Deterministic Equations for Stochastic Spatial Evolutionary Games[1]

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Abstract

Spatial evolutionary games model individuals who are distributed in a spatial domain and update their strategies upon playing a normal form game with their neighbors. We derive integro-differential equations as deterministic approximations of the microscopic updating stochastic processes. This generalizes the known mean-field ordinary differential equations and provide a powerful tool to investigate the spatial effects in populations evolution. The deterministic equations allow to identify many interesting features of the evolution of strategy profiles in a population, such as standing and traveling waves, and pattern formation, especially in replicator-type evolutions.

Keywords: Evolutionary games, mean-field interactions, deterministic approximation, Kac potentials, pattern formation, traveling wave solutions.

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1 Introduction

Many macroeconomic phenomena occur as the aggregate results of the actions and interactions of many and unrelated agents. Interactions among agents are inherently local because they are separated by spatial location, language, culture, and etc. As an example the occurrence of depression or inflation might depend on the decentralized decisions of many agents to save or consume based on the local economic conditions. Another example of a social phenomenon where spatial considerations matter is the decision on where to live, which frequently depends on the neighbors living there and this, in turn, can induce the spatial patterns of segregation in residential areas (Schelling, 1971). Our main goal in this paper is to develop tools to understand the implications of spatial interactions in evolutionary games.

We consider a class of spatial stochastic processes in which agents are located on vertices of a graph and update their strategy after observing the strategy of a distinguished set of their neighbors. Such stochastic models have been studied in evolutionary game theory by, among others, Kandori, Mailath, and Rob (1993), Ellison (1993), Blume (1993), Blume (1995), Young (1998). In *mean-field* models without any consideration of geometrical proximity (e.g., see Kandori, Mailath, and Rob (1993)), every player is considered a neighbor and is given the same weight in evaluating payoffs and the model can be described using only the aggregate quantities such as the proportion of players with a given strategy.

The popular use of mean-field stochastic models in evolutionary game theory is mainly due to the simple structure of the stochastic processes. While these models demonstrate effectively how the combination of the myopic strategy revision and the random experiments (noise or mutation) of individuals can lead to a selection of an equilibrium from among multiple Nash equilibria, they assume that everyone in a population interacts *uniformly* with the entire population of players, neglecting the local structure of interaction. Because of this, these models fail to address the importance of the locality of interactions in forming the globally observed phenomenon. The range of interaction itself is a critical factor in determining the speed of convergence to equilibrium and the time that the society locked in a "bad" state such as *Pareto-inefficient* state. For instance, Ellison (1993) shows that the speed of convergence in the system with the uniform interaction in a large population is very slow and so the question of equilibrium selection in the mean field models may not yield a useful insight.

By contrast, in models with local interactions players interact with a fixed finite set of neighbors (e.g., see Blume (1993), Blume (1995), Young (1998, Chapter 5) and see Szabo and Fath (2007) for a comprehensive survey of spatial evolutionary games). Some important questions about equilibrium selection and speed of convergence to equilibrium have been addressed for special models using tools such as stochastic stability and Gibbs measures (Young (1993),Ellison (1993),Blume (1993), Young (1998), See also Freidlin and Wentzell (1984)). These results are however limited to either potential games and logit dynamics or coordinations games and perturbed best response dynamics; many other important games addressing economic problems – such as the Rock-paper-scissors games – are neither potential games nor coordination games. Also another important behavioral rule like imitative updating, which may arise from the limited information or poor computational ability of agents, might exhibit very different phenomena and should be compared to the (perturbed) best response rule (See Levine and Pesendorfer (2007); Bergin and Bernhardt (2009) for the importance of imitative behaviors). However, these questions using stochastic and probabilistic methods lead to very difficult problems (e.g. see Durrett (1999)).

We concentrate here on an intermediate case, *local mean-field models* where a given player interacts with a substantial proportion of the population, but where spatial variations in the strength of interactions are nonetheless allowed. By considering the intermediate model, we are able to rigorously derive deterministic equations which approximate the original stochastic processes. The distinctive advantages of our approaches lie in that under our assumptions, the questions of pattern formations, equilibrium selection, and the speed of convergence at the level of complex stochastic processes can be directly translated into those at the level of the differential equations - a level which is more tractable and also incorporates some underlying stochastic details. An attractive feature of the derived differential equations is that it can provide tractable and systematic tools which can examine the relationship between the local heterogeneity and the globally observed phenomena; for example, one can easily study how a given set of preferences of individuals at a certain neighborhood can lead to segregation patterns in the residential areas.

