Math 623: Homework 3

1. In class we first proved the Bounded Convergence Theorem (using Egorov Theorem). We then proved Fatou's Lemma (using the Bounded Convergence theorem) and deduced from it the Monotone Convergence Theorem. Finally we prove the Dominated Convergence Theorem (using both the Monotone Convergence Theorem and the Bounded Convergence Theorem).

   There are other ways to prove this sequence of results, for example:

   (a) Deduce Fatou's Lemma from the Monotone Convergence Theorem by showing that for any sequence of nonnegative measurable functions $\{f_n\}$ we have

   $$\int \liminf_n f_n \, dm \le \liminf \int f_n \, dm$$

   Hint: Note that $\inf_{n\geq k} f_n \le f_j$ for any $j \ge k$ and thus $\int \inf_{n\geq k} f_n \, dm \le \inf_{j\geq k} \int f_j \, dm$.

   (b) Deduce the dominated Convergence Theorem from Fatou's Lemma. *Hint:* Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.

2. In the Monotone Convergence Theorem we assumed that $f_n \ge 0$. This can be generalized in the following ways:

   (a) Assume that $\{f_n\}$ is a *decreasing* sequence of nonnegative measurable, i.e., $f_n \ge 0$ for a.e x, and $f_{n+1}(x) \le f_n(x)$ for a.e. x and suppose that $\int f_1 dx < \infty$. Show that $\lim_{n\to\infty} \int f_n dx = \int \lim_{n\to\infty} f_n dx$. *Hint:* Set $g_n = f_1 - f_n$.

   (b) Suppose that $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \le f_{n+1}(x)$ for a.e. x and $\int f_1 dx < \infty$. Show that hat $\lim_{n\to\infty} \int f_n dx = \int \lim_{n\to\infty} f_n dx$. *Hint:* Assume first that $f = \lim_{n\to\infty} f_n \le 0$ and use (a). Then decompose $f = f^+ - f^-$ into the difference of two non-negative functions.

3. Prove the following Theorem which deals with exchanging limit and differentiation with integrals.

   Theorem 0.0.1 *Let $-\infty < a < b < \infty$ and let $f : \mathbf{R}^d \times [a, b] \to \mathbf{R}$ be such that $f(x, t)$ is integrable for any $t \in [a, b]$. Let*

   $$F(t) = \int f(x, t) \, dx \, .$$

   (a) Suppose that $f(x, \cdot)$ is continuous in t for every x and that there exists an integrable function g such that $|f(x, t)| \le g(x)$ for all x, t. Then the function $F(t)$ is continuous.

(b) *Suppose that $\frac{\partial f}{\partial t}$ exist and that there exists and integrable function h such that $|\frac{\partial f}{\partial t}(x,t)| \le h(x)$ for all x, t. Then the function $F(t)$ is differentiable and $F'(t) = \int \frac{\partial f}{\partial t}(x,t)dx$*

Hint: Given t_0 let $\{t_n\}$ be an arbitrary sequence such that $\lim_n t_n = t_0$ and apply the Dominated Convergence Theorem.

4. Exercise 16, p. 92

5. (Computing $\int_0^\infty e^{-x^2} dx$). Consider the functions

$$f(t) = \left(\int_{[0,t]} e^{-x^2} dx \right)^2, \quad g(t) = \int_{[0,1]} \frac{e^{-t^2(1+x^2)}}{1+x^2} dx$$

   (a) Show that $f'(t) + g'(t) = 0$.
   (b) Show that $f(t) + g(t) = \frac{\pi}{4}$.
   (c) Conclude that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Hint: Use the previous problem and the fact that Riemann and Lebesgue integral coincide for nice functions defined on a finite interval and so you can use calculus. Justify all computations carefully.

6. Exercise 6, p. 91

7. Exercise 8, p. 91

8. Exercise 9, p. 91

9. Exercise 10, p. 91

10. Exercise 11, p. 91

11. Exercise 15, p.92

12. Compute and justify your computations

   (a) $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\, dm$.
   (b) $\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)}\, dm$.
   (c) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1}\, dm$. Distinguish between $a > 0$, $a = 0$ and $a < 0$.