Suggested Solutions to HW 2

1. First we note that from the multi-linearity of payoff functions,
$$\frac{\partial}{\partial x_{j,\xi}}\boldsymbol{\pi}_\gamma\left(\mathbf{p}\right)=\boldsymbol{\pi}_\gamma\left(\mathbf{e}_{j,\xi},\mathbf{p}_{-\xi}\right) \tag{1}$$
So if $\boldsymbol{\pi}_\gamma=\boldsymbol{\pi}$ for all $\gamma\in\Gamma$, then
$$\frac{\partial}{\partial x_{i,\gamma}}\boldsymbol{\pi}_\xi(\mathbf{e}_{j,\xi},\mathbf{p}_{-\xi})=\frac{\partial}{\partial x_{i,\gamma}}\frac{\partial}{\partial x_{j,\xi}}\boldsymbol{\pi}_\xi(\mathbf{p})=\frac{\partial}{\partial x_{j,\xi}}\frac{\partial}{\partial x_{i,\gamma}}\boldsymbol{\pi}_\gamma(\mathbf{p})=\frac{\partial}{\partial x_{j,\xi}}\boldsymbol{\pi}_\gamma(\mathbf{e}_{i,\gamma},\mathbf{p}_{-\xi}) \tag{2}$$
for all $\gamma\in\Gamma$, $\xi\in\Gamma$, $i\in S_\gamma$, $j\in S_\xi$. Conversely, let γ,ξ be fixed. For $\mathbf{p}\in\Delta$, we can write $\mathbf{p}=(\sum_{l=1}^{n_\gamma}a_{i_l}\mathbf{e}_{i_l,\gamma},\sum_{m=1}^{n_\xi}b_{j_m}\mathbf{e}_{j_m,\xi},\mathbf{p}_{-(\xi,\gamma)})$. So we have
$$\begin{aligned}\boldsymbol{\pi}_\xi\left(\mathbf{p}\right) &= \sum_{l=1}^{n_\gamma}\sum_{m=1}^{n_\xi}a_{i_l}b_{j_m}\boldsymbol{\pi}_\xi(\mathbf{e}_{i,\gamma},\mathbf{e}_{j,\xi},\mathbf{p}_{-(\xi,\gamma)})=\sum_{l=1}^{n_\gamma}\sum_{m=1}^{n_\xi}a_{i_l}b_{j_m}\frac{\partial}{\partial x_{i,\gamma}}\boldsymbol{\pi}_\xi(\mathbf{e}_{j,\xi},\mathbf{p}_{-\xi})\\ &= \sum_{l=1}^{n_\gamma}\sum_{m=1}^{n_\xi}a_{i_l}b_{j_m}\frac{\partial}{\partial x_{j,\xi}}\boldsymbol{\pi}_\gamma(\mathbf{e}_{i,\gamma},\mathbf{p}_{-\xi})=\sum_{l=1}^{n_\gamma}\sum_{m=1}^{n_\xi}a_{i_l}b_{j_m}\boldsymbol{\pi}_\gamma(\mathbf{e}_{i,\gamma},\mathbf{e}_{j,\xi},\mathbf{p}_{-(\xi,\gamma)})\ =\boldsymbol{\pi}_\gamma\left(\mathbf{p}\right)\end{aligned}$$
and thus $\boldsymbol{\pi}_\gamma=\boldsymbol{\pi}$ for all $\gamma\in\Gamma$. Note that you can use theorems about the conservative fields or exact forms from vector calculus and show that the integrability condition there is equivalent to the condition given in the problem (the consequence of the multi-linearity of the payoffs).