In the current literature of evolutionary game theory, (e.g. Hofbauer and Sigmund (2003); Weibull (1995); Sandholm (2010)), the time evolution of the proportion of agents with strategy i at time t, $f_t(i)$, is specified by an ordinary differential equation of the type

$$\frac{\partial}{\partial t} f_t(i) = \sum_{k\in S} \mathbf{c}^M(k,i,f_t) f_t(k) - f_t(i) \sum_{k\in S} \mathbf{c}^M(i,k,f_t) \quad \text{for } i \in S. \tag{1}$$

Examples of such equations include the well-known replicator dynamics, the von-Neumann Nash dynamics, and the logit dynamics. The first term in equation (1) describes the rate at which agents switch to strategy i from some strategy other than i, while the second term describes the rate at which agents switch to some other strategy from strategy i. For this reason equation (1) is also called an *input-output* equation.

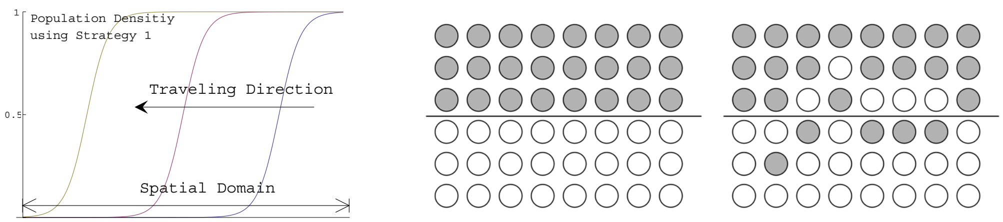
It is well-known (Kurtz, 1970; Benaim and Weibull, 2003; Darling and Norris, 2008) that the solution of equation (1) $f_t(i)$ approximates, on finite time intervals, a suitable mean-field stochastic process, in the limit of infinite population, and $f_t(i)$ is the average, over the entire spatial domain, of the proportion of player with strategy i. In our local mean-field model where spatial structures survive, we will describe instead the state of the system by a local density function $f_t(u,i)$. Here u belongs to the spatial domain $\mathbb{A} \subset \mathbb{R}^n$ where agents are continuously distributed and $f_t(u,i)$ represents the proportion of agents with strategy i at the spatial location u. Our main result is that local mean-field stochastic processes are approximated, on finite time intervals and in the limit of infinite population, by equations of the type

$$\frac{\partial}{\partial t} f_t(u,i) = \sum_{k\in S} \mathbf{c}(u,k,i,f_t) f_t(u,k) - f_t(u,i) \sum_{k\in S} \mathbf{c}(u,i,k,f_t) \quad \text{for } i \in S\,, \tag{2}$$

which provides a natural generalization of equation (1). For example, the term $\mathbf{c}(u,k,i,f)$ describes the rate at which agents at spatial location u switch from strategy k to i. This rate depends on the strategies of agents at other spatial locations and typically, $\mathbf{c}(u,k,i,f)$ will have the functional form

$$\mathbf{c}(u,k,i,f) = G(k,i,\mathcal{J} * f(u,i)), \text{ where } \mathcal{J} * f(u,i) := \int \mathcal{J}(u-v) f(v,i) dv\,.$$

Here $\mathcal{J} * f$ is the convolution product of $\mathcal{J}$ with f and $\mathcal{J}(u)$ is a non-negative probability kernel which describes the interaction strength between players whose relative distance is u. When $\mathcal{J}$ is constant equation (2) reduces to equation (1). Note that the rate of increases in f_t at u depends on $f_t(v,i)$ for all v in the spatial domain



(a) Traveling front

(b) Imitative update vs perturbed best response

Figure 1: **Traveling front and strategy choices** Panel (a) illustrates how a traveling front solution can describe the propagation of a strategy to the whole spatial domain for a two strategy game. Panel (b) shows the configurations of strategy choices of individuals at each site (white circle: strategy 1, black circle: strategy 2) In the replicator dynamics (the left: an example of imitative updating rules) the interface between the choices of strategies is sharp; in the logit dynamics (the right: an example of perturbed best response rules) the interface is not sharp

$\mathbb{A}$ and that equation (2) is non-local; so, the equation is called an *integro-differential equation* (IDE).

