Computer Vision

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Textbooks

Multiple View Geometry in Computer Vision, Hartley, R., and Zisserman

Richard Szeliski, **Computer Vision: Algorithms and Applications**, 2^{nd} edition, 2022

Reference books

Readings for these lecture notes:

- Hartley, R., and Zisserman, A. **Multiple View Geometry in Computer Vision**, Cambridge University Press, 2004, Chapters 1-3.

- Forsyth, D., and Ponce, J. **Computer Vision: A Modern Approach**, Prentice-Hall, 2003, Chapter 2.

These notes contain material c Hartley and Zisserman (2004) and Forsyth and Ponce (2003).

References

These notes are based

- Dr. Matthew N. Dailey's course: AT70.20: Machine Vision for Robotics and HCI

- Dr. Sohaib Ahmad Khan CS436 / CS5310 Computer Vision Fundamentals at LUMS

Grading breakup

I. Midterm = 35 points
II. Final term = 40 points
III. Quizzes = 6 points (A total of 6 quizzes)
IV. Group project = 15 points
   a. Pitch your project idea = 2 points
   b. Research paper presentation relevant to your project = 3 points
   c. Project prototype and its presentation = 5 points
   d. Research paper in IEEE conference template = 5 points
V. OpenCV based on Python presentation = 2.5 points
VI. Matlab presentation = 2.5 points

Some top tier conferences of computer vision

I. Proceedings of the IEEE International Conference on Computer Vision and Pattern Recognition **(CVPR).**
II. Proceedings of the European Conference on Computer Vision **(ECCV).**
III. Proceedings of the Asian Conference on Computer Vision **(ACCV).**
IV. Proceedings of the International Conference on Robotics and Automation **(ICRA).**
V. Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems **(IROS).**

Some well known Journals

I. International Journal of Computer Vision **(IJCV).**
II. IEEE Transactions on Pattern Analysis and Machine Intelligence **(PAMI).**
III. Image and Vision Computing.
IV. Pattern Recognition.
V. Computer Vision and Image Understanding.
VI. IEEE Transactions on Robotics.
VII. Journal of Mathematical Imaging and Vision

Homogeneous Objects in Mathematics

If we take the coefficient form of the equation of a line (ax + by + c = 0):

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and multiply it by any scalar k, we still have the same line:

$$\begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$$

$\begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ **remains unchanged**, demonstrating that it is a **homogeneous object**.

Definition of Homogeneous Objects:

- Homogeneous objects are mathematical entities that are determined **only up to scale**. This means that scaling them by a **nonzero factor** does not change their fundamental nature.

- In this case, the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{T}$ represents the **same line** as $\mathbf{k}(\mathbf{a}, \mathbf{b}, \mathbf{c})^{T}$ for any **nonzero constant k**.

What are Homogeneous Coordinates?

- In projective geometry, we extend **2D coordinates** to **3D** for easier transformations.

- A **point (x, y) in Cartesian coordinates** is represented **as (x, y, 1) in homogeneous coordinates**.

- This allows for **uniform representation of transformations** like scaling and translation.

Homogeneous Representation of Points and Lines in $\mathbb{P}^2$

- A **point in homogeneous coordinates** is represented as

$\vec{\boldsymbol{p}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})^T$

- A **line in homogeneous coordinates** is represented as

$\vec{l} = (a, b, c)^T$

- The equation of a line in homogeneous coordinates is given by:

a x + b y + c w = 0

- This equation defines the set of points (x, y, w) that lie on the line.

  For example If a $\vec{\boldsymbol{p}} = (\boldsymbol{x}, \boldsymbol{y}, \mathbf{1})^T$ lies on $\vec{l} = (a, b, c)^T$ if their dot product is zero.

Dot Product Interpretation

- The **dot product** between **the homogeneous point $\vec{p}$** and **the homogeneous line $\vec{l}$** is:

$$\vec{l} \cdot \vec{p} = (a, b, c)^T . (x, y, w)^T \cdot$$

$$\vec{l} \cdot \vec{p} = \text{a } x + \text{b } y + \text{c } w = 0$$

Thus, we conclude:

$$\vec{l} \cdot \vec{p} = 0$$

Conclusion

If a point $\vec{p}$ lies on a line $\vec{l}$, then their **dot product** in homogeneous coordinates is **zero**. This result follows directly from the definition of a homogeneous line equation in projective space.

Example: Consider the line given by the homogeneous equation:

$2\,x + 3\,y$ −5c = 0 means the line is represented in homogeneous coordinates as:

$$\vec{l} = (2, 3, -5)^T$$

Now, let's take a point in homogeneous coordinates:
$\vec{p} = (1, 1, 1)^T$

$$\vec{l}.\vec{p} = (2, 3, -5)^T.(1, 1, 1)^T$$
$$= 2 \times 1 + 3 \times 1 - \ 5 \times 1$$
$$= 0$$

Since the dot product is **zero**, the point (1, 1, 1) lies on the line.

