ME5302

Computational Fluid Mechanics

CA2

Professor: Dr Zhang Mengqi

Done by: Lu Ziyue, A0200058W

February 22, 2023

Chapter 1

Crack-Nicolson

The Crack-Nicolson methos is given as,

$$u^{n+1} = u^n + \frac{1}{2}h\left(u'^{n+1} + u'^{n}\right) \tag{1.1}$$

1.1 Solving diffusion equation using CN method

The 1D diffusion equation, $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$, where $\nu = \frac{C}{10000}$ with $C = 58$. The equation therefore becomes, $\frac{\partial u}{\partial t} = \frac{58}{10000}\frac{\partial^2 u}{\partial x^2}$. Applying the CN time and central space scheme to the 1D diffusion equation, we get the following,

$$u_i^{n+1} = u_i^n + \frac{\nu h}{2\Delta x^2}\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1} + u_{i-1}^{n} - 2u_i^{n} + u_{i+1}^{n}\right) \tag{1.2}$$

From the equation, $\tau = \frac{\nu h}{2\Delta x^2}$. Substituting $\nu = \frac{58}{100}$, $h = 0.02$, and $dx = 0.005 \implies \tau = 232$. For points 0 and 202, using the periodic boundary condition, $u(x = 0) = u(x = 1) = 0$, $u_a = u_b = 0$, bc matrix $= 0$. Using 200 interior grid points, i.e 202 $grid\, points$ in total, $dt = 0.02$, we iterate the time step to $t = 2$,

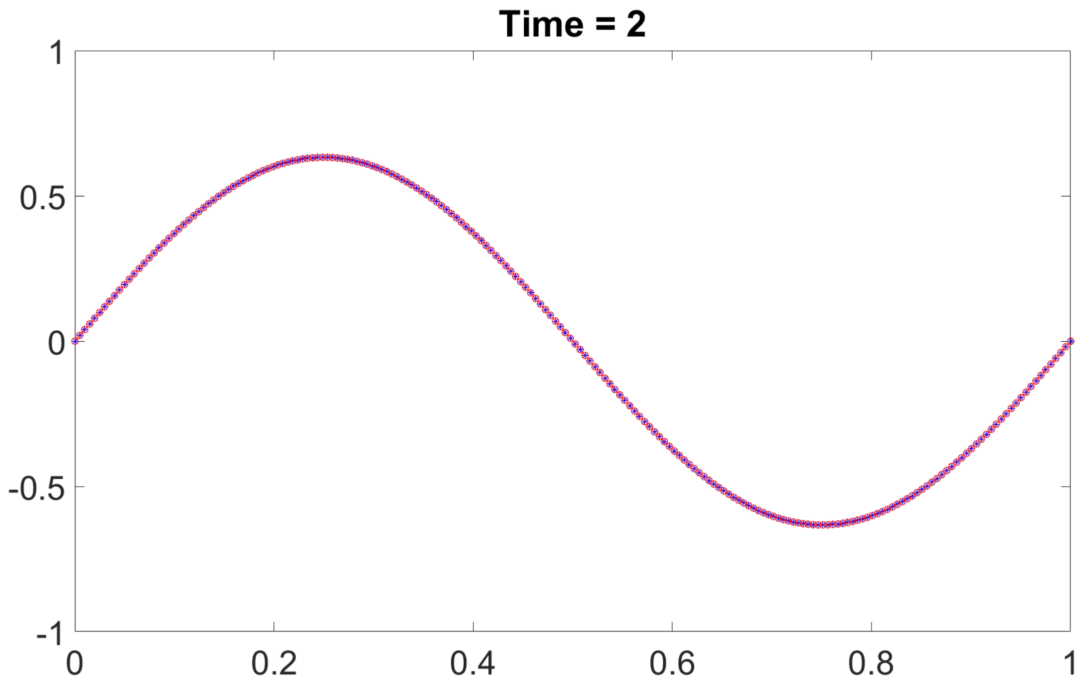

Figure 1.1: Diffusion equation using CN and central schemes at $t = 2$ with periodic bc.

