z=0) = \frac{V^-}{V^+}$$ ⇨ Substitute it into $Z(z)$:

$$Z\left(z=0\right) = Z_0 \frac{1+\Gamma(z=0)}{1-\Gamma(z=0)}$$ That is, $$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}$$

Therefore, we have $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ and we can write down $\Gamma(z) = \Gamma_L e^{2\gamma z}$

# Input Impedance of Lossy Lines (3/3)



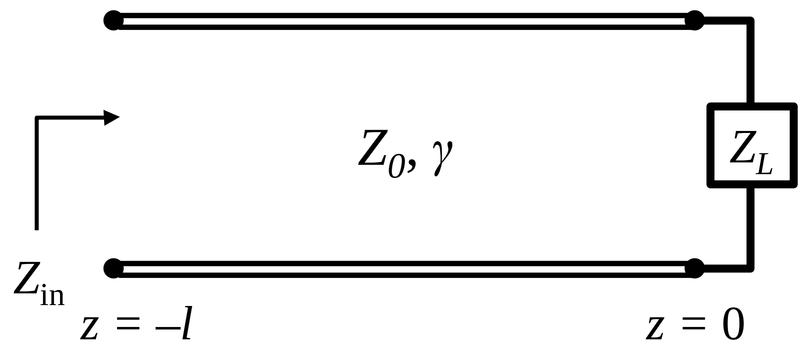

Step 3: Derive the input impedance $Z_{in} = Z(z = -l)$

Since $Z(z) = Z_0 \dfrac{1+\Gamma(z)}{1-\Gamma(z)}$ and $\Gamma(z) = \dfrac{Z_L - Z_0}{Z_L + Z_0} e^{2\gamma z}$

$$\Rightarrow \quad Z(z) = Z_0 \frac{(Z_L + Z_0)e^{-\gamma z} + (Z_L - Z_0)e^{\gamma z}}{(Z_L + Z_0)e^{-\gamma z} - (Z_L - Z_0)e^{\gamma z}} = Z_0 \left( \frac{Z_L + Z_0 \tanh(-\gamma z)}{Z_0 + Z_L \tanh(-\gamma z)} \right)$$

At the input terminal $(z = -l)$, $Z_{in} = Z_0 \left( \dfrac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right)$

# Time-Average Power in a Lossy Line (1/2)




- The time-average power flow along the line at the point $z$:

$$P_{av} = \frac{1}{2}\text{Re}\left[V(z)I(z)^*\right]$$

- Substituting $V(z) = V^+e^{-\gamma z}[1+\Gamma(z)]$, $I(z) = \dfrac{V^+e^{-\gamma z}}{Z_0}\left[1-\Gamma(z)\right]$ into $P_{av}$:

(The derivation is more complicated than the lossless case)

# Time-Average Power in a Lossy Line (2/2)

$$P_{av} = \frac{1}{2}\text{Re}\left[V(z)I(z)^*\right] = \frac{1}{2}\text{Re}\left\{V^+e^{-\gamma z}\left[1+\Gamma(z)\right]\frac{V^{+*}}{Z_0^*}e^{-\gamma^* z}\left[1-\Gamma^*(z)\right]\right\}$$

$$= \frac{1}{2}\text{Re}\left\{\frac{\left|V^+\right|^2}{Z_0^*}e^{-2\alpha z}\left[1-\left|\Gamma(z)\right|^2 + j2\,\text{Im}\left(\Gamma(z)\right)\right]\right\}$$

$$= \frac{1}{2}\left|V^+\right|^2 e^{-2\alpha z}\left\{G_0\left(1-\left|\Gamma(z)\right|^2\right)+2B_0\times\text{Im}\left(\Gamma(z)\right)\right\}$$

where $Y_0 = G_0 + jB_0$

- If the imaginary part of $Z_0$ can be neglected, then

$$P_{av}(z) = \frac{\left|V^+\right|^2}{2Z_0}\left(1-\left|\Gamma(z)\right|^2\right)e^{-2\alpha z}$$

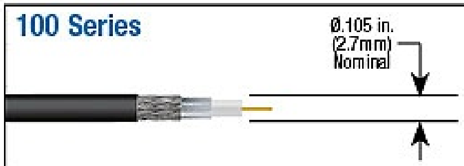
Now $P_{av}(z)$ is a function of $z$

- The attenuation of the line (or we say the insertion loss "IL" of the line):

$$-10\log e^{-2\alpha l} \approx \alpha l \times 8.69 \text{ (dB)}$$

# Attenuation in Coaxial Cables

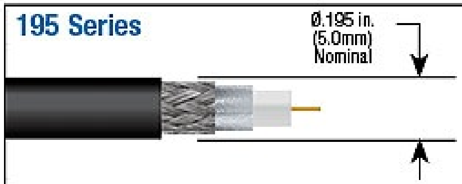
### 100 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 22.8 | 74.8 |
| 1800 | 33.2 | 108.8 |
| 2500 | 39.8 | 130.6 |

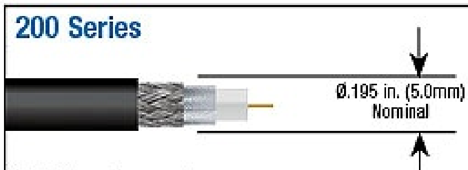
### 195 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 11.1 | 36.5 |
| 1800 | 16.0 | 52.5 |
| 2500 | 19.0 | 62.4 |
| 5800 | 29.9 | 98.1 |

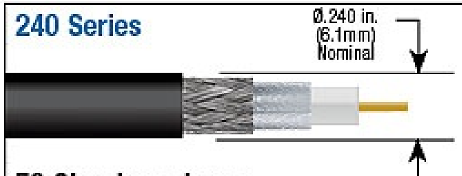
### 200 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 9.9 | 32.6 |
| 1800 | 14.2 | 46.6 |
| 2500 | 16.9 | 55.4 |
| 5800 | 26.4 | 86.5 |

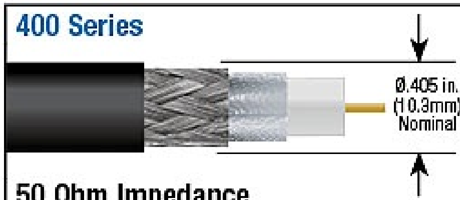
### 240 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 7.6 | 24.8 |
| 1800 | 10.9 | 35.6 |
| 2500 | 12.9 | 45.4 |
| 5800 | 20.4 | 66.8 |

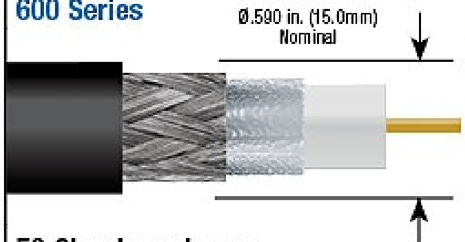
### 400 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 3.9 | 12.8 |
| 1800 | 5.7 | 18.6 |
| 2500 | 6.8 | 22.2 |
| 5800 | 10.8 | 35.5 |