We will obtain this equation in a suitable *spatial* scaling where agents are continuously distributed in a spatial domain $\mathbb{A}$ and is often called a *mesoscopic scaling* in the physics literature. For this reason equations of the form (2) are often called mesoscopic equations but are sometimes also referred to as local mean-field equations. Mesoscopic limits similar to ours have been derived in several models in statistical mechanics by Comets (1987), DeMasi, Orlandi, Presutti, and Triolo (1994), Katsoulakis, Plechac, and Tsagkarogiannis (2005), and others. We generalize these results to the spatial stochastic processes arising in evolutionary game theory. Other scaling limits, such as hydrodynamic limits, where space and time are scaled simultaneously, giving rise to evolving fronts and interfaces, e.g. Katsoulakis and Souganidis (1997), are also potentially relevant to game theory but will not be discussed here.

Specifically, using this approximated equation we study the effect of a given initial condition and behavioral update rule on pattern formations and observe that when individuals behave by imitation of their close neighbors, the segregation of choices of strategy may develop and persist (see Figure 2). But in a society where (perturbed) best response rule is the dominating behavior, we observe that the system, instead, converges everywhere to a "rational" equilibrium exponentially fast (see Figure 1(b) for comparison of interfaces). The traveling front solutions shows a propagation of a local strategy profile into a global domain of the space and plays an important role in examining the selection of equilibrium from among multiple Nash equilibria and

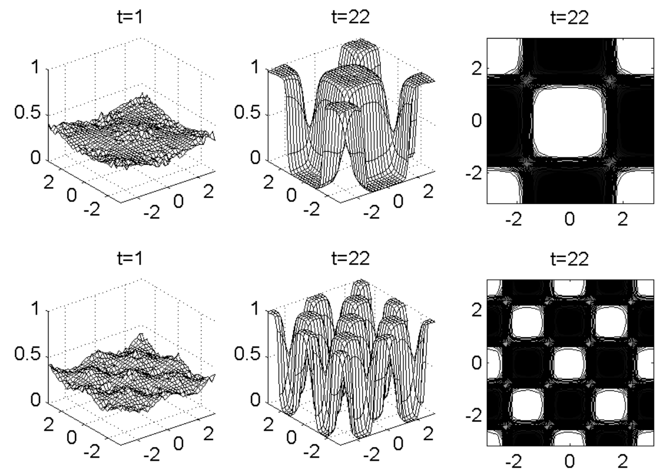


Figure 2: **Pattern Formation in the replicator dynamics** We use two-player coordination game with $a_{11} = 2/3, a_{22} = 1/3, a_{12} = a_{21} = 0$. The left and middle panels show the time evolutions of population densities of strategy 1 in the spatial domain $T^d = [-\pi, \pi]^2$. The number of nodes is 64 and the time step is 0.0175. The initial conditions are $1/3 + \text{rand} \cos(x)\cos(y)$ (upper panel) and $1/3 + \text{rand} \cos(2x)\cos(2y)$ (lower panel), where rand denotes a realization of the uniform random variable $[0, 1]$ at each node. For the interaction kernel, $\mathcal{J}(r) = \exp\left(-bx^2\right) / \int \exp(-bx^2)dx, b = 15$. The right panels show the contours of the densities at $t = 22$.

the speed of convergence to equilibrium (see Figure 1 (a), Hofbauer, Hutson, and Vickers (1997)). In our models, in imitative dynamics, we observe an extremely slow transition to the better equilibrium in contrast to standard beliefs on equilibrium selection.

Current approaches to address the pattern formation and the existence of traveling front solutions in evolutionary games have traditionally employed reaction-diffusion partial differential equations; such models are typically obtained by adding a constant coefficient diffusion term to the mean-field equations, which in turn models fast but homogeneous spatial diffusion of agents(Hutson and Vickers, 1992; Vickers, Hutson, and Budd, 1993; Hofbauer, Hutson, and Vickers, 1997; Hofbauer, 1997; Durrett, 1999):

$$\frac{\partial}{\partial t} f_t(u,i) = \sum_{k \in S} \mathbf{c}^M(k,i,f) f_t(u,k) - f_t(u,i) \sum_{k \in S} \mathbf{c}^M(i,k,f) + \epsilon \Delta f. \tag{3}$$

In contrast, in our scaling-limit approach, the spatial effects are introduced at the microscopic level and lead to diffusive effects which differ markedly from equation to equation and are, in general, density dependent. This introduces a number of new interesting spatial structures which are absent in reaction-diffusion equations.

