Math 597/697: Solution 4

1. Let T_1 and T_2 denote the lifetimes of the machines and let A denote the event that the first machine fails first. Conditioning on whether the first machine fails before or after t we find

$$P\{A\} = P\{A\,|T_1 \le t\}P\{T_1 \le t\} + P\{A\,|\,T_1 > t\}P\{T_1 > t\} \qquad (1)$$

Clearly $P\{A\,|T_1 \le t\} = 1$ and by the memoryless property

$$\begin{aligned} P\{A\,|\,T_1 > t\} &= P\{T_2 > T_1\} \\ &= \int_0^\infty ds_1 \int_{s_1}^\infty ds_2 \lambda_1 \lambda_2 e^{-\lambda_1 s_1} e^{-\lambda_2 s_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned} \qquad (2)$$

Hence

$$P(A) = (1 - e^{-\lambda_1 t}) + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 t} \qquad (3)$$

2. A radioactive source emits particles according to a Poisson process with rate $\lambda = 2$ particles per minute.

   (a) The probability to have 4 particles emitted between the first and second minute is

   $$P\{N_2 - N_1 = 4\} = e^{-2}\frac{2^4}{4!} = \frac{2}{3e^2}\,. \qquad (4)$$

   (b) The probability that the first particle emitted is between the second and third minutes is

   $$P\{2 \le T_1 \le 3\} = \int_2^3 ds 2e^{-2s} ds = e^{-4} - e^{-6}\,. \qquad (5)$$

   (c) The probability that the fourth particles is emitted less than two minutes after the second one is

   $$P\{T_3 + T_4 \le 2\} = \int_0^2 dse^{-2s} 4s = 1 - 5e^{-4}\,. \qquad (6)$$

   (d) The expected time between the emission of the third particle and the emission of the seventh particle is

   $$E[T_4 + T_5 + T_6 + T_7] = 4\frac{1}{2} = 2\,. \qquad (7)$$

(e) The conditional probability that 3 particles are emitted in the first minute given that 5 particles are emitted in the first two minutes is

$$P\{N_1 = 3 \,|N_2 = 5\} \;=\; \frac{P\{N_1 = 3\,,\, N_2 - N_1 = 2\}}{P\{N_2 = 5\}} \;=\; \frac{e^{-2}\frac{2^3}{3!}e^{-2}\frac{2^2}{2!}}{e^{-4}\frac{4^5}{5!}} \;=\; \binom{5}{2}\frac{1}{2^5}\,. \tag{8}$$

(f) The expected number of particles emitted between the third and fourth minutes given that 5 particle were emitted in the first two minutes is by the property of independent increments

$$E[N_4 - N_3 \,|N_2 = 5] \;=\; E[N_4 - N_3] \;=\; 2\,. \tag{9}$$

(g) The expectation of the arrival time of the fifth particle given that $N_t = 3$ is, by the memoryless property

$$E[S_5 \,|\, N_t = 3] \;=\; t + E[T_4 + T_5] \;=\; t + 1 \tag{10}$$

3. Let T_1 and T_2 be your service times at servers 1 and 2 and T_A and T_B the service times of A and B at server 2. By the memoryless property one might assume that you and A enter your respective servers at the same time.

(a) The probability P_A that A is still in service when you move over to server 2 is given by $P_A = P\{T_1 < T_A\}$ which is, as in Problem 1, $\mu_1/(\mu_1 + \mu_2)$.

(b) The probability P_B that B is still in service when you move over to server 2 is given by $P\{T_1 > T_A\,,\, T_1 < T_A + T_B\} = P\{T_A < T_1 < T_A + T_B\}$. Conditioning on your service time one finds

$$\begin{aligned} &P\{T_A < T_1 < T_A + T_B\} \\ &= \int_0^\infty dt P\{T_A < T_1 < T_A + T_B \,|\, T_1 = t\} f_{T_1}(t) \end{aligned} \tag{11}$$

and

$$\begin{aligned} &P\{T_A < T_1 < T_A + T_B \,|\, T_1 = t\} \;=\; P\{T_A < t < T_A + T_B\} \\ &= \int_0^t ds \int_{t-s}^\infty du \mu_2^2 e^{-\mu_2 s} e^{-\mu_2 u} \;=\; \mu_2 t e^{-\mu_2 t} \end{aligned} \tag{12}$$