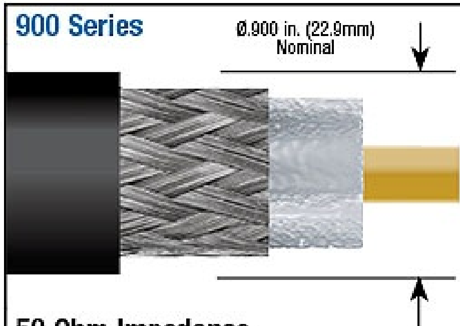
### 600 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 2.5 | 8.2 |
| 1800 | 3.7 | 12.1 |
| 2500 | 4.4 | 14.5 |
| 5800 | 7.3 | 23.8 |

### 900 Series




50 Ohm Impedance

| NOMINAL ATTENUATION | | |
|---|---|---|
| MHz | db/100ft | db/100m |
| 900 | 1.7 | 5.6 |
| 1800 | 2.5 | 8.2 |
| 2500 | 2.9 | 9.8 |
| 5800 | 4.9 | 16.0 |

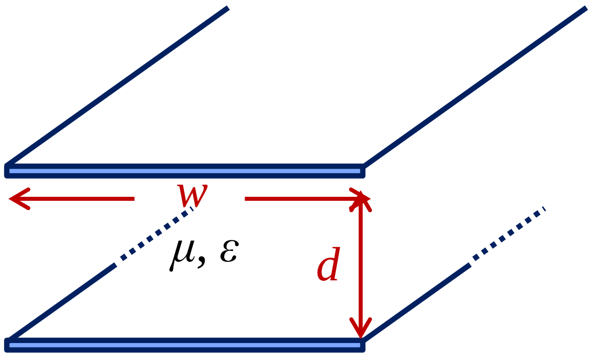


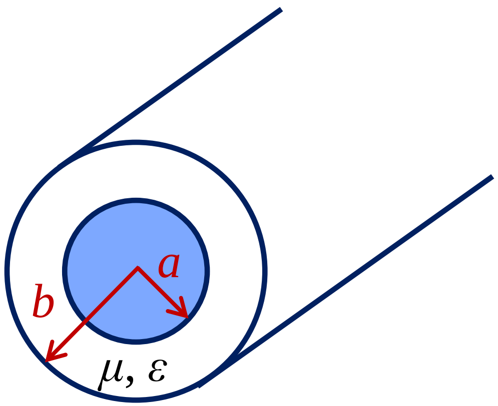
Low Loss Coaxial Cable for Wireless Applications. [Online] Available: http://www.l-com.com/blog/post/2013/06/26/Low-Loss-Coaxial-Cable.aspx

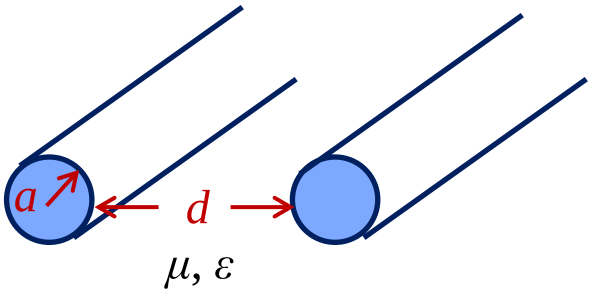
# Three-Dimensional Structures (1/2)

(Good for long distance communication)

| Parallel-plate waveguide | Coaxial cable | Parallel cylindrical wires |
| --- | --- | --- |




$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon'(1 - j\tan\delta) = \varepsilon' + j(\sigma/\omega)$ is the complex permittivity

Parallel-plate waveguide:

$$C = \varepsilon' \frac{w}{d}$$

$$L = \mu \frac{d}{w}$$

$$Z_0 \approx \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon'}} \frac{d}{w}$$

Coaxial cable:

$$C = \frac{2\pi\varepsilon'}{\ln(b/a)}$$

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

$$Z_0 \approx \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon'}} \ln \frac{b}{a}$$

Parallel cylindrical wires:

$$C = \frac{\pi\varepsilon'}{\cosh^{-1}(d/2a)}$$

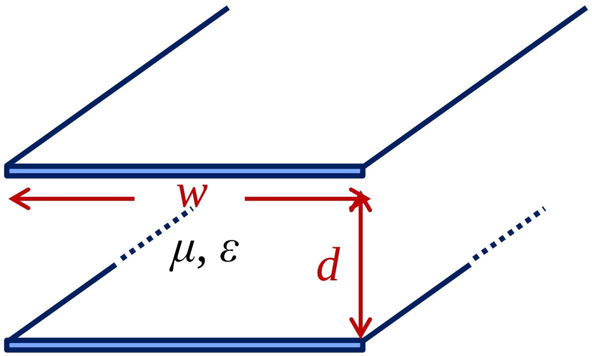
$$L = \frac{\mu}{\pi} \cosh^{-1}(d/2a)$$

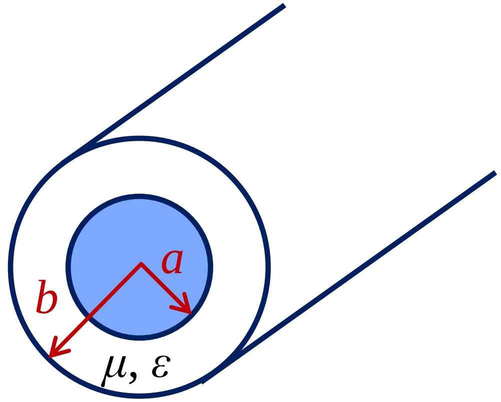
$$Z_0 \approx \sqrt{\frac{L}{C}} = \frac{1}{\pi} \sqrt{\frac{\mu}{\varepsilon'}} \cosh^{-1}(d/2a)$$

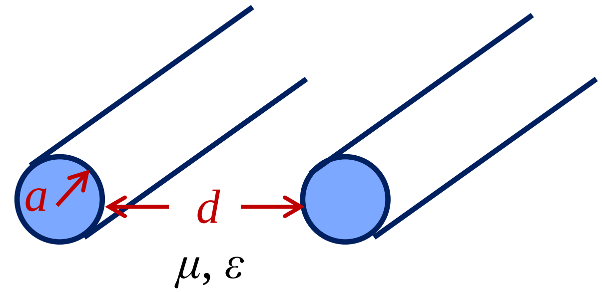
# Three-Dimensional Structures (2/2)

(Good for long distance communication)

| Parallel-plate waveguide | Coaxial cable | Parallel cylindrical wires |
| --- | --- | --- |