Mathematical Formulation

- A line in Cartesian coordinates is given by: $ax + by + c = 0$

- This can be written in homogeneous form as:

  $L = [a\ b\ c]^T$ **(column vector representation)**

- **The dot product with a point in homogeneous coordinates:**

  $[a\ b\ c] \bullet [x\ y\ 1]^T = 0$

Example: Converting to Homogeneous Form

- Given a line equation: $3x + 4y - 5 = 0$
- **Coefficient vector representation**: $L = [3\ 4\ -5]^T$
- **Homogeneous point** (2, 1) represented as $[2\ 1\ 1]^T$

Dot product:

$[3\ 4\ -5] \bullet [2\ 1\ 1]^T$

$= 3(2) + 4(1) - 5(1)$

$= 6 + 4 - 5$

$= 5$

Since $5 \neq 0$, the point does not lie on the line.

Object only determined up to scale

- We can multiply the coefficient vector $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^{\boldsymbol{T}}$ by any scalar, and the resulting vector still represents the same line, even though the numerical values change.

- This is because lines in **homogeneous coordinates** are **defined up to scale**, making them **homogeneous objects.**

- The **coefficient vector** representation of a **line in 2D** is a **homogeneous entity.**

Projective Space $\mathbb{P}^2$

- Every coordinate is defined by a 3-vector **x** = $(x_1,\ x_2,\ x_3)^T$
- The **first two elements** of the vector define its **direction** only (outward from origin).

- **Point (0,0,0)^T** does not **define a direction**, hence is excluded from $\mathbb{P}^2$

  $\mathbb{P}^2 = \mathbb{R}^3 - (0, 0, 0)^T$

- The set of points in $\mathbb{P}^2$ can be thought of as the set of points in $\mathbb{R}^3$ **augmented by ideal points**.

2D projective geometry

The 2D projective plane: homogeneous coordinates and $\mathbb{P}^2$

- $(a, b, c)^T$ represents the same line as k $(a, b, c)^T$ for any **non-zero constant k**.

- A **homogeneous vector** is an **equivalence class of vectors** denoted by **scaling**.

- The set of homogeneous equivalence classes of vectors in $\mathbb{R}^3$- $(0, 0, 0)^T$ is called the **projective space** $\mathbb{P}^2$.

Homogenous representation of three vectors [1]

- If we have vectors with three components, then we talk about the homogenous representation of three vectors.

- **"The set of homogeneous equivalence classes of vectors in $\mathbb{R}^3$ - $(0, 0, 0)^T$ is called the projective space $\mathbb{P}^2$ "**

- One way to think about this is to think $\mathbb{P}^2$ as actually the set of all **unique lines in 2D** i.e.,

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ = k $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ or $(\mathrm{a}, \mathrm{b}, \mathrm{c})^T = k\ (\mathrm{a}, \mathrm{b}, \mathrm{c})^T$ for any non-zero scalar k.

Homogenous representation of three vectors [2]

- This type of equality differs from the standard **"="** used in **Euclidean geometry**. In this context, a vector represents a 3D point and is only equal to another vector if they correspond to the exact same point in space.

- However, in $\mathbb{P}^2$ **any point** represented by a three-component vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ or **(a, b, c)^T**, as long as it is not **(0, 0, 0)^T**.

- Such a point belongs to an **equivalence class** that includes **all scalar multiples of the vector**, meaning any scaled version of the same vector represents the **same point in projective space**.

Why we exclude the origin in the projective space? [1]

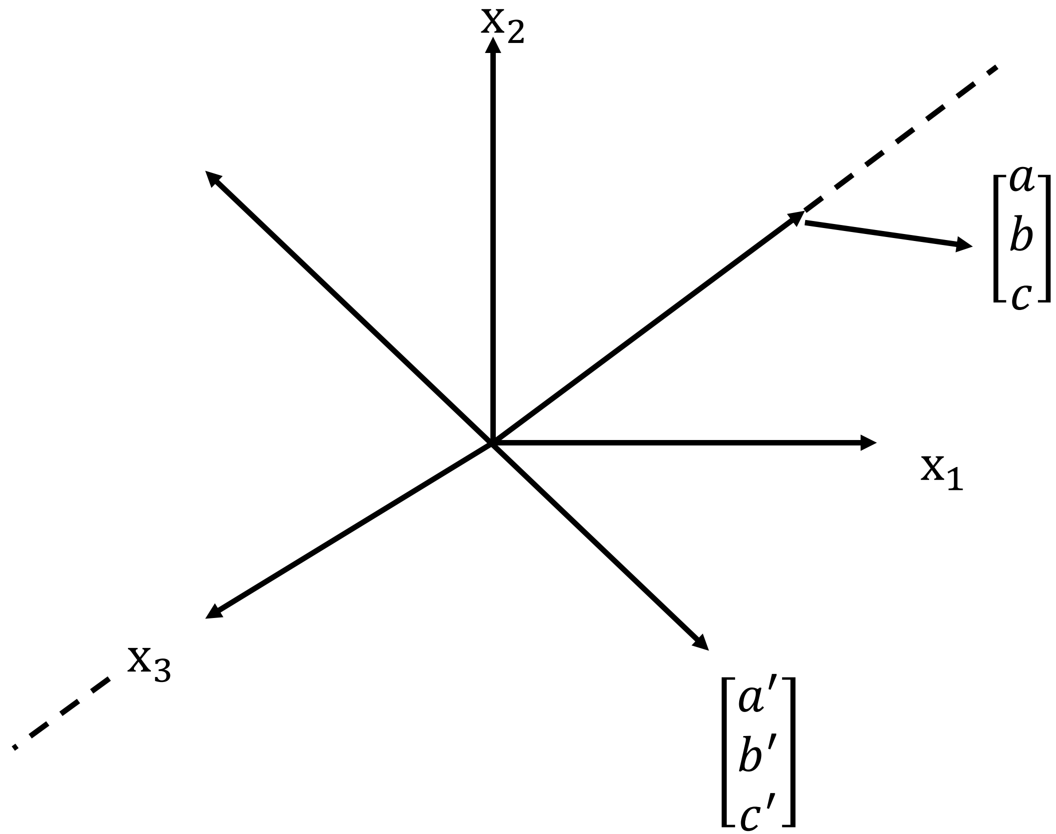
- If we multiply a point $(\mathbf{0}, \mathbf{0}, \mathbf{0})^{\boldsymbol{T}}$ by any scalar k, we are not introducing anything new. The result remains $(\mathbf{0}, \mathbf{0}, \mathbf{0})^{\boldsymbol{T}}$, meaning we are still at the same point.

