1 Characterisation of Metric Continuity

Definition 1. **(Open set in a metric space)** *A set $U \subseteq X$ in a metric space (X, d) is open if U contains some ε-ball $B_\varepsilon(x)$ centered at every $x \in U$. In other words $\forall x \in U, \exists \varepsilon > 0: B_\varepsilon(x) \subseteq U$.*

Definition 2. **(ε-Ball)** *Let (X, d) be a metric space. An ε-Ball centered at $x \in X$ is the set*

$$B_\varepsilon(x) = \{y \in X: d(x, y) < \varepsilon\}$$

Proposition 3. *Let X, Y be metric spaces and $f: X \to Y$, a map. The following statements are equivalent.*

a) For all $x \in X$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for arbitrary $\varepsilon > 0$.

b) For all $y \in Y$, the preimage $f^{-1}(B_\varepsilon(y)) \subseteq X$ is open for arbitrary $\varepsilon > 0$.

c) The preimage $f^{-1}(U) \subseteq X$ is open for arbitrary open $U \subseteq Y$.

d) If the sequence $(x_n \in X)_{n \in \mathbb{N}}$ converges to $x \in X$, then $(f(x_n) \in X)_{n \in \mathbb{N}}$ converges to $f(x) \in Y$.

Note that we use the metric definition of open here.

Proof. $\boldsymbol{b \Rightarrow a}$

Let $x \in X$ and $B_\varepsilon(f(x))$ be some ε-ball centered at $f(x)$. By our assumption, the preimage $f^{-1}(B_\varepsilon(f(x)))$ is open. In particular, this implies that there exists a δ-ball centered at x, such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$, which implies $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

$\boldsymbol{a \Rightarrow c}$

Let $U \subseteq Y$ be some open set in Y and let $x \in f^{-1}(U)$. Then, by a, there exists a $\delta > 0$, such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for arbitrary $\varepsilon > 0$. For sufficiently small ε, $B_\varepsilon(f(x)) \subseteq U$, since U is open and $f(x) \in U$. Therefore, $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq U$, which means $B_\delta(x) \subseteq f^{-1}(U)$.

$\boldsymbol{c \Rightarrow b}$

If $f^{-1}(U)$ is open for an arbitrary set $U \subseteq Y$, it is also open for the case $U = B_\varepsilon(y)$.

$\boldsymbol{a \Rightarrow d}$

Let $(x_n \in X)_{n \in \mathbb{N}}$ be a sequence which converges to $x \in X$. Then, $\{x_n \in X: n > N\}$ is contained in any δ-ball $B_\delta(x)$ for some sufficiently large N. By a, this means we may find a $\delta > 0$, such that $\{f(x_n) \in Y: n > N\} \subseteq f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for any $\varepsilon > 0$. In other words, for any $\varepsilon > 0$, $d(f(x_n), f(x)) < \varepsilon$ for all $n > N$, which implies $(f(x_n) \in X)_{n \in \mathbb{N}}$ converges to $f(x) \in Y$.

$\boldsymbol{\neg a \Rightarrow \neg d}$

Let us assume that for some $x \in X$ and some ε-ball $B_\varepsilon(f(x))$, there exists no $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since $f(B_\delta(x)) \nsubseteq B_\varepsilon(f(x))$ for all $\delta > 0$, for all upper bounds $d > d(x, z)$, there exists at least some $z_d \in X$, such that $d(f(x), f(z_d)) \geqslant \varepsilon$. Now we define the sequence $(z_{d(n)} \in X)_{n \in \mathbb{N}}$, where the upper bound $d(n)$ is a monotonically decreasing function of n. This sequence converges to x, but $(f(z_{d(n)}) \in Y)_{n \in \mathbb{N}}$ does not converge to $f(x)$ since $d(f(x), f(z_{d(n)})) \geqslant \varepsilon$ for some ε and for all d. $\square$

2 Neighbourhoods

Lemma 4. *Let X be a topological space and let U_i be open sets with $1 \leqslant i \leqslant N$ for some N. The intersection of all U_i is open.*

Proof. Let V_n be the intersection of all U_i with $i \leqslant n$. $V_1 = U_1$ is open. If V_k is open, the intersection of V_k and U_{k+1} is open. Therefore, V_{k+1} is open. By induction, V_N is open and so the intersection of all U_i is open. □

Proposition 5. *Let $V_1, \ldots, V_n$ be a finite collection of neighbourhoods of $x \in X$. Their intersection is also a neighborhood of $x \in X$.*

Proof. Each neighbourhood V_i contains an open set U_i, which contains x. The intersection U of all U_i is open and must be contained in the intersection V of all V_i. U and V must also contain x. Thus, V is also a neighbourhood. □

Proposition 6. *Let X be a topological space. If U is a neighbourhood of $x \in X$ then U is the neighbourhood of all $y \in V$ for some neighbourhood V of x.*

Proof. U contains an open set V, which contains x. V is a neighbourhood for all elements in V, since V contains an open set V, which contains all elements in V. □

Proposition 7. *A subset $U \subseteq X$ is open if and only if it is a neighbourhood of all its points*

Proof. If U is open it contains an open set U which also contains every element of U. Therefore U is a neighbourhood of all its points.

