Computer Vision

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Textbook

- Linear Algebra and its application
by David C Lay

Reference books

- Elementary Linear Algebra
  by Howard Anton and Chris Rorres

- Linear Algebra and its application
  by David C Lay

References

Readings for these lecture notes:

- **Linear Algebra and its application**

by David C Lay

- **Elementary Linear Algebra**

by Howard Anton and Chris Rorres

- https://en.wikipedia.org/wiki/Rule_of_Sarrus
- https://www.tutorialspoint.com/computer_graphics/3d_computer_graphics.htm

These notes contain material from the above recourses.

Factorizing Transformations

- Opposite of **Concatenation of Transformations**
- Given a **transformation matrix**, decompose it into a sequence of **simpler transformations**
- Example:

$$\begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

- Question: How to factorize the multiplicative part?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- Is the factorization unique?

Singular Value Decomposition

- Let **M** be a *m*-by-*n* matrix whose entries are real numbers. Then **M** may be decomposed as

$$\mathbf{M} = \mathbf{U}\ \mathbf{S}\ \mathbf{V}^T$$

where:

- **U** is an *m*-by-*m* **orthonormal matrix**
- **S** is an *m*-by-*n* matrix with **non-negative numbers** on the **main diagonal** and zeros elsewhere
- **V** is an *n*-by-*n* **orthonormal matrix**

- **Example**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

http://en.wikipedia.org/wiki/Singular_value_decomposition

Singular Value Decomposition

- Implication: **We can take the multiplicative part of any** transform and describe it as a sequence of a **rotation, scaling and another rotation**
- 2D Example: **Decomposing an Affine Transformation**

M =	**0.95**	**0.49**	0.46
	0.23	**0.89**	0.02
	0	0	1

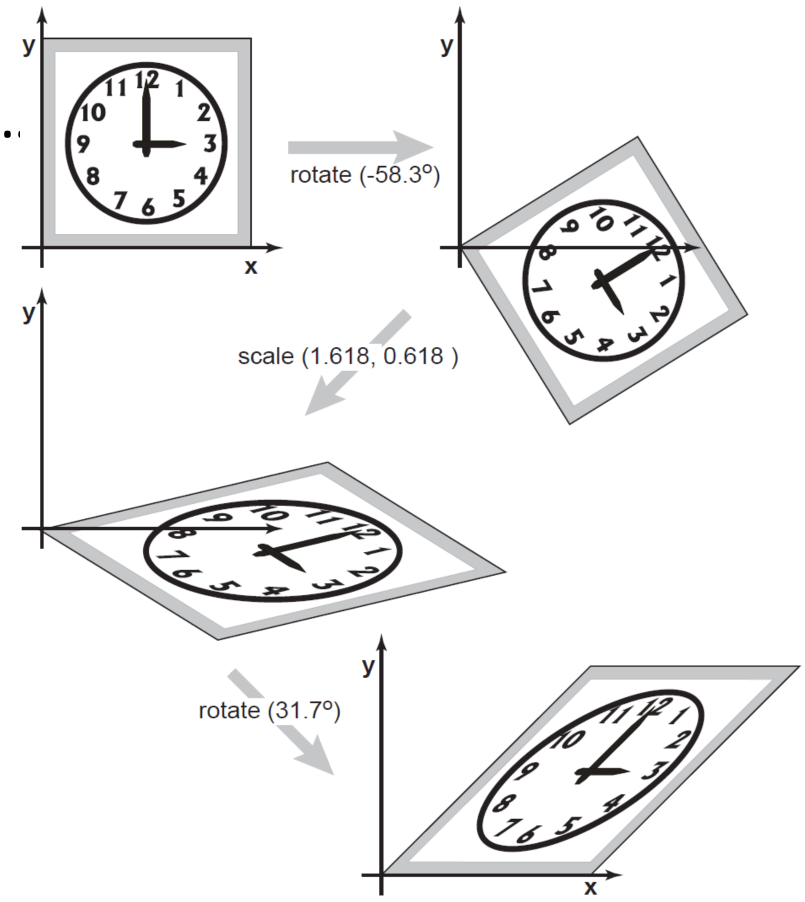
```
>> U * S * V'

ans =

      0.95      0.49
      0.23      0.89
```

```
>> [U, S, V] = svd(M(1:2, 1:2))

U =
   -0.78156   -0.62384
   -0.62384    0.78156

S =
    1.2904          0
         0    0.56789
V =
   -0.68658   -0.72705
   -0.72705    0.68658
```