- Origin of $\mathbb{R}^3$ has some **special interpretation**. It is not a part of $\mathbb{P}^2$

- To be element of $\mathbb{P}^2$, we have to be a **line** in three space.

- One way to understand this is as follows: Take any point $(\mathrm{a}, \mathrm{b}, \mathrm{c})^T$ in $\mathbb{R}^3$. This point belongs to the **same equivalence class as all other points** that lie along the same line passing through the origin.

Why we exclude the origin in the projective space? [2]



Why we exclude the origin in the projective space? [3]

- The **origin** is a part of **every line** passing through it. In projective space $\mathbb{P}^2$, each point represents an entire line through the origin, meaning the **origin** itself **cannot uniquely represent any point** in $\mathbb{P}^2$

- This becomes clearer when considering projections from **3D to 2D**. In a camera model, the **origin corresponds to the focal point of the lens or the pinhole**.

- At the **focal point**, the **entire scene is compressed** to **a single point**, losing all directional information. This means that **at the origin,** the **image collapses into an infinitely small projection**, making it impossible to reconstruct the original image.

2D projective geometry

The 2D projective plane: homogeneous point representations

- Since a point **x** = $(x, y)^T$ lies on a line **l** =$(a, b, c)^T$iff
  ax + by + c = 0, we can equivalently write the **inner product**
  $(x, y, 1)\,(a, b, c)^T = (x, y, 1)\mathbf{l} = 0$

- If $(x, y, 1)\mathbf{l} = 0$, it is also true that $(kx, ky, k)\mathbf{l} = 0$
- This makes it convenient to represent a point **x** = $(x, y)^T$ in $\mathbb{R}^2$ with the **homogeneous vector $(x, y, 1)^T$** .
- This means the arbitrary point **x** = $(x_1, x_2, x_3)^T$ in $\mathbb{P}^2$ can be used to represent the point $(\frac{x_1}{x_3}, \frac{x_2}{x_3})$ in $\mathbb{R}^2$
- This is nice! Why? Because now we can say that a homogeneous point **x** = $(x_1, x_2, x_3)^T$ lies on line **l** iff $x^T\mathrm{l} = 0$.

Relationship between a point and a line [1]

ax + by + c = 0 or $[x\ y\ 1]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

i.e., the point $[x\ y\ 1]$ *lies on the line* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Notice we can replace $a\ point\ [x\ y\ 1]$ by scalar times say k a point, i.e., k$[\ x\ y\ 1]$.

If a point $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ satisfies the line, so does $\begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$

So, we can treat **lines** as well as **points** as **homogeneous quantity** i.e.,

$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$ (these points are equivalence class in $\mathbb{P}^2$)

Relationship between a point and a line [2]

These all points are related to the original point. They are in equivalence class in $\mathbb{P}^2$.

Note:

- If a **three vector (or three-component vector)** is a **homogenous representation** of a **2D point** then we should be thinking, we are taking about a point in $\mathbb{P}^2$ but not a about point in $\mathbb{R}^3$.

- When **we are talking about a point in** $\mathbb{P}^2$ then it can be represented by a **homogenous three-component vector**.

- If we have a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ representing a point in $\mathbb{P}^2$, then how can we recover the corresponding point in $\mathbb{R}^2$?