$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon'\left(1 - j\tan\delta\right) = \varepsilon' + j\left(\sigma/\omega\right)$ is the complex permittivity

| Parallel-plate waveguide | Coaxial cable | Parallel cylindrical wires |
| --- | --- | --- |
| $R = \dfrac{2R_S}{w}$ | $R = \dfrac{R_S}{2\pi}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ | $R = \dfrac{R_S}{\pi a}$ |
| $G = \omega\varepsilon''\dfrac{w}{d}$ | $G = \omega\varepsilon''\dfrac{2\pi}{\ln\dfrac{b}{a}}$ | $G = \omega\varepsilon''\dfrac{\pi}{\cosh^{-1}\left(d/2a\right)}$ |

$R_S = \dfrac{1}{\sigma\delta_S}$ is the surface resistance of the conductor ($\sigma$: conductivity (S/m); $\delta_S = \sqrt{\dfrac{2}{\omega\mu\sigma}}$: skin depth (m))

# EM Properties in Some Dielectrics

## EM characteristics in dielectrics:

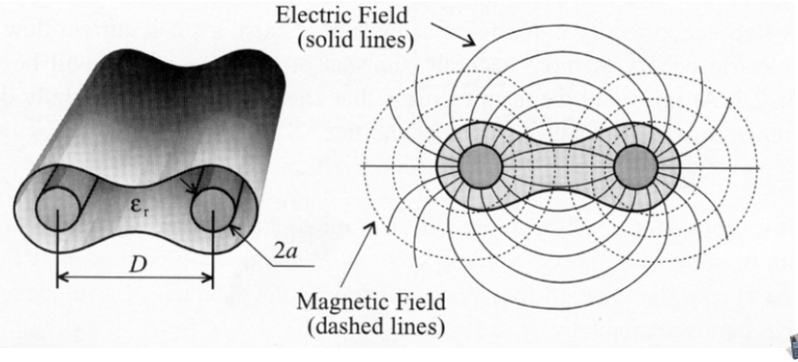
- Phase velocity: $v_p = \sqrt{\frac{1}{\varepsilon\mu}} = \sqrt{\frac{1}{\varepsilon_0\varepsilon_r\mu_0}} = \frac{C}{\sqrt{\varepsilon_r}}$
- So, the wavelength in the material: $\lambda_g = \frac{v_p}{f} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$  $\lambda_0$: the free-space wavelength
- The larger the dielectric constant, the smaller the associated wavelength is; hence, the microwave component can be fabricated smaller

| Material | Dielectric constant ($\varepsilon_r$) |
|---|---|
| Air | 1.0054 |
| FR4 (玻璃纖維板) | 4.4 |
| Epoxy (環氧樹脂) | 3.9 |
| Polyimide (聚亞醯胺) | 4.5 |
| Polytetrafluoroethylene (聚四氟乙烯) | 2.1 |

| Material | Dielectric constant ($\varepsilon_r$) |
|---|---|
| Polystyrene (聚苯乙烯) | 2.5 – 2.6 |
| Polypropylene (聚丙烯) | 2.25 |
| Ceramic (陶瓷) | 3 – 7 |
| Duroid RO 5880 | 2.2 |
| Duroid RO 4003 | 3.55 |

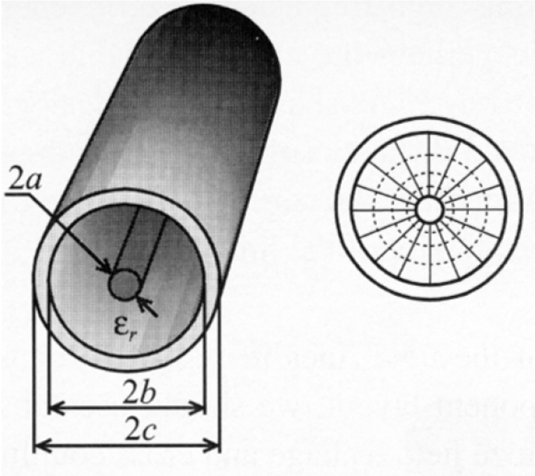
# Practical Issues of 3-D TLs

## Parallel cylindrical wires:




- A bad example of transmitting RF voltage and current waves
  - Electric and magnetic field lines emanating from the conductors extend to infinity and thus tend to influence electronic equipment in the vicinity of the lines
  - Radiation loss tends to be very high
- Applications: connecting TV set to receiving antenna, 50-60 Hz power line, and local telephone connection

## Coaxial cable:




- Used for almost all cases of RF systems or measurement equipment (up to 18 GHz)
- The outer conductor is grounded, so the radiation loss and field interference are minimized
- Coaxial connectors: a special structure of coaxial cable

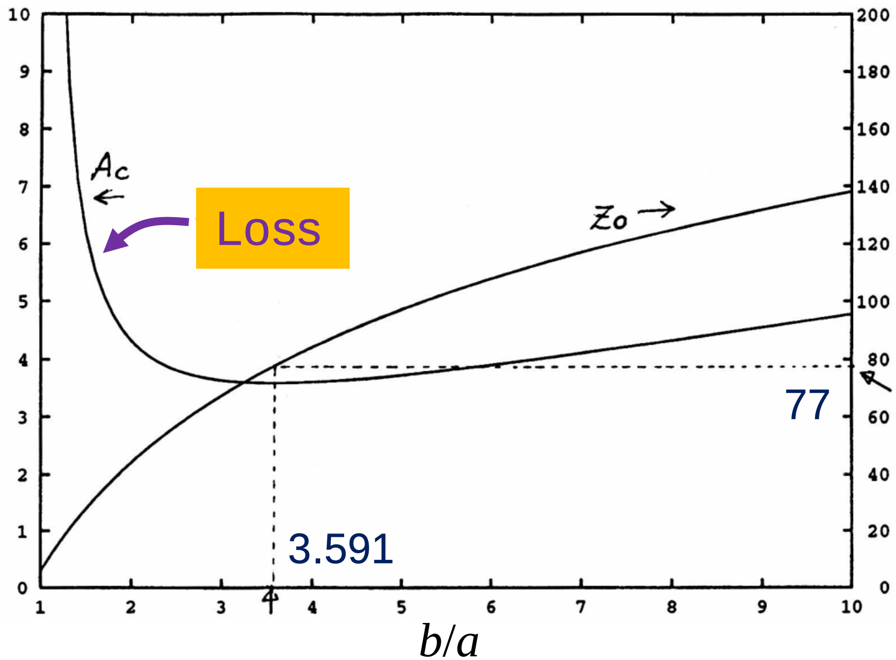
R. Ludwig, *RF Circuit Design: Theory & Applications*, 2nd edition, Pearson, 2009.

