Math 597/697: Homework 5

1. Machine 1 is currently working and machine 2 will be put in use at a time T from now. If the lifetimes of the machines 1 and 2 are exponential random variables with parameters λ_1 and λ_2, what is the probability that machine 1 is the first machine to fail?

2. Cars cross a certain point on the highway according to a Poisson process with a rate $\lambda = 3$ per min. Running blindly across the highway, what is the probability that you will be injured if it takes you S seconds to cross the road ? (Assume that you are injured whenever a car passes by.) What is the probability to be injured if you suppose that you are agile enough to avoid one car, but that you will be injured if you encounter two or more cars while crossing the highway?

3. Let N_t be a Poisson process with rate λ and let $0 < s < t$. Compute

   (a) $P\{N_t = n + k | N_s = k\}$

   (b) $P\{N_s = k | N_t = n + k\}$

   (c) $E[N_t N_s]$

4. A radioactive source emits particles according to a Poisson process with rate $\lambda = 2$ particles per minute.

   (a) What is the probability that 4 particles are emitted between the first and second minute?

   (b) What is the probability that the first particle emitted is between the second and third minutes?

   (c) What is the probability that the fourth particles is emitted less than two minutes after the second one?

   (d) What is the expected time between the emission of the third particle and the emission of the seventh particle?

   (e) What is the conditional probability that 3 particles are emitted in the first minute given that 5 particle are emitted in the first two minutes?

   (f) What is the expected number of particles emitted between the third and fourth minutes given that 5 particle were emitted in the first two minutes?

   (g) What is the expectation of the arrival time of the fifth particle given that $N_t = 3$?

5. Consider a two-server system in which a customer is first served by server 1, then by server 2 and then departs. The service times at server i are exponential random variables with parameter μ_i. $i = 1, 2$. When you enter the system you find server 1 free and two customers at server 2, customer A in service and customer B waiting in line.

   (a) Find the probability P_A that A is still in service when you move over to server 2.

   (b) Find the probability P_B that B is still in service when you move over to server 2.

   (c) Compute $E[T]$, where T is the total time you spend in the system. *Hint:* Write $T = S_1 + S_2 + W_A + W_B$ where S_i is your service time at server i, W_A the amount of time you wait in queue when while A is being served, and W_B the amount of time you wait in queue when while B is being served.

6. *A model for the price of a security*: Let $S(t)$ denote the price of a security at time $t \geq 0$. The price remains unchanged until a "shock" occurs, at which time the price of the stock is multiplied by a random factor. If the initial price of the security is S_0, then the price after the first shock will be $S_0 X_1$, the price after the second shock will be $S_0 X_1 X_2$, and so on... Assume that the shocks occurs according to a Poisson process N_t with rate λ, that the multiplicative factors X_0, X_1, etc.. are i.i.d exponential random variables with parameter μ and that $N_t, t \geq 0$ is independent of X_i. Compute the expected value and the variance of the security price $S(t)$ at time t.

7. In winter the weather in Noho can be in the states 0 (sunny), 1 (foggy), 2 (gray and rainy) or 3 (cold and snowy). The transition between the these different states is governed by the transition matrix

$$\begin{pmatrix} 0 & .1 & .6 & .3 \\ .6 & 0 & .2 & .2 \\ .4 & .1 & 0 & .5 \\ .5 & .1 & .4 & 0 \end{pmatrix} \tag{1}$$

   The weather remains in state i with a mean time (in days) 4, 0.2, 3 and 5 respectively. Compute the mean time elapsed between two episodes of snow as well as the mean time between a sunny episode and a snowy episode.

8. (a) An airline reservation system has two computers, one on-line and one backup. The operating computer fails after an exponentially distributed time with parameter μ and is replaced by the backup. There is one repair facility and the repair time is exponentially distributed with parameter λ. Let X_t denote the number of computers in operating condition at time t. Write down the generator

A of X_t. In the long run, what is the proportion of the time when the reservation system is on.

(b) Answer the same questions in the case where the two machines are simultaneously online if they are in operating condition.

(c) Starting with two machines in operation, compute in both cases, the expected value of the time T until both are in the repair facility.

9. A component is in two possible states 0=on or 1=off. A system consists of two components A and B which are independent of each other. Each component remains on for an exponential time with rate λ_i, $i = A, B$ and when it is off it remains off for an exponential time with rate $\mu_i, i = A, B$. Determine the long run probability that the system is operating if

   (a) They are working in parallel, i.e. at least one must be operating for the system to be operating.

   (b) They are working in series, i.e. both must work for the system to be operating.

   Hint: You might either construct a 4-state Markov chain or use the independence of both components.

10. Consider a birth and death chain with parameters λ_n and μ_n and assume it is irreducible and positive recurrent and that π is the stationary distribution. Show that, in the long run, the rate at which people die is the same as the rate at which people are born, i.e.,

$$\sum_n \lambda_n \pi(n) = \sum_n \mu_n \pi(n) .$$