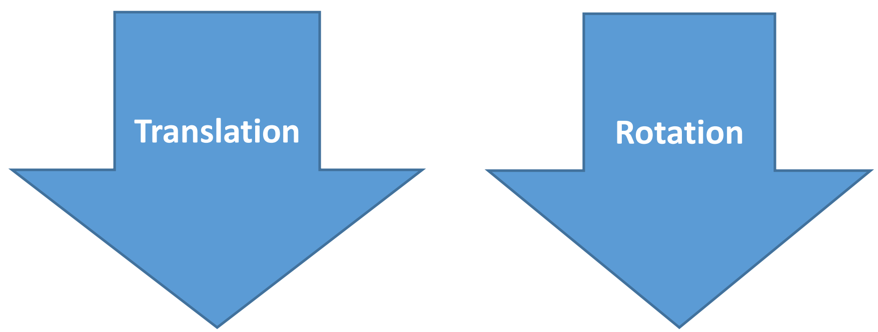
Interpretation in terms of angles?

Singular Value Decomposition

- Implications: Even a simple **shear** can be written as a **rotation ⟶ scaling⟶rotation**
- Try visualizing it to understand how..



$$\begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$



$$= \begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

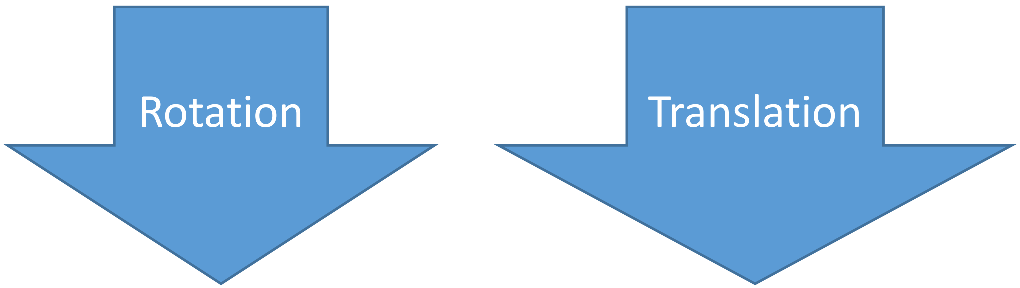
SVD

- We want to make rotation $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as simpler transformation

- **Orthonormal** is always a **rotation** but it could be a **reflection**

- SVD = Sequence of **rotation**, **scaling** and then **rotation**

$$\begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$



$$= \begin{bmatrix} a_1 & a_2 & a_1 b_1 + a_2 b_2 \\ a_3 & a_4 & a_3 b_1 + a_4 b_2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Order of transformation matters

SVD

$$a = \begin{matrix} -0.7816 & -0.6238 \\ -0.6238 & 0.7816 \end{matrix}$$

det(a) = -1.0000 (It is a reflection)

Note: Interchange columns then it becomes rotation

$$a = \begin{matrix} -0.6238 & -0.7816 \\ 0.7816 & -0.6238 \end{matrix}$$

det(a) = 1.0000 (It is a rotation)

The Singular Value Decomposition

The **decomposition** of **A** involves an **m × n** **"diagonal"** matrix Σ or **D** **(sometimes)** of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

m - r rows

---------------(1)

n - r columns

- where **D** is an **r × r diagonal matrix** for **some r not exceeding** the **smaller of m and n**. (If r equals m or n or both, some or all of the zero matrices do not appear.)

