Math 597/697: Solution 3

1. (a) The number of papers in the pile in the evening is $0, 1, 2, 3, 4$ and the transition matrix is

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 1/3 & 0 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

(b) The Markov chain is aperiodic and irreducible, and the state space is finite. Therefore there is a unique stationary distribution π_j and $\lim_{n\to\infty} P^n_{ij} = \pi_j$. Solving $\pi P = \pi$ one finds $\pi = (\frac{81}{211}, \frac{54}{211}, \frac{36}{211}, \frac{24}{211}, \frac{16}{211})$. In the long run $P\{X_n = j\} \approx \pi_j$ and so, in the long run the expected number of papers is $54/211 + 2 \times 36/211 + 3 \times 24/211 + 4 \times 16/211 = 262/211$.

(c) Starting with 0 papers the expected time until the pile will have again 0 papers is $E[\tau_0|X_0 = 0] = \pi_0^{-1} = 211/81$.

(d) To obtain $E[\tau_0|X_0 = 2]$ we must compute $M = (I - Q)^{-1}$ where Q is the 4×4 matrix obtained by erasing the first row and first column of P:

$$Q = \begin{pmatrix} 0 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

It is easy to see that

$$Q^2 = \begin{pmatrix} 0 & 0 & 4/9 & 0 \\ 0 & 0 & 0 & 4/9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q^3 = \begin{pmatrix} 0 & 0 & 0 & 8/27 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

and $Q^n = 0$ if $n \geq 4$ so that

$$M = (1 - Q)^{-1} = 1 + Q + Q^2 + Q^3 = \begin{pmatrix} 1 & 2/3 & 4/9 & 8/27 \\ 0 & 1 & 2/3 & 4/9 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Starting from 2 the expected number of visits to 1, 2, 3, 4 are respectively 0, 1, 2/3, 4/9 and therefore the expected time to reach 0 is 19/9.

2. Using conditional expectation and the Markov property we have

$$P\{\tau_j = n, X_1 = k|X_0 = i\} = P\{\tau_j = n|X_1 = k\}P_{ik} \tag{5}$$

and $P\{\tau_j = 1|X_1 = j\} = 1$ and $P\{\tau_j = n|X_1 = j\} = 0$ if $n \geq 2$. We also have that for $k \neq j$

$$E[\tau_j|x_1 = k] = 1 + E[\tau_j|x_0 = k]\,. \tag{6}$$

Using this we obtain

$$\begin{aligned}
M_{ij} &= E[\tau_j|X_0 = i] \\
&= \sum_k \sum_{n\geq 1} nP\{\tau_j = n, X_1 = k|X_0 = i\} \\
&= \sum_{n\geq 1} nP\{\tau_j = n|X_1 = j\}P_{ij} + \sum_{k\neq j}\sum_{n\geq 1} nP\{\tau_j = n|X_1 = k\}P_{ik} \\
&= P_{ij} + \sum_{k\neq j} E[\tau_j|x_1 = k]P_{ik} \\
&= P_{ij} + \sum_{k\neq j}(1 + E[\tau_j|x_0 = k])P_{ik} \\
&= 1 + \sum_{k\neq j} P_{ik}M_{kj}\,.
\end{aligned} \tag{7}$$

Since the chain is irreducible with finite state space, it is positive recurrent and there is a unique stationary distribution π_i. Multiplying Eq. (7) with π_i, summing over i, using that $\pi P = P$ and $\sum_i \pi_i = 1$ we obtain

$$\begin{aligned}
\sum_i \pi_i M_{ij} &= \sum_i \pi_i + \sum_{k\neq j}\sum_i \pi_i P_{ik} M_{kj} \\
&= 1 + \sum_{k\neq j} \pi_k M_{kj}
\end{aligned}$$

and therefore

$$\pi_j M_{jj} = 1 \tag{8}$$

or

$$\pi_j = \frac{1}{M_{jj}} = \frac{1}{E[\tau_j|X_0 = j]} \tag{9}$$

3. *The Ehrenfest urn model*, continued from Problem 3 of HWK #2.

