Computer Vision

Dr. Syed Faisal Bukhari
Associate Professor
Department of Data Science
Faculty of Computing and Information Technology
University of the Punjab

Textbook

Multiple View Geometry in Computer Vision, Hartley, R., and Zisserman

Richard Szeliski, **Computer Vision: Algorithms and Applications**, 2nd edition, 2022

Reference books

Readings for these lecture notes:

- Hartley, R., and Zisserman, A. **Multiple View Geometry in Computer Vision**, Cambridge University Press, 2004, Chapters 1-3.

- Forsyth, D., and Ponce, J. **Computer Vision: A Modern Approach**, Prentice-Hall, 2003, Chapter 2.

- **Linear Algebra and its application**
by David C Lay

These notes contain material c Hartley and Zisserman (2004), Forsyth and Ponce (2003), an Linear Algebra and its application by David C Lay

References

These notes are based

- Dr. Matthew N. Dailey's course: AT70.20: Machine Vision for Robotics and HCI

- Dr. Sohaib Ahmad Khan CS436 / CS5310 Computer Vision Fundamentals at LUMS

2D projective geometry
A model for the projective plane

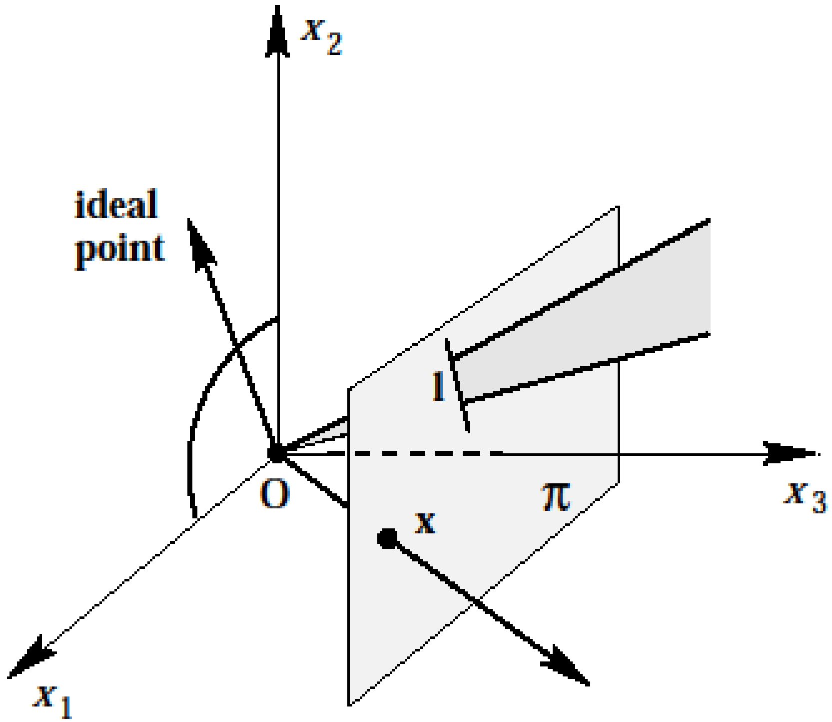


Fig 2.1 **A model of the projective plane**. **Points** and **lines** of $\mathbb{P}^2$ are represented by **rays** and **planes**, respectively, through the origin in $\mathbb{R}^3$. Lines lying in the $x_1\ x_2$ -plane represent **ideal points**, and the $x_1\ x_2$ **-plane** represents $\vec{l}_\infty$ or $\mathbf{l}_\infty$.

2D projective geometry
A model for the projective plane

As illustrated in Fig 2.1 **the rays** representing **ideal points** and **the plane** representing $\vec{\mathrm{l}}_\infty$ or $\mathbf{l}_\infty$ are parallel to the **plane plane $x_3 = 1$**

Review $\mathbb{P}^2$

- **A point** is represented as a **homogeneous 3 vector** $(x_1, x_2, x_3)^T$ where $(\frac{x_1}{x_3}, \frac{x_2}{x_3})^T$ gives the **corresponding point** in the **plane x_3 = 1**