The paper is organized as follows. In Section 2 we introduce the stochastic process and the scaling limits, and present our main results (Section 2.3). A heuristic derivation of the equation is given in Section 2.4 and the relation with ODE such as (1) is elucidated in Section 2.5. In Section 3 we analyze equilibrium selection and pattern formation in two player coordination games using a combination of linear analysis and numerical simulations. In the appendix we prove our main results.

2 Spatial Evolutionary Games

2.1 Strategy Revision Processes

In models of spatial evolutionary games, agents are located at the sites of a graph and play a normal form game with their neighbors. The graph Λ is assumed here to be a subset of the integer lattice $\mathbb{Z}^d$. We consider a single population playing a normal form game, but the generalization to multiple population games is straightforward. A normal form game is specified by a finite set of strategies S and a payoff function $a(i,j)$ which gives the payoff for a player using strategy $i \in S$ against strategy $j \in S$. Here we view a strategy as a type of behavior and so terms, "strategy" and "type", are used interchangeably (Maynard Smith, 1982; Hofbauer and Sigmund, 1998).

The strategy of the agent at site $x \in \Lambda$ is $\sigma_\Lambda(x) \in S$, and we denote by $\sigma_\Lambda = \{\sigma_\Lambda(x) : x \in \Lambda\}$ the configuration of strategies for every agent in the population. With these notations, the state space, i.e., the set of all possible configurations, is S^Λ. The subscript of $\sigma = \sigma_\Lambda$ will be suppressed, whenever no confusion arises. As in Young (1998, chapter 6), we assign positive weights $\mathcal{W}(x-y)$ to any two sites x and y to capture the importance or intensity of the interaction among neighbors. Note that we assume that these weights depend only on the relative location $x-y$ between the players (i.e., we assume translation invariance). It is convenient to assume that total weight that site x attaches to all its neighbors is normalized to 1, i.e.,

$$\sum_{y \in \Lambda} \mathcal{W}(x-y) \approx 1. \tag{4}$$

We say y is a neighbor of x whenever $\mathcal{W}(x-y) > 0$. An individual agent, at site x

with strategy i given a configuration σ, receives an average payoff

$$u(x,\sigma,i) := \sum_{y\in\Lambda} \mathcal{W}(x-y)a(i,\sigma(y)). \tag{5}$$

If we think of $\mathcal{W}$ as the probability with which an agent samples his neighbors, then $u(x,\sigma,i)$ is the expected payoff for an agent at x choosing strategy i if the population strategy profile is σ. Or we can think that an agent receives an instantaneous payoff flow from his interactions with other neighbors (Blume, 1993; Young, 1998; Young and Burke, 2001).

In the special case where $\mathcal{W}(x-y)$ is constant, the interaction is uniform and there is no spatial structure and if there are a total of n^d agents in the population, then $\mathcal{W}(x-y) \approx \frac{1}{n^d}$ because of (4). On the other hand, when $\mathcal{W}(x-y) = \frac{1}{2d}$ if $\|x-y\| = 1$ and 0 otherwise, interactions only arise between nearest sites (Blume, 1995; Szabo and Fath, 2007).

In this paper we concentrate on long range interactions where each agent interacts with as many other agents as in the mean-field case, but the interaction is not uniform. This limit is known as "local mean field model" (Comets, 1987) or "Kac potential" (Lebowitz and Penrose, 1966; DeMasi, Orlandi, Presutti, and Triolo, 1994; Presutti, 2009). More specifically, let $\mathcal{J}(x)$ be a non-negative, compactly supported, and integrable function such that $\int \mathcal{J}(x)dx = 1$. We assume that $\mathcal{W}$ has the form:

$$\mathcal{W}^\gamma(x-y) = \gamma^d \mathcal{J}(\gamma(x-y)), \tag{6}$$

and we will take the limit $\Lambda \nearrow \mathbb{Z}^d$ and $\gamma \to 0$ in such a way that $\gamma^{-d} \approx |\Lambda| \approx n^d$. Here n^d is the size of the population and $|\ |$ denotes the cardinality. Hence the factor γ^d is chