Math 623: Problem set 1

This problem set (as well as the next ones) will contain several problems meant to refresh your memory about a number of facts you should know from your undergraduate analysis class. Go back to your favorite book as needed!

1. **(About** $\limsup$ **and** $\liminf$**).** Let $\{x_n\}_{n\geq 1}$ be a *bounded sequence of real numbers* (i.e., there exists $M > 0$ such that $|x_n| \leq M$ for all $n \geq 1$. Recall that b is an **accumulation point** of the sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}_{j\geq 1}$ such that $\lim_{j\to\infty} x_{n_j} = b$

   Consider the sets

   $$X = \{x\,;\ \text{infinitely many } x_n \text{ are } > x\}\,, \quad Y = \{x\,;\ \text{infinitely many } x_n \text{ are } < x\}\,.$$

   and define

   $$\xi := \sup X\,, \quad \eta := \inf Y\,.$$

   (a) Prove that ξ is the largest accumulation point of $\{x_n\}$ and that η is the smallest accumulation point of $\{x_n\}$. We then write

   $$\xi = \limsup_{n\to\infty} x_n \qquad \text{the limit superior of the sequence } \{x_n\}\,.$$

   $$\eta = \liminf_{n\to\infty} x_n \qquad \text{the limit inferior of the sequence } \{x_n\}\,.$$

   (b) Show the formulas

   $$\limsup_{n\to\infty} x_n \;=\; \lim_{n\to\infty} \sup_{k\geq n} x_k \;=\; \inf_{n\geq 1} \sup_{k\geq n} x_k\,.$$

   $$\liminf_{n\to\infty} x_n \;=\; \lim_{n\to\infty} \inf_{k\geq n} x_k \;=\; \sup_{n\geq 1} \inf_{k\geq n} x_k\,.$$

   (c) Prove that

   $$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

   $$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$$

   and show that the inequalities can be strict (find such examples).

   (d) Exhibit a sequence $\{x_n\}$ with $0 \leq x_n \leq 1$ such that any number in $[0,1]$ is an accumulation point of $\{x_n\}$. *Hint:* The rational are dense in $[0,1]$.

2. **(Closed sets)** Let us define a set $E \subset \mathbb{R}^d$ to be closed if its complement E^c is open. Show that the following are equivalent.

   (a) E is closed

(b) E contains all its limit points (x is a limit point of E is

(c) For any convergent sequence $\{x_n\}$ with $x_n \in E$ and the limit $\lim_{n\to\infty} x_n = x$ belongs to E.

3. **(Compact sets)** Let us define a set $E \subset \mathbb{R}^d$ to be compact if E is closed and bounded. Show that the following are equivalent.

   (a) E is compact

   (b) Any cover of E by open sets i.e., $E \subset \cup_\alpha O_\alpha$ with O_α open for all α contain a finite subcover $E \subset \cup_{i=1}^M O_i$.

   (c) Any sequence $\{x_n\}$ with $x_n \in E$ contains a convergent subsequence.

4. Show that a countable union of set of exterior measure 0 has exterior measure 0 using directly the definition. In particular any countable set has measure 0, e.g. the rational numbers in $[0, 1]$.

5. Exercise 1, p. 37

6. Exercise 2, p. 37

7. Exercise 3, p. 38

8. Exercise 4, p. 38

9. Exercise 9, p. 40

10. Exercise 11, p. 41