Computer Vision

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Textbooks

Multiple View Geometry in Computer Vision, Hartley, R., and Zisserman

Richard Szeliski, **Computer Vision: Algorithms and Applications**, 2nd edition, 2022

Reference books

Readings for these lecture notes:

- ❑Hartley, R., and Zisserman, A. **Multiple View Geometry in Computer Vision**, Cambridge University Press, 2004, Chapters 1-3.

- ❑Forsyth, D., and Ponce, J. **Computer Vision: A Modern Approach**, Prentice-Hall, 2003, Chapter 2.

These notes contain material c Hartley and Zisserman (2004) and Forsyth and Ponce (2003).

References

These notes are based

- Dr. Matthew N. Dailey's course: AT70.20: Machine Vision for Robotics and HCI

- Dr. Sohaib Ahmad Khan CS436 / CS5310 Computer Vision Fundamentals at LUMS

Grading breakup

I. Midterm = 35 points
II. Final term = 40 points
III. Quizzes = 6 points (A total of 6 quizzes)
IV. Group project = 15 points
    a. Pitch your project idea = 2 points
    b. Research paper presentation relevant to your project = 3 points
    c. Project prototype and its presentation = 5 points
    d. Research paper in IEEE conference template = 5 points
V. OpenCV based on Python presentation = 2.5 points
VI. Matlab presentation = 2.5 points

Some top tier conferences of computer vision

I. Proceedings of the IEEE International Conference on Computer Vision and Pattern Recognition **(CVPR).**
II. Proceedings of the European Conference on Computer Vision **(ECCV).**
III. Proceedings of the Asian Conference on Computer Vision **(ACCV).**
IV. Proceedings of the International Conference on Robotics and Automation **(ICRA).**
V. Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems **(IROS).**

Some well known Journals

I. International Journal of Computer Vision **(IJCV).**
II. IEEE Transactions on Pattern Analysis and Machine Intelligence **(PAMI).**
III. Image and Vision Computing.
IV. Pattern Recognition.
V. Computer Vision and Image Understanding.
VI. IEEE Transactions on Robotics.
VII. Journal of Mathematical Imaging and Vision

Intersection of Two Lines

Two lines will intersect at a point

Let **l** and **l'** intersect at point, **x**

Then

$\mathbf{x} = \mathbf{l_1} \times \mathbf{l_2}$

or

$\vec{x} = \vec{l}_1 \times \vec{l}_2$

Proof:

The point **x** passes through both $\mathbf{l_1}$ and $\mathbf{l_2'}$.

Therefore

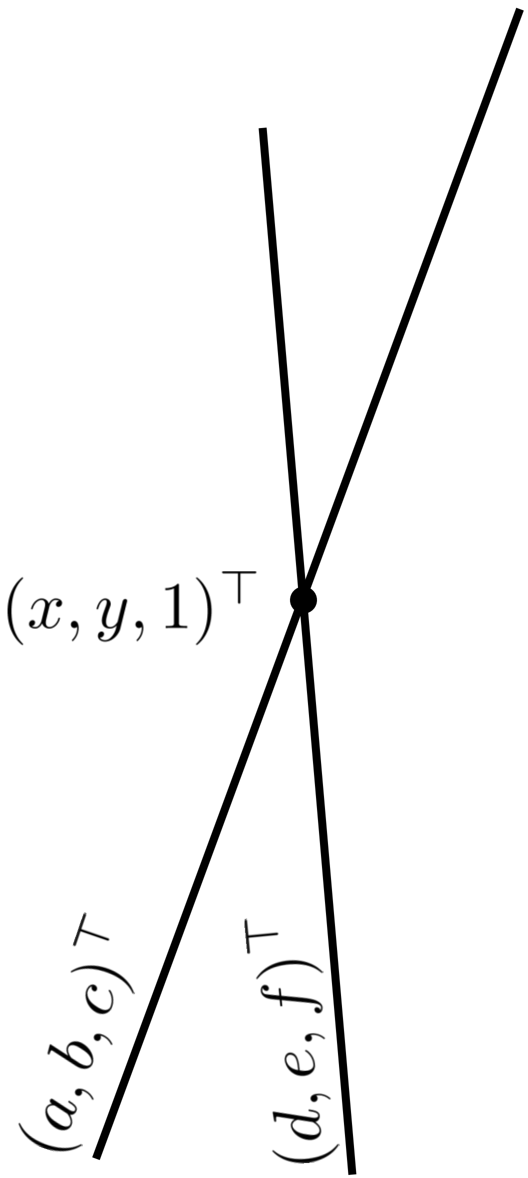
$\mathbf{x^T l_1 = 0}$

$\mathbf{x^T l_2 = 0}$

This is non trivially possible when **x** is orthogonal to both $\mathbf{l_1}$ and $\mathbf{l_2}$