The Singular Value Decomposition

Theorem 10: The Singular Value Decomposition Let *A* be an $m \times n$ **matrix** with rank r. Then there exists an $m \times n$ **matrix** Σ as in (1) for which the **diagonal entries** in ***D*** are the first ***r* singular values** of *A*, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $m \times m$ **orthogonal matrix *U*** and an $n \times n$ **orthogonal matrix *V*** such that

$$A = U\Sigma V^T$$

$$A_{m \times n} = U_{m\times m}\, \Sigma_{m\times n}\, V^T{}_{n\times n}$$

The Singular Value Decomposition

- Any factorization $A = U\Sigma V^T$, with U and V orthogonal, Σ as in **(1),** and positive diagonal entries in D, is called a **singular value decomposition** (or **SVD**) of A.

- The columns of U in such a decomposition are called **left singular vectors** of A, and the columns of V are called **right singular vectors** of A.

The Singular Value Decomposition

- The diagonalization theorems in Sections 5.3 and 7.1 play a part in many interesting applications. Unfortunately, as we know, not all matrices can be factored as **$A = QDP^{-1}$** with **D diagonal.**

- However, a factorization **$A = QDP^{-1}$** *is* possible for any **$m \times n$ matrix** A! A **special factorization** of this type, called the ***singular value decomposition***, is one of the most useful matrix factorizations in applied linear algebra.

The Singular Value Decomposition

The **singular value decomposition** is based on the following property of the **ordinary diagonalization** that can be imitated for **rectangular matrices**: The absolute values of the eigenvalues of a **symmetric matrix A** measure the **amounts that A stretches** or **shrinks** certain vectors (the eigenvectors).

If $A\mathbf{x} = \lambda\mathbf{x}$ and $\|x\| = 1$, then

$\|Ax\| = \|\lambda\mathbf{x}\| = |\lambda|\,\|x\| = |\lambda|$ --------------(1)