- **A line** in the **plane x_3 = 1** is represented by a homogeneous vector $(\mathrm{a}, \mathrm{b}, \mathrm{c})^{\mathrm{T}}$ where **ax + by + c = 0**

- The vector $(\mathrm{a}, \mathrm{b}, \mathrm{c})^{\mathrm{T}}$ **can be interpreted as the normal to a plane in $\mathbb{R}^3$** ax_1 + bx_2 + cx_3 = 0

- The **intersection** of the **plane** ax_1 + bx_2 + cx_3 = 0 with the **plane** x_3 = 1 is the **line** ax + by + c = 0

Review $\mathbb{P}^2$

- If $\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$ implies point $\boldsymbol{x}^{\boldsymbol{T}} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)^{\boldsymbol{T}}$ lies on the line $\boldsymbol{l} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^{\boldsymbol{T}}$

- Since $\boldsymbol{l}$ is a **line** but it is also **interpreted** as a **normal vector** to the **plane** that forms that **line**.

OR

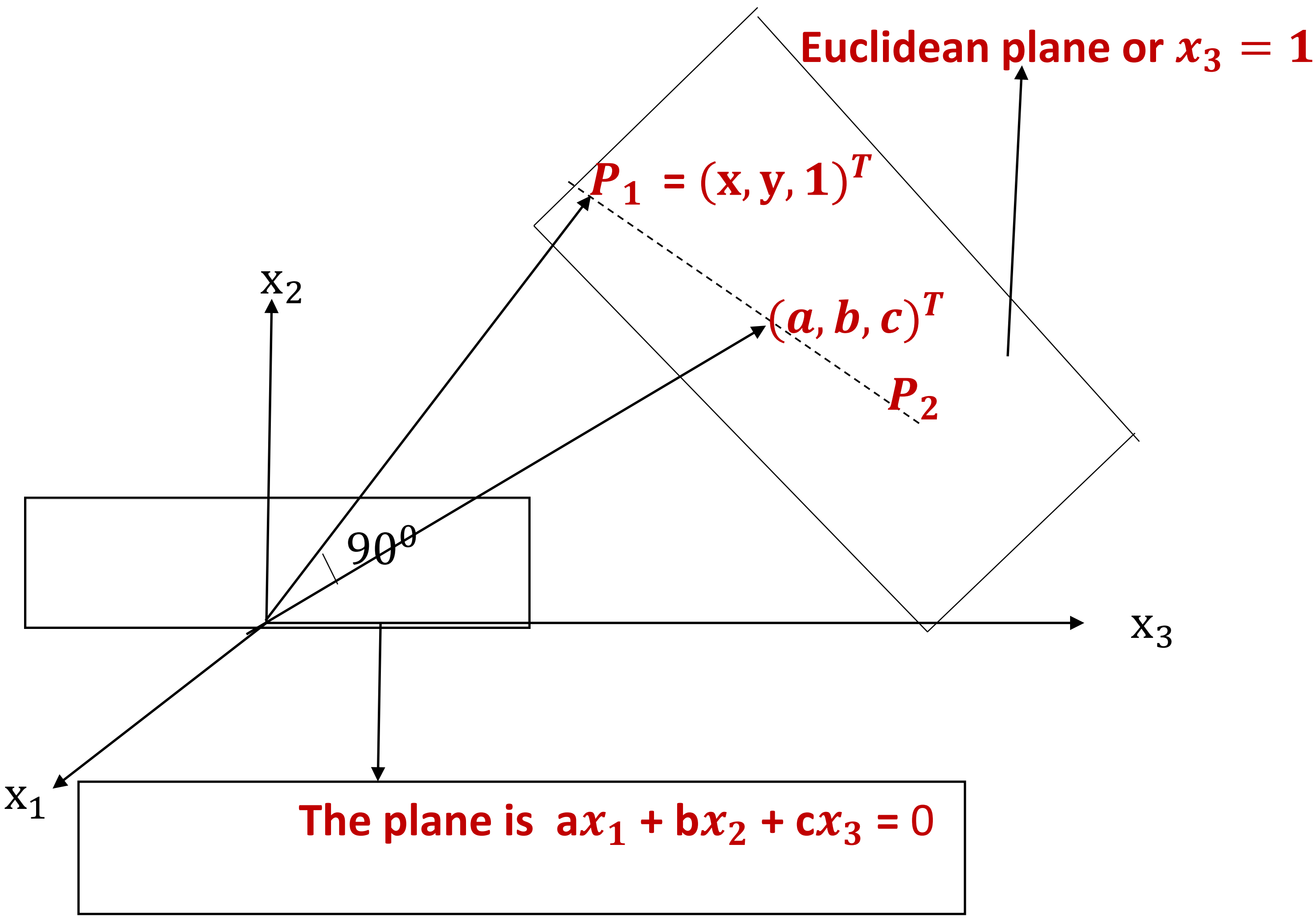
- **Points** on **a line** must be **orthogonal** to the **vector** that is **orthogonal** to the **plane** containing **that line**.

The vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ representing a **line** in the **Euclidean plane (i.e., $\boldsymbol{x_3 = 1}$),** when **interpreted as a vector** in $\mathbb{R}^{\mathbf{3}}$ is **orthogonal** to the $\mathbb{R}^{\mathbf{3}}$ plane representing the line in $\mathbb{P}^{\mathbf{2}}$.

The **line** representation $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ in $\mathbb{R}^{\mathbf{3}}$ is a **vector orthogonal** to the **plane** formed by the **points** on line $\boldsymbol{l}$ and the **origin.**

Proof:

Lines in $\mathbb{P}^2$ is described by $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. The vector, $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ in $\mathbb{R}^3$ is interpreted as being a **normal vector** to some plane in $\mathbb{R}^3$ through the **origin.**


Euclidean plane or $x_3 = 1$
$P_1 = (x, y, 1)^T$
$(a, b, c)^T$
P_2
x_2
90^0
x_3
x_1
The plane is $ax_1 + bx_2 + cx_3 = 0$

- The vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is **orthogonal** to the **plane** $\mathbf{ax_1 + bx_2 + cx_3 = 0}$. Because $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is describing a **normal vector** of the **plane** $\mathbf{ax_1 + bx_2 + cx_3 = 0}$ to the **origin**.

- It means $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ describes **some plane**. We would like to establish $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is also a **line** describing the **points**.

- Suppose we take some arbitrary point $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$. If the point $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$ lies on $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ then their **dot product** must be **zero** i.e., $\mathbf{(x, y, 1)}\ (\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}} = \mathbf{0}$

- If we **think geometrically** in $\mathbb{R}^3$ we must ask: *what is the* ***relationship between these vectors*** $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$ ***and*** $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$

  in three-dimensional space?.

- Let us consider the vector $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$**.** If this vector is **orthogonal** to the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ **it implies that:**

  $(\boldsymbol{x}, \boldsymbol{y}, \mathbf{1})^{\boldsymbol{T}} . (\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}} = 0$,

  which leads to the equation:

  a$\boldsymbol{x}$ + b$\boldsymbol{y} + \mathbf{1}.\boldsymbol{c} = \mathbf{0}$.

- This is precisely the equation of a **plane through the origin** in $\mathbb{R}^3$:

  $\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = \mathbf{0}$.

- It also means the point $(x, y, 1)^T$ must lies on the plane **$\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = \mathbf{0}$**.

- We already know $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ is **orthogonal to the plane** **$\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = \mathbf{0}$**.

- Since $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ represents **the normal vector**, it defines **a plane** that passes **through the origin**.

- **Note:** If a plane can be described by its **normal vector**, then the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ uniquely defines a **plane passing through the origin**, with $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ being **orthogonal to every point on that plane**.

- If we take any point $(x_1, x_2, x_3)^T$ on the **plane** $\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = 0$ and we **normalize** it then what we will get?

- We get a point that is **orthogonal** to the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. It means the **point** satisfy the equation. That means the **point** is on the **line** in the **Euclidean plane**.

- Take any **two points** on the **line** described by $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. These two points (i.e., $\mathbf{P_1}$ and $\mathbf{P_2}$) that lie on the **line** $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ **described by the plane** x_3=1.

- If a **point** lies on the **line** $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ then it must lie on the **plane through the origin** that is described by the $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is **orthogonal** to **every point** on the **plane**.