2. Here we consider strategy dependent linear payoff rescaling (see classnotes p.35) for Nash equivalence. Then (only if) part is trivial. Suppose that there exists $\boldsymbol{\pi}$ such tat for $\gamma\in\Gamma$,
$$\boldsymbol{\pi}_\gamma(\mathbf{p}_\gamma,\mathbf{p}_{-\gamma})-\boldsymbol{\pi}_\gamma(\mathbf{q}_\gamma,\mathbf{p}_{-\gamma})=\boldsymbol{\pi}(\mathbf{p}_\gamma,\mathbf{p}_{-\gamma})-\boldsymbol{\pi}(\mathbf{q}_\gamma,\mathbf{p}_{-\gamma}). \tag{3}$$
Let $\gamma\in\Gamma$ be fixed and i_1 be the first strategy of γ. We define
$$d_\gamma\left(\mathbf{s}_{-\gamma}\right):=\boldsymbol{\pi}_\gamma(\mathbf{e}_{i_1,\gamma},\mathbf{s}_{-\gamma})-\boldsymbol{\pi}(\mathbf{e}_{i_1,\gamma},\mathbf{s}_{-\gamma}) \tag{4}$$
From (3) since $\boldsymbol{\pi}_\gamma(\mathbf{e}_{i_1,\gamma},\mathbf{s}_{-\gamma})-\boldsymbol{\pi}(\mathbf{e}_{i_1,\gamma},\mathbf{s}_{-\gamma})=\boldsymbol{\pi}_\gamma(\mathbf{q}_\gamma,\mathbf{s}_{-\gamma})-\boldsymbol{\pi}(\mathbf{q}_\gamma,\mathbf{s}_{-\gamma})$ for all $\mathbf{q}_\gamma\in\Delta_\gamma$ and (4) does not depend on the choice of i_1 and, hence, is well defined. Then for $j\in S_\gamma$, we have, by construction,
$$\boldsymbol{\pi}\left(\mathbf{e}_{j,\gamma},\mathbf{s}_{-\gamma}\right)=\boldsymbol{\pi}_\gamma\left(\mathbf{e}_{j,\gamma},\mathbf{s}_{-\gamma}\right)+d_\gamma\left(\mathbf{s}_{-\gamma}\right)$$
So $\boldsymbol{\pi}_\gamma\overset{NE}{\sim}\boldsymbol{\pi}$ for all $\gamma\in\Gamma$ and $(\Gamma,S,(\boldsymbol{\pi}_\gamma)_{\gamma\in\Gamma})\overset{NE}{\sim}(\Gamma,S,(\boldsymbol{\pi})_{\gamma\in\Gamma})$. For example, suppose that we have the following two games
$$(\boldsymbol{\pi})_{\gamma\in\Gamma}=\begin{pmatrix}0,0 & 2,2\\ 2,2 & 0,0\end{pmatrix},\quad (\boldsymbol{\pi}_\gamma)_{\gamma\in\Gamma}=\begin{pmatrix}4,-2 & 3,0\\ 6,1 & 1,-1\end{pmatrix}$$
We see that all differences between rows (columns resp.) to player α (player β resp.) are same for these games, so the condition (3) is satisfied. To show NE equivalence of $(\boldsymbol{\pi}_\gamma)_{\gamma\in\Gamma}$ to $(\boldsymbol{\pi})_{\gamma\in\Gamma}$, we need to specify how to transform the payoff of $(\boldsymbol{\pi}_\gamma)_{\gamma\in\Gamma}$ to obtain $(\boldsymbol{\pi})_{\gamma\in\Gamma}$. Then it is easy to see that we need to subtract 4 from player α 's payoffs against the first strategy of β and 1 from player α 's payoffs against the second strategy of β. So $d_\alpha(\{\beta\text{'s the first strategy}\})=4$ and $d_\alpha(\{\beta$'s the second

strategy$\}) = 1$. We also see that the first number 4 equals to $(4-0)$ and $(6-2)$ which shows that $d_\gamma(\mathbf{s}_{-\gamma})$ is independent of γ's strategy.

