ME5302

Computational Fluid Mechanics

CA1

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Chapter 1

Finite Difference Scheme for Numerical Solution

1.1 Numerically Verifying Order of Accuracy

From lecture, we went through the derivation of the the 4th order compact scheme for $\frac{du}{dx}$. A simple periodic function is then chosen to verify the scheme. Starting with $u = sin(2\pi x) \implies \frac{du}{dx}_{ana} = 2\pi cos(2\pi x)$ and $u = cos(2\pi x) \implies \frac{du}{dx}_{ana} = -2\pi sin(2\pi x)$. We get the following plot,

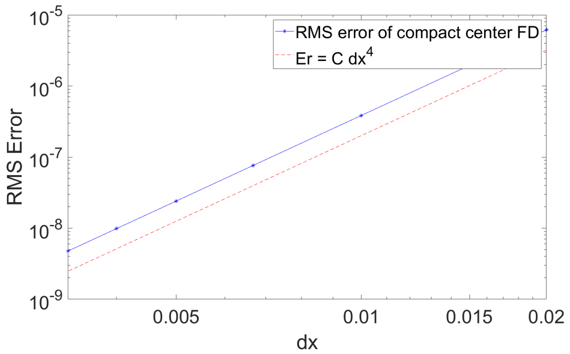

Figure 1.1: Log-Log plot for $y = sin(2\pi x)$

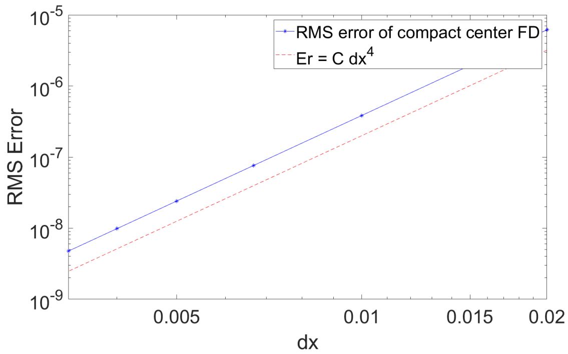

Figure 1.2: Log-Log plot for $y = cos(2\pi x)$

The graphs shows similar results that the log of both the RMS and the scheme are parallel.

1.2 Derive and Verify OoA of Central Finite Difference Scheme of $\frac{d^2u}{dx^2}$

For deriving the 4th order OoA convectional (centre) finite difference scheme for $\frac{d^2u}{dx^2}$, we start with the following 5 points, u_j and its surrounding 4 points, and the taylor series expansion,

$$
\begin{aligned}
u_{j-2} &= u_j - u_j' 2\Delta x + u_j'' \frac{(2\Delta x)^2}{2!} - u_j''' \frac{(2\Delta x)^3}{3!} + u_j^{(4)} \frac{(2\Delta x)^4}{4!} - u_j^{(5)} \frac{(2\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \\
u_{j-1} &= u_j - u_j' \Delta x + u_j'' \frac{(\Delta x)^2}{2!} - u_j''' \frac{(\Delta x)^3}{3!} + u_j^{(4)} \frac{(\Delta x)^4}{4!} - u_j^{(5)} \frac{(\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \\
u_{j+1} &= u_j + u_j' \Delta x + u_j'' \frac{(\Delta x)^2}{2!} + u_j''' \frac{(\Delta x)^3}{3!} + u_j^{(4)} \frac{(\Delta x)^4}{4!} + u_j^{(5)} \frac{(\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \\
u_{j+2} &= u_j + u_j' 2\Delta x + u_j'' \frac{(2\Delta x)^2}{2!} + u_j''' \frac{(2\Delta x)^3}{3!} + u_j^{(4)} \frac{(2\Delta x)^4}{4!} + u_j^{(5)} \frac{(2\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6
\end{aligned}
$$