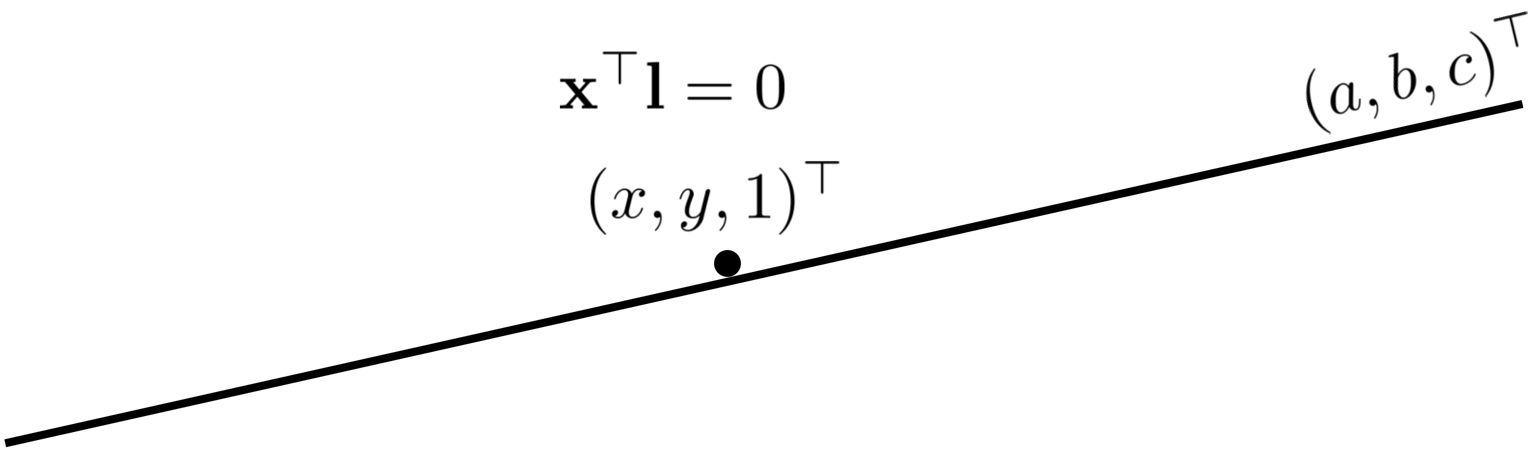
$x = \frac{x_1}{x_3}$ and $y = \frac{x_2}{x_3}$

- The 3D representation of a point i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in a plane is called the **homogenous representation** of that point but whenever we want to convert it into a corresponding point in $\mathbb{R}^2$ the actual point in the Cartesian plane then we have to **normalize** this point
- How do we normalize a homogenous point?
- We just divide the **first two elements** by the **last element** and we get the original point in $\mathbb{R}^2$

Point on Line

- Point **x** lies on line **l** iff

$$\mathbf{x}^\top \mathbf{l} = 0$$



- Even though **x** and **l** are 3-vectors, they have 2 degrees of freedom each
- **Note:** In computer vision, **degrees of freedom (DoF)** refer to the number of **independent parameters** that define an object's position, orientation, or movement in a given space.

Points and lines are dual

- If a point $\vec{x} = (x_1, x_{2,} x_3)^T$ lies on $\vec{\boldsymbol{l}} = (a, b, c)$, then $\vec{x}^T . \vec{l}$ = 0

- $\vec{x}^T$ takes our column vector as row vector.

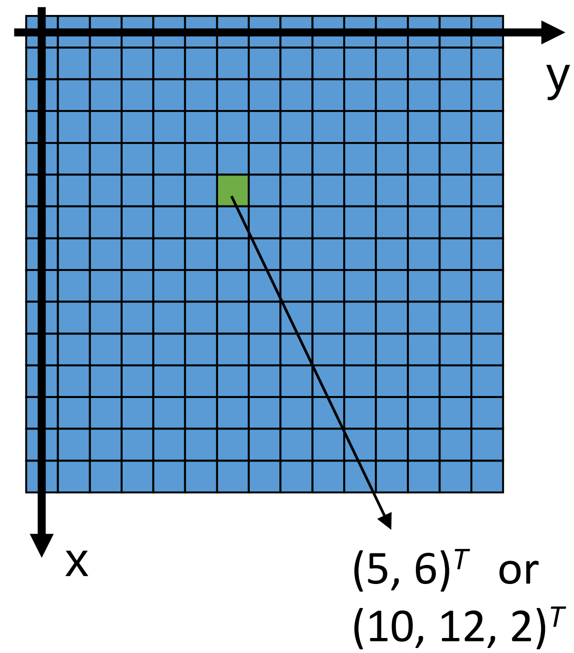
- Representation of a **line** is in a same form as a representation of a **point**. Somehow **lines** and **points** are similar to each other. This will have a very powerful consequence as we see later.