# Standard Coaxial Cables

| RG or M17 | Characteristic impedance (Ω) | Dielectric Type | Loss at 1 GHz (dB/100 ft) | Capacitance (pF/ft) |
|---|---|---|---|---|
| RG 8A/U | 50 | PE | 9.0 | 29.5 |
| RG 213A/U | 50 | Foam PE | 9.0 | 30.8 |
| RG 188A/U | 50 | Solid TFE | 30.0 | 29.4 |
| RG 179B/U | 75 | Solid TFE | 25.0 | 19.5 |
| RG 141A/U | 50 | Solid TFE | 13.0 | 29.4 |
| M17/90 | 93 | Air space PE | | 13.5 |
| M17/56 | 95 | PE | | 17.0 |
| M17/6 | 75 | PE | | 20.6 |
| M17/95 | 95 | Solid TFE | | 15.4 |

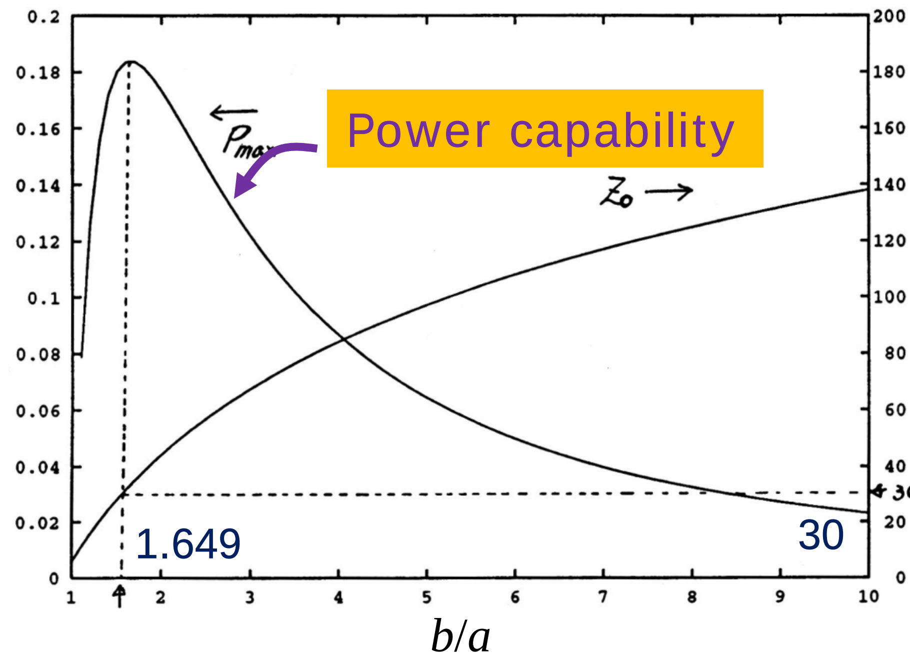
# Why Do We Use 50-Ω Coaxial Lines?

For the aspect of loss:




- Different geometric parameters result in different $Z_0$ and different $\alpha$
- For an air-filled coaxial line, loss has the minimum value at $Z_0 = 77\ \Omega$

For the aspect of breakdown:




- Different geometric parameters result in different power capacity (that is, the time-average power at breakdown)
- For an air-filled coaxial line, power capability has the maximum value at $Z_0 = 30\ \Omega$

To compromise between $77\ \Omega$ and $30\ \Omega$, we choose $Z_0 = 50\ \Omega$

# Coaxial Connectors



(Invax System Technology Corp)

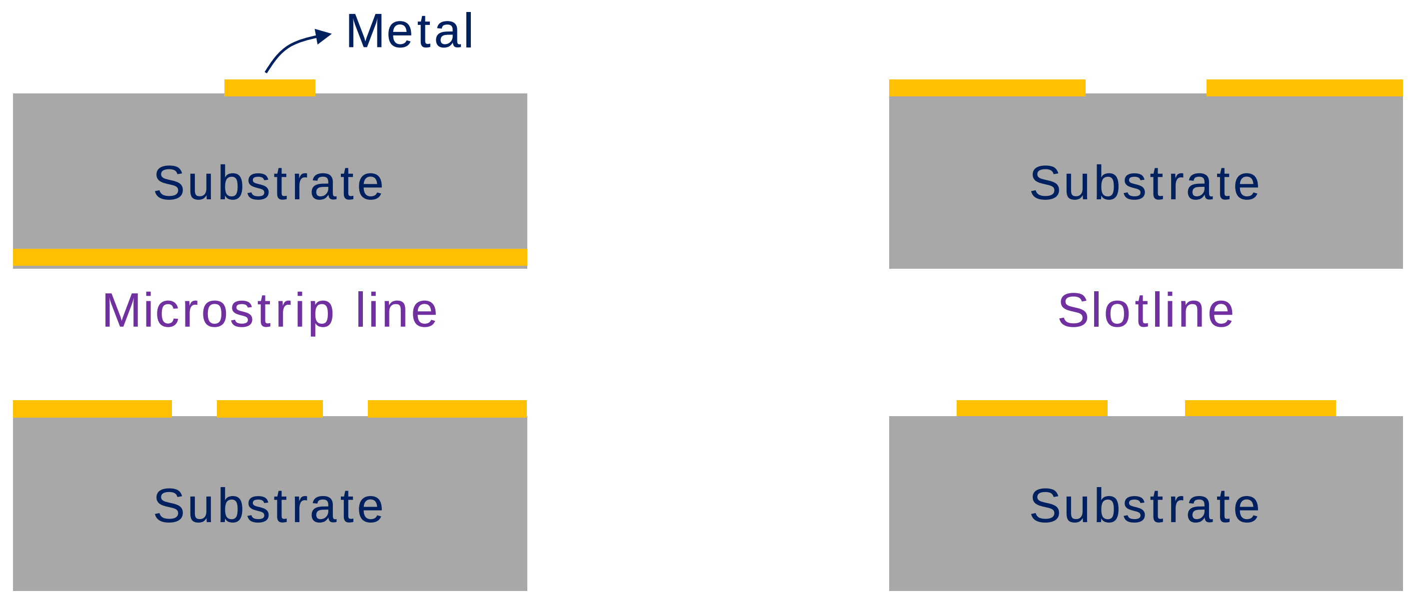
- **Type-N connector**: The outer diameter of the female end is 0.625 in. (passband: 11-18 GHz)
- **TNC connector**: This connector is similar to the very common BNC connector (passband: up to 1 GHz)
- **APC-7 connector**: This connector is sexless (passband: up to 18 GHz)
- **SMA connector**: The outer dimension of the female end is 0.25 in. This connector can be used up to 18-25 GHz and is the most popular coaxial connector for RF and microwave measurement applications



WENTEQ Microwave Corp.. [Online] Available: http://www.wenteq.com/Products/sma.html

# Planar Structures

(Good for microwave integrated circuit applications)