Trivial conditions: x is a zero vector

$\mathbf{l_1}$ and $\mathbf{l_2}$ are same lines



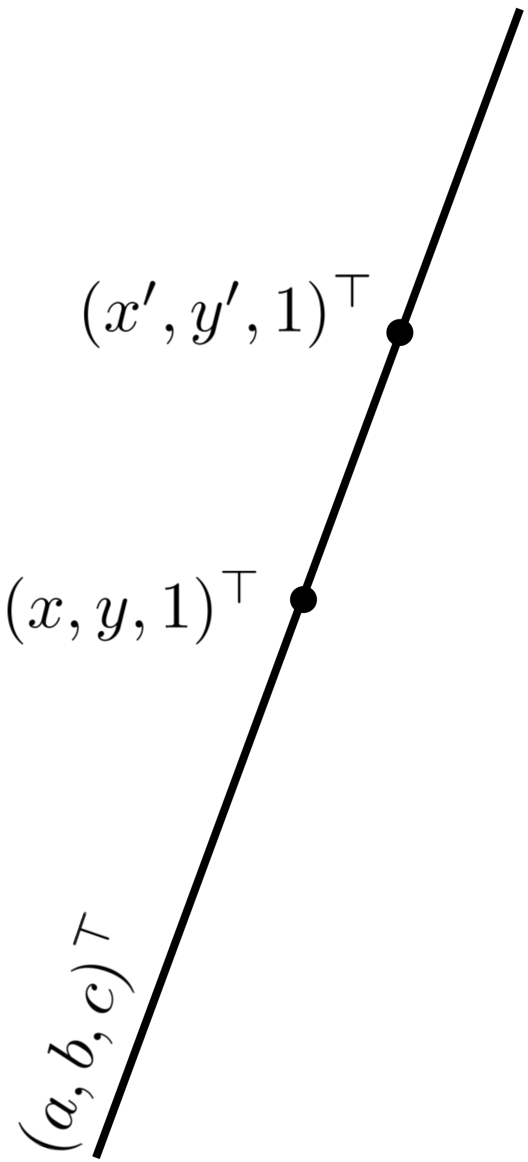
Line Joining Two Points

Two points lie on a line

Let $\mathbf{x}$ and $\mathbf{x}'$ lie on line $\mathbf{l}$

Then

l = $\mathbf{x_1} \times \mathbf{x_2}$



Proof:

The line **l** passes through both $\mathbf{x_1}$ and $\mathbf{x_2'}$

Therefore

$\mathbf{l}^{\mathbf{T}}\mathbf{x_1}$ = 0

$\mathbf{l}^{\mathbf{T}}\mathbf{x_2}$ = 0

This is non trivially possible when **l** is orthogonal to both $\mathbf{x_1}$ and $\mathbf{x_2}$

Duality

$$\mathbf{x} \quad \longleftrightarrow \quad \mathbf{l}$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{l} = 0 \quad \longleftrightarrow \quad \mathbf{l}^{\mathrm{T}}\mathbf{x} = 0$$

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 \quad \longleftrightarrow \quad \mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

Duality Theorem: To any theorem of 2-dimensional projective geometry, there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

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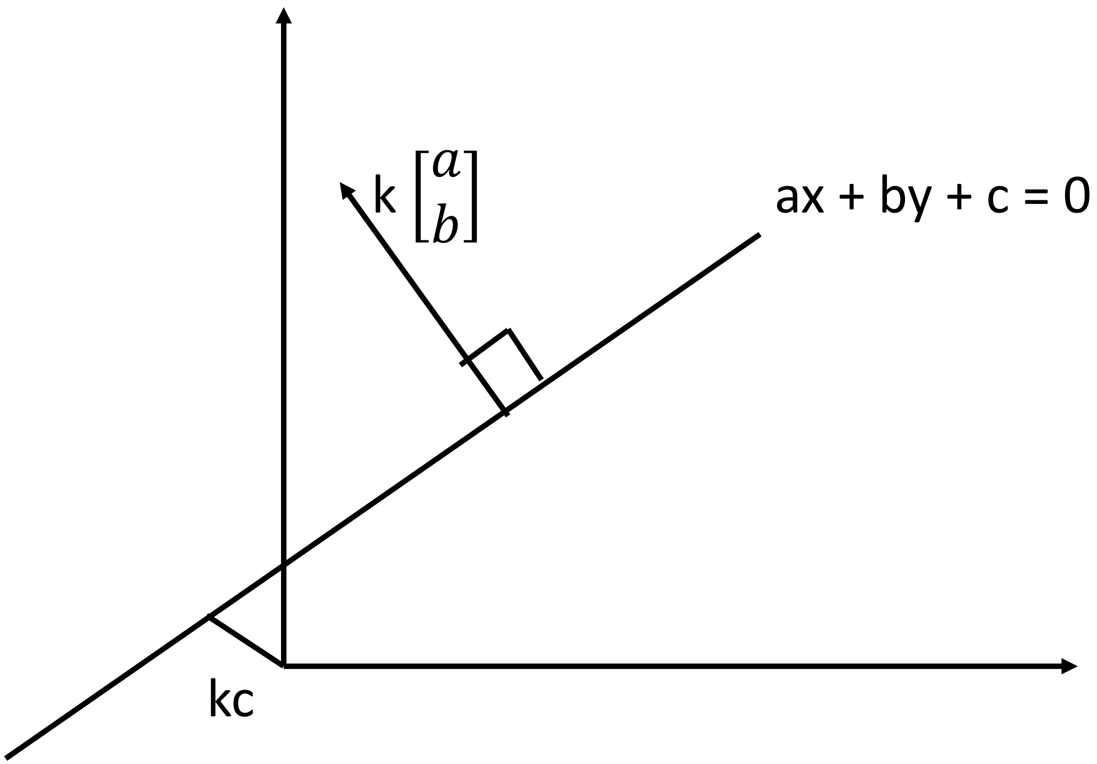
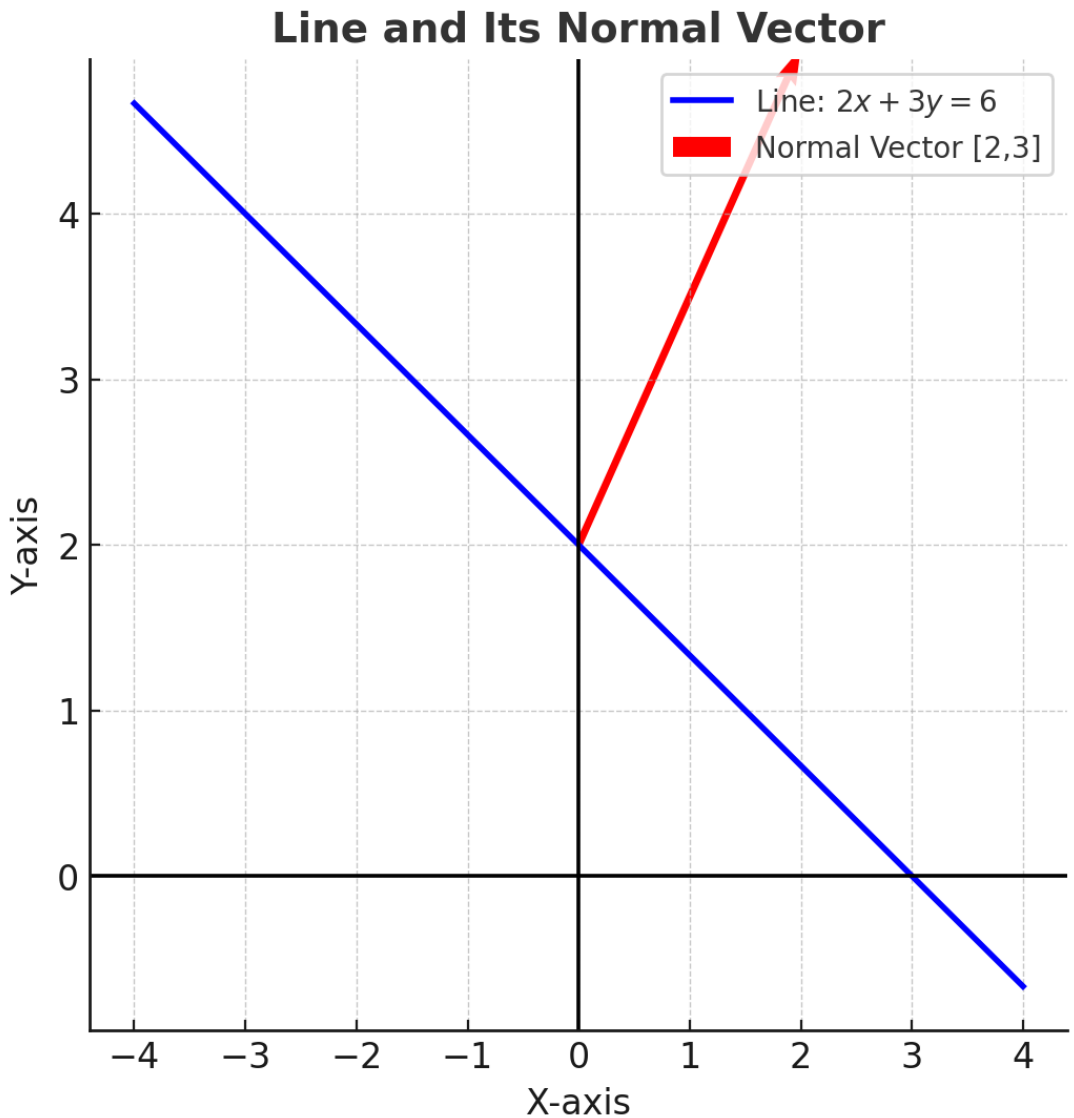
Geometrical interpretation of the vector $(a, b, c)^T$




Line and Its Normal Vector
Line: $2x + 3y = 6$
Normal Vector [2,3]
4
3
2
1
0
Y-axis
−4
−3
−2
−1
0
1
2
3
4
X-axis

Geometrical interpretation of the vector $(a, b, c)^T$

- **a and b will give a vector that should be normal to the line.**
  - The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is proportional to a and b.
  - Equation of a line is a **homogenous object**. If we multiply it by any scalar, then we should be getting the same line.
  - If we have the **same line**, then we should have the **same normal vector**.
  - If we **scale** the **normal vector** in the positive or negative direction, then we still have the same vector orthogonal to the line.
  - So, a and b will give us a vector that is **normal** to the **line**.

Geometrical interpretation of the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$

Normal Vector of a Line

Given two coefficients, a and b, they define a vector that is **normal (perpendicular)** to the line. This can be represented as:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

Since this vector is proportional to a and b, **scaling** it by any **nonzero scalar** will still result in a **normal vector**

Geometrical interpretation of the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$

Homogeneity of a Line Equation

- The equation of a line represents a **homogeneous object**, meaning that if we multiply the equation by any scalar, the **geometric representation** of the **line remains unchanged.**

- If the **line remains the same**, its **normal vector must also remain the same**.

- Scaling the normal vector, whether **positively** or **negatively**, does not change its **orthogonality** to the line.

- Therefore, **a** and **b define a vector that is always normal to the given line**.

Geometrical interpretation of the vector $(a, b, c)^T$

- What is c?

  - c is going to be **proportional** to the **distance of the origin to the line**.

  - If we **normalize** a, b, and c by appropriate amount, then c will give us a **minimum orthogonal signed distance** to **origin** from the line and a and b will us a **normal vector** to the line.

Understanding the Role of c in a Line Equation

- The **parameter c** in the equation of a line is **proportional to the perpendicular distance from the origin to the line**.

- To **interpret c** in terms of distance:
  - If we appropriately **normalize a, b,** and **c**, then **c represents** the **minimum orthogonal signed distance** from the **origin to the line.**
  - The values of **a** and **b** together define a **normal vector** to **the line**, indicating the **direction perpendicular to it**.

- Thus, **normalizing** these terms provides **a direct geometric interpretation** of the **equation of a line**.

Geometrical interpretation of the vector $(a, b, c)^T$

- What should be the scale factor k in order the third component of the vector be the **distance of the line** to the **origin**?

- We use the scale factor k such that the length of this vector $\left\|\begin{bmatrix} ka \\ kb \end{bmatrix}\right\| = 1$

- If we **normalize,** ax + by + c = 0 i.e., multiply through by a scalar such that $\left\|\begin{bmatrix} ka \\ kb \end{bmatrix}\right\| = 1$ then we have this relationship,
  - we have a **normal vector** of **unit length**
  - c is the **actual Euclidean distance** of **origin to a line**.

Geometrical interpretation of the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$

Determining the Scale Factor k

- To express the **third component** of the vector as the **distance of the line from the origin**, we must determine an appropriate **scale factor k**.

- Choosing the Scale Factor k
- We select k such that the length of the vector

$$\left\| \begin{bmatrix} ka \\ kb \end{bmatrix} \right\| = 1$$

- This ensures that the normal vector is of **unit length**.

Normalization of the Line Equation

Given the line equation:

ax + by + c = 0

We normalize it by multiplying through by a scalar such that:

$\left\| \begin{bmatrix} ka \\ kb \end{bmatrix} \right\| = 1$

- This results in the following key properties:
- The vector (a, b) represents a **unit normal vector** to the line.
- The value of c corresponds to the **actual Euclidean distance from the origin to the line**.

Example

Given Line Equation:

3x + 4y - 10 = 0

Step 1: Compute the Normalization Factor

$\sqrt{3^2+4^2} = \sqrt{25} = 5$

Step 2: Compute the Unit Normal Vector

(3/5, 4/5) = (0.6, 0.8)

Thus, the unit normal vector is (0.6, 0.8)

Step 3: Compute the Distance from the Origin

$|c| \,/\, \sqrt{a^2+b^2} =$ = | -10 | / 5 = 2

- The unit normal vector to the line is (0.6, 0.8)
- The Euclidean distance from the origin to the line is 2 units.

2D projective geometry
Ideal points and the line at infinity

- Parallel lines $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ and $(\mathbf{a}, \mathbf{b}, \mathbf{c}')^{\mathrm{T}}$ intersect in projective space $\mathbb{P}^{2}$.

- In **Euclidean space** ($\mathbb{R}^{2}$), parallel lines do not intersect, but in $\mathbb{P}^{2}$, they do at infinity.

- **Their intersection** is given by **the cross product**:
  $(c' - c)(b, -a, 0)^{\mathrm{T}} = (b, -a, 0)^{\mathrm{T}}$

- This point has no **inhomogeneous** representation in $\mathbb{R}^{2}$ since the third coordinate is zero.

- What does $(b/0, a/0)^{\mathrm{T}}$ represent?

2D projective geometry
Ideal Points and the Line at Infinity

- Any point of the form $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{0})^{\mathbf{T}}$ in $\mathbb{P}^2$ is called an **ideal point** or or a **point at infinity.**

  - These points exist along the direction $(\mathrm{x}_1, \mathrm{x}_2)^{\mathrm{T}}$.

  - They represent directions rather than finite locations in Euclidean space.

- Such points define the **line at infinity** in projective geometry.

2D projective geometry
Points at Infinity and the Line at Infinity

- All **points at infinity** lie on the **line at infinity $\mathbf{l}_\infty$= (0, 0,1)$^{\mathrm{T}}$**

- In projective space $\mathbb{P}^2$, **lines** correspond to planes in $\mathbb{R}^3$.

