Math 624: Problem set 2

1. Consider a function $f \in L^2([-\pi,\pi])$ with Fourier coefficients c_n and Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}\,, \quad c_n = \frac{1}{2\pi}\int f(x)e^{-inx}\,dx$$

   (a) Show that if f is real-valued then its Fourier series can be written as

$$\frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

   for suitable coefficients a_n, b_n. What happens to the coefficents a_n, b_n, if f is even, respectively odd?

   (b) Prove that one can write for any $0 \leq x \leq \pi$

$$\sin(x) \quad = \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

   for suitable coefficients a_n which you will compute. Justify the covergence in the formula.
   Hint: Note that the formula holds for $0 \leq x \leq \pi$ not $-\pi \leq x \leq \pi$. Extend the function $\sin(x)$ on $0 \leq x \leq \pi$ to an even function on $[0, 2\pi]$.

2. Exercise 2, p.312

3. Exercise 3, p. 312

4. Let $\mu_\star$ be an exterior measure. Show that if E is Carathéodory measurable and if A is an arbitrary subset of X we have

$$\mu_\star(E \cup A) + \mu_\star(E \cap A) \;=\; \mu_\star(E) + \mu_\star(A)\,.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Define for any $A \subset X$

$$\mu_\star(A) \;=\; \inf\left\{ \sum_i \mu(E_i), E_i \in \mathcal{M}, A \subset \bigcup_i E_i \right\}\,.$$

   Show that $\mu_\star$ is an exterior measure which extends μ, i.e. $\mu(E) = \mu_\star(E)$ for any $E \in \mathcal{M}$. It is called the *exterior measure generated* by μ.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mu_\star$ be the exterior measure generated by μ (see problem 5). Show that the following are equivalent

(a) The set E is measurable in the sense of Caratheodory.

(b) $\mu(A) = \mu_\star(A \cap E) + \mu_\star(A \cap E^c)$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$.

(c) $\mu(A) \geq \mu_\star(A \cap E) + \mu_\star(A \cap E^c)$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$.

(d) $\mu_\star(A) \geq \mu_\star(A \cap E) + \mu_\star(A \cap E^c)$ for all $A \subset X$.

Use this to conclude that every set $E \in \mathcal{M}$ is measurable in the sense of Caratheodory.

Hint: Prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$

Remark: Note that if $\mu_\star$ is obtained from a premasure on an algebra $\mathcal{A}$ then a similar characterization holds with $A \in \mathcal{M}$ replaced by $A \in \mathcal{A}$ (with the same proof).

7. Suppose $(X, \mathcal{N}, \mu)$ is a finite measure space (i.e. $\mu(X) < \infty$). Show that a set is measurable in the sense of Caratheodory if and only if

$$\mu_\star(E) + \mu_\star(E^c) = \mu_\star(X)$$

Hint: Use Problem 6(c). Pick a measurable A and apply the definition of measurability using both the sets E and E^c.

8. Exercise 5, p. 313. In addition deduce from this fact the amusing fact that the volume of the d-dimensional ball of radius 1 tends to 0 as $d \to \infty$. Recall that the Gamma function $\Gamma(x)$ is given, for $x \geq 0$, by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

and that by integration parts you can prove that $\Gamma(x+1) = x\Gamma(x)$.

9. Exercise 14, p.315

10. Exercise 15, p.316