Microstrip line

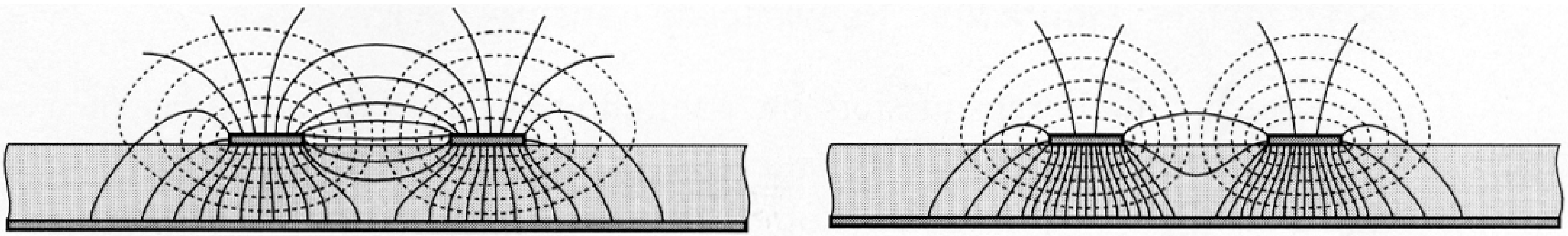
Slotline

Coplanar waveguide (CPW)

Coplanar strip (CPS)

- Their parameters (characteristic impedance $Z_0$, attenuation constant $\alpha$, phase constant $\beta$) depend on geometric characteristics and dielectric properties $\varepsilon_r$
- By carefully determining the geometric characteristics (such as width, height, spacing, etc.), a number of desired $Z_0$ are thus obtained; thus, planar structures are popular for microwave IC applications

# Problem of Field Leakage



(a) Teflon epoxy ($\varepsilon_r = 2.55$) (b) Alumina ($\varepsilon_r = 10.0$)

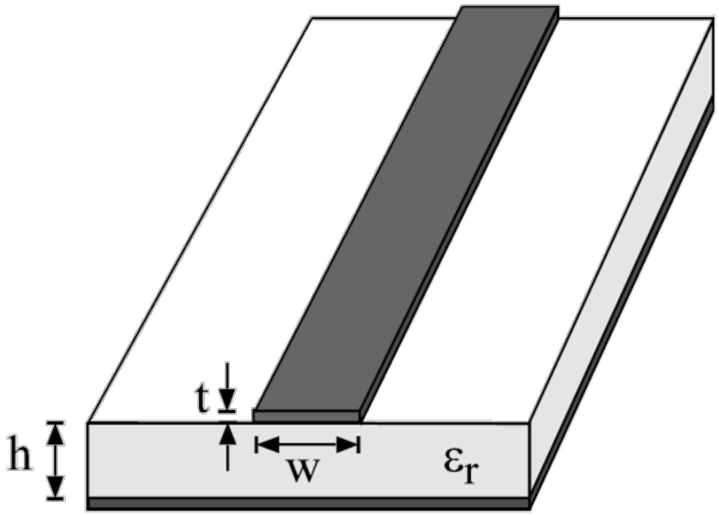
## PCB transmission line:

- They have rather high radiation loss (in comparison with 3-D TLs)
- They are prone to "crosstalk" (interfere) between neighboring conductor systems
- The severity of field leakage depends on the relative dielectric constants
- Substrates having high dielectric constants can minimize field leakage and cross coupling



R. Ludwig, *RF Circuit Design: Theory & Applications*, 2nd edition, Pearson, 2009.

# Microstrip Lines




## EM characteristics in MSL:

- Phase velocity in MSL:

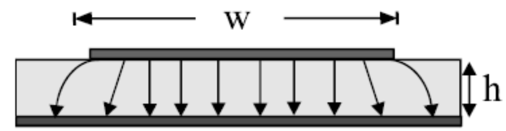
$$v_p = \frac{C}{\sqrt{\varepsilon_{eff}}}$$

- Wavelength in MSL:

$$\lambda = \frac{v_p}{f} = \frac{\lambda_0}{\sqrt{\varepsilon_{eff}}}$$

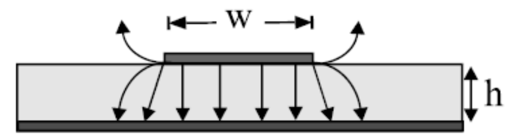
- Effective dielectric constant: the dielectric constant of a homogeneous medium that replaces the air and dielectric regions
- For very wide lines (w/h >> 1):

$$\varepsilon_{eff} = \varepsilon_r$$




- For very narrow lines (w/h << 1):

$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2}$$




Example: We need a 65-Ω MSL to feed a 65-Ω patch antenna. If we choose FR4 ($\varepsilon_r = 4.4$) as a substrate, and the associated height is 0.6 mm, what is the required width for 2-GHz operation?

# MSL Experienced Formulation (Analysis)

## Analysis:

1. If the line width is narrow ($w/h \le 1$)

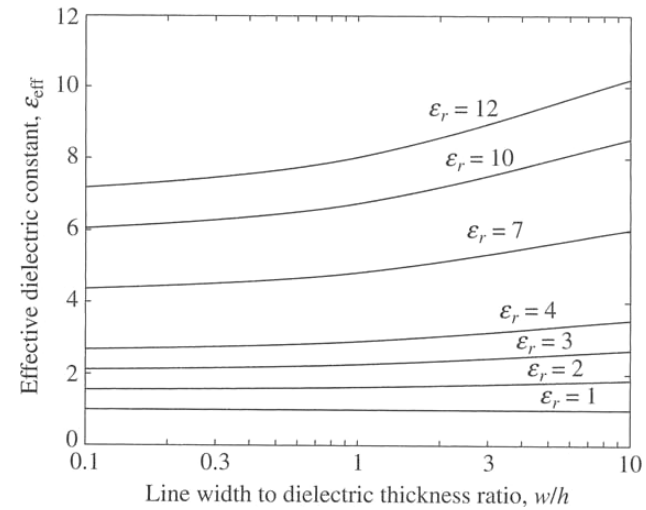
$$Z_0 = \frac{60}{\sqrt{\varepsilon_{eff}}} \ln\left(8\frac{h}{w} + \frac{w}{4h}\right)$$

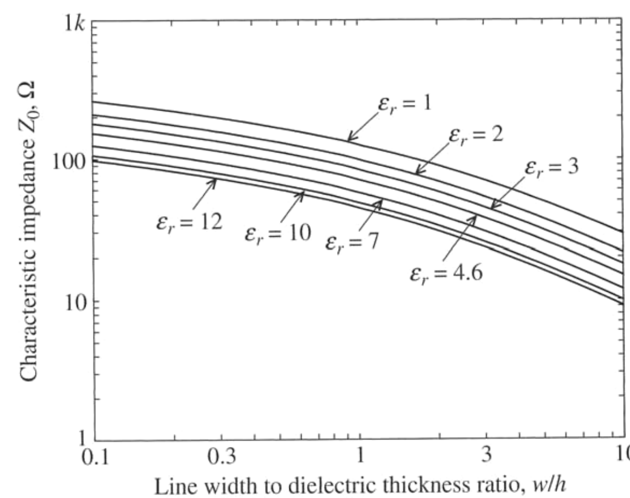
$$\text{where } \varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left[\left(1 + 12\frac{h}{w}\right)^{-0.5} + 0.04\left(1 - \frac{w}{h}\right)^2\right]$$