- **Points at infinity** lie on the **plane x_3 = 0,** so we represent the **plane x_3 = 0** by its normal vector **(0, 0, 1)^T**

2D projective geometry
Intersection of Parallel Lines in $\mathbb{P}^2$

- In $\mathbb{P}^2$ any **two lines intersect**, even if they are **parallel in Euclidean space**.
- This follows from the inclusion of **ideal points at infinity.**
- Parallel lines in $\mathbb{P}^2$ meet at a **unique deal point** on the **line at infinity.**

Intersection of two lines [1]

Intersection of two lines **(a, b, c)^T** and $(a', b', c')^T$ is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

- A **parameter-based representation** of a line allows us to easily determine the **intersection of two lines.** In elementary geometry, we learned that **parallel lines never intersect**. However, in **projective geometry**, parallel lines intersect at a **point at infinity**.

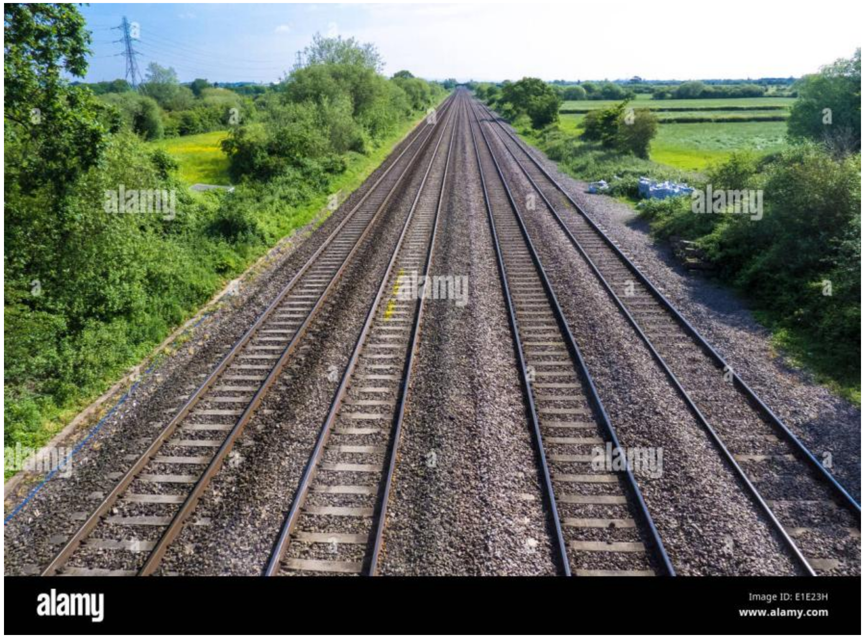
Visualizing Parallel Lines in Projective Geometry

- Consider looking at the **floor** where you may notice **two parallel lines**. At first glance, they appear to **never intersect**.

- However, if you observe these lines in an **orthogonal coordinate system**, where the camera is aligned perpendicular to the plane containing the lines, they seem truly parallel.

- Now, if you **lower your viewpoint—for** example, by kneeling down—and trace these lines towards the horizon, you will notice that they appear to **converge as they extend farther away**.