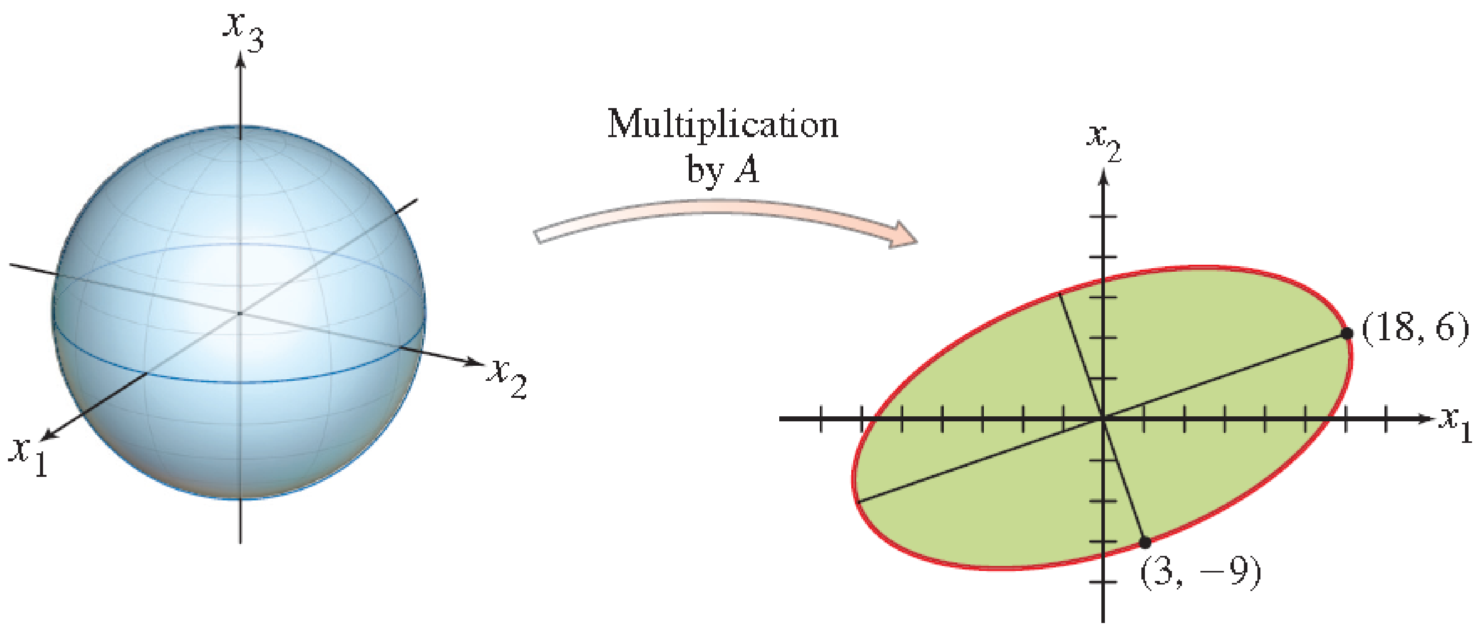
The Singular Value Decomposition

- If λ_1 is the **eigenvalue** with the **greatest magnitude**, then a corresponding **unit eigenvector** $\mathbf{v}_1$ identifies a direction in which the **stretching effect of A** is **greatest**. That is, the length of $A\mathbf{x}$ is maximized when $\mathbf{x} = \mathbf{v}_1$, and $\|A\mathbf{v}_1\| = |\lambda_1|$, by (1).

- This description of $\mathbf{v}_1$ and $|\lambda_1|$ has an analogue for **rectangular matrices** that will lead to the singular value decomposition

Example 1 If A = $\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$, then the linear transformation **x** $\mapsto$A**x** maps the **unit sphere**

{x: ||x || = 1} in $\mathbb{R}^3$ onto an ellipse in $\mathbb{R}^2$, shown in the figure below. Find a unit vector **x** at which the length $\|A\mathbf{x}\|$ is **maximized**, and compute this **maximum length**.



A transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Solution The quantity $\|A\mathbf{x}\|^2$ is maximized at the same **x** that maximizes $\|A\mathbf{x}\|$, and $\|A\mathbf{x}\|^2$ is easier to study. Observe that

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T(A^TA)\mathbf{x}.$$

- Also A^TA is a symmetric matrix, since $(A^TA)^T = A^TA^{TT} = A^TA$. So the problem now is to maximize the quadratic form $\mathbf{x}^T(A^TA)\mathbf{x}$ subject to the constraint $\|\mathbf{x}\| = 1$.

- By Theorem 6 in Section 7.3, the **maximum value** is **greatest eigenvalue** λ_1 of A^TA.

- Also, the **maximum value** is attained at a **unit eigenvector** of A^TA corresponding to λ_1.

For the **matrix A** in this example,

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3}$$

$$A^TA = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}_{3\times 3}$$

Characteristic equation: det **$(A^T A - \lambda I)x = 0$**

$$A^T A - \lambda I = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}$$

$$\det (A^T A - \lambda I) = \det \begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}$$

First approach 1

$$= \textbf{det}\begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}$$

$$= (80-\lambda)\begin{vmatrix} 170-\lambda & 140 \\ 140 & 200-\lambda \end{vmatrix}$$

$$- 100\begin{vmatrix} 100 & 140 \\ 40 & 200-\lambda \end{vmatrix} + 40\begin{vmatrix} 100 & 170-\lambda \\ 40 & 140 \end{vmatrix}$$

$= (80-\lambda)\{(170-\lambda)(200-\lambda) - (140)(140)\} - 100\{(100)(200-\lambda) - (140)(40)\} + 40\{(100)(140) - (170-\lambda)(40)\}$

$= (80-\lambda)\{(170-\lambda)(200-\lambda)-(140)(140)\} - 100\{(100)(200-\lambda)-(140)(40)\} + 40\{(100)(140)-(170-\lambda)(40)\}$

$= (80-\lambda)(3400-170\lambda-200\lambda+\lambda^2) - 100(2000-100\lambda-5600)$

$+(40)(14000-6800+40\lambda)$

$= (80-\lambda)(3400-370\lambda+\lambda^2) - 100(-100\lambda-3600) + 40(40\lambda+7200)$

$\vdots \qquad \vdots$

Rule of Sarrus

Sarrus' rule or **Sarrus' scheme** is a method and a **memorization scheme** to compute the **determinant** of a **3 × 3 matrix**. It is named after the French mathematician **Pierre Frédéric Sarrus**.