1.2 Dirichlet Boundary Condition

Changing the periodic boundary condition to a Dirichlet boundary condition with $u_x|_{x=1} = 0$, the last, or $202th\ grid\ point$ will use a backward difference scheme. Using points u_{200} and u_{201},

$$\begin{aligned}
u_{200} &= u_{202} - 2\Delta x u'_{202} + \frac{1}{2!}(2\Delta x)^2 u''_{202} + \mathcal{O}(\Delta x)^3 \\
u_{201} &= u_{202} - \Delta x u'_{202} + \frac{1}{2!}(\Delta x)^2 u''_{202} + \mathcal{O}(\Delta x)^3 \\
\implies u'_{202} &= \frac{u_{200} - 4u_{201} + 3u_{202}}{2\Delta x} = u'_b \qquad (1.3) \\
\implies u_{202} &= \frac{1}{3}\left(4u_{201} - u_{200}\right) + \frac{1}{3}\left(2\Delta x u'_b\right) \qquad u'_b = 0 \\
u_{202} &= \frac{1}{3}\left(4u_{201} - u_{200}\right) \qquad (1.4)
\end{aligned}$$

Substituting u_{202} into the CN numberical equation at $i = 201$,

$$
\begin{aligned}
u_{201}^{n+1} &= u_{201}^{n} + \tau\left[u_{200}^{n+1} - 2u_{201}^{n+1} + u_{202}^{n+1} + u_{200}^{n} - 2u_{201}^{n} + u_{202}^{n}\right] \\
u_{201}^{n+1} &= u_{201}^{n} + \tau\left[u_{200}^{n+1} - 2u_{201}^{n+1} + \frac{1}{3}\left(4u_{201}^{n+1} - u_{200}^{n+1}\right) + u_{200}^{n} - 2u_{201}^{n} + \frac{1}{3}\left(4u_{201}^{n} - u_{200}^{n}\right)\right] \\
u_{201}^{n+1} &= u_{201}^{n} + \tau\left[\frac{2}{3}\left(u_{200}^{n+1} - u_{201}^{n+1}\right) + \frac{2}{3}\left(u_{200}^{n} - u_{201}^{n}\right)\right] \\
u_{201}^{n+1} &= u_{201}^{n} + \frac{2}{3}\tau\left[u_{200}^{n+1} - u_{201}^{n+1} + u_{200}^{n} - u_{201}^{n}\right]
\end{aligned}
$$

Shifting all the $n+1$ time step u to the left,

$$
u_{201}^{n+1} - \frac{2}{3}\tau\left[u_{200}^{n+1} - u_{201}^{n+1}\right] = u_{201}^{n} + \frac{2}{3}\tau\left[u_{200}^{n} - u_{201}^{n}\right] \tag{1.5}
$$

The last row of both the **RHS** and the **LHS** matrices will be,

$$
\begin{aligned}
\text{Last row of }\textbf{RHS} &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \frac{2}{3}\tau & 1-\frac{2}{3}\tau \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 154\frac{2}{3} & -153\frac{2}{3} \end{bmatrix} \\
\text{Last row of }\textbf{LHS} &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{2}{3}\tau & 1+\frac{2}{3}\tau \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -154\frac{2}{3} & 155\frac{2}{3} \end{bmatrix}
\end{aligned}
$$