Adding multiplier to 3 equations, they become,

$$
\begin{aligned}
u_{j-2} &= u_j - u_j' 2\Delta x + u_j'' \frac{(2\Delta x)^2}{2!} - u_j''' \frac{(2\Delta x)^3}{3!} + u_j^{(4)} \frac{(2\Delta x)^4}{4!} - u_j^{(5)} \frac{(2\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \\
\alpha[u_{j-1}] &= \alpha \left[u_j - u_j' \Delta x + u_j'' \frac{(\Delta x)^2}{2!} - u_j''' \frac{(\Delta x)^3}{3!} + u_j^{(4)} \frac{(\Delta x)^4}{4!} - u_j^{(5)} \frac{(\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \right] \\
\beta[u_{j+1}] &= \beta \left[u_j + u_j' \Delta x + u_j'' \frac{(\Delta x)^2}{2!} + u_j''' \frac{(\Delta x)^3}{3!} + u_j^{(4)} \frac{(\Delta x)^4}{4!} + u_j^{(5)} \frac{(\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \right] \\
\gamma[u_{j+2}] &= \gamma \left[u_j + u_j' 2\Delta x + u_j'' \frac{(2\Delta x)^2}{2!} + u_j''' \frac{(2\Delta x)^3}{3!} + u_j^{(4)} \frac{(2\Delta x)^4}{4!} + u_j^{(5)} \frac{(2\Delta x)^5}{5!} + \mathcal{O}(\Delta x)^6 \right]
\end{aligned}
$$

Adding them up and setting u_j', u_j''', $u_j^{(4)}$ and $u_j^{(5)}$to be zero, we get,

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -\frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{15} \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -\frac{1}{6} \\ \frac{1}{24} \\ -\frac{1}{120} \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{24} \\ \frac{1}{120} \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ \frac{4}{3} \\ \frac{2}{3} \\ \frac{4}{15} \end{bmatrix}
$$

After solving, we get $\alpha = -16$, $\beta = -16$, and $\gamma = 1$. This gives us the following equation,

$$
\begin{aligned}
u_{j-2} - 16u_{j-1} - 16u_{j+1} + u_{j+2} &= -30u_j - 12u_j''(\Delta x)^2 + \mathcal{O}(\Delta x)^6 \\
-12u_j''(\Delta x)^2 &= u_{j-2} - 16u_{j-1} + 30u_j - 16u_{j+1} + u_{j+2} + \mathcal{O}(\Delta x)^6 \\
u_j'' &= -\frac{1}{12(\Delta x)^2}(u_{j-2} - 16u_{j-1} + 30u_j - 16u_{j+1} + u_{j+2}) + \mathcal{O}(\Delta x)^4
\end{aligned}
$$

Then, construct the **A** matrix and suppose $u = sin(2\pi x) \implies \frac{d^2u}{dx^2} = -(2\pi)^2 sin(2\pi x)$, the verification plot of the RMS and the log scale is,