2D Points

- Image points are 2-dimensional

  $\vec{x} = (x_1, x_2)^T \in \mathbb{R}^2$

  or $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$



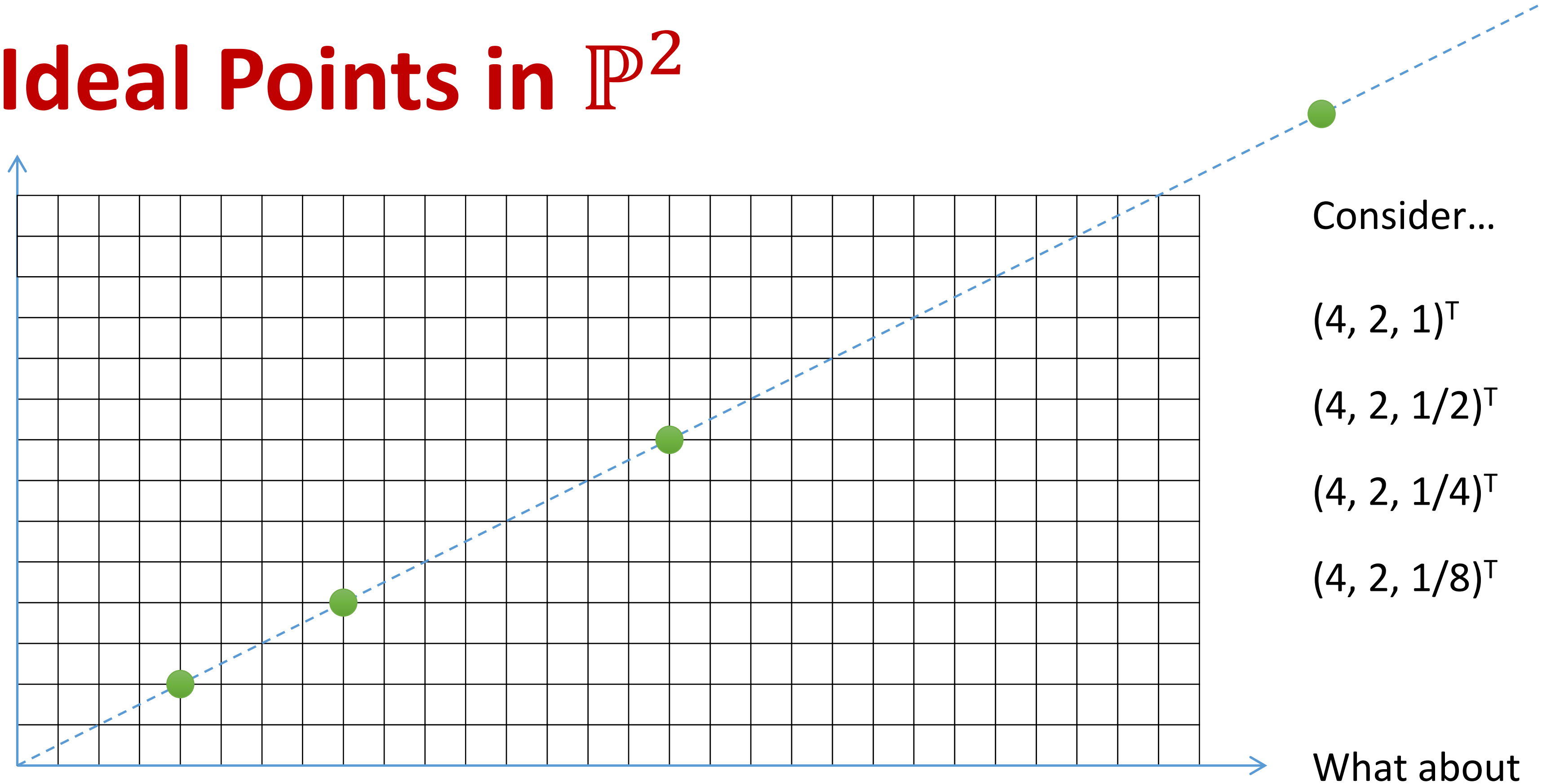
- **Homogeneous Coordinates**
  - Vectors that **differ** only by **scale** are **equivalent**

    $\vec{x} = (x_1, x_2, x_3)^T \in \mathbb{P}^2$
  - **$(10, 20, 1)^T \equiv (30, 60, 3)^T \equiv (5, 10, ½)^T$**
  - Every point has **infinite representations**
  - $\mathbb{P}^2$ is the **2D projective space**
- A **homogenous vector** $\mathbf{x}$ or $\vec{x}$ can be converted to **inhomogeneous coordinates** by dividing by k

$$\vec{x} = (x_1, x_2, x_3)^T = k(x_1, x_2, x_3)^T = k\,\vec{x}$$

$$x_1 = \frac{x_1}{k}, x_2 = \frac{x_2}{k}$$

Ideal Points in $\mathbb{P}^2$



Consider…

$(4, 2, 1)^T$

$(4, 2, 1/2)^T$

$(4, 2, 1/4)^T$

$(4, 2, 1/8)^T$

What about $\lim_{k \to 0}(4, 2, k)^T$

- Points with k = 0 are **ideal points**, or points at infinity
- They define a **direction** from ***origin***
- They do not have a **non-homogeneous** equivalent

Projective Space $\mathbb{P}^2$

- **Every coordinate** is defined by a **3-vector** $\vec{x} = (x_1, x_{2,}\, x_3)^T$

- The **first two elements** of the vector define its **direction** only (outward from origin)

- **Point (0,0,0)^T** does not define a **direction,** hence is excluded from $\mathbb{P}^2$
  $\mathbb{P}^2 = \mathbb{R}^3$ - **(0, 0, 0)^T**

- The set of points in $\mathbb{P}^2$ can be thought of as the set of points in $\mathbb{R}^2$ **augmented** by **ideal points**

Why Projective Geometry?

There are four reasons:

1. **Camera** is a **projective engine**
2. **Points at infinity** are handled
3. **Algebra is simpler than usual**
4. Is the most general framework to work in

   Affine or **Euclidean** upgrades can be made if required

What is a PTZ Camera?

A Pan-Tilt-Zoom (PTZ) camera is a surveillance or robotic camera that allows remote control over its movement and zoom functionalities.

Key Features of a PTZ Camera

1. Pan (Horizontal Movement) - **Rotates left and right** to cover a wide area.

2. Tilt (Vertical Movement) - **Moves up and down** to track objects at different heights.

3. Zoom (Optical & Digital Zoom) - **Adjusts focus to zoom** in or out.

Applications of PTZ Cameras

Surveillance & Security: Used in public spaces, traffic monitoring, and commercial buildings.

Broadcast & Live Events: Dynamic camera angles in news broadcasting and sports.

Industrial & Research: Automation, robotics, and remote monitoring.

A Pan-Tilt-Zoom (PTZ) Camera

Why Projective Geometry?