   (a) Conditioning $E[X_n] = E[E[X_n|X_{n-1}]]$, we have

$$\begin{aligned}
E[X_n|X_{n-1} = k] &= \sum_j jP\{X_n = j|X_{n-1} = k\} \\
&= (k+1)\frac{a-k}{2a} + (k-1)\frac{a+k}{2a} \\
&= \frac{a-1}{a}k\,.
\end{aligned} \tag{10}$$

Therefore we have $E[X_n] = \frac{a-1}{a} E[X_{n-1}]$ and so $E[X_n] = (\frac{a-1}{a})^n E[X_0]$. We have then $\lim_{n\to\infty} E[X_n] = 0$ and this means that, in the long run, we can expect to have to have the same number of balls in urns A and B.

(b) We note first that, by the binomial theorem

$$\sum_{j=-a}^{a} \binom{2a}{a+j} = \sum_{k=0}^{2a} \binom{2a}{k} = 2^{2a} \tag{11}$$

and thus π_i is normalized. Furthermore we have

$$\begin{aligned}
\sum_i \pi_i P_{ij} &= \pi_{j-1} P_{j-1j} + \pi_{j+1} P_{j+1j} \\
&= 2^{-2a} \left(\frac{2a!}{(a+j-1)!(a-j+1)!} \frac{a-j+1}{2a} \right. \\
&\qquad \left. + \frac{2a!}{(a+j+1)!(a-j-1)!} \frac{a+j+1}{2a} \right) \\
&= 2^{-2a} \frac{2a!}{(a+j-1)!(a-j-1)!} \left(\frac{1}{(a-j)2a} + \frac{1}{(a+j)2a} \right) \\
&= 2^{-2a} \frac{2a!}{(a+j)!(a-j)!} = \pi_j
\end{aligned} \tag{12}$$

4. To prove positive recurrence we use the theorem which states that if an irreducible Markov chain has a stationary distribution, then it is postive recurrent. We have

$$\begin{aligned}
&P_{00} = (1-p)\,, \quad P_{01} = p \\
&P_{ii-1} = (1-p)q \quad P_{ii} = pq + (1-p)(1-q) \quad P_{ii+1} = p(1-q)\,.
\end{aligned} \tag{13}$$

We consider the equation for the stationary distribution $\pi P = \pi$. For $n \geq 2$ we have

$$\pi_n = p(1-q)\pi_{n-1} + (pq + (1-p)(1-q))\pi_n + q(1-p)x_{n+1}\,. \tag{14}$$

Making the ansatz $\pi_n = x^n$ we find the equation

$$q(1-p)x^2 - (q(1-p) + p(1-q))x + p(1-q) = 0 \tag{15}$$

It has the form

$$ax^2 - (a+b)x + b = 0\,, \quad a, b > 0 \tag{16}$$

with roots 1 and b/a. So here the general solution is $C_1 + C_2 \frac{p(1-q)}{q(1-p)}$. The solution will be a probability distribution (i.e. normalizable) provided

$$\frac{p}{1-p} \frac{1-q}{q} < 1\,, \text{ or } 0 < p < q < 1. \tag{17}$$

that is if the rate at which one enters the queue is smaller than the rate at which one exits it. Note that we excluded the case $p = 0$ and $q = 1$, indeed one verifies in these two special cases that the Markov chain is not irreducible (there are absorbing states).

5. The transition matrix has the form

$$P = \begin{pmatrix} (1-p) & p & 0 & 0 & \dots \\ (1-p) & 0 & p & 0 & \dots \\ (1-p) & 0 & 0 & p & \dots \\ (1-p) & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} . \tag{18}$$

From this one sees easily that $P\{\tau_0 = n | X_0 = 0\} = p^{n-1}(1-p)$ for $n = 1, 2, \dots$, and this is a geometric distribution with parameter $(1-p)$ and so $E[\tau_0 | x_0 = 0] = (1-p)^{-1}$. The equations for the stationary distribution is $\pi_0 = (1-p)\sum_i \pi_i$ and $p\pi_n = \pi_{n+1}$ and so $\pi_n = (1-p)p^n$.

6.
$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .3 & 0 & .7 & 0 \\ 0 & .3 & 0 & .7 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{19}$$

Relabelling the states as $0, 3, 1, 2$ we have

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .3 & 0 & 0 & .7 \\ 0 & .7 & .3 & 0 \end{pmatrix} . \tag{20}$$

So we have

$$Q = \begin{pmatrix} 0 & .7 \\ .3 & 0 \end{pmatrix} \quad S = \begin{pmatrix} .3 & 0 \\ 0 & .7 \end{pmatrix} \tag{21}$$

and

$$M = (I - Q)^{-1} = \begin{pmatrix} 100/79 & 70/79 \\ 30/79 & 100/79 \end{pmatrix}, \ A = MS = \begin{pmatrix} 30/79 & 49/79 \\ 9/79 & 70/79 \end{pmatrix} \tag{22}$$

So we have

(a) The expected number of visits to the state 1 starting from state 2 is $M_{21} = 30/79$.