3. From the fact (s_i, t_k) and (s_j, t_l) are NEs, we have

$$\begin{array}{llll} \pi_\alpha(s_i,t_k) \geq \pi_\alpha(s_j,t_k) & (a) & \pi_\beta(s_i,t_k) \geq \pi_\beta(s_i,t_l) & (b) \\ \pi_\alpha(s_j,t_l) \geq \pi_\alpha(s_i,t_l) & (c) & \pi_\beta(s_j,t_l) \geq \pi_\beta(s_j,t_k) & (d) \end{array}$$

and from the fact the game is a zero sum game, the above inequalities imply

$$\begin{array}{llll} \pi_\beta(s_i,t_k) \leq \pi_\beta(s_j,t_k) & (e) & \pi_\alpha(s_i,t_k) \leq \pi_\alpha(s_i,t_l) & (f) \\ \pi_\beta(s_j,t_l) \leq \pi_\beta(s_i,t_l) & (g) & \pi_\alpha(s_j,t_l) \leq \pi_\alpha(s_j,t_k) & (h) \end{array}$$

Then by applying $(f) \to (a) \to (h)$ and $(g) \to (d) \to (e)$ respectively we obtain

$$\pi_\alpha(s_i,t_l) \geq \pi_\alpha(s_j,t_l) \text{ and } \pi_\beta(s_i,t_l) \geq \pi_\beta(s_j,t_l)$$

Hence, (s_i, t_l) is NE. Similarly, by applying $(h) \to (c) \to (f)$ and $(e) \to (b) \to (g)$ we have

$$\pi_\alpha(s_j,t_k) \geq \pi_\alpha(s_i,t_k) \text{ and } \pi_\beta(s_j,t_k) \geq \pi_\beta(s_j,t_l)$$

So (s_j, t_k) is NE. Alternatively we can use the second part of theorem 3.30. Since the maxima of lower envelope function $\underline{\pi}_\alpha(q_a) := \min_{q_\beta} \pi_\alpha(q_\alpha, q_\beta)$ are achieved at s_i, s_j and the minima of upper envelope function $\bar{\pi}_\alpha(q_\beta) := \max_{q_\alpha} \pi_\alpha(q_\alpha, q_\beta)$ are achieved at t_k, t_l, (s_i,t_l) and (s_j,t_k) are NEs. Note that this argument extends to the case of mixed strategy equilibrium case, and problem 5 provides an example of this.

4. The claim follows from the following chain of equalities:

$$\begin{aligned} \min_{\mathbf{q}_\beta}\max_{\mathbf{q}_\alpha} < \mathbf{q}_\alpha, \boldsymbol{\pi}_\alpha\mathbf{q}_\beta > &= \max_{\mathbf{q}_\alpha}\min_{\mathbf{q}_\beta} < \mathbf{q}_\alpha, \boldsymbol{\pi}_\alpha\mathbf{q}_\beta > \text{ by Minimax theorem (Theorem 3.30)} \\ &= \max_{\mathbf{q}_\alpha}\min_{\mathbf{q}_\beta} \quad < \boldsymbol{\pi}_\alpha^{\mathbf{T}}\mathbf{q}_\alpha, \mathbf{q}_\beta > \\ &= \max_{\mathbf{q}_\alpha}\min_{\mathbf{q}_\beta} \quad - < \boldsymbol{\pi}_\alpha\mathbf{q}_\alpha, \mathbf{q}_\beta > \text{ by anti-symmetry of payoff matrix} \\ &= -\min_{\mathbf{q}_\alpha}\max_{\mathbf{q}_\beta} \quad < \mathbf{q}_\beta, \boldsymbol{\pi}_\alpha\mathbf{q}_\alpha > \\ &= -\min_{\mathbf{q}_\beta}\max_{\mathbf{q}_\alpha} \quad < \mathbf{q}_\alpha, \boldsymbol{\pi}_\alpha\mathbf{q}_\beta > \text{ by symmetric game} \end{aligned}$$

So we conclude $\min_{\mathbf{q}_\beta}\max_{\mathbf{q}_\alpha} < \mathbf{q}_\alpha, \boldsymbol{\pi}_\alpha\mathbf{q}_\beta > = 0$. The second way to show this is as follows: since we know that $<\mathbf{q}, \pi_\alpha\mathbf{q}>=0$ and every symmetric game has at least one symmetric NE, the payoff of the symmetric NE is 0. Because the payoffs for all NEs for zero sum games are the same, we conclude that the value of the game is 0. The third way is to notice that when the value of game is not equal to 0, one of players necessarily obtains negative payoff because of the zero-sum. However this player can just choose the same strategy as the opponent and obtain 0 payoff (because $<\mathbf{q}, \pi_\alpha\mathbf{q}>=0$), so the original equilibrium is not NE which is a contradiction. Hence the value must be 0.