2. If the line width is wide ($w/h \ge 1$)

$$Z_0 = \frac{120\pi}{\sqrt{\varepsilon_{eff}}\left(1.393 + \frac{w}{h} + 0.667\ln\left(\frac{w}{h} + 1.444\right)\right)}$$

$$\text{where } \varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left(1 + 12\frac{h}{w}\right)^{-0.5}$$






R. Ludwig, *RF Circuit Design: Theory & Applications*, 2nd edition, Pearson, 2009.

# MSL Experienced Formulation (Design)

## Synthesis or design:

1. If the line width is narrow ($w/h < 2$)

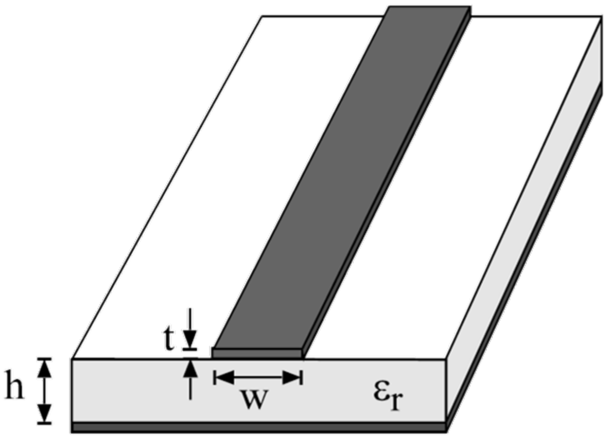
$$\frac{w}{h} = \frac{8e^{A}}{e^{2A} - 2}$$

$$\text{where } A = \frac{Z_0}{60}\sqrt{\frac{\varepsilon_r + 1}{2}} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(0.23 + \frac{0.11}{\varepsilon_r}\right)$$

2. If the line width is wide ($w/h \geq 2$)

$$\frac{w}{h} = \frac{2}{\pi}\left\{B - 1 - \ln\left(2B - 1\right) + \frac{\varepsilon_r - 1}{2\varepsilon_r}\left[\ln\left(B - 1\right)\right] + 0.39 - \frac{0.61}{\varepsilon_r}\right\}$$

$$\text{where } B = \frac{377\pi}{2Z_0\sqrt{\varepsilon_r}}$$






R. Ludwig, *RF Circuit Design: Theory & Applications*, 2nd edition, Pearson, 2009.

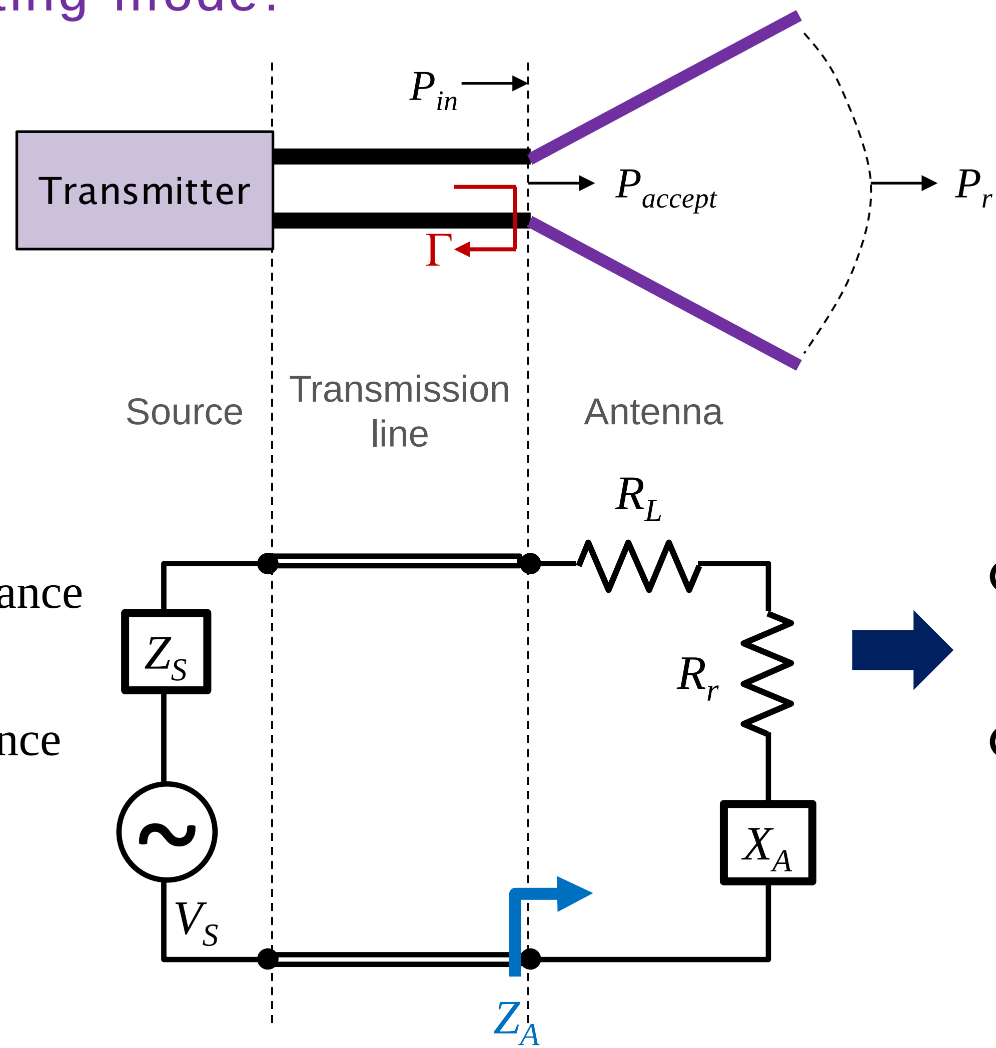


# Contents

## 2.5 Antenna Impedance

# Equivalent Circuit of Transmitting Antennas

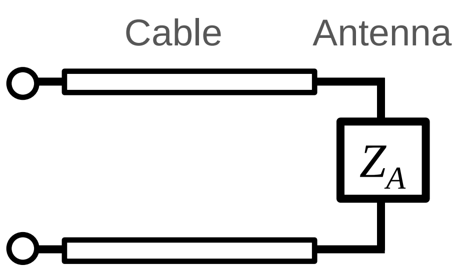
## In transmitting mode:




$R_r$: radiation resistance

$R_L$: loss resistance

$X_A$: antenna reactance




**Antenna impedance**

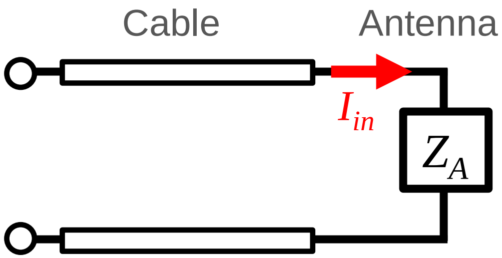
$$Z_A = R_{total} + jX_A$$
$$= (R_r + R_L) + jX_A$$

# Antenna Impedance (1/4)

## Antenna impedance $Z_A$:

- Suitable terminals must be defined for an antenna

$$Z_A = R_{total} + jX_A = (R_r + R_L) + jX_A$$




- The input resistance, $R_{total}$, represents dissipation

$$P_{in} = \frac{1}{2} R_{total} \left| I_{in} \right|^2$$

- $I_{in}$: the current at the input terminals
- $|I_{in}|$: the amplitude of the current at the input terminal of the antenna

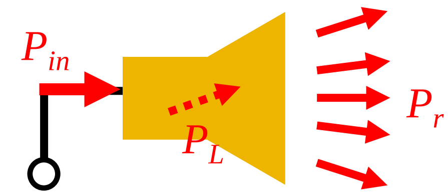
- Power can be dissipated by two ways

## Radiation resistance $R_r$:

- Power leaves the antenna and never returns ⇒ radiation

## Loss resistance $R_L$:

- Heating losses on the antenna structure



# Antenna Impedance (2/4)

| | Radiation resistance $R_r$ |
|---|---|
| Formula | $R_r = \dfrac{2P_r}{\left\|I_{in}\right\|^2}$ |
| How do we calculate the radiated power? | $P_r = \dfrac{1}{2}\iint_{s_{ff}} \left(\mathbf{E}\times\mathbf{H}^*\right)\cdot d\mathbf{s}$ |
| Example (short dipoles) | Given $\mathbf{E} = \dfrac{jk\eta I_{in} L\sin\theta}{4\pi r} e^{-jkr}\hat{\theta},\ \mathbf{H} = \dfrac{jkI_{in} L\sin\theta}{4\pi r} e^{-jkr}\hat{\phi} = \dfrac{\left\|\mathbf{E}\right\|}{\eta} e^{-jkr}\hat{\phi}$ <br> $\Rightarrow \mathbf{E}\times\mathbf{H}^* = \dfrac{1}{\eta}\left\|\dfrac{k\eta I_0 L\sin\theta}{4\pi r}\right\|^2\hat{r} = \dfrac{\eta\sin^2\theta}{4}\left\|\dfrac{I_{in}L}{\lambda}\right\|^2\hat{r},\ P_r = \dfrac{\eta k^2\left\|I_{in}\right\|^2 L^2}{12\pi}$ <br> $\Rightarrow R_r = \dfrac{\eta k^2 L^2}{6\pi} = 80\pi^2\left(\dfrac{L}{\lambda}\right)^2$ |

| $L$ ($\lambda$) | $R_r$ ($\Omega$) |
|---|---|
| 0.05 | 2 |
| 0.1 | 7.9 |

→ Imprecise

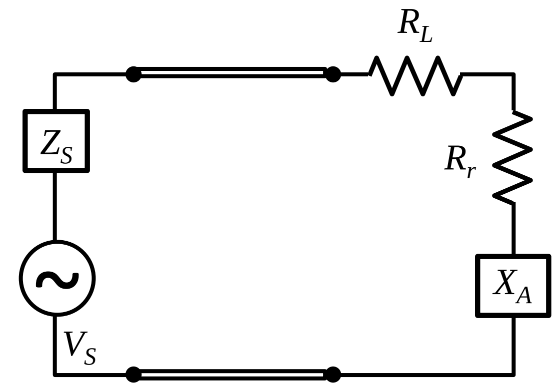
# Antenna Impedance (3/4)

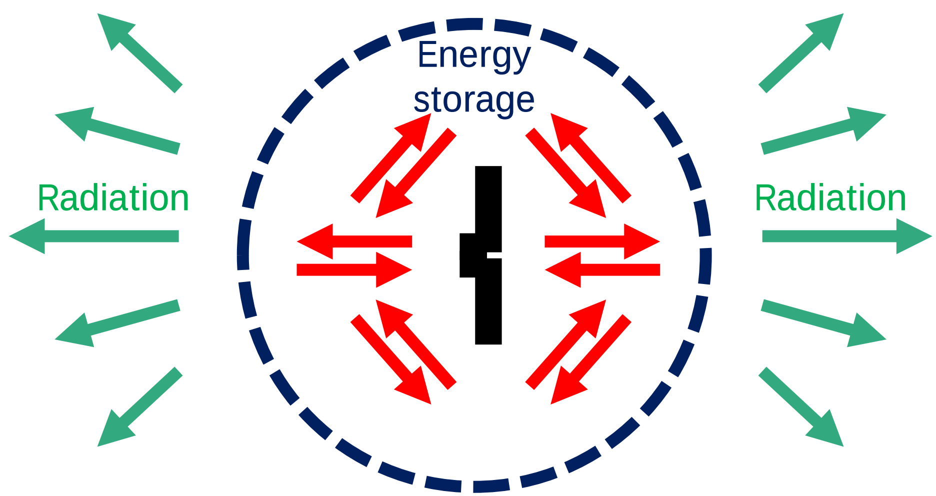
| | Loss resistance $R_L$ |
|---|---|
| What results in ohmic loss? | 1. Conductor loss<br>2. Dielectric loss |
| Formula | 1. Conductor loss:<br><br>$R_{L_{conductor}} = \frac{L}{\sigma A}$<br>• $\sigma$: conductivity<br>• $A$: cross-section area<br>• $L$: length of conductor<br>2. Dielectric loss:<br>Much more complex!<br>Full-wave analysis is necessary |
| Example (thin-wire antennas) | The cross-section area: $A = \pi d \delta$<br>• $d$: conductor diameter<br>(The current is almost confined to the surface of the conductor; we can approximate the depth by $\delta$, namely, the skin depth)<br>$\delta = \frac{1}{\sqrt{f \pi \mu \sigma}}$<br>• $f$: operational frequency<br>• $\mu$: permeability<br>➡ $R_L = \frac{L}{d}\sqrt{\frac{f\mu}{\pi\sigma}}$ |

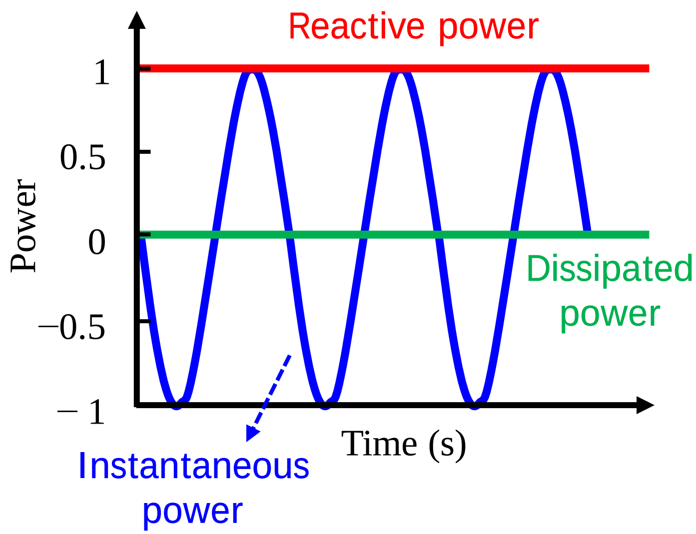
# Antenna Impedance (4/4)