For the reverse, let U_i be some neighbourhood of all its elements. The intersection $U_i \cap U_j$ is a neighbourhood since U_i and U_j are both neighborhoods of some point in the intersection (using our previous result). A union V of U_i is a neighbourhood of all its points, since we can find an open set contained in V that contains every point. Let $x \in V$. x must also be an element of some U_i. Therefore an open set exists in $U_i \subseteq V$ which contains x and so V is a neighbourhood of all its points. If we define a neighborhood, such that $\varnothing$ is a neighborhood of all its points and we realise that since X is open, it is a neighbourhood of all its points, we can see that U_i is in fact open. □

3 Maps Between Topological Spaces

3.1 Bijective Maps

In this subsection, let X and Y be topological spaces and $f: X \to Y$ a bijective map.

Proposition 8. *The preimage of a set U under f is equal to the inverse of f applied to U.*

Proof. Let the preimage of U be the set $A = \{x \in X: f(x) \in U\}$ and the inverse applied to U be the set $B = \{f^{-1}(y): y \in U\}$. If $x \in A$ is an element of the preimage, $f(x) \in U$ is an element of U. Since f is one-to-one, $x = f^{-1}(y)$ for $y = f(x) \in U$ and so x is also an element of B. If x is now taken to be an element of B, we know that $x = f^{-1}(y)$ for some $y \in U$. Because f is injective, there is only one such pair (x, y) with $x = f^{-1}(y)$ and because f is surjective, the unique x must be in the domain if y is in the codomain. Therefore $x \in X$ and $f(x) \in U$, meaning x is also an element of A if it is an elemnt of B. □

Proposition 9. *f is a homeomorphism if and only if $f: X \to Y$ is open*

Proof. Notice that $f(U)$ is the preimage of U under the map f^{-1}, since

$$f(U) = \{f(x): x \in U\} = \{(f^{-1})^{-1}(x): x \in U\} = \{y \in Y: f^{-1}(y) \in U\}$$

where the last equality used proposition 8 to equate the inverse applied to a set with the preimage of U under the map f^{-1}. Therefore, if f is open all preimages of open sets under f^{-1} are open, meaning f^{-1} is continuous, which implies f is a homeomorphism. Conversely, if f is a homeomorphism, preimages of open sets under f^{-1} are open, which is equivalent to saying f maps open sets to open sets and so f is open. □

Lemma 10. *$f(X \setminus U) = Y \setminus f(U)$ for any subset $U \subseteq X$.*

Proof. Any point $y \in f(X \setminus U)$ must be in Y and cannot be in $f(U)$. If y were an element of $f(U)$, too, there would be some point in U and some point in $X \setminus U$ which both map to y. However, this contradicts f being an injective function. Therefore y is an element of $Y \setminus f(U)$.

On the other hand, any point $y \in Y \setminus f(U)$ must be □

Proposition 11. *f is open if and only if f is closed*

Proof. Let f be open (or closed for the converse) and $U \subseteq X$ be an open (closed) set. Then $f(U) \subseteq Y$ is open (closed) and the compliment $Y \setminus f(U)$ is closed (open). as shown in lemma 10, $f(X \setminus U) = Y \setminus f(U)$, so f also maps the closed (open) set $X \setminus U$ to the closed (open) set $Y \setminus f(U)$. Because every closed (open) set is the compliment of some open (closed) set, the map f is closed (open). □

3.2 Generic Maps

In this subsection, let X and Y be topological spaces and $f: X \to Y$ a map between them.

Proposition 12. *Arbitrary f are continuous if and only if the topology on X is discrete or the topology on Y is coarse.*

Proof. If the topology on X is discrete any subset of the domain is open, in particular every preimage is open. Therefore, f is continuous.

If the topology on Y is coarse, we need only look at preimages of $\varnothing$ and Y, since those are the only open sets. The preimage of $\varnothing$ is $\varnothing$, which is open; the preimage of Y is X, which is open, since all elements in X map into Y.

For the converse, suppose all maps $f: X \to Y$ are continuous, but the topology on X is not discrete and the topology on Y is not coarse. Then, there exists a subset $U \subseteq X$ which is not open and an open subset $V \subset Y$ different from $\varnothing$ and Y. If we pick f, such that $f(U) = V$, the preimage of the open set V would be the set U, which is not open. Therefore, we have a contradiction since and f ought to be continuous. □

Proposition 13. *Let $\mathcal{S}$ be a subbasis of the topology on Y. The, the map $f: X \to Y$ is continuous if and only if the preimage $f^{-1}(U)$ is open in X for any $U \in \mathcal{S}$.*

Proof. If the map f is continuous, preimages of any open sets in Y are open. In particular, since elements of the subbasis are themselves open, the preimages $f^{-1}(U)$ for $U \in \mathcal{S}$ must be open.

For the converse, let preimages of sets in the subbasis be open. Since any set $U \subseteq Y$ may be expressed as a union of finite intersections of sets in $\mathcal{S}$, we have

$$\begin{aligned} f^{-1}(U) &= \{x \in X: f(x) \in U\} \\ &= \left\{x \in X: f(x) \in \bigcup \left\{\bigcap V_i: V_i \in \mathcal{S}\right\}\right\} \\ &= \bigcup \left\{x \in X: f(x) \in \bigcap V_i\right\} \\ &= \bigcup \left\{\bigcap \{x \in X: f(x) \in V_i\}\right\} \\ &= \bigcup \left\{\bigcap f^{-1}(V_i)\right\} \end{aligned}$$

By our assumption, preimages of the subbasis elements V_i are open and because a union of finite intersections of open sets must be open, $f^{-1}(U)$ is open, which implies f is continuous. □