At $t = 2$, the following plot is produced,

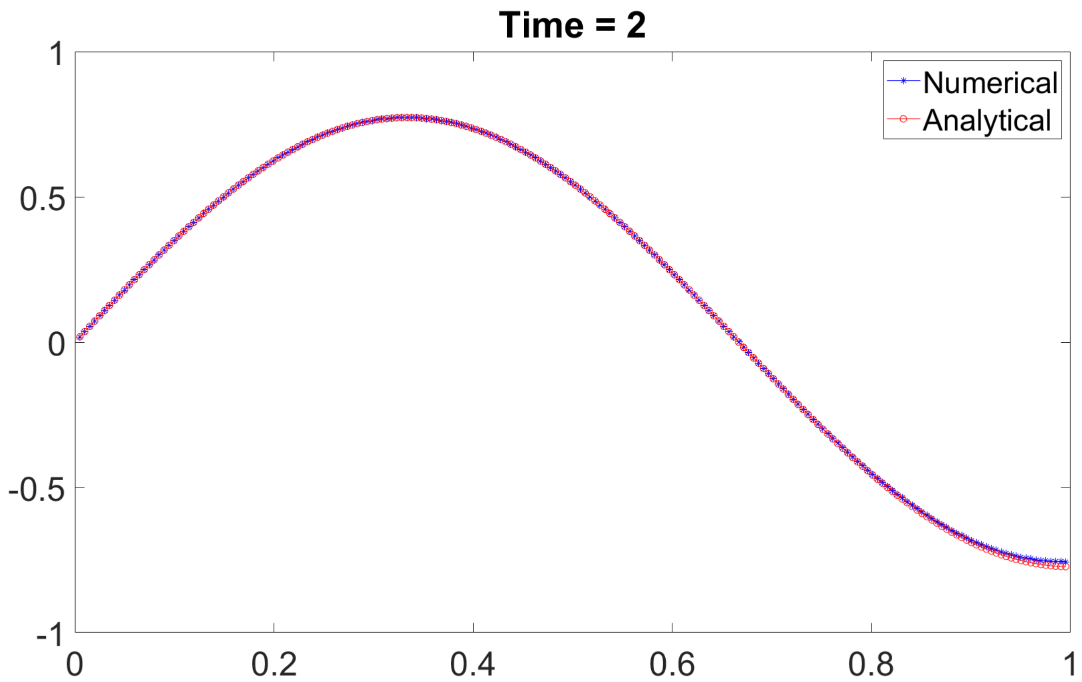

Figure 1.2: Diffusion equation using CN and central schemes at $t = 2$ with Dirichlet bc.

1.3 Stability of numerical method

The stability of the numerical method used in section 1.1 can be analysed by finding the **G** matrix.

$$\mathbf{G} = \mathbf{LHS} \setminus \mathbf{RHS} \tag{1.6}$$

Finding the eigen values σ_i of the **G** matrix, and if $|\sigma_i| < 1$, the numerical method is stable. By looking at the matlab code, the numerical method in section 1.1 is stable for a significant numbers of values fordt and dx.

Chapter 2

Convection-Diffusion Equation

2.1 6 Numerical methods

Using 6 different methods of discretisation to discretise the convection diffusion equation,

$$Explicit\ upwind: \ \frac{u_j^{n+1}-u_j^n}{dt}+V\frac{u_j^n-u_{j-1}^n}{dx}=\frac{\nu}{2}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{dx^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{dx^2}\right) \tag{2.1}$$

$$Explicit\ downwind: \ \frac{u_j^{n+1}-u_j^n}{dt}+V\frac{u_{j+1}^n-u_j^n}{dx}=\frac{\nu}{2}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{dx^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{dx^2}\right) \tag{2.2}$$

$$Explicit\ centre: \ \frac{u_j^{n+1}-u_j^n}{dt}+V\frac{u_{j+1}^n-u_{j-1}^n}{2dx}=\frac{\nu}{2}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{dx^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{dx^2}\right) \tag{2.3}$$

$$Implicit\ upwind: \ \frac{u_j^{n+1}-u_j^n}{dt}+V\frac{u_j^{n+1}-u_{j-1}^{n+1}}{dx}=\frac{\nu}{2}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{dx^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{dx^2}\right) \tag{2.4}$$

$$Implicit\ downwind: \ \frac{u_j^{n+1}-u_j^n}{dt}+V\frac{u_{j+1}^{n+1}-u_j^{n+1}}{dx}=\frac{\nu}{2}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{dx^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{dx^2}\right) \tag{2.5}$$

$$Implicit\ centre: \ \frac{u_j^{n+1}-u_j^n}{dt}+V\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2dx}=\frac{\nu}{2}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{dx^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{dx^2}\right) \tag{2.6}$$

