Math 624 Spring 2012: Midterm exam

Problem 1 Suppose $f : \mathbf{R} \to \mathbf{C}$ is a C^k function (i.e., k-times continuously differentiable) and periodic of period 2π. Show that the Fourier coefficients of f,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx},$$

satisfy the bounds

$$c_n = o(n^{-k})$$

(Recall that we say that $c_n = o(g(n))$ if $\lim_{n\to\infty} \frac{c_n}{g(n)} = 0$.) *Hint:* Induction

Problem 2 Let $(X, \mathcal{M})$ be a measurable space. Suppose $\{\nu_n\}$ is a sequence of measures on $(X, \mathcal{M})$ which is increasing in the sense that

$$\nu_n(A) \leq \nu_{n+1}(A)$$

for all $n = 0, 1, \cdots$ and all $A \in \mathcal{M}$. Define ν by

$$\nu(A) = \lim_{n\to\infty} \nu_n(A)$$

Show that ν is a measure.

Problem 3 Let $(X, \mathcal{M}, \mu)$ be a *finite* measure space $\mu(X) < \infty$. Let $\mathcal{F}$ be the set of all complex valued measurable functions on X (finite valued but not necessarily integrable). For $f, g \in \mathcal{F}$ let us define

$$\rho(f, g) = \int_X \frac{|f-g|}{1+|f-g|} \, d\mu \, .$$

Prove the following assertions

1. $0 \leq \rho(f,g) < \infty$ and $\rho(f,g) = 0$ if and only if $f = g$ a.e.
2. $\rho(f,g) = \rho(g,f)$
3. $\rho(f,h) \leq \rho(f,g) + \rho(g,h)$
4. If $\{f_n\}$ satisfies $\lim_{m,n\to\infty} \rho(f_n, f_m) = 0$, then there exists a complex-valued measurable function g such that $\lim_{n\to\infty} \rho(f_n, g) = 0$.
5. For a sequence $\{f_n\}$ in $\mathcal{F}$ and $f \in \mathcal{F}$ we have $\lim_{n\to\infty} \rho(f_n, f) = 0$ if and only if f_n converges to f in measure.

Note that this problem shows that the set of measurable functions on a finite metric space can be seen as a complete metric space, and that the metric ρ is the metric of convergence in measure.

Problem 4 Let ν be a Borel measure on the positive real line $[0, \infty)$ such that

$$\Phi(t) = \nu([0, t))$$

is finite for every $t > 0$.

Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function. For every t consider the level set

$$S(t) \ = \ \{x \in X; f(x) > t\}\,.$$

1. Prove that
$$\int_X \Phi(f(x)) d\mu \ = \ \int_{[0,\infty)} \mu(S(t))\, d\nu$$
2. Compute this formula for (a) $d\nu = dt$, (b) $d\nu = pt^{p-1}dt$ and (c) $\nu = \delta_{t_0}$ (the delta measure at t_0).

Problem 5 Consider the function $g : \mathbf{R}^2 \to \mathbf{R}$ given by

$$g(x, y) \ = \ \begin{cases} 2 & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let μ be the measure on $\mathbf{R}^2$ which is absolutely continuous with respect to the Lebesgue measure $m \times m$ on $\mathbf{R}^2$ with Radon Nikodym derivative

$$\frac{d\mu}{d(m \times m)} = g$$

Let $T : \mathbf{R}^2 \to \mathbf{R}$ be the map given by $T(x, y) = x$ and and let $\tau = \mu \circ T^{-1}$ be the measure on $\mathbf{R}$ given by

$$\tau(A) \ = \ \mu(T^{-1}(A))$$

Find the Lebesgue decomposition of the Lebesgue measure m on R with respect to τ, $m = m_{ac} + m_{sing}$ and compute the Radon-Nykodym derivative $\frac{dm_{ac}}{d\tau}$.