Rule of Sarrus

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$+ \quad + \quad +$

$$\det A = \left| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$- \quad - \quad -$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{31}\, a_{22}\, a_{13} - a_{32}\, a_{23}\, a_{11} - a_{33}\, a_{21}\, a_{12}$$

det (A - λI) = det $\begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}$

= **det**

$$\begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix} \begin{matrix} 80-\lambda & 100 \\ 100 & 170-\lambda \\ 40 & 140 \end{matrix}$$

= $(80-\lambda)(170-\lambda)(200-\lambda)$ + (100)(140)(40) + (40)(100)(140) - (40)$(170-\lambda)$(40) - (140)(140)$(80-\lambda)$ - $(200-\lambda)$(100)(100)

= (13600 - 80 λ - 170 λ + λ^2)$(200-\lambda)$ + 560000 + 560000 – 272000 + 1600 λ – 1568000

+ 19600 λ- 2000000 + 10000λ

= $(\lambda^2 - 250\lambda + 13600)(200 - \lambda)$+560000 + 560000 – 272000 + 1600 λ – 1568000 + 19600 λ- 2000000 + 10000λ

= $(\lambda^2 - 250\lambda + 13600)(200 - \lambda)$ + 31200λ - 2720000

= 200λ^2 - λ^3 - 50000 λ + 250λ^2 + 2720000 - 13600λ - 31200λ – 2720000

det **(**A **-** $\lambda\boldsymbol{I}$**)** = - $\boldsymbol{\lambda^3}$ **+ 450**$\boldsymbol{\lambda^2}$ **-** $\mathbf{32400}\boldsymbol{\lambda}$

$\Longrightarrow$ - $\boldsymbol{\lambda^3}$ **+ 450**$\boldsymbol{\lambda^2}$ **-** $\mathbf{32400}\boldsymbol{\lambda}$ **= 0**

$\boldsymbol{\lambda}$**(**$\boldsymbol{\lambda^2}$ **- 450**$\boldsymbol{\lambda}$ **+** $\mathbf{32400}$**)= 0**

$$\lambda(\lambda^2 - 450\lambda + 32400) = 0$$

$$\Longrightarrow \lambda(\lambda^2 - 360\lambda - 90\lambda + 32400) = 0$$

$$\Longrightarrow \lambda\{\lambda(\lambda - 360) - 90(\lambda - 360)\} = 0$$

$$\Longrightarrow \lambda(\lambda - 360)(\lambda - 90) = 0$$

The eigenvalues of A^TA are

$$\Longrightarrow \boldsymbol{\lambda = 0, \ \lambda = 360, \lambda = 90}$$

1. **Eigen vector** against the eigen value **$\lambda = 360$**:

$$A^TA - 360\, I = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} - 360 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -280 & 100 & 40 \\ 100 & -190 & 140 \\ 40 & 140 & -160 \end{bmatrix}$$

$$\frac{100}{280} \times R_1 + R_2,\ \frac{40}{280} \times R_1 + R_3 \longrightarrow$$

$$= \begin{bmatrix} -280 & 100 & 40 \\ 0 & -1080/7 & 1080/7 \\ 0 & 1080/7 & -1080/7 \end{bmatrix}$$

$$\begin{bmatrix} -280 & 100 & 40 \\ 0 & -1080/7 & 1080/7 \\ 0 & 1080/7 & -1080/7 \end{bmatrix}$$

$$\mathbf{1 \times R_2 + R_3} \longrightarrow \begin{bmatrix} \mathbf{-280} & 100 & 40 \\ 0 & \mathbf{-1080/7} & 1080/7 \\ 0 & 0 & 0 \end{bmatrix}$$

Basic variables = x_1 and x_2 and **free variable** = x_3

$\mathbf{-\ 280\ x_1 + 100x_2 + 40\ x_3 = 0}$

$\mathbf{280\ x_1 - 100x_2 - 40\ x_3 = 0}$ **---------------(1)**

$-\left(\frac{1080}{7}\right) x_2 + \left(\frac{1080}{7}\right) x_3 = 0 \Longrightarrow \mathbf{x_2 = x_3}$