By rearranging the variables such that $\mathbf{LHS}u^{n+1}=\mathbf{RHS}u^n$, and $\mathbf{G}=\mathbf{LHS}\setminus\mathbf{RHS}$,

$$-\frac{\nu}{2dx^2}u_{j-1}^{n+1}+\left(\frac{1}{dt}+\frac{2\nu}{2dx^2}\right)u_j^{n+1}-\frac{\nu}{2dx^2}u_{j+1}^{n+1}=\left(\frac{V}{dx}+\frac{\nu}{2dx^2}\right)u_{j-1}^n+\left(\frac{1}{dt}-\frac{V}{dx}-\frac{2\nu}{2dx^2}\right)u_j^n+\frac{\nu}{2dx^2}u_{j+1}^n \tag{2.7}$$

$$-\frac{\nu}{2dx^2}u_{j-1}^{n+1}+\left(\frac{1}{dt}+\frac{2\nu}{2dx^2}\right)u_j^{n+1}-\frac{\nu}{2dx^2}u_{j+1}^{n+1}=\frac{\nu}{2dx^2}u_{j-1}^n+\left(\frac{1}{dt}+\frac{V}{dx}-\frac{2\nu}{2dx^2}\right)u_j^n+\left(-\frac{V}{dx}+\frac{\nu}{2dx^2}\right)u_{j+1}^n \tag{2.8}$$

$$-\frac{\nu}{2dx^2}u_{j-1}^{n+1}+\left(\frac{1}{dt}+\frac{2\nu}{2dx^2}\right)u_j^{n+1}-\frac{\nu}{2dx^2}u_{j+1}^{n+1}=\left(\frac{V}{2dx}+\frac{\nu}{2dx^2}\right)u_{j-1}^n+\left(\frac{1}{dt}-\frac{2\nu}{2dx^2}\right)u_j^n+\left(-\frac{V}{2dx}+\frac{\nu}{2dx^2}\right)u_{j+1}^n \tag{2.9}$$

$$-\left(\frac{V}{dx}+\frac{\nu}{2dx^2}\right)u_{j_1}^{n+1}+\left(\frac{1}{dt}+\frac{V}{dx}+\frac{2\nu}{2dx^2}\right)u_j^{n+1}-\frac{\nu}{2dx^2}u_{j+1}^{n+1}=\frac{\nu}{2dx^2}u_{j-1}^n+\left(\frac{1}{dt}-\frac{2\nu}{2dx^2}\right)u_j^n+\frac{\nu}{2dx^2}u_{j+1}^n \tag{2.10}$$

$$-\frac{\nu}{2dx^2}u_{j_1}^{n+1}+\left(\frac{1}{dt}-\frac{V}{dx}+\frac{2\nu}{2dx^2}\right)u_j^{n+1}+\left(\frac{V}{dx}-\frac{\nu}{2dx^2}\right)u_{j+1}^{n+1}=\frac{\nu}{2dx^2}u_{j-1}^n+\left(\frac{1}{dt}-\frac{2\nu}{2dx^2}\right)u_j^n+\frac{\nu}{2dx^2}u_{j+1}^n \tag{2.11}$$

$$-\left(\frac{V}{2dx}+\frac{\nu}{2dx^2}\right)u_{j_1}^{n+1}+\left(\frac{1}{dt}+\frac{2\nu}{2dx^2}\right)u_j^{n+1}+\left(\frac{V}{2dx}-\frac{\nu}{2dx^2}\right)u_{j+1}^{n+1}=\frac{\nu}{2dx^2}u_{j-1}^n+\left(\frac{1}{dt}-\frac{2\nu}{2dx^2}\right)u_j^n+\frac{\nu}{2dx^2}u_{j+1}^n \tag{2.12}$$

2.1.1 Explicit Upwind

The **G** matrix is calculated and the eigen values are plotted,

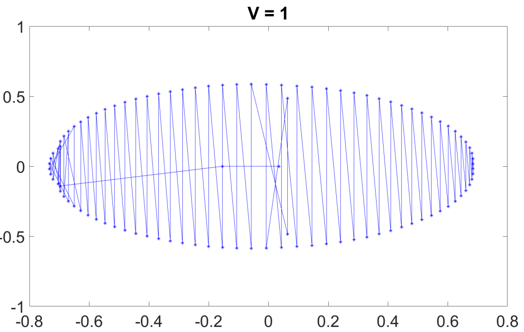

Figure 2.1: Eigenvalue plot at $V = 1$

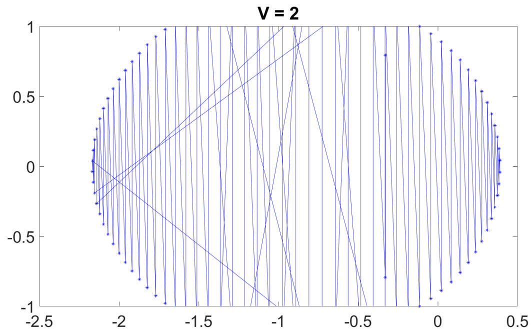

Figure 2.2: Eigenvalue plot at $V = 2$

The numerical solution at $V = 1$ is stable as all the eigan values are within the unit circle. Whereas, when the value of V is increased to 2 onwards, the solution is no longer stable. The explicit upwind method is stable when $V < 1.2$.