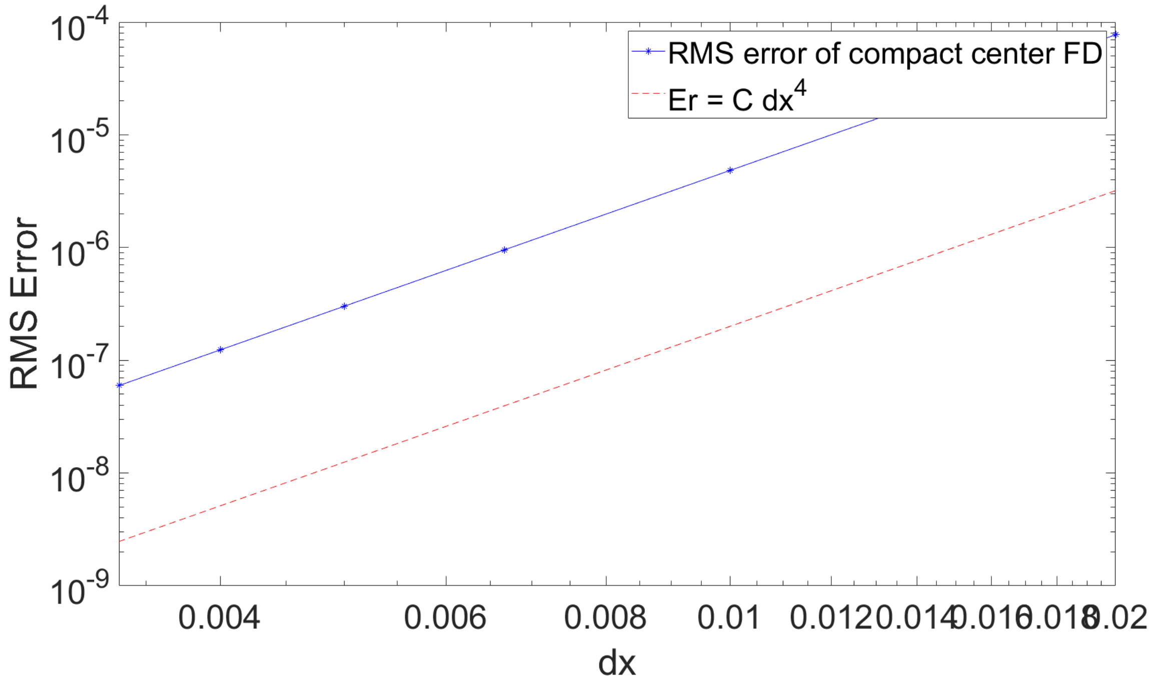

Figure 1.3: Log-Log plot for $y = sin(2\pi x)$, $\frac{d^2u}{dx^2}$, 5 points

1.3 Derive and Verify Compact Central Finite Difference Scheme of $\frac{d^2u}{dx^2}$

For the compact scheme, we will only be using the points u_{j-1}, u_j, and u_{j+1}. We will be utilising the following taylor series expansions,

$$\begin{aligned}
u_{j-1} &= u_j - u_j'\Delta x + u_j''\frac{(\Delta x)^2}{2!} - u_j'''\frac{(\Delta x)^3}{3!} + u_j^{(4)}\frac{(\Delta x)^4}{4!} - u_j^{(5)}\frac{(\Delta x)^5}{5!} + \cdots \\
u_{j+1} &= u_j + u_j'\Delta x + u_j''\frac{(\Delta x)^2}{2!} + u_j'''\frac{(\Delta x)^3}{3!} + u_j^{(4)}\frac{(\Delta x)^4}{4!} + u_j^{(5)}\frac{(\Delta x)^5}{5!} + \cdots \\
u_{j-1}' &= u_j' - u_j''\Delta x + u_j'''\frac{(\Delta x)^2}{2!} - u_j^{(4)}\frac{(\Delta x)^3}{3!} + u_j^{(5)}\frac{(\Delta x)^4}{4!} - u_j^{(6)}\frac{(\Delta x)^5}{5!} + \cdots \\
u_{j+1}' &= u_j' + u_j''\Delta x + u_j'''\frac{(\Delta x)^2}{2!} + u_j^{(4)}\frac{(\Delta x)^3}{3!} + u_j^{(5)}\frac{(\Delta x)^4}{4!} + u_j^{(6)}\frac{(\Delta x)^5}{5!} + \cdots
\end{aligned}$$