So

$$\begin{aligned} P_B &= \int_0^\infty dt \mu_1 \mu_2 t e^{-(\mu_1+\mu_2)t} \\ &= \frac{\mu_1\mu_2}{\mu_1+\mu_2}\int_0^\infty dt\, t(\mu_1+\mu_2)e^{-(\mu_1+\mu_2)t} = \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2} \end{aligned}$$

(c) The total time T you spend in the system is $T = T_1 + T_2 + W_A + W_B$ where W_A the amount of time you wait in queue when while A is being served, and W_B the amount of time you wait in queue when while B is being served. The R.V. W_A is zero if $T_1 > T_A$ (with probability $1-P_A$) and non zero if $T_1 < T_A$ (this occurs with probability P_A). In the latter case, by the memoryless property W_A conditioned on $T_1 < T_A$ has the same distribution as T_A. More formally

$$\begin{aligned} P\{W_A > s\} &= P\{W_A > s \,|T_1 < T_A\}P\{T_1 < T_A\} \\ &= P\{T_A > T_1 + s\,|T_A > T_1\}P_A = P\{T_A > s\}P_A \\ &= e^{-\mu_2 s}\frac{\mu_1}{\mu_1+\mu_2}\,. \end{aligned} \tag{13}$$

Therefore

$$E[W_A] = \int_0^\infty ds P\{W_A > s\} = \frac{\mu_1}{(\mu_1+\mu_2)}\frac{1}{\mu_2} \tag{14}$$

To determine $E[W_B]$ we proceed similarly. Note that W_B is zero if $T_1 > T_A + T_B$, $W_B = T_B$ if $T_1 < T_A$ and W_B has the same distribution as T_B if conditioned on $T_A < T_1 < T_A + T_B$ (by the memoryless property). So by conditioning one finds

$$P\{W_B > s\} = \left(\frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}\right)e^{-\mu_2 s} \tag{15}$$

and

$$E[W_B] = \left(\frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}\right)\frac{1}{\mu_2} \tag{16}$$

Therefore we find that

$$E[T] = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}\frac{1}{\mu_2} + \left(\frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}\right)\frac{1}{\mu_2} \tag{17}$$

4. Let N_t be a Poisson process with rate λ and let $0 < s < t$.

(a) Since N_t has independent increments

$$P\{N_t = n + k | N_s = k\} = P\{N_t - N_s = n | N_s = k\}$$
$$= P\{N_t - N_s = n\} \; = \; e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \, . \tag{18}$$

(b)

$$P\{N_s = k | N_t = n + k\} \; = \; \frac{P\{N_s = k \, , \, N_t - N_s = n\}}{P\{N_t = n + k\}}$$
$$= \; \frac{e^{-\lambda s} \frac{(\lambda s)^k}{k!} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!}}{e^{-\lambda t} \frac{(\lambda t)^{n+k}}{(n+k)!}} \; = \; \binom{n+k}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^n \tag{19}$$

and this is a binomial distribution with parameter $n + k$ and s/t.

(c) Using that the Poisson process has independents increments and that for a Poisson R.V. X with parameter μ, $E[X] = \mu$ and $E[X^2] = \mu + \mu^2$ we find

$$E[N_t N_s]$$
$$= \; E[(N_t - N_s)N_s] + E[N_s^2] \; = \; E[N_t - N_s]E[N_s] + E[N_s^2]$$
$$= \; \lambda(t-s)\lambda s + \lambda^2 s^2 + \lambda s \; = \; \lambda^2 ts + \lambda s \tag{20}$$

5. The probabilities to be injured are respectively $P\{N_{t+S} - N_t \geq 1\} = 1 - e^{-\frac{S}{20}}$ and $P\{N_{t+S} - N_t \geq 2\} = 1 - e^{-\frac{S}{20}} - \frac{S}{20}e^{-\frac{S}{20}}$.