Substitute $x_2 = x_3$ in (1), we get

$280\ x_1 - 100x_3 - 40\ x_3 = 0 \Longrightarrow 280\ x_1 = 140\ x_3$

$\Longrightarrow \mathbf{x_1 = \frac{1}{2} x_3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$\textbf{Norm} = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + 1^2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$\textbf{Unit eigen vector, } v_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

2. Eigen vector against the eigen value **$\lambda = 90$**:

$$A^T A - 90I = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} - 90 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 100 & 40 \\ 100 & 80 & 140 \\ 40 & 140 & 110 \end{bmatrix}$$

$10 \times R_1 + R_2,\ 4 \times R_1 + R_3 \longrightarrow$

$$= \begin{bmatrix} -10 & 100 & 40 \\ 0 & 1080 & 540 \\ 0 & 540 & 270 \end{bmatrix}$$

$$\begin{bmatrix} -10 & 100 & 40 \\ 0 & 1080 & 540 \\ 0 & 540 & 270 \end{bmatrix}$$

$$-\frac{1}{2} \times R_2 + R_3 \longrightarrow \begin{bmatrix} \mathbf{-10} & 100 & 40 \\ 0 & \mathbf{1080} & 540 \\ 0 & 0 & 0 \end{bmatrix}$$

Basic variables = x_1 and x_2 and **free variable** = x_3

$- 10x_1 + 100x_2 + 40x_3 = 0$

$\mathbf{10x_1 - 100x_2 - 40x_3 = 0}$ **--------------(2)**

$1080x_2 + 540x_3 = 0 \Longrightarrow \mathbf{x_2 = -\frac{1}{2} x_3}$

Substitute $\mathbf{x_2 = -\frac{1}{2} x_3}$ in (2), we get

$10x_1 + 50x_3 - 40x_3 = 0 \Longrightarrow 10x_1 = -10x_3 \Longrightarrow \mathbf{x_1 = - x_3}$

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \end{bmatrix} = \begin{bmatrix} -x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \mathbf{-1} \\ \mathbf{-\frac{1}{2}} \\ \mathbf{1} \end{bmatrix}$$

$$\text{Norm} = \sqrt{(-1)^2 + \left(-\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

Unit eigen vector, $\mathbf{v_2} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

3. **Eigen vector** against the eigen value $\lambda = 0$:

$$A^TA - 0\boldsymbol{I} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} - 0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$$-\frac{100}{80} \times R_1 + R_{2,} -\frac{40}{80} \times R_1 + R_3 \longrightarrow$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 0 & 45 & 90 \\ 0 & 90 & 180 \end{bmatrix}$$

$$\begin{bmatrix} 80 & 100 & 40 \\ 0 & 45 & 90 \\ 0 & 90 & 180 \end{bmatrix}$$

$$-2 \times \mathbf{R_2} + \mathbf{R_3} \longrightarrow \begin{bmatrix} \mathbf{80} & 100 & 40 \\ 0 & \mathbf{45} & 90 \\ 0 & 0 & 0 \end{bmatrix}$$

Basic variables = x_1 and x_2 and **free variable** = x_3

$\Rightarrow \mathbf{80x_1 + 100x_2 + 40x_3 = 0}$ **---------------(3)**

$\Rightarrow 45x_2 + 90x_3 = 0 \Longrightarrow \mathbf{x_2 = -2x_3}$

Substitute $\mathbf{x_2 = -2x_3}$ in (3), we get

$80x_1 - 200x_3 + 40x_3 = 0 \Longrightarrow 80x_1 = 160x_3 \Longrightarrow \mathbf{x_1 = 2x_3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\textbf{Norm} = \sqrt{(2)^2 + (-2)^2 + 1^2} = \mathbf{3}$$

$$\textbf{Unit eigen vector, } \mathbf{v_3} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

The maximum value of $\|A\mathbf{x}\|^2$ is **360**, attained when $\mathbf{x}$ is the unit vector $\mathbf{v}_1$. The vector $A\mathbf{v}_1$ is a point on the ellipse in Figure 1 farthest from the origin,namely

$$A\mathbf{v}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

For $\|\mathbf{x}\| = 1$, **the maximum** value of $\|A\mathbf{x}\|$ is

$$\|A\mathbf{v}_1\| = \sqrt{360} = 6\sqrt{10}$$

Example 1 suggests that the effect of A on the unit sphere in $\mathbb{R}^3$ is related to the quadratic form $\mathbf{x}^T(A^TA)\mathbf{x}$.