5. The payoff matrix is

	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$(1,1)$	$0,0$	$2,-2$	$-3,3$	$0,0$
$(1,2)$	$-2,2$	$0,0$	$0,0$	$3,-3$
$(2,1)$	$3,-3$	$0,0$	$0,0$	$-4,4$
$(2,2)$	$0,0$	$-3,3$	$4,-4$	$0,0$

and there are four mixed strategy NEs; $((0,\frac{4}{7},\frac{3}{7},0),(0,\frac{3}{5},\frac{2}{5},0))$, $((0,\frac{4}{7},\frac{3}{7},0),(0,\frac{4}{7},\frac{3}{7},0))$, $(0,\frac{3}{5},\frac{2}{5},0),(0,\frac{3}{5},\frac{2}{5},0))$, $((0,\frac{3}{5},\frac{2}{5},0),(0,\frac{4}{7},\frac{3}{7},0))$ and the value of the game is 0. Note this game provide an example of problem 3 as well as problem 4.

6. Two NEs :$(0,0,1)$ and $\left(\frac{1}{2},\frac{1}{2},0\right)$, ESS: $(\frac{1}{2},\frac{1}{2},0)$. To check $(\frac{1}{2},\frac{1}{2},0)$ is ESS, the first order condition (p.51) is automatically satisfied because it is NE. For the second order condition, note $\pi(\mathbf{p}^*,\mathbf{p}^*)=5$ and $\pi(\mathbf{q},\mathbf{p}^*)=5q_1+5q_2+q_3$ and since $\mathbf{q}\in\Delta$, $\pi(\mathbf{p}^*,\mathbf{p}^*)=\pi(\mathbf{q},\mathbf{p}^*)$ if and only if $q_3=0$. Then since we are looking at $(q_1,q_2,0)$, we can delete the third row and column and obtain 2 by 2 Hawk-Dove subgame. From here, it is easy to see $(\frac{1}{2},\frac{1}{2},0)$ is ESS. See the figure in problem 7.

7. A closed nonempty set E of Δ is ES set if for each $\mathbf{p}\in E$, there exists a neighborhood W of $\mathbf{p}$ such that

$$\begin{aligned}\pi(\mathbf{p},\mathbf{q}) &\geq \pi(\mathbf{q},\mathbf{q}) \quad \text{for } q\in W\cap E \\ \pi(\mathbf{p},\mathbf{q}) &> \pi(\mathbf{q},\mathbf{q}) \quad \text{for } q\in W\cap E^C\end{aligned} \tag{5}$$

First we show that if $\mathbf{p}\in E$ then $\mathbf{p}$ is a NE. Suppose that $\mathbf{p}$ is not a NE. Then there exists $\mathbf{d}\in\Delta$ such that

$$\pi(\mathbf{d},\mathbf{p})>\pi(\mathbf{p},\mathbf{p}) \tag{6}$$

Then by definition of ES set, there exists a neighborhood $W_{\mathbf{p}}$ satisfying (5). We choose $\epsilon_{\mathbf{d}}$ such that for all ϵ such that $0<\epsilon<\epsilon_{\mathbf{d}}$, $(1-\epsilon)\mathbf{p}+\epsilon\mathbf{d}\in W_{\mathbf{p}}$. Then from (5)

$$\pi(\mathbf{p},(1-\epsilon)\mathbf{p}+\epsilon\mathbf{d})\geq\pi((1-\epsilon)\mathbf{p}+\epsilon\mathbf{d},(1-\epsilon)\mathbf{p}+\epsilon\mathbf{d}) \tag{7}$$

By rearranging (5) and using $\epsilon>0$, we obtain

$$\pi(\mathbf{p},(1-\epsilon)\mathbf{p}+\epsilon\mathbf{d})\geq\pi(\mathbf{d},(1-\epsilon)\mathbf{p}+\epsilon\mathbf{d}) \text{ for all } \epsilon \text{ such that } 0<\epsilon<\epsilon_{\mathbf{d}} \tag{8}$$