"Algebra is simpler than usual" EXAMPLE

- The camera model for a **typical surveillance camera** which can **pan and tilt** is given by:

$$x = f\frac{(X-X_0)\cos\theta + (Y-Y_0)\sin\theta - r_1}{-(X-X_0)\sin\theta\sin\phi + (Y-Y_0)\cos\theta\sin\phi - (Z-Z_0)\cos\phi + r_3 + f},$$

$$y = f\frac{-(X-X_0)\sin\theta\cos\phi + (Y-Y_0)\cos\theta\cos\phi + (Z-Z_0)\sin\phi - r_2}{-(X-X_0)\sin\theta\sin\phi + (Y-Y_0)\cos\theta\sin\phi - (Z-Z_0)\cos\phi + r_3 + f}.$$

OR

$x = f \times ((X - X_0) \cos\theta + (Y - Y_0) \sin\theta - r_1) / (-(X - X_0) \sin\theta \sin\phi + (Y - Y_0) \cos\theta \sin\phi - (Z - Z_0) \cos\phi + r_3 + f')$

$y = f \times (-(X - X_0) \sin\theta \cos\phi + (Y - Y_0) \cos\theta \cos\phi + (Z - Z_0) \sin\phi - r_2) / (-(X - X_0) \sin\theta \sin\phi + (Y - Y_0) \cos\theta \sin\phi - (Z - Z_0) \cos\phi + r_3 + f')$

Why Projective Geometry?

- Using projective mathematics, it is simple, where **P** is a **3 x 4** matrix of camera parameters. $x_{3\times 1} = P_{3\times 4} X_{4\times 1}$
- In computer vision a **camera matrix or (camera) projection** matrix is a **3 x 4 matrix** which describes the **mapping of a pinhole camera from 3D points in the world to 2D points in an image**.
- **Properties of the camera** can be derived simply from **P**. For example the location of the camera is just the **right null vector** of **P.**
- **Note:** A **pan–tilt–zoom camera (PTZ camera)** is a camera that is capable of remote directional and zoom control.

Projection Equations in PTZ Camera

The transformation is given by:

$$x = f \times \left((X - X_0)\cos\theta + (Y - Y_0)\sin\theta - r_1 \right) / \left(-(X - X_0)\sin\theta\sin\phi + (Y - Y_0)\cos\theta\sin\phi - (Z - Z_0)\cos\phi + r_3 + f' \right)$$

$$y = f \times \left(-(X - X_0)\sin\theta\cos\phi + (Y - Y_0)\cos\theta\cos\phi + (Z - Z_0)\sin\phi - r_2 \right) / \left(-(X - X_0)\sin\theta\sin\phi + (Y - Y_0)\cos\theta\sin\phi - (Z - Z_0)\cos\phi + r_3 + f'\right)$$

- The given equations describe the **perspective projection** in a **PTZ camera**, mapping **3D world coordinates (X, Y, Z) to 2D image coordinates (x, y)**.
- This helps the **camera adjust** for **pan, tilt, and zoom dynamically**.

Explanation of Terms

(X_0, Y_0, Z_0): **Camera position in world coordinates**.

θ (Theta): **Pan angle (horizontal rotation).**

ϕ (Phi): **Tilt angle (vertical rotation).**

r_1, r_2, r_3: **Transformation adjustments** (translation or rotation offsets).

f: **Focal length, affecting zoom**.

f': **Optical center shift**.

Lines in 2D

- Equation of line in 2D

  $ax + by + c = 0$

- Thus, a **line** can be represented by **vector** $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^{\boldsymbol{T}}$.

- $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^{\boldsymbol{T}}$ and **k**$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^{\boldsymbol{T}}$ mean the same line for $k \neq 0$.

- Thus lines can be represented by **equivalence classes of vectors** in $\mathbb{R}^3$ - $\mathbf{(0, 0, 0)^T}$ i.e., Projective space $\mathbb{P}^2$.

Point on a line

- 2D point $\vec{x} = (x_1, x_2)^T$

- Point will lie on line iff ax + by + c = 0

- This can be written as inner product

  (x, y, 1) $(a, b, c)^T$ = 0

  (x, y, 1)**l** = 0

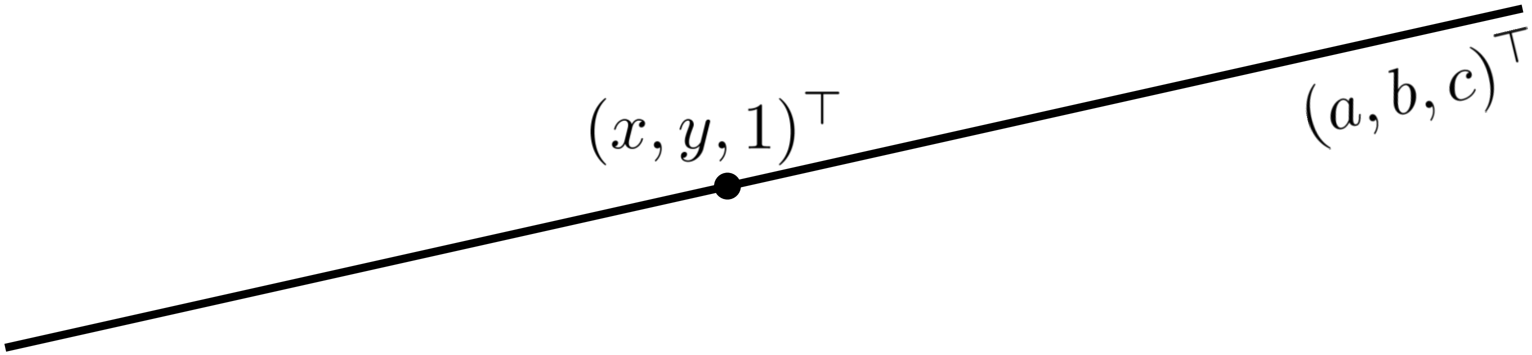
- Any non-zero k can be multiplied to the point, **without loss of generality**

