Math 597/697: Homework 1

1. Consider a Markov chain with state space $S = \{0, 1\}$ and transition matrix
$$P = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix}. \tag{1}$$
Compute the following probabilities
   (a) If the chain is initially in state 0 what is the probability that it is in state 1 at time 3?
   (b) If the chain is initially in state 1 what is the probability that it is in state 1 at time 1 and in state 0 at time 3?
   (c) If the chain starts in state 0 what is the probability that the chain first visit in state 1 occurs at time 3.
   (d) At time 0 the chain is in state 0 with probability 1/4 and in state 1 with probability 3/4. What is the probability distribution of X_3?

2. The weather is in two states: rainy or not rainy and it depends on the weather in the previous two days. If it was rainy yesterday and and the day before yesterday it will be rainy today with probability 0.8. If it was not rainy yesterday and the day before yesterday it will be rainy today with probability 0.2. If it was rainy yesterday but not rainy the day before yesterday, then the weather will be rainy today with probability 0.5. If it was not rainy yesterday but rainy the day before yesterday, then the weather will be rainy today with probability 0.4.
   (a) Explain why it is not reasonable to model the weather on a single day as a markov chain.
   (b) Show that one can construct a Markov chain by taking as a state the weather in two consecutive days. Write the corresponding transition probabilities.
   (c) If it rained yesterday and the day before yesterday, what is the probability it will rain tomorow?

3. Three white balls and three red balls are distributed among two urns in such a way that each balls contains three urns. We say that the system in the state $j = 0, 1, 2, 3$ if the first urn contains j white balls. At each step one draws one ball from each of the urns and one place the ball drawn in the first urn in the second one and conversely. Let X_n denote the state of the system after the n-th step. Explain why X_n is a Markov chain, calculate its transition matrix and the stationary distribution.

4. Let X_n and Y_n be two independent Markov chains, each with the same discrete finite state space $S = \{1, \cdots, N\}$ and transition probabilities P_{ij}. Define the process $Z_n = (X_n, Y_n)$ with state space $S \times S$. Show that Z_n is a Markov chain and determine the state space and the transition probabilities.

5. Consider the Markov chain with state space $S = \{0, 1, 2\}$ and transition matrix
$$P = \begin{pmatrix} .4 & .4 & .2 \\ .3 & .4 & .3 \\ .2 & .4 & .4 \end{pmatrix}.$$

Show that this chain has a unique stationary distribution π and compute π.

6. (a) Suppose that the Markov chain X_n has two distinct stationary distribution π_1 and π_2. Show that $\alpha\pi_1 + (1-\alpha)\pi_2$ is a stationary distribution for any real number α with $0 \le \alpha \le 1$.
   Remark: A set C of vectors in $\mathbf{R}^N$ is said to be convex if for any x, y in C, and any real number α with $0 \le \alpha \le 1$, the vector $\alpha x + (1-\alpha)y$ is also in C. An example of a convex set is the set of all stationary distributions for a Markov chain X_n.

   (b) Consider the Markov chain X_n with four states and the transition matrix

$$P = \begin{pmatrix} 2/5 & 3/5 & 0 & 0 \\ 3/10 & 7/10 & 0 & .0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/6 & 5/6 \end{pmatrix}.$$

   Find all the stationary distributions for this Markov chain.

7. A $N \times N$ transition probability matrix P is said to be doubly stochastic if the sum over each column is equal to 1. Show that in this case the uniform distribution π given by

$$\pi(j) = \frac{1}{N}, \quad 1 \le j \le N,$$

   is a stationary distribution for the Markov chain with transition matrix P.

8. Consider the following method of card shuffling. To shuffle a deck of, say 52 cards, one picks at random one of the 52 cards and place it on top of the deck. Repeat this procedure. This defines a Markov chain whose state space consists of the 52! permutation of the deck. Use the previous exercise show that the uniform distribution is a stationary distribution for this Markov chain.

9. Let P be a transition probability matrix and assume that there exists a stationary distribution $\pi = (\pi(1), \cdots, \pi(N))$ with $\pi(j) > 0$ for all j. We now define a new matrix, called the *adjoint* of P and denoted by P^T:

$$P^T_{ij} = \frac{\pi(j)}{\pi(i)} P_{ji}.$$

   (a) Show that P^T_{ij} is a stochastic matrix and that π is again a stationary distribution for P^T.

   (b) Let X_n be the Markov chain with transition matrix P and initial distribution π. Let $\bar{X}_n$ be the Markov chain with transition matrix P^T and initial distribution π. Show that

$$P\{\bar{X}_0 = i_0, \bar{X}_1 = i_1, \cdots, \bar{X}_n = i_n\} = P\{X_n = i_0, X_{n-1} = i_1, \cdots X_0 = i_n\}.$$

   As a consequence of this property the Markov chain $\bar{X}_n$ is called the *time reversed* chain for the Markov chain X_n.

10. This homework should be done with the help of a computer, with maple, mathematica, or whatever program you prefer. It will illustrate the various long term behavior of Markov chains. Students who are not familiar with such programs are strongly encouraged to work with one of your colleagues who is. And conversely.

For the following transition matrices you should computer P^n to a sufficiently high power n and interpret the results. Does $\lim_{n\to\infty} P_{ij}^n$ exists? Is $\lim_{n\to\infty} P_n^{ij} = \pi_j$ independent of of i?

(a)

$$P = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 1/8 & 1/8 & 3/4 \\ 1/12 & 1/4 & 2/3 \end{pmatrix}.$$

(b)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/5 & 4/5 & 0 & 0 \\ 1/6 & 0 & 1/6 & 2/3 \\ 0 & 1/10 & 3/5 & 3/10 \end{pmatrix}.$$

(c) Random walk on $\{0, 1, 2, 3, 4\}$ with reflecting boundary conditions.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(d) Random walk on $\{0, 1, 2, 3, 4\}$ with absorbing boundary conditions.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(e)

$$P = \begin{pmatrix} 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/8 & 0 & 2/3 & 0 & 5/24 \\ 0 & 5/6 & 0 & 1/6 & 0 \\ 0 & 0 & 1/7 & 0 & 6/7 \end{pmatrix}.$$