Math 645: Midterm

This a take home exam. Closed books, but you can use class notes. Don't talk to each other, but talk to me if you have questions. Each problem is worth 20 points.

1. Solve the Cauchy problem

$$x' = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

2. Show that the Cauchy problems

$$\begin{aligned} x' &= 2 - x - \frac{4xy}{1+x^2}, \\ y' &= 3x\left(1 - \frac{y}{1+x^2}\right), \end{aligned} \tag{2}$$

and

$$\begin{aligned} x' &= x\left(3 - x^2 - 2y^2\right), \\ y' &= x^2 - y^3, \end{aligned} \tag{3}$$

with $x(0) = x_0$ and $y(0) = y_0$ have a unique solution for all $t > 0$.

3. Consider the linear differential equation

$$x' = A(t)x, \quad A(t) = S(t)^{-1}BS(t) \tag{4}$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, \quad S(t) = \begin{pmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{pmatrix} \tag{5}$$

(a) Show that, for any t, all eigenvalues of $A(t)$ have a negative real part.

(b) Show, that for a suitable choice of a, the differential equation (4) has solutions $x(t)$ which satisfy $\lim_{t\to\infty} \|x(t)\| = \infty$.

Hint: Consider the transformation $y(t) = S(t)x(t)$.

4. (a) Let A and $B(t)$ be $n \times n$ matrices where A does not depend on t and $B(t)$ is a continuous function of t for $t \in [0, \infty)$. Show that $x(t)$ is a solution of the Cauchy problem

$$x' = (A + B(t))\, x\,, \quad x(0) = x_0\,, \tag{6}$$

if and only if $x(t)$ is a solution of the integral equation

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} B(s) x(s)\, ds\,. \tag{7}$$

(b) Assume that the eigenvalues λ_i of A have negative real part and that

$$\int_0^\infty \|B(s)\|\, ds < \infty\,. \tag{8}$$

Show that all the solutions of (6) remains bounded, i.e. there exists a constant $K < \infty$

$$\sup_{0 \le t < \infty} \|x(t)\| \le K \tag{9}$$

Hint: Use (a) and Gronwall Lemma (in the version of Homework #2).

5. Let $A = \{(t, x) \in \mathbf{R} \times \mathbf{R}\,;\, 0 \le t \le a\,, |x| \le b\}$ and let $f : A \to \mathbf{R}$ be a continuous function with $M = \sup_{(t,x) \in A} \|f(t, x)\|$. Consider the Cauchy problem

$$x' = f(t, x)\,, \quad x(0) = 0 \tag{10}$$

(a) Assume that

i. $f(t, x) \ge 0$.

ii. f is an increasing function of x, i.e. $f(t, x_1) \le f(t, x_2)$ if $x_1 \le x_2$.

Show that the Picard-Lindelöf iteration $x_n(t)$ converges for $t \in [0, \alpha]$ where $\alpha = \min(a, b/M)$.

(b) Show that the limit $x(t) = \lim_{n \to \infty} x_n(t)$ in (b) is a solution of the Cauchy problem. Is it the unique solution?