- Conversely **every point** on the **plane** ($\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = 0$). is describing a **homogeneous representation** of a point that is on the **line in the Euclidean plane (plane** x_3=1**)**.

If [x y 1] $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ = 0 is true then it tells us two things

- **[x y 1]** lies on the line $[\mathbf{a} \;\; \mathbf{b} \;\; \mathbf{c}]^{\mathbf{T}}$ in the Euclidean plane.

- The vector that represents this point in $\mathbb{R}^3$ is necessarily orthogonal the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\boldsymbol{T}}$ in $\mathbb{R}^{\mathbf{3}}$.

- Therefore, it is lying on the **plane through the origin** and through the **line in the Euclidean plane.** These are some interesting facts about lines and points in $\mathbb{p}^{\mathbf{2}}$

- If the equation

$$[x\ y\ 1]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

holds true, it reveals two key insights:

1. The point **[x y 1]** lies on the **line** defined by the vector $[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]^{\mathbf{T}}$ in the **Euclidean plane.**

2. The vector representing this point in $\mathbb{R}^3$ is orthogonal to the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^T$ in $\mathbb{R}^3$.

This implies that **the point** lies on the **plane through the origin** in $\mathbb{R}^3$**,** which contains the **entire line** in the Euclidean plane.

Now we start seeing these relationship:

- Vector describing **points**
- Vectors describing **lines**
- Vectors describing **planes**

They are closely related to each other.

Conic section

What is conic section?

- In mathematics, a **conic section** (or simply **conic**) is the curve formed by the intersection of a **plane** with the **surface of a cone**. There are three fundamental types of conic sections: the **hyperbola**, the **parabola**, and the **ellipse**. The **circle** is a special case of the ellipse, and due to its unique properties and historical importance, it is sometimes considered a fourth distinct type of conic section.

OR

- A conic section is the intersection of **a plane** and **a cone**.

OR

- A **conic** is a two-dimensional shape formed by slicing a cone in a particular way. In Euclidean geometry, the resulting curves—known as **conic sections** that **include circles, parabolas, hyperbolas**, and **ellipses**.

Conic section

conic sections can be written in matrix form as:

$\vec{x}^T C\, \vec{x} = 0$

$$\text{Where } C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

We can represent any **conic section** by the **3×3 matrix**

- Matrix C is a **symmetric matrix**, meaning that it contains **nine elements** in total, but only **six** of them are unique **due to symmetry. As a result, the matrix has six degrees of freedom (DoF).**

- Since C is a **homogeneous matrix**, we could scale it by any factor, and we still satisfy the equation, i.e., $\mathbf{x}^T C\, \mathbf{x} = 0$. Intuitively, it means we **lose one dof**.

Conic section

- **For example**, if we fix the scale of the conic say, by normalizing the matrix C such that all its elements are divided by a **nonzero scalar f**, then the conic effectively has **five degrees of freedom instead of six**.

- Understanding and managing **degrees of freedom (DoF)** is central to computer vision, as it governs the number of **independent parameters** we must estimate for various objects of interest.

Conic section

- It means, if we want to **estimate the parameters** of a **conic**, then we have to **constraint five parameters**.

- We have **five parameters** based linear object, and we like to **estimate the parameters** of this object.

Conic section

- For a conic, we **need six equations** to fully constraint the **five dof of this conic**.

- If we have **three points** on a parabola, then that's all we need to define it. In the case of a circle, three points fully define it. We need 3 points to fit an ellipse and so on.

- All conic sections can be defined by finding **three points** on that **particular conic**. We need **five equations** to define a conic section up to **scale uniquely**.

Inhomogeneous representation of conic section:

Any conic section in **inhomogeneous coordinates** can be represented as

$ax^2 + bxy + cy^2 + dx + ey + f = 0$------------(1) (any quadratic equation in xy-plane)

Trick: Homogenizing, we write the point

$(x_1, x_2, x_3)^T$ as $(\frac{x_1}{x_3}, \frac{x_2}{x_3})^T$.