- Hence **points** can also be represented as **homogeneous vectors**

Point on Line

- Point $\mathbf{x}$ lies on line $\mathbf{l}$ iff

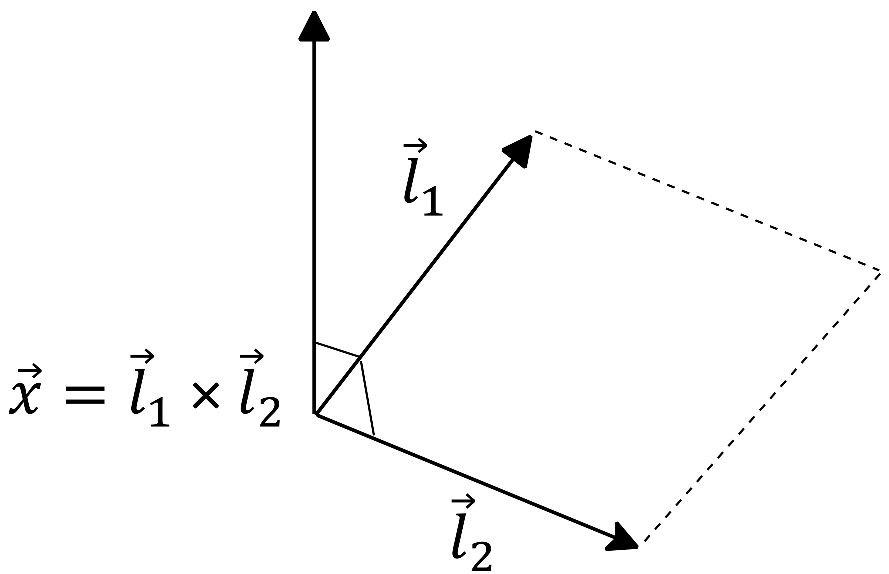
$$\mathbf{x}^\top \mathbf{l} = 0$$



- Even though $\mathbf{x}$ and $\mathbf{l}$ are 3-vectors, they have 2 degrees of freedom each

Recall: Cross product of two vectors

- **Cross product between two vectors** i.e., $\vec{l}_1 \times \vec{l}_2$. Consider these vectors in 3 space. If **x** = $\mathbf{l} \times \mathbf{l}'$, **then x** is the **vector normal** to $\mathbf{l}$ and $\mathbf{l}'$ with **magnitude** equal to the **area of the parallelogram** formed by $\mathbf{l}$ and $\mathbf{l}'$.



2D projective geometry

The 2D projective plane: intersection of two lines

- **Another nice property:** the **intersection of two lines** $\mathbf{l}$ and $\mathbf{l}'$ is the point **x** = $\mathbf{l} \times \mathbf{l}'$.

- **Reminder:** the cross product of two vectors **x** = $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)^{\mathrm{T}}$ and $\mathbf{x}'$ = $(\mathrm{x}_1', \mathrm{x}_2', \mathrm{x}_3')^{\mathrm{T}}$ is defined as

$$\mathbf{l} \times \mathbf{l}' = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \\ \mathrm{x}_1' & \mathrm{x}_2' & \mathrm{x}_3' \end{bmatrix}$$

Another reminder: If **x** = $\mathbf{l} \times \mathbf{l}'$, **then x** is the **vector normal** to $\mathbf{l}$ and $\mathbf{l}'$ with **magnitude** equal to the **area of the parallelogram** formed by $\mathbf{l}$ and $\mathbf{l}'$.

2D projective geometry

The 2D projective plane: intersection of two lines

Proof: Since $\mathbf{x}$ is **orthogonal** to both $\mathbf{l}$ and $\mathbf{l}$, so

$$\mathbf{l}^{\mathrm{T}}\mathbf{x} = 0$$

and

$$\mathbf{l}'^{\mathrm{T}}\mathbf{x} = 0,$$ meaning $\mathbf{x}$ lies on both $\mathbf{l}$ and $\mathbf{l}'$

Similarly, the line $\mathbf{l}$ joining two points is just $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

Intersection of two lines ($\vec{l}_1$ and $\vec{l}_2$)[1]

$\vec{l}_1 : a_1$x + b_1y + c_1 = 0 ----------(1)

$\vec{l}_2 : a_2$x + b_2y + c_2 = 0 ----------(2)

If a **point** lies on the **intersection of these two lines**, then how can we find it?

(We can solve the system of linear equations. If we represent these lines in parametric vector form, then we have powerful consequences.)

