Math 523H–Homework 8

This homework has two parts. The first part should be completed on a separate piece of paper (stapled separately as well please). Write a short summary of the most important concepts theorems on the following two topics: (i) Uniform convergence, continuity and uniform continuity and (ii) The Riemann integral.

For the second part do the following problems:

1. (a) Use a geometric series to write down a series for $\frac{1}{1+x^2}$.

   (b) Use your result in (a) to write down a series expansion for $\arctan(x)$ for $-1 < x < 1$. Justify carefully all the steps.

2. Consider the sequences of functions on $[0, 1]$:

$$(a)\ \ f_n(x) = \frac{nx}{(1+n^2x^2)^2}\,, \qquad (b)\ \ f_n(x) = \frac{n^2x}{(1+n^2x^2)^2}$$

   Compute the limits $\lim_{n\to\infty} f_n(x)$. Determine if the convergence is uniform (compute the maximum of f_n). Finally determine whether

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \; = \; \int_0^1 \lim_{n\to\infty} f_n(x)dx\,.$$

3. Find a series for the function $g(t) = \int_0^x e^{-t^2} dt$ by using the series for the exponential function. Justify carefully all steps.

4. Suppose that $f_n(x)$ converges uniformly on $[a, b]$. Show that $F_n(x) = \int_a^x f_n(t)dt$ converges uniformly on $[a, b]$.