- This observation reflects how **parallel lines meet at a point at infinity** in projective geometry.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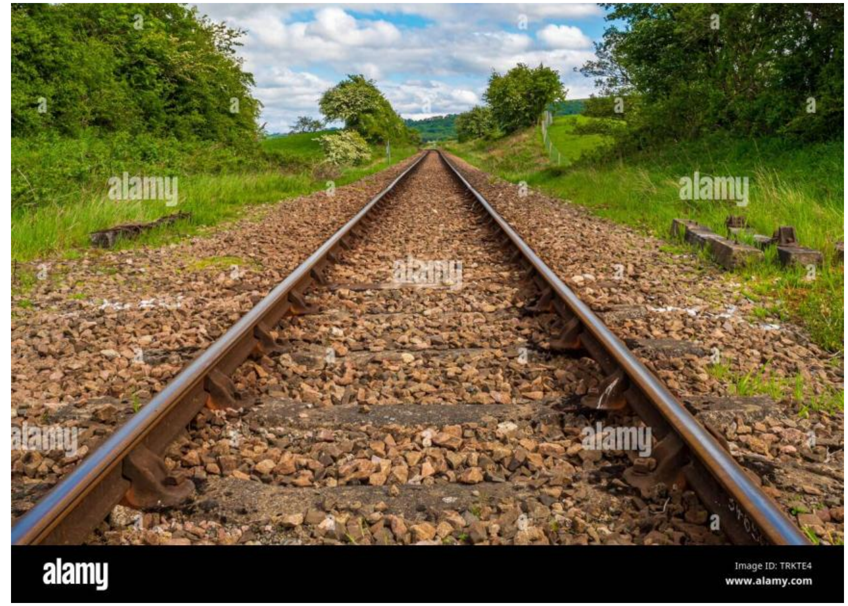
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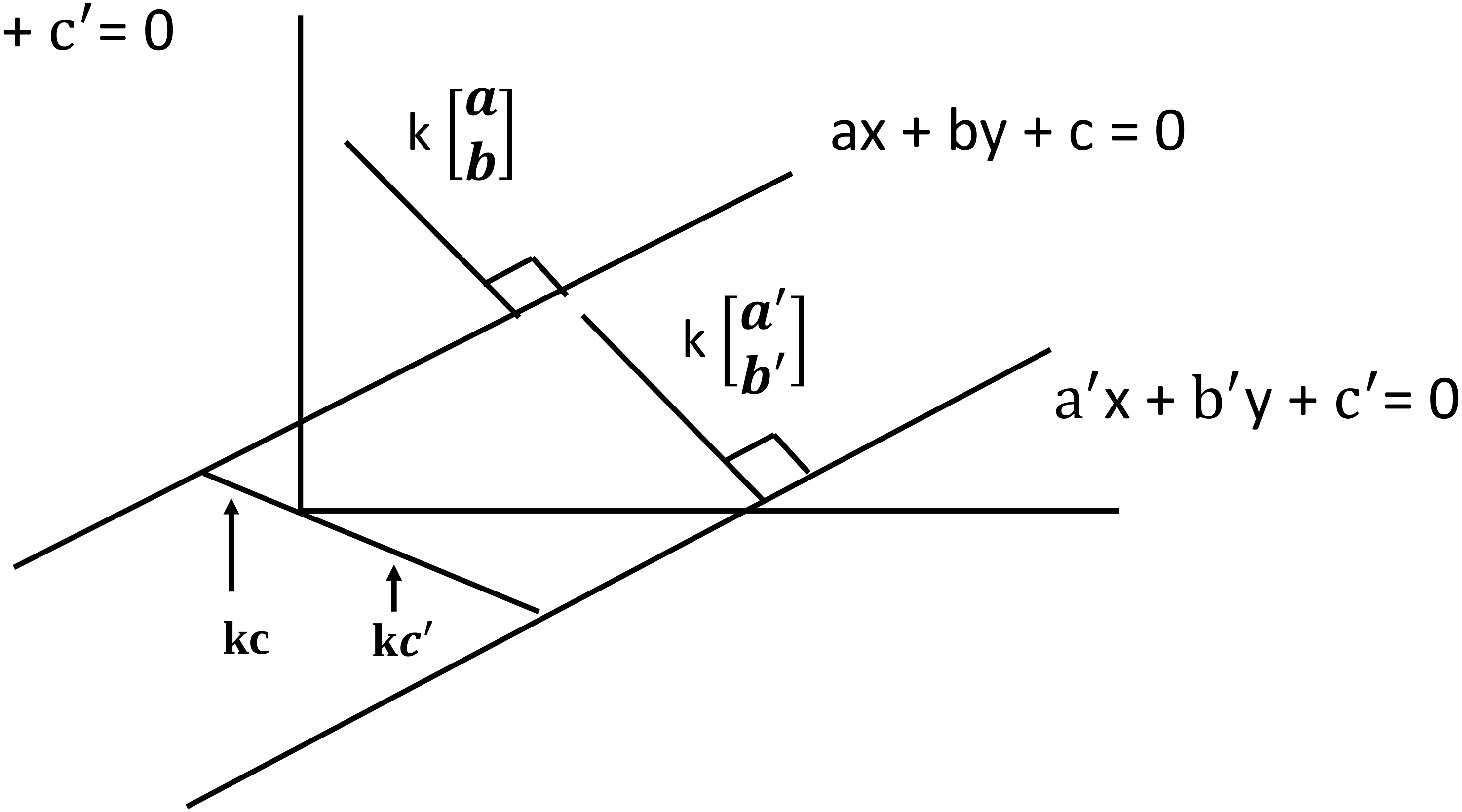
Relationship between the parameters of two parallel lines a, b, c and a′, b′, c′

Suppose we have two parallel lines

$ax + by + c = 0$

$a'x + b'y + c' = 0$



Relationship between the parameters of two parallel lines a, b, c and a′, b′, c′

- These **normal vectors** i.e., $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} a' \\ b' \end{bmatrix}$ should be the **same** or at least in the **same direction**, when **two lines are parallel.**
- If $\begin{bmatrix} a \\ b \end{bmatrix}$ *and* $\begin{bmatrix} a' \\ b' \end{bmatrix}$ are normalized then they are the same normal vectors. That's why we are writing two parallel lines as **$(a, b, c)^T$** and **$(a, b, c')^T$**
- These two parallel lines may have **different distances** to the **origin**.

If we assume the two parallel lines are normalize then

$ax + by + c = 0$

$ax + by + c' = 0$

Intersection of two lines [3]

Intersection of two parallel lines $(a, b, c)^T$ and $(a, b, c')^T$ is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ \mathrm{b} \\ \mathrm{c}' \end{bmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ a & \mathrm{b} & \mathrm{c}' \end{vmatrix}$$

Rule of Sarrus

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ a & b & c & a & b \\ a & \mathrm{b} & \mathrm{c}' & a & \mathrm{b} \end{matrix}$$

Rule of Sarrus

Sarrus' rule or **Sarrus' scheme** is a method and a **memorization scheme** to compute the **determinant** of a **3 × 3 matrix**. It is named after the French mathematician **Pierre Frédéric Sarrus**.

Rule of Sarrus

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{array}{ccccc} + & & + & & + \end{array}$$

$$\det A = \left|\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$

$$\begin{array}{ccccc} - & & - & & - \end{array}$$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{31}\, a_{22}\, a_{13} - a_{32}\, a_{23}\, a_{11} - a_{33}\, a_{21}\, a_{12}$$

Intersection of two lines [4]

Rule of Sarrus

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ a & b & c & a & b \\ a & \mathrm{b} & \mathrm{c}' & a & \mathrm{b} \end{matrix}$$

= $\hat{\imath}$ bc′ + $\hat{\jmath}$ac + $\hat{k}ab$ - $\hat{k}ab$ - bc$\hat{\imath}$ - ac′$\hat{\jmath}$

= (bc′- bc) $\hat{\boldsymbol{\imath}}$ + (ac - ac′) $\hat{\boldsymbol{\jmath}}$ + 0$\hat{\boldsymbol{k}}$

$$= \begin{bmatrix} \mathrm{bc}' - \mathrm{bc} \\ \mathrm{ac} - \mathrm{ac}' \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{b}(\mathrm{c}' - \mathrm{c}) \\ \mathrm{a}(\mathrm{c} - \mathrm{c}') \\ 0 \end{bmatrix}$$

Intersection of two lines [5]

$$= \begin{bmatrix} b(c' - c) \\ a(c - c') \\ 0 \end{bmatrix}$$

- This is the point where these **parallel lines intersect**. This is a point in the projective plane but it does not exist in the Euclidean plane.
- **Inhomogeneous** representation of ideal point

Normalizing the above point

$$\begin{bmatrix} b(c' - c)/0 \\ a(c - c')/0 \end{bmatrix}$$

"This is the **point at infinity** or an **ideal point**"

Intersection of two lines [6]

- Institutively, it makes sense. What is the point, where parallel lines intersect?

- The **parallel lines** intersect at **point at infinity.** It is a point but it is a special point. In $\mathbb{P}^{2}$, it is a normal point except its 3rd component is zero i.e.,

$$\begin{bmatrix} b(c' - c) \\ -a(c' - c) \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

Intersection of two lines [7]

- It means this homogeneous representation is quite convenient. In **Euclidean geometry**, the solution of this **problem does not exist**. But in $\mathbb{P}^2$, we have the solution and we can work with these points.

- In general, points with homogeneous coordinates **$(x, y, 0)^T$** do not correspond to any finite in $\mathbb{R}^2$. This observation agrees with the usual idea that **parallel lines** meet at **infinity**.

Example

Find the point of intersection of two parallel lines

$x = 1$ and $x = 2$.

Solution

The general equation of a line is

$ax + by + c = 0$

$x = 1$

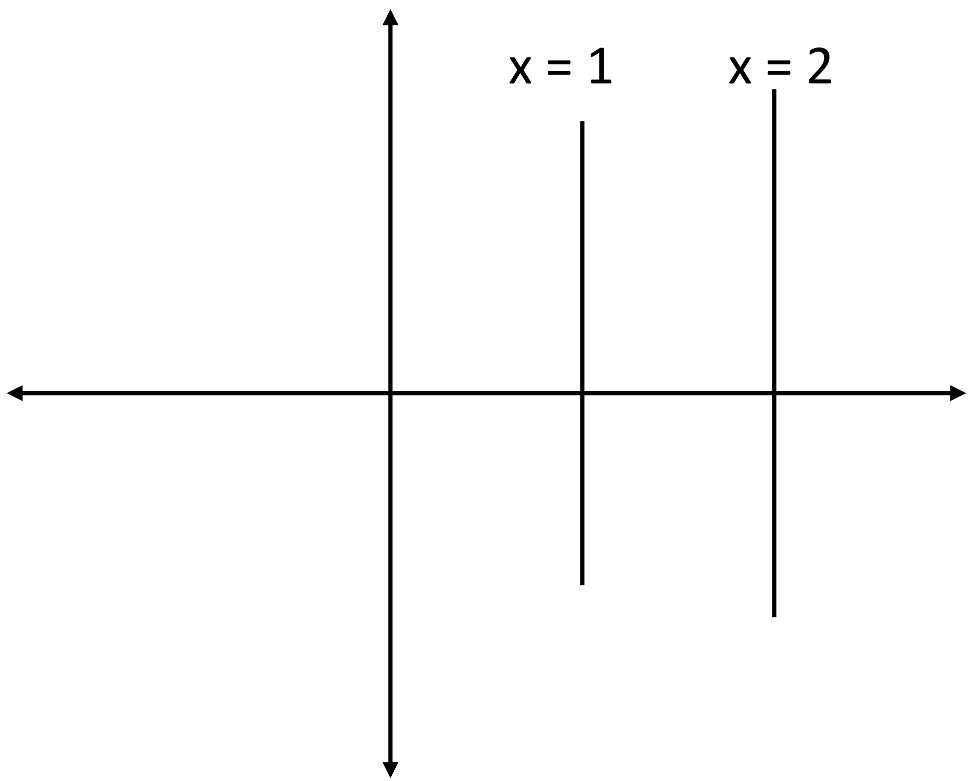
$\Longrightarrow 1.x + 0.y - 1 = 0$

$\Longrightarrow [1\ 0\ {-1}]^T = \vec{l}_1$

$x = 2$

$\Longrightarrow 1.x + 0.y - 2 = 0$

$\Longrightarrow [1\ 0\ {-2}]^T = \vec{l}_2$


x = 1
x = 2

$$\vec{x} = \vec{l}_1 \times \vec{l}_2$$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$\begin{matrix} & i & j & k & i & j \\ = & 1 & 0 & -1 & 1 & 0 \\ & 1 & 0 & -2 & 1 & 0 \end{matrix}$$