Getting the extra steps to generate more data to be utilised,

$$\begin{aligned}
u_{j-1} &= u_j - u_j'\Delta x + u_j''\frac{(\Delta x)^2}{2!} - u_j'''\frac{(\Delta x)^3}{3!} + u_j^{(4)}\frac{(\Delta x)^4}{4!} - u_j^{(5)}\frac{(\Delta x)^5}{5!} + \cdots \\
u_{j+1} &= u_j + u_j'\Delta x + u_j''\frac{(\Delta x)^2}{2!} + u_j'''\frac{(\Delta x)^3}{3!} + u_j^{(4)}\frac{(\Delta x)^4}{4!} + u_j^{(5)}\frac{(\Delta x)^5}{5!} + \cdots \\
\beta u_{j-1}'' &= \beta u_j'' - \beta u_j'''\Delta x + \beta u_j^{(4)}\frac{(\Delta x)^2}{2!} - \beta u_j^{(5)}\frac{(\Delta x)^3}{3!} + \beta u_j^{(6)}\frac{(\Delta x)^4}{4!} - \beta u_j^{(7)}\frac{(\Delta x)^5}{5!} + \cdots \\
\beta u_{j+1}'' &= \beta u_j'' + \beta u_j'''\Delta x + \beta u_j^{(4)}\frac{(\Delta x)^2}{2!} + \beta u_j^{(5)}\frac{(\Delta x)^3}{3!} + \beta u_j^{(6)}\frac{(\Delta x)^4}{4!} + \beta u_j^{(7)}\frac{(\Delta x)^5}{5!} + \cdots
\end{aligned}$$

Adding them together,

$$u_{j-1} + u_{j+1} + \beta u_{j-1}'' + \beta u_{j+1}'' = 2u_j + 2\beta u_j'' + 2u_j''\frac{(\Delta x)^2}{2!} + 2\beta u_j^{(4)}\frac{(\Delta x)^2}{2!} + 2u_j^{(4)}\frac{(\Delta x)^4}{4!} + 2\beta u_j^{(6)}\frac{(\Delta x)^4}{4!} + \cdots$$

Setting $2\beta\frac{(\Delta x)^2}{2!} + 2\frac{(\Delta x)^4}{4!} = 0$, we get $\beta = -\frac{(\Delta x)^2}{12}$. Substituting it back,

$$
\begin{aligned}
u_{j-1} + u_{j+1} - \frac{(\Delta x)^2}{12}[u''_{j-1} + u''_{j+1}] &= 2u_j - u''_j\frac{(\Delta x)^2}{6} + u''_j(\Delta x)^2 - u_j^{(6)}\frac{(\Delta x)^6}{144} + \cdots \\
u_{j-1} - 2u_j + u_{j+1} &= \frac{(\Delta x)^2}{12}[u''_{j-1} + u''_{j+1}] - u''_j\frac{(\Delta x)^2}{6} + u''_j(\Delta x)^2 - u_j^{(6)}\frac{(\Delta x)^6}{144} + \cdots \\
\frac{144}{(\Delta x)^2}[u_{j-1} - 2u_j + u_{j+1}] &= 12u''_{j-1} + 120u''_j + 12u''_{j+1} - u_j^{(6)}(\Delta x)^4 \\
12u''_{j-1} + 120u''_j + 12u''_{j+1} &= \frac{144}{(\Delta x)^2}[u_{j-1} - 2u_j + u_{j+1}] + \mathcal{O}(\Delta x)^4
\end{aligned}
$$

Running the code for verification, we get the following graph,

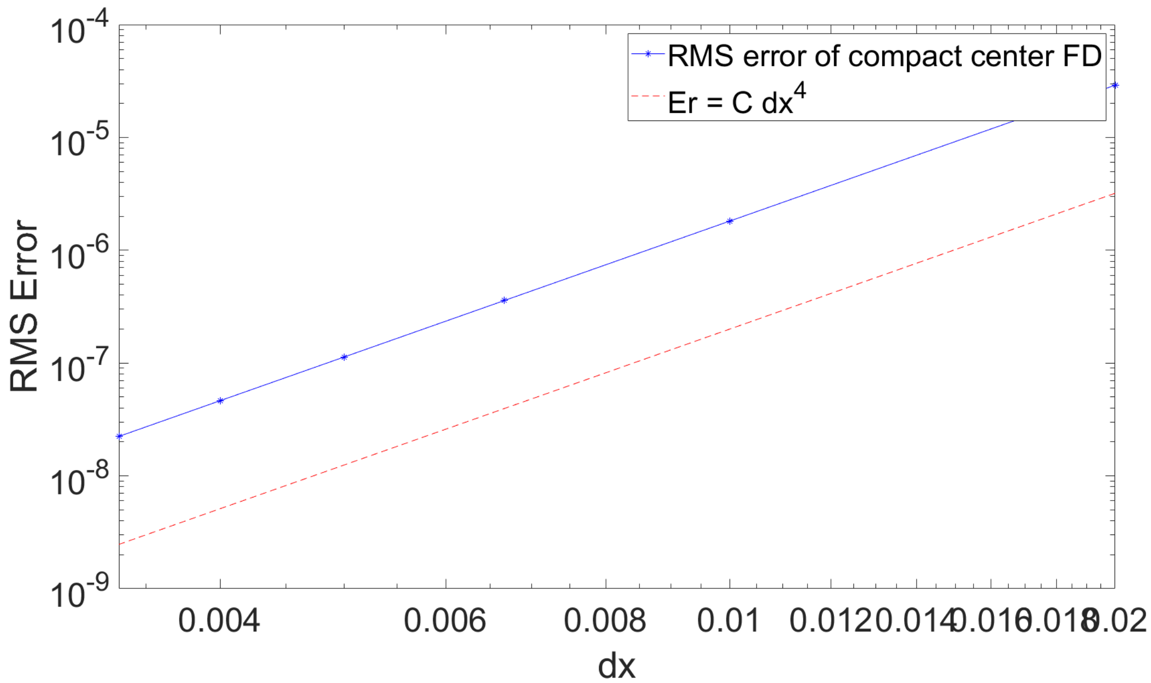

Figure 1.4: Log-Log plot for $y = sin(2\pi x)$, $\frac{d^2u}{dx^2}$, compact

1.4 Solving the ODE

The ODE given is, $u'' + u' = -4\pi^2 C sin(2\pi x) + 2\pi C cos(2\pi x)$ with $C = 58$. This ODE then becomes, $u'' + u' = -232\pi^2 sin(2\pi x) + 116\pi cos(2\pi x)$. Using the above finite difference schemes, D1 and D2 matrices are for the differential part, and the vector matrix will be the right hand side.