(b) The expected number of visits to states 1 and 2 prior to absorption is $170/79$ starting from state 1 and $130/79$ starting from state 2.

(c) The probability of absorption into state 0 starting from state 1 is $30/79$.

7. The chain ξ_n has state space $(0,0),(1,1),(0,1),(1,0)$. With this ordering of the state the transition matrix is

$$\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/9 & 4/9 & 2/9 & 2/9 \\ 1/6 & 1/3 & 1/3 & 1/6 \\ 1/6 & 1/3 & 1/6 & 1/3 \end{pmatrix}. \tag{23}$$

We have $X_n = Y_n$ if and only if $\xi_n = (0,0)$ or $(1,1)$ and therefore we set

$$Q = \begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix}, S = \begin{pmatrix} 1/6 & 1/3 \\ 1/6 & 1/3 \end{pmatrix}, \tag{24}$$

and therefore

$$M = (I-Q)^{-1} = \begin{pmatrix} 8/5 & 2/5 \\ 2/5 & 8/5 \end{pmatrix}, A = MS = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}. \tag{25}$$

(a) From the matrix Q we see that $E[T] = 2/5 + 8/5 = 2$

(b) From the matrix A we see that $P\{X_T = 1\} = 2/3$

(c) The stationary distribution for X_n or Y_n is $(\pi_0, \pi_1) = (2/5, 3/5)$, therefore the stationary distribution for ξ_n is $\pi_{(0,0)} = 4/25$, $\pi_{(0,1)} = 6/25$, $\pi_{(1,0)} = 6/25$, $\pi_{(1,1)} = 9/25$. Therefore, in the long run, the chains spend $4/25 + 9/25 = 13/25$ of the time in the same state.

8. This is just the Gambler's ruin problem with $p = 0.6$. If you start with \$5 then $j = 5$ and $N = 25$ since the probability that you wipe out your friend is the probability that your fortune reaches \$25 before it reaches \$0 hence it is

$$A_{5,25} = \frac{1 - (\frac{2}{3})^5}{1 - (\frac{2}{3})^{25}} \approx 0.868 \tag{26}$$

and if you start with \$10 it is

$$A_{10,30} = \frac{1 - (\frac{2}{3})^{10}}{1 - (\frac{2}{3})^{30}} \approx 0.982 \tag{27}$$

9. For a branching process, the probability a that the population eventually dies out starting with one individual is given by the smallest positive root of the equation $a = \sum_{k=1}^{\infty} p_k a^k$.

(a) $p_0 = \frac{1}{4}$, $p_2 = \frac{3}{4}$, here $\mu > 1$ and the roots are $1/3$ and 1 and so $a = 1/3$.

(b) $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$, here $\mu = 1$ and the root is 1 (multiplicity 2) and so $a = 1$.

(c) $p_0 = \frac{1}{6}$, $p_1 = \frac{1}{2}$, $p_3 = \frac{1}{3}$. Here $\mu > 1$ and the equation is $a^3/3 - a/2 + 1/6$. Using that 1 ia always a root and factorizing one find the roots 1 and $(-1 \pm \sqrt{3})/2$. Hence $a = (-1 + \sqrt{3})/2$.

10. Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers. If $\lim_n a_n = a$, then for any $\epsilon > 0$ there exists $N \geq 0$ such that $|a_n - a| < \epsilon$ if $n \geq N$. So for $m > N$ we have

$$\begin{aligned}
|b_m - a| &= \left| \frac{1}{m+1} \sum_{k=0}^{m} (a_k - a) \right| \\
&= \left| \frac{1}{m+1} \sum_{k=0}^{N} (a_k - a) + \frac{1}{m+1} \sum_{k=N+1}^{m} (b_k - a) \right| \\
&\leq \frac{1}{m+1} |a_0 + \ldots + a_N - Na| + \frac{m-N-1}{m+1} \epsilon \qquad (28)
\end{aligned}$$

If we choose m large enough the r.h.s. of (28) is less than 2ϵ.