Example 2 Let A be the matrix in Example 1. Since the **eigenvalues** of **A^TA** are **360**, **90**, and **0**, the **singular values** of A are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{360} = 6\sqrt{10},$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{90} = 3\sqrt{10},$$

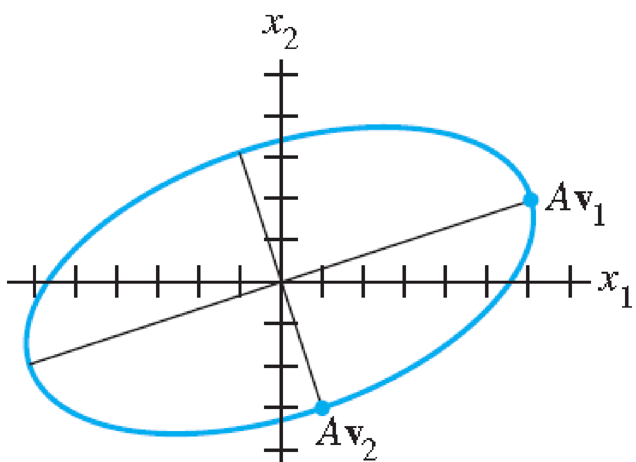
$$\sigma_3 = \sqrt{\lambda_3} = \sqrt{0} = 0$$

From **Example 1**, the first **singular value** of A is the maximum of **$\|Ax\|$ over all unit vectors**, and the **maximum** is attained at the **unit eigenvector** v_1.

Theorem 7 in Section 7.3 shows that **the second singular value** of **A** is the maximum of $\|\mathbf{Ax}\|$ over all **unit vectors** that are ***orthogonal*** *to* v_1, and this **maximum** is attained at the **second unit eigenvector**,

$\mathbf{v}_2$ For the $\mathbf{v}_2$ in **Example 1**

$$A\mathbf{v}_2 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$



Recall: Orthogonal Matrix

Orthogonal Matrix: A **square invertible matrix U** such that $\mathbf{U^{-1} = U^T}$.

THEOREM 6 An $\mathbf{m \times n}$ **matrix U** has **orthonormal columns** if and only if $\mathbf{U^T U} = \boldsymbol{I}$.

Example 3 Use the results of Examples 1 and 2 to construct a **singular value decomposition** of

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{\mathbf{2\times 3}}$$

$$A_{2\times 3} = \boldsymbol{U}_{\mathbf{2\times 2}}\, \boldsymbol{\Sigma}_{\mathbf{2\times 3}}\, \boldsymbol{V^T}_{\mathbf{3\times 3}}$$

Solution A construction can be divided into **three steps**.

- ***Step 1. Find an orthogonal diagonalization of A^TA***. That is, find the **eigenvalues** of ***A^TA*** and a corresponding **orthonormal set** of **eigenvectors**.

- If ***A*** had only **two columns**, the **calculations** could be done by **hand**. **Larger matrices** usually require a **matrix program**. However, for the matrix *A* here, the **eigendata** for ***A^TA*** are provided in Example 2.

