Math 623: Problem set 5

1. Consider a function $f \in L^1([a,b])$ and let us extend the function f to be 0 outside of $[a,b]$. For $h > 0$ define

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt$$

   (a) Show that f is a continuous function.

   (b) Show that $\|f_h\|_{L^1([a,b]} \leq \|f\|_{L^1([a,b]}$ for any $h > 0$

   (c) Show that $\lim_{h \to 0} \|f_h - f\|_{L^1([a,b]} = 0$.

   Hint: Fubini.

2. Problem 7, page 147

3. Show that if a function is of absolutely continuous on $[a,b]$ then it is of bounded variation on $[a,b]$.

4. Problem 10, page 147

5. Problem 13, p. 147

6. Prob 16, page 147.

   Hint: For (a) use corollary 3.7. For (b) write $F' = g + h$ where g is a *step* function and $\int |g| dx \leq \epsilon$. Consider then $F = G + H$ where $G = \int_a^x g(t)dt$ and $H = \int_a^t h(t)dt$.

7. (a) Show that the function f given by $f(0) = 0$ and $f(x) = x^a \sin(x^{-b})$ for $x \in (0,1]$ with $a, b > 0$ is absolutely continuous iff $a > b$.

   (b) Consider the function f given by $f(0) = 0$ and $f(x) = x^2 |\sin(1/x)|$ for $x \in (0,1]$ and $g(x) = \sqrt{x}$. Show that f and g and $f \circ g$ are absolutely continuous but that $g \circ f$ is not absolutely continuous.

8. Compute the positive and negative variation of $f(x) = x^3 - |x|$, $-1 \leq x \leq 1$ and $f(x) = \cos(x)$ for $0 \leq x \leq 2\pi$.

9. Problem 24, page 150

10. Problem 19, page 148