Math 597/697: Homework 1

You are encouraged to work in groups. But please write your solutions yourself. If you have questions about the homework, please ask them in class, the other students will profit from them too.

1. (a) Suppose X is a discrete random variable which takes only integer values $0, 1, 2, \cdots$. Show that

$$E[X] = \sum_{n\geq 1} P\{X \geq n\} = \sum_{n \geq 0} P\{X > n\} \tag{1}$$

   (b) (**Optional!** For the students with a good analysis background) Show that if X is a continous random variable taking only positive values, then

$$E[X] = \int_0^\infty P\{X > x\} \tag{2}$$

   Hint: Integrate by parts.

2. The *moment generating function* $f(t)$ of a random variable X is defined as

$$f(t) = E[e^{tX}] \tag{3}$$

   (a) Express the moments of X, $E[X^n]$, $n = 0, 1, 2, \cdots$, as well as the variance of X, $Var(X)$ in terms of $f(t)$.

   (b) Compute the moment generating function $f(t)$ of a geometric random variable X with paramater p. Use the expression for $f(t)$ to compute mean and variance of X.

   (c) Same question as in (b) for the uniform random variable on $[a, b]$.

   Remark: The moment generating function, mean and variance of the most common distribution are summarized on page 68 and69 of Section 2.6 of your book.

3. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \quad 0 < x < \infty \quad 0 < y < \infty \tag{4}$$

   Compute $E[X|Y = y]$.

4. Let X and Y be discrete random variables with joint probability distribution $p(x, y)$ given

$$\begin{array}{lll} p(1,1) = \frac{1}{9}\,, & p(2,1) = \frac{1}{3}\,, & p(3,1) = \frac{1}{9}\,, \\ p(1,2) = \frac{1}{9}\,, & p(2,2) = 0\,, & p(3,2) = \frac{1}{18}\,, \\ p(1,3) = 0\,, & p(2,3) = \frac{1}{6}\,, & p(3,3) = \frac{1}{9}\,, \end{array} \tag{5}$$

   (a) Are the random variables X and Y independent?

   (b) Compute $E[X|y = 1]$.

5. (Sampling without replacement) An urn contains four white balls, five black balls and two red balls. Three of these balls are randomly selected from the urn. Let X and Y denote respectively the number of white and black balls selected.

   (a) Compute $P\{Y = 1\}$

   (b) Compute the conditional distribution of X given that $Y = 1$

   (c) Compute the $E[X|Y = 1]$

6. (Sampling with replacement) Repeat the previous exercise under the assumption that when a ball is selected its color is noted and put back in the urn.

7. A prisoner is in front of three doors. The first one leads him to a tunnel which bring him outside in 1 hours. The second and the third lead him to tunnels which come back to the starting point after 2 and 4 hours respectively. Assume the prisoner choose each door with equal probability every time. Let X denote the time until the prisoner is outside of the prison.

   (a) Compute $E[X]$.

   (b) Compute $E[X]$ under the assumption that the prisoner remembers the doors he chose before and doest not choose them ever again.

8. Same setting as the previous problem and choosing each door with equal probability. Let N denote the numbers of doors selected befores the miner reaches safety. let T_i denote the travel time corresponding to the ith choice, $i \geq 1$.

   (a) Given an identity which relates X to N and the T_i.

(b) What is $E[N]$?

(c) What is $E[T_N]$?

(d) What is $E[\sum_{i=1}^{N} T_i | N = n]$?

(e) Using the preceeding, what is $E[X]$?

9. A gambler wins each game with probability p. In the following two cases determine the expected total number of wins $E[N]$.

   (a) The gambles plays n games. If he wins X of these n games, then he will play an additional X games before stopping.

   (b) The gambles plays until he wins; if it takes him Y games to get this win, he will play an additional Y games.

10. Twelve people plus you arrive at a party, each one at a time which has a uniform distribution on $[0, 1]$. Let N denote the number of people who arrive before you. Compute $E[N]$.

11. The number of storms N hitting the coast of Florida during every rainy season is a Poisson randon variable with a parameter Λ which is itself a random variable uniformly distributed on $[0, 0.5]$. Find the probability that at least three stroms occur during the next season.

12. Two player rolls a pair of dice in turn, with A rolling first. The games ends either when A rolls a total of 6 or B rolls a total of 7.

    (a) Find the probability that A is the winner

    (b) Find the expected number of rolls of the dice until the game ends.