## Antenna reactance $X_A$:

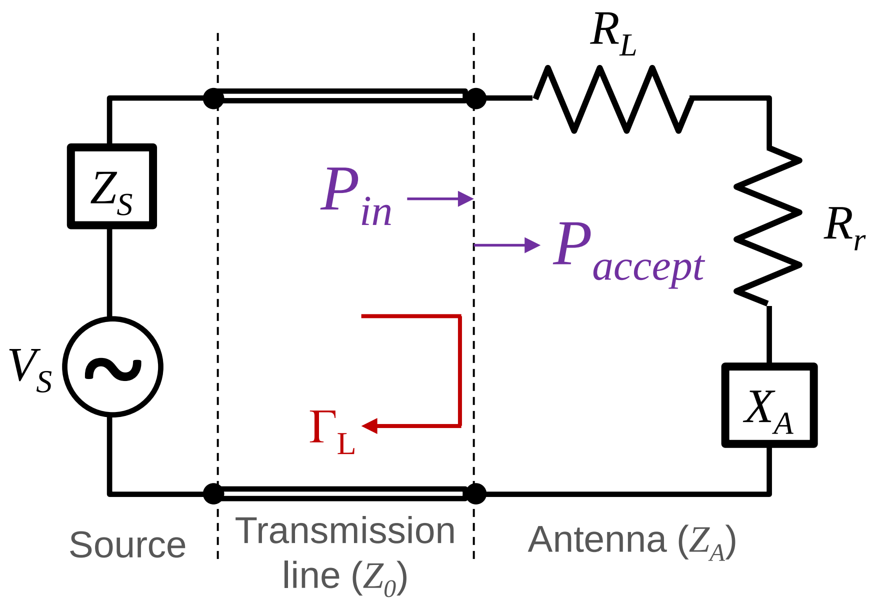
- $X_A$ represents the power stored in the near field of the antenna
- $X_A \neq 0$: The voltage and current supplied from source result in reactive power
- $X_A = 0$: resonance
- The frequency with $X_A = 0$: resonant frequency



# Return Loss (1/2)




- Reflection coefficient $\Gamma_L$:

$$\Gamma_L = \frac{Z_A - Z_0}{Z_A + Z_0}$$

- $|\Gamma_L|^2$ is a measure of how the input power is delivered to the antenna
- The accepted power of the antenna:

$$P_{accept} = P_{in}\left(1 - |\Gamma_L|^2\right)$$

# Return Loss (2/2)

$$P_{accept} = P_{in}\left(1-|\Gamma_L|^2\right),\ \Gamma_L = \frac{Z_A - Z_0}{Z_A + Z_0}$$

Some special cases of antenna impedance:

1. If $Z_A = \infty$, then $|\Gamma_L|^2 = 1$ and $P_{accept} = 0$
2. If $Z_A = 0$, then $|\Gamma_L|^2 = 1$ and $P_{accept} = 0$
3. If $Z_A = Z_0$, then $|\Gamma_L|^2 = 0$ and $P_{accept} = P_{in} \Longrightarrow$ **Matching case**

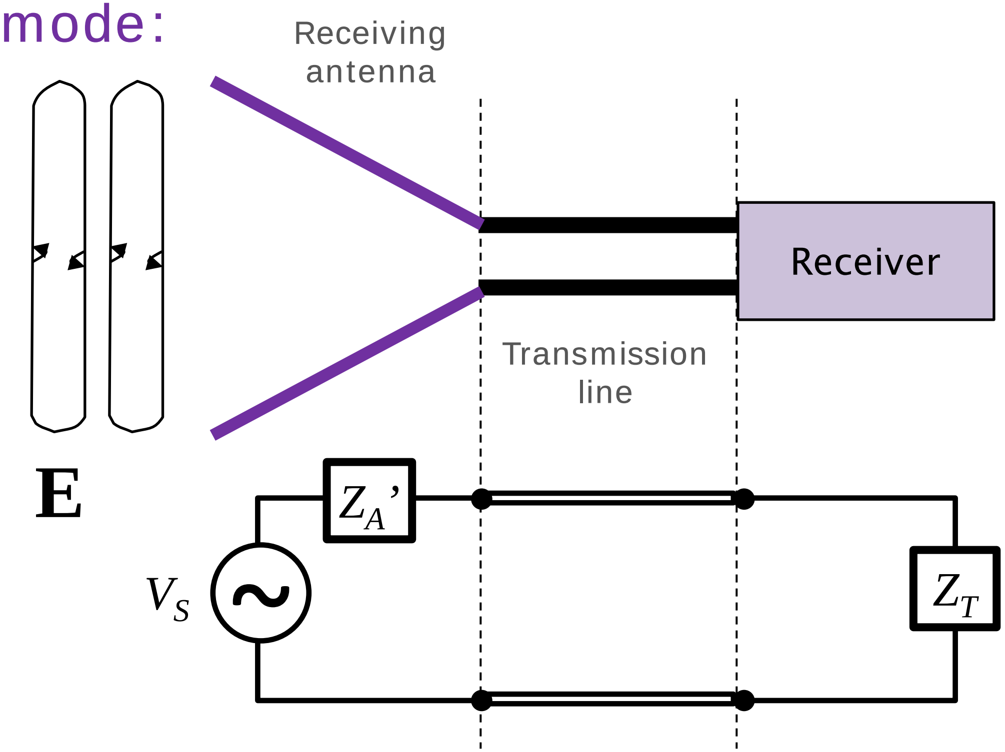
Return loss ($RL$): to further quantify the relation between $P_{accept}$ and $P_{in}$

$$RL = -10\log\left(\frac{P_{in} - P_{accept}}{P_{in}}\right) = -10\log|\Gamma_L|^2 \quad \text{(Unit: dB)}$$

| Return loss (RL) | $\lvert\Gamma_L\rvert^2$ | Meaning |
|---|---|---|
| 10 dB | 0.1 | 90% of power is accepted by the antenna |
| 6 dB | 0.25 | 75% of power is accepted by the antenna |
| 3 dB | 0.5 | Only 50% of power is accepted by the antenna |

# Equivalent Circuit of Receiving Antennas

In Receiving mode:




- $V_s$: The induced voltage of the receiving antenna
- If the antenna is a short dipole, then $V_S = \int_L \mathbf{E} \cdot d\mathbf{l}$ ($L$: length of dipole)
- By reciprocal theorem, the impedance of the receiving antenna $Z_A$' is identical to $Z_A = (R_r + R_L) + jX_A$