We have a homogeneous point $(x_1, x_2, x_3)^T$. This point is mapped to the **plane $x_3 = 1$.**

Replace x by x_1/x_3 and y by x_2/x_3 in equation (1)

$a(x_1/x_3)^2 + b(x_1/x_3)(x_2/x_3) + c(x_2/x_3)^2 + d(x_1/x_3) + e(x_2/x_3) + f = 0$

$\Rightarrow ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$

This is the equation of conic section in **homogenous coordinates**.

Example 1

Let the equation of parabola is

$y = (x - 2)^2 + 1$

$x^2 - y - 4x + 5 = 0$ --(1)

We know equation of conic section in **homogeneous coordinates**

$\mathbf{a x_1^2 + b x_1 x_2 + c x_2^2 + d x_1 x_3 + e x_2 x_3 + f x_3^2 = 0}$ ----------------(2)

Replace **x** by $\mathbf{x_1/x_3}$ and **y** by $\mathbf{x_2/x_3}$ in equation (1)

$(x_1/x_3)^2 - x_2/x_3 - 4(x_1/x_3) + 5 = 0$

$\Rightarrow x_1^2 - x_2 x_3 - 4x_1 x_3 + 5x_3^2 = 0$ ------------------------------------(3)

Comparing equations (2) and (3), we get

$\Rightarrow a = 1, b = 0, c = 0, d = -4, e = -1, f = 5$

Given

a = 1, b = 0, c = 0, d = -4, e = -1, f = 5

$\vec{x}^T C \vec{x} = 0$

where

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 0/2 & -4/2 \\ 0/2 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix}$$

$\vec{x}^T C \vec{x} = 0$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_{\mathbf{1\times 3}} \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix}_{\mathbf{3\times 3}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathbf{3\times 1}} = 0$$

$$\begin{bmatrix} x_1 - 2x_3 & -\frac{1}{2}x_3 & -2x_1 - \frac{1}{2}x_2 + 2x_3 \end{bmatrix}_{\mathbf{1\times 3}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathbf{3\times 1}} = 0$$

$$x_1^2 - 2x_1x_3 - \frac{1}{2}x_2x_3 - 2x_1\,x_3 - \frac{1}{2}x_2x_3 + 5x_3^2 = 0$$

$$x_1^2 - 4x_1x_3 - x_2x_3 + 5x_3^2 = 0.$$

Which is the same equation.

Note: We need 5 equations to define a conic section upto scale.

Show that the point $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ satisfy the $\vec{x}^T C \vec{x} = 0$

$\vec{x}^T C \vec{x} = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$-\frac{1}{2} + \frac{1}{2} = 0$$

$$0 = 0$$

The point, $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ satisfies the equation.

Note: Any point satisfies the equation $\vec{x}^T C \vec{x} = 0$ lies on the conic

Example. Write the equation of circle about the origin with radius r in conic matrix form and verify the point $\begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{bmatrix}$ lies on the circle $x^2 + y^2 = r^2$. Let r = 5.

$x^2 + y^2 = r^2$ --(1)

Replace x by $\frac{x_1}{x_3}$ and y by $\frac{x_2}{x_3}$ in (1), we get

$\Rightarrow (\frac{x_1}{x_3})^2 + (\frac{x_2}{x_3})^2 = r^2$

$\Rightarrow x_1^2 + x_2^2 = r^2 x_3^2$

$\Rightarrow x_1^2 + x_2^2 - r^2 x_3^2 = 0$ ------------------------------------(2)

We know equation of conic section in homogeneous coordinates

$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$-------(3)

Comparing (2) and (3), we get

$\Rightarrow a = 1, b = 0, c = 1, d = 0, f = - r^2$

$\vec{x}^T C \vec{x} = 0$

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\Rightarrow [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Suppose $x^2 + y^2 - 5^2 = 0$

$$\Rightarrow [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow [x_1 \quad x_2 \quad -25x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1^2 + x_2^2 - 25x_3^2 = 0$

$$\vec{x}^T C\, \vec{x} = 0$$

$$\Rightarrow \begin{bmatrix} \frac{5\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{bmatrix} = 0$$

$$\vec{x}^T C\, \vec{x} = \begin{bmatrix} \frac{5\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} & -25 \end{bmatrix} \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$= \frac{50}{4} + \frac{50}{4} - 25$$

$$= 0$$