Math 697 Fall 2014: Week 5

Exercise 1 ***(Card shuffling)*** Suppose you have a deck of 52 cards. At each time you shuffle the cards by picking a card at random and placing it on top of the deck. The state space of this Markov chain consists of all permutation of the 52 cards. Show that Markov chain defined in this way is irreducible and aperiodic and find the stationary distribution. Suppose you shuffle the deck of cards every second. What is the average time (measured in years) until the deck returns to the original order?

Exercise 2 Consider a Markov chain with state $\{1, \cdots, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. Is the chain irreducible?
2. What is the period of the chain?
3. Let $\tau^{(1)}$ be the first return time to state 1. Compute directly $E[\tau^{(1)}|X_0 = 1]$ by computing the pdf of $\tau^{(1)}$.
4. Compute the stationary distribution π.

Exercise 3 ***(Inventory model)*** We consider the following inventory model for a certain item. Stock levels are inspected at fixed time intervals, say every evening. The following restocking policy is followed. Fix two number $0 \le s \le S$, if the inventory level is $\le s$ then the stock is replenished up to level S. If the inventory level is $> s$ and $\le S$, then no action is taken. Assume furthermore that the demand for the item during successive inspections and replenishments of the stocks is given by i.i.d random variables D_n. If the demand during a single day exceeds the number of items in stock, it is assumed that this demand remains unfulfilled. Let X_n to be the stock level at the inspection time just before restocking.

1. Describe this model using the state space $\{0, 1, \cdots S\}$ and determine the transition matrix P if $s = 1$, $S = 3$ and

$$P\{D_i = 0\} = .4\,, \quad P\{D_i = 1\} = .3\,,$$
$$P\{D_i = 2\} = .2\,, \quad P\{D_i = 3\} = .1$$

2. Compute the expected amount of items in stock at the moment of inspection in the long run.

3. If you want to determine if your restocking policy is efficient an important quantity to compute for this model is the average amount of command per day which goes unfulfilled. To do this consider the Markov chain Y_n where Y_n is defined similarly as in X_n but is also allowed to take negative value. For example $Y_n = -1$ means that 1 command was unfilled that day, this happens for example if $X_n = 2$ and $D_n = 3$. Determine the state space for Y_n and the transition matrix with the same distribution for D_n. Compute the average number of unfulfilled commands per day in the long run.

Exercise 4 ***(An algorithm for the stationary distribution)*** If the state space not very small it can be tedious to compute the stationary distribution, i.e., to solve $\pi P = \pi$. We give an alternative way to compute π which is very easy to implement on a computer. We let I be the identity matrix and we let M be the matrix whose all entries are $M(i,j) = 1$.

Algorithm: Suppose X_n is an irreducible Markov chain with transition probabilities matrix P. Then the unique stationary distribution π is given by

$$\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1} .$$

To justify this

1. Assume first that the matrix $(I - P + M)$ is invertible. Show then π is given by by $\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}$.

2. You need next to prove that $(I - P + M)$ is invertible. This is equivalent to prove that $(I - P + M)x = 0$ implies $x = 0$. To do this

   (a) Multiply $(I - P + M)x = 0$ by on the left by π and deduce from this that $Mx = 0$. Thus $Px = x$.

   (b) Use that the only solutions of $Px = x$ are of the form $x = c(1 \cdots, 1)^T$. Conclude.

Exercise 5 ***(Computer exercise)*** Consider the Markov chain with transition probabilities

$$P = \begin{pmatrix} .5 & .4 & 0 & .1 \\ .3 & ..3 & .3 & .1 \\ 0 & 0 & .4 & .6 \\ .25 & .25 & 0 & .5 \end{pmatrix} .$$

Write down a program which generates a sample of size n (chosen by you) of the Markov chain.

1. Use now the sample you have produced to give an estimate for $\pi(4)$.

2. Suppose the sample you have just generated is the product of some experiment which you assume to be the product of an irreducible Markov chain whose transition probabilities you do not know. Use this to produce an estimate of $P(4, 2)$?