Suppose $\vec{x}$ is the **intersection** of $\vec{l}_1$ and $\vec{l}_2$ that means

$\vec{x}.\vec{l}_1$ = 0 -------------------------(3)

$\vec{x}.\vec{l}_2$ = 0 -------------------------(4)

Intersection of two lines ($\vec{\mathrm{l}}_1$ and $\vec{\mathrm{l}}_2$)[2]

If the **dot product** of **two vectors** is **zero** then what does it means? It means that the two vectors are **orthogonal** i.e.,

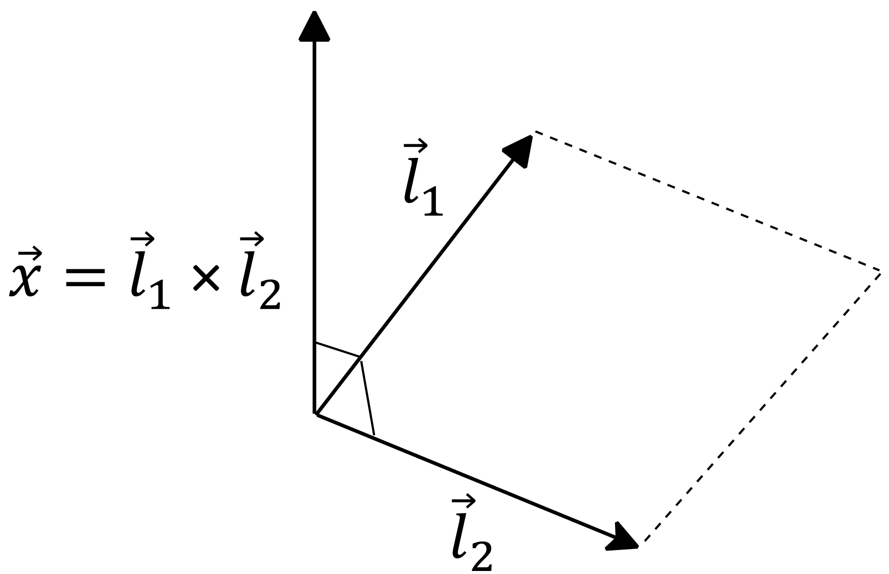
$\vec{x} \perp \vec{l}_1$ and

$\vec{x} \perp \vec{l}_2$

We want to find a vector $\vec{x}$ that is **orthogonal** to these two vectors (3) and (4)

Intersection of two lines ($\vec{l}_1$ and $\vec{l}_2$)[3]

- ❑Cross product between two vectors i.e., $\vec{l}_1 \times \vec{l}_2$. Consider these vectors in 3 space



Intersection of two lines ($\vec{l}_1$ and $\vec{l}_2$)[4]

- What is the **cross product** between these two vectors?
  - It is a **vector** that is **orthogonal** to the **plane span by these two vectors** and the **magnitude** is equal to **the area of parallelogram**.

  - If we substitute the value of $\vec{x}$ (i.e., $\vec{l}_1 \times \vec{l}_2$) in equation (3) then we get a vector orthogonal to $\vec{l}_1$ and the result must be equal to zero

$$\vec{x}.\vec{l}_1 = 0$$

$$\Rightarrow \vec{l}_1 \times \vec{l}_2 . \vec{l}_1 = 0$$

Intersection of two lines ($\vec{l}_1$ and $\vec{l}_2$)[5]

Similarly, if we substitute $\vec{x}$ in equation (4), we get

$$\vec{x} \,.\, \vec{l}_2 = 0$$

$$\Rightarrow \vec{l}_1 \times \vec{l}_2 \,.\, \vec{l}_2 = 0$$

- So, $\vec{x} = \vec{l}_1 \times \vec{l}_2$ satisfies both the equations. Here $\vec{x}$ must be **homogenous representation** of the point that is the **intersection** of $\vec{l}_1$ and $\vec{l}_2$

- So the **cross product** of **two lines** will give us **a point** that is at the **intersection** between those **lines**.

How we differentiate between points and lines?

- In $\mathbb{P}^2$, the **points** and the **lines** are dual to each other, so that the **three vectors** can be called a **point** or a **line**.

- Given 3-vector points, you can go back to the **line** that contains those **points** and given **lines** you can get back to **point** on the lines.

- This is beautiful thing about points in plane. So, points are and lines are **dual** to each other.

- If we have t**wo points**, we aim to determine the **equation of the line that passes through both of them**. The general form of a line equation is:

  ax_1 + by_1 + c = 0

  ax_2 + by_2 + c = 0

- Since both $(x_1,y_1, 1)^{\mathrm{T}}and(x_2,y_2, 1)^{\mathrm{T}}$are points lying on the same line, they satisfy the same equation.

- This system consists of **two linear equations** with **three unknowns (a, b, c).** As a result, it has **infinitely many solutions**. However, this is acceptable because all possible solutions are proportional to one another by a scaling factor

What is the kicker?

- **Finding the Intersection of Two Lines**
  - To determine the point of intersection of two lines, compute the cross product of their equations.

- **Finding a Line Through Two Points**
  - To find the equation of a line that passes through two given points, take the cross product of the two points.

- **Understanding the Duality**
  - In projective geometry, a three-component vector can represent either a point or a line.

  - This symmetry holds in 2D projective geometry, but in 3D, lines and points are not dual in the same way

  - **Projective geometry** follows a **structured duality in 2D**, but this relationship **changes in higher dimensions**.