= 0i - j + 0k - 0k + 0i + 2j

$= \hat{j}$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Which is the **point at infinity** in the direction of y-axis.

Ideal points and the line at infinity

- The homogeneous vector $\vec{x} = (x_1, x_2, x_3)^T$ such that $x_3 \neq 0$ correspond to finite points in $\mathbb{R}^2$.

- If the 3rd component is zero i.e., $x_3 = 0$ then the resulting space is the set of all homogeneous three vectors, namely the projective space $\mathbb{P}^2$.

- The points with the last coordinate $x_3 = 0$ are called the **ideal points** with a particular point specified by the ratio $x_1 : x_2$ is $(x_1, x_2, 0)^T$

Ideal points and the line at infinity

- The set of points $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{0})^{\mathbf{T}}$ lies on a **single line** called the **line at infinity** denoted by $\vec{\mathrm{l}}_{\infty} = (0, 0, 1)^{\mathrm{T}}$

$$\vec{\mathrm{x}}^{\mathrm{T}}\vec{\mathrm{l}} = 0$$

$$\Rightarrow [\, \mathrm{x}_1 \; \mathrm{x}_2 \; 0] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow 0 = 0$$

Line at infinity

- If we look at the **intersection of the parallel lines** in the plane i.e., all possible parallel lines and find their intersection then **all of them are going** to lie on a **line**.

- All points on that line are **"ideal points"** but this line is only valid in $\mathbb{P}^2$. This line has a name called the line **at infinity** i.e., $\vec{l}_\infty = [0\ 0\ 1]^T$.

- The **line at infinity $[0\ 0\ 1]^T$** does not make sense in the **Euclidean plane**.

- We can write it as a line in the plane as $\vec{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Line at infinity in the Euclidean plane

$\vec{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or $[0\ 0\ 1]^T$

$\implies \vec{x}^T \vec{l} = 0$

$\implies [x\ y\ 1]_{1\times3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1} = 0$

$\implies$ 0x + 0y + 1 = 0 (This is an invalid line and it will crash)

This equation is inconsistent (i.e., $1 \neq 0$), meaning it represents an invalid line in the Euclidean plane.

Key Insight:

- The line **does not exist in** the Euclidean plane ($\mathbb{R}^2$**).**
- However, it **does exist** in **projective space** $\mathbb{P}^2$, where it represents the **line at infinity.**

Intersection of Parallel Lines

- Consider two parallel lines

$\vec{l}_1$: ax + by + c = 0

$\vec{l}_2$: ax + by + c′= 0

$\vec{l}_1 \times \vec{l}_2 = (c' - c)(b, -a, 0)^T$

- Computing intersection (as before)

$$(b, -a, 0)^T$$

- Thus, point of intersection is

$$(b, -a, 0)^T$$

- Converting to **inhomogeneous coordinates**:

$$(\frac{b}{0}, \frac{-a}{0})^T$$

- **Hence Parallel lines intersect at ideal points**

Ideal Points lie on a line

- Recall that all parallel lines intersect at an ideal point or point at infinity, of the form $(x, y, 0)^T$
- Consider two such ideal points:

  $\vec{x}: (\mathrm{x}, y, 0)^T$

  $\vec{x}': (x', y', 0)^T$
- The line joining them is given by:

  $$\vec{\mathrm{x}} = \vec{\mathrm{l}}_1 \times \vec{\mathrm{l}}_2$$

  or $\vec{\mathrm{x}} = (0, 0, xy' - yx')^T \equiv (0, 0, 1)^T$

Thus, <u>all **points at infinity** lie on a single line, the **line at infinity**</u>

$$\vec{l}_\infty = (0, 0, 1)^T$$

Line at Infinity

- Any line $\vec{l}: (\mathrm{a}, \mathrm{b}, c)^T$ intersects $\vec{l}_\infty$ at: $(\mathrm{b}, -\mathrm{a}, 0)^T$

- Any line parallel to $\vec{l}$, i.e. $\vec{l}': (\mathrm{a}, \mathrm{b}, c')^T$ will intersects $\vec{l}_\infty$ also at: $(\mathrm{b}, -\mathrm{a}, 0)^T$

- In **inhomogeneous coordinates**, $(\mathrm{b}, -\mathrm{a})^T$ represents line direction.

- Hence, as line direction varies, its intersection with $\vec{l}_\infty$ varies.

- Line at infinity is the set of directions for lines in a plane