Math 623: Homework 4

1. Exercise 4, p.90

2. Exercise 19, p.93

3. Let $I = [0, 1]$ and consider the following functions on $I \times I$

   (a) $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

   (b) $f(x, y) = (x - \frac{1}{2})^{-3}$ if $0 < y < |x - \frac{1}{2}|$, $f(x, y) = 0$ otherwise.

   Investigate the existence and equality of the integrals

$$\int_{I \times I} f(x, y) d(x, y) \,, \int_I \int_I f(x, y) dx dy \,, \quad \int_I \int_I f(x, y) dy dx$$

   (a) $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

   (b) $f(x, y) = (x - \frac{1}{2})^{-3}$ if $0 < y < |x - \frac{1}{2}|$, $f(x, y) = 0$ otherwise.

4. Exercise 21, p. 94

5. Exercise 22, p. 94

6. Exercise 23, p. 94

7. Exercise 25, p. 95.

8. Let E be subset of $\mathbf{R}^d$ with finite measure. For any two functions f, g measurable on E let $\rho_E(f, g) = \int_E \frac{|f-g|}{1+|f-g|} dm$.

   (a) Show that $\rho_E(f, g)$ defines a metric on the set of measurable functions defined on E.

   (b) Let $\{f_n\}$ be a sequence of measurable functions defined on E . Show that $\lim_{n\to\infty} \rho_E(f_n, f) = 0$ if and only if f_n converges to f in measure.

   (c) Show that the assumption that E has finite measure is necessary. *Hint:* Consider $f_n(x) = \frac{1}{nx}$.

9. The following is a variation of the Dominated Convergence Theorem: Suppose that f_n converge to f in measure and that there exists a function $g \in L^1$ such that $|f_n| \leq g$ a.e. for all n. Show that f is integrable and $\lim_{n\to\infty} \int |f - f_n| dm = 0$. *Hint:* You may either argue by contradiction or you may use the following elementary fact that a sequence $\{a_n\}$ converges to a if and only if every subsequence of $\{a_n\}$ has a convergent subsubsequnce which converges to a.

10. Suppose that the sequence $\{f_n\}$ converges in measure to a function f on the finite interval $[a, b]$. Let $g : \mathbf{R} \to \mathbf{R}$ be a bounded and uniformly continuous function. Show that

$$\lim_{n\to\infty} \int_{[a,b]} g(f_n)dx = \int_{[a,b]} g(f)dx.$$

Hint: You may use the previous problem.