6. By the order statistics property of $S_1, \ldots, S_n$ conditioned on the event $\{N_t = n\}$ we have

$$P\{S_1 > s | N_t = n\} \;\; = \;\; P\{U_{(1)} > s\} \; = \; P\{U_1 > s, \ldots U_n > s\}$$
$$= \;\; \left(\frac{(t-s)}{t}\right)^n . \tag{21}$$

where U_i are iid uniform R.V on $[0, t]$. Therefore

$$E[S_1 | N_t = n] \; = \; E[U_{(1)}] \; = \; \int_0^t P\{U_{(1)} > s\} ds \; = \; \frac{t}{n+1} \, . \tag{22}$$

and so the answer of (a) is $1/n + 1$ and the answer of (c) is $t/3$. Furthermore

$$E[S_1 S_2 | N_1 = 2] \; = \; E[U_{(1)} U_{(2)}] \; = \; E[U_1 U_2] \; = \; E[U_1]E[U_2] \; = \; \frac{1}{4} \, . \tag{23}$$

7. As in the $M/G/\infty$ queue, the distribution of the particles which have been emitted but not annihilated at time t is a Poisson distribution with parameter $\lambda \int_0^t (1 - G(y))\, dy$ As $t \to \infty$ the parameter converges to $\lambda \int_0^\infty P\{S \geq y\}\, dy = \lambda\mu$ for large t the distribution is Poisson with a parameter $\approx \lambda\mu$.

8. (a) Using the order statistics property

$$\begin{aligned} E\left[\sum_{i=1}^{N_t} f(S_i) \,|\, N_t = n\right] &= E\left[\sum_{i=1}^{n} f(U_{(i)})\right] \\ = E\left[\sum_{i=1}^{n} f(U_i))\right] &= \frac{n}{t}\int_0^t f(s)\, ds \end{aligned} \tag{24}$$

Hence

$$\begin{aligned} E\left[\sum_{i=1}^{N_t} f(S_i)\right] &= E\left[E\left[\sum_{i=1}^{N_t} f(S_i) \,|\, N_t\right]\right] \\ = E\left[N_t \frac{1}{t}\int_0^t f(s)\, ds\right] &= \lambda \int_0^t f(s)\, ds \end{aligned} \tag{25}$$

(b) Using the order statistics property one finds, with $\mu = E[X]$,

$$\begin{aligned} E\left[Z_t \,|\, N_t = n\right] &= E\left[\sum_{k=1}^{n} X_k e^{-\alpha(t - S_k)} \,|\, N_t = n\right] \\ = E\left[\sum_{k=1}^{n} X_k e^{-\alpha(t - U_{(k)})}\right] &= E\left[\sum_{k=1}^{n} X_k e^{-\alpha(t - U_k)}\right] \\ = n\mu \frac{1}{t}\int_0^t e^{-\alpha(t-s)}\, ds &= n\mu \frac{1 - e^{\alpha t}}{\alpha t} \end{aligned} \tag{26}$$

and so

$$E\left[Z_t\right] = \lambda\mu \frac{1 - e^{\alpha t}}{\alpha} \tag{27}$$

9. We have $S(t) = S_0 \prod_{i=1}^{N_t} X_i$, where X_i is an exponential R.V with $E[X_i] = 1/\mu$ and $E[X_i^2] = 2/\mu^2$. Conditioning on the event $\{N_t = n\}$ we have,

$$\begin{aligned} E\left[\prod_{i=1}^{N_t} X_i \,|\, N_t = n\right] &= \left(\frac{1}{\mu}\right)^n \\ E\left[\left(\prod_{i=1}^{N_t} X_i\right)^2 \,|\, N_t = n\right] &= \left(\frac{2}{\mu^2}\right)^n . \end{aligned} \tag{28}$$

Note that the moment generating function of a Poisson R.V. Y with parameter κ is $E[s^Y] = e^{\kappa(s-1)}$. Therefore

$$\begin{aligned} E[S(t)] &= S_0 e^{\lambda t(\frac{1}{\mu}-1)}, \\ Var(S(t)) &= S_0^2 \left(e^{\lambda t(\frac{2}{\mu^2}-1)} - e^{2\lambda t(\frac{1}{\mu}-1)} \right). \end{aligned} \tag{29}$$

10. One finds $m(9) = \int_0^9 \lambda(s)\, ds = 70$, therefore the number of customers entering the store on a given day is a Poisson distribution with parameter 60.