2.1.2 Explicit Downwind

The **G** matrix is calculated and the eigen values are plotted,

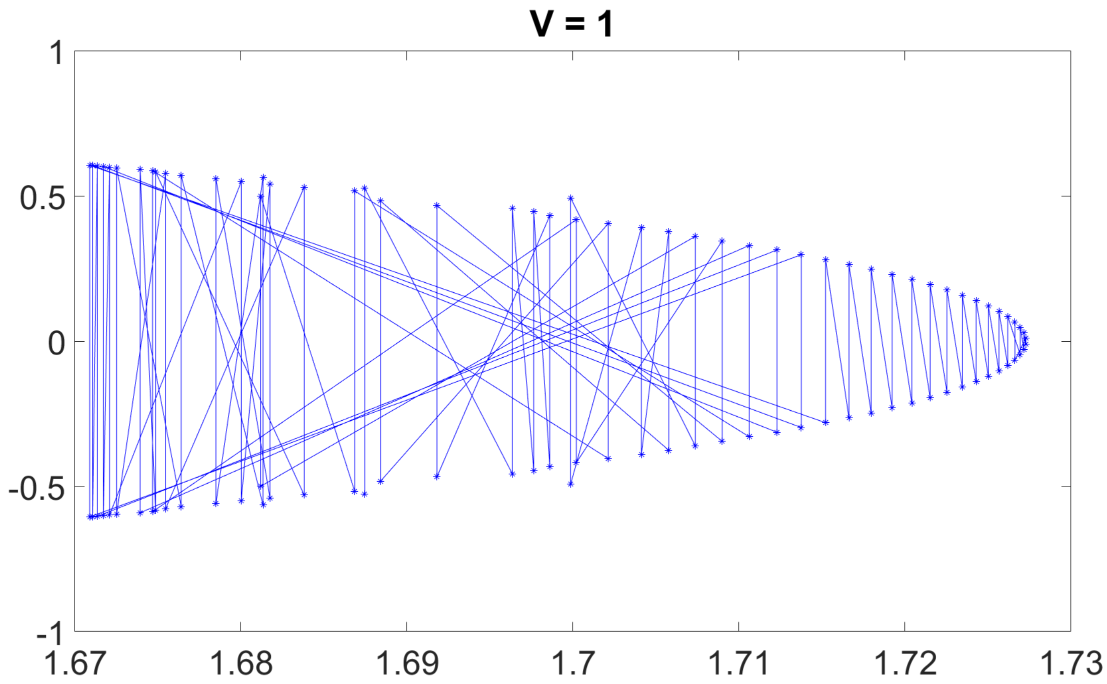

Figure 2.3: Eigenvalue plot at $V = 1$

The value of the graph for the real value of the eigenvalues are more than 1, it shows that for all values of V, the numerical solution is unstable.

2.1.3 Explicit Centre

The **G** matrix is calculated and the eigen values are plotted,

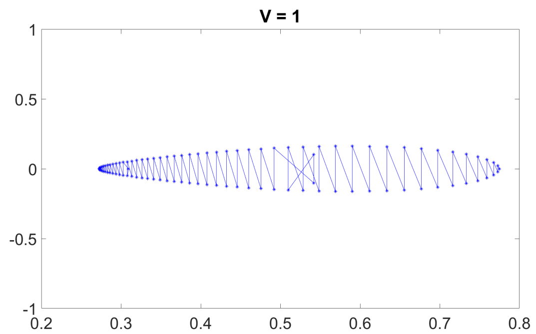

Figure 2.4: Eigenvalue plot at $V = 1$

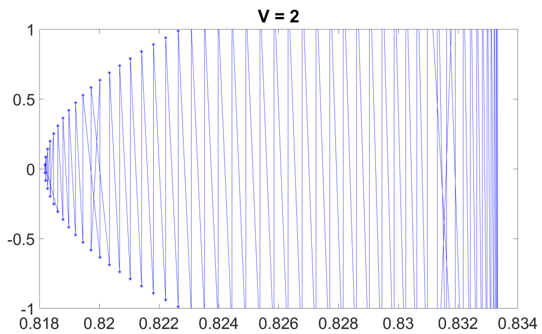

Figure 2.5: Eigenvalue plot at $V = 2$

The numerical solution at $V = 1$ is stable as all the eigan values are within the unit circle. Whereas, when the value of V is increased to 2 onwards, the solution is no longer stable. The explicit centremethod is stable when $V \leq 1$.

2.1.4 Implicit Upwind

The **G** matrix is calculated and the eigen values are plotted,

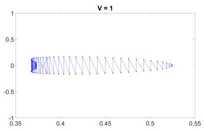

Figure 2.6: Eigenvalue plot at $V = 1$

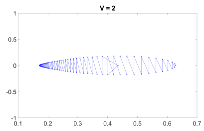

Figure 2.7: Eigenvalue plot at $V = 2$

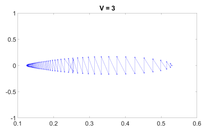

Figure 2.8: Eigenvalue plot at $V = 3$

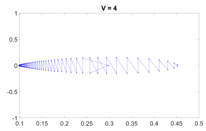

Figure 2.9: Eigenvalue plot at $V = 4$

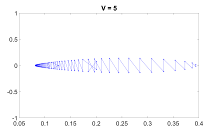

Figure 2.10: Eigenvalue plot at $V = 5$

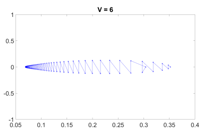

Figure 2.11: Eigenvalue plot at $V = 6$

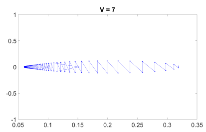

Figure 2.12: Eigenvalue plot at $V = 7$

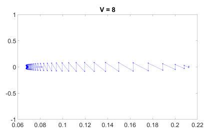

Figure 2.13: Eigenvalue plot at $V = 8$

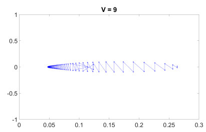

Figure 2.14: Eigenvalue plot at $V = 9$

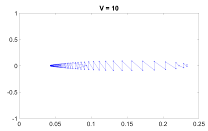

Figure 2.15: Eigenvalue plot at $V = 10$

The numerical solution is stable for all values of V, as the eigenvalues goes closer and closer to 0.