Since ϵ is independent of $\mathbf{p}$ and $\mathbf{d}$, we can take $\epsilon\to 0$ and from the continuity of the payoff function, we obtain

$$\pi(\mathbf{p},\mathbf{p})\geq\pi(\mathbf{d},\mathbf{p})$$

which is a contradiction to (6) To show that the equivalence to ESS when $E=\{p\}$, it is enough to note that in this case (5) becomes

$$\begin{aligned}\pi(\mathbf{p},\mathbf{q}) &\geq \pi(\mathbf{q},\mathbf{q}) \quad \text{for } q\in W\cap\{p\} \\ \pi(\mathbf{p},\mathbf{q}) &> \pi(\mathbf{q},\mathbf{q}) \quad \text{for } q\in W\setminus\{p\}\end{aligned}$$

Then the equivalence follows from proposition 4.7 in the classnotes. The set of all NEs is $\Delta^{NE}=\{(p_1,p_2,p_3):p_1=p_2\}$, ES set is the same as the set of all NEs, since $\pi(\mathbf{p}^*,\mathbf{q})-\pi(\mathbf{q},\mathbf{q})=$

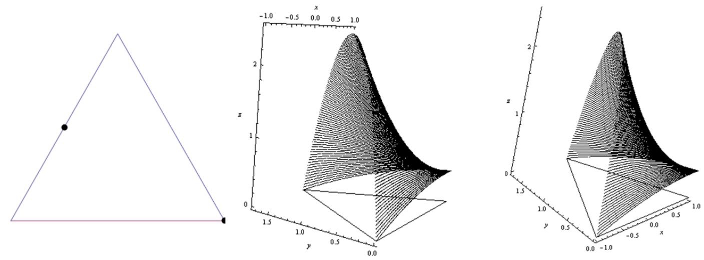

Figure 1: Problem 6. NE and ESS, Energy Function

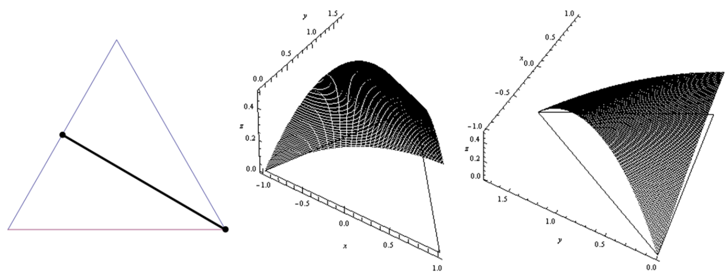

Figure 2: Problem 7. NE and ES set and Energy Function

$(q_1 - q_2)^2 \geq 0$ for all $\mathbf{p} \in \Delta^{NE}$, $\mathbf{q} \in \Delta$, and there is no ESS. Note that we have

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \overset{NE}{\sim} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and the similarity of these payoffs to one in the problem 6. (See the figures below). However, the argument for ESS in problem 6 is not valid here because $\pi(\mathbf{p}^*, \mathbf{p}^*) = 1$ and $\pi(\mathbf{q}, \mathbf{p}^*) = q_1 + q_2 + q_3$.

8. First note that since $\mathbf{p} \in int\Delta$, $\boldsymbol{\pi}(\mathbf{e}_1, \mathbf{p}) = \boldsymbol{\pi}(\mathbf{e}_2, \mathbf{p}) = \boldsymbol{\pi}(\mathbf{p}, \mathbf{p})$. So $\boldsymbol{\pi}(\mathbf{e}_1, \mathbf{q}) - \boldsymbol{\pi}(\mathbf{p}, \mathbf{q}) = \epsilon \boldsymbol{\pi}(\mathbf{e}_1 - \mathbf{p}, \mathbf{r})$. Since
$$\boldsymbol{\pi}(\mathbf{e}_1 - \mathbf{p}, \mathbf{r}) = (\frac{1}{3}, -\frac{1}{3}) \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{6} > 0$$
we have $\boldsymbol{\pi}(\mathbf{e}_1, \mathbf{q}) > \boldsymbol{\pi}(\mathbf{p}, \mathbf{q})$.