Step 2. Set up V and Σ. Arrange the **eigenvalues** of ***A^TA*** in **decreasing order**. In **Example 1**, the **eigenvalues** are already listed in **decreasing order**:

360, 90, and **0**. The **corresponding unit eigenvectors**, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, are the **right singular vectors** of *A*. Using Example 1, construct

$$V = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

- The **square roots** of the **eigenvalues** are the **singular values**:

$$\sigma_1 = 6\sqrt{10}, \quad \sigma_2 = 3\sqrt{10}, \quad \sigma_3 = 0$$

- The **nonzero singular values** are the **diagonal entries** of ***D***.
- The matrix $\boldsymbol{\Sigma}$ is the **same size** as ***A***, **with *D* in its upper left corner** and with **0's elsewhere**.

$$A_{2\times3} = \boldsymbol{U}_{\mathbf{2\times2}}\, \boldsymbol{\Sigma}_{\mathbf{2\times3}}\, \boldsymbol{V}^{\boldsymbol{T}}{}_{\mathbf{3\times3}}$$

$$\boldsymbol{D} = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}_{\mathbf{2\times2}}$$

$$\boldsymbol{\Sigma} = [D \;\; 0] = \begin{bmatrix} \underbrace{\begin{matrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{matrix}}_{D} & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}_{\mathbf{2\times3}}$$

- ***Step 3. Construct U***. When ***A*** has **rank *r***, the **first *r* columns** of ***U*** are the **normalized vectors** obtained from $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$.

- In this example, A has **two nonzero singular values**, so **rank A = 2**. Recall from equation (2) and the paragraph before Example 2 that $\|A v_1\| = \sigma_1$ and $\|A v_2\| = \sigma_2$.

$\mathrm{A}\boldsymbol{v}_1 = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$, $\mathrm{A}\boldsymbol{v}_2 = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

Note that $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is already a **basis** for $\mathbb{R}^2$.
Thus no additional vectors are needed for *U*, and
***U* = [$\boldsymbol{u}_1\ \boldsymbol{u}_2$].**

$$U = [u_1 \; u_2]$$

$$= \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}_{2\times 2}$$

$$V = [v_1 \;\; v_2 \;\; v_3]$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}_{3\times 3}$$

The **singular value decomposition (SVD)** of ***A*** is

$A_{2\times3} = \boldsymbol{U}_{2\times2}\,\boldsymbol{\Sigma}_{2\times3}\,\boldsymbol{V^T}_{3\times3}$

$\boldsymbol{A = U\Sigma V^T}$

$$\therefore \mathrm{A} = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}_{2\times2}$$

$$\times \begin{bmatrix} 6/\sqrt{10} & 0 & 0 \\ 0 & 3/\sqrt{10} & 0 \end{bmatrix}_{2\times3}$$

$$\times \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}_{3\times3}$$

SVD: MATLAB

Find **SVD** of the given matrix

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

Solution:

```
A = [4, 11, 14; 8, 7, -2]

[U,S,V] = svd(A)
```

$$U = \begin{matrix} -0.9487 & -0.3162 \\ -0.3162 & 0.9487 \end{matrix}$$

$$S = \begin{matrix} 18.9737 & 0 & 0 \\ 0 & 9.4868 & 0 \end{matrix}$$

$$V = \begin{matrix} -0.3333 & 0.6667 & -0.6667 \\ -0.6667 & 0.3333 & 0.6667 \\ -0.6667 & -0.6667 & -0.3333 \end{matrix}$$

Rough Method for finding λ :

$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$ ----------------------(1)

- To solve this equation, we shall begin by searching for **integer solutions**.

- This task can be **greatly simplified** by exploiting the fact that **all integer solutions** (if there are any) to a polynomial equation with **integer coefficients**

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

must be **divisors** of the constant term, c_n . Thus, the only **possible integer** solutions of **(1)** are the **divisors of −4**, that is, **±1, ±2, ±4**.

❑Rough Method for finding λ :

❑Successively substituting these values in **(1)** shows $\boldsymbol{\lambda = 4}$ that is an integer solution.

❑As a consequence λ - 4, must be a factor of the left side of **(1).**

❑Dividing $\lambda - 4$ **into** $\boldsymbol{\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0}$ shows that **(1)** can be rewritten as $(\lambda - 4)(\lambda^2 - \boldsymbol{4\lambda + 1}) = 0$

Thus the **eigenvalues of *A* are**

$\boldsymbol{\lambda_1 = 4, \lambda_2 = 2 + \sqrt{3}, \lambda_3 = 2 - \sqrt{3}}$