2.1.5 Implicit Downwind

The **G** matrix is calculated and the eigen values are plotted,

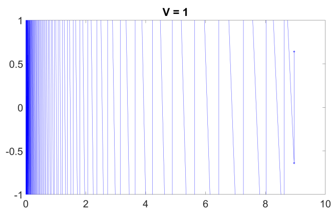

Figure 2.16: Eigenvalue plot at $V = 1$

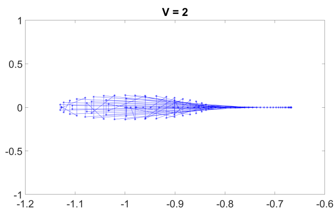

Figure 2.17: Eigenvalue plot at $V = 2$

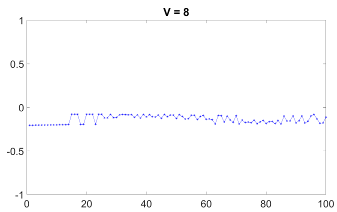

Figure 2.18: Eigenvalue plot at $V = 8$

The solution becomes stable after $V = 2$, however, at $V = 8$, there is another unstable range.

2.1.6 Implicit Centre

The **G** matrix is calculated and the eigen values are plotted,

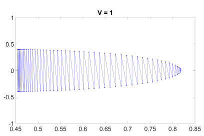

Figure 2.19: Eigenvalue plot at $V = 1$

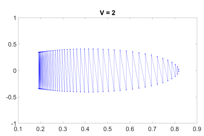

Figure 2.20: Eigenvalue plot at $V = 2$

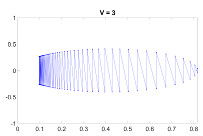

Figure 2.21: Eigenvalue plot at $V = 3$

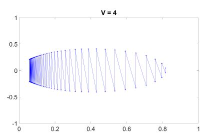

Figure 2.22: Eigenvalue plot at $V = 4$

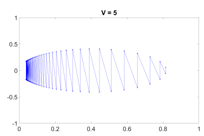

Figure 2.23: Eigenvalue plot at $V = 5$

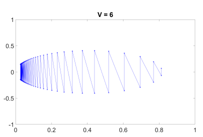

Figure 2.24: Eigenvalue plot at $V = 6$

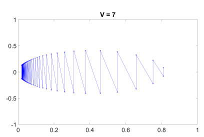

Figure 2.25: Eigenvalue plot at $V = 7$

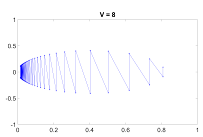

Figure 2.26: Eigenvalue plot at $V = 8$

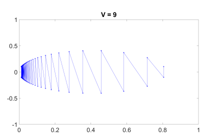

Figure 2.27: Eigenvalue plot at $V = 9$

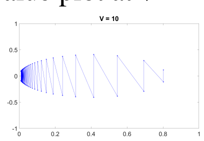

Figure 2.28: Eigenvalue plot at $V = 10$

The numerical solution is stable for all values of V, as the eigenvalues are all within the unit circle.

2.2 Discussion

The numerical solution for explicit methods are either stable when the value of V is small, or not stable at all. However, the implicit methods are all mostly stable, or they will eventaully become stable. The interesting thing is the implicit downwind method, where it starts off unstable, but becomes stable as V grows. However around $V = 8$, there is another unexpected unstable region. The eigenvalues exploded till 100 before converging back again.