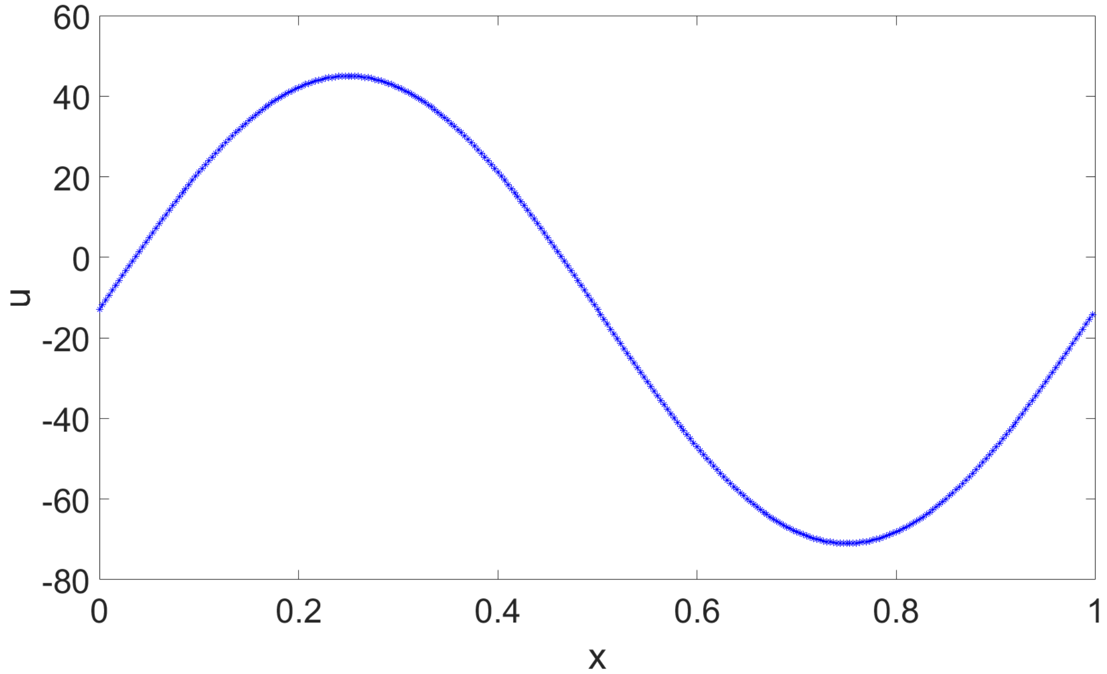

Figure 1.5: Solution without fixed constant

Along with the graph, is the following warning message, *Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND = 1.367751e-19.* Setting $u(1) = 0$ for the initial condition, the solution becomes,

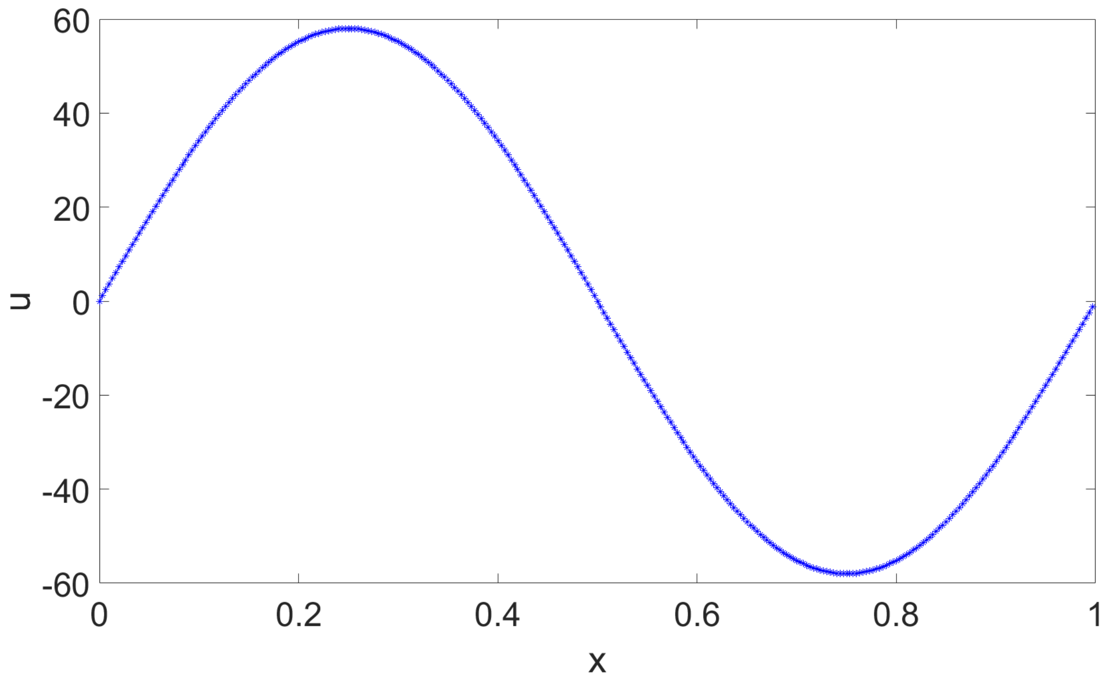

Figure 1.6: Solution with fixed constant

The solution corresponds with the original ODE, and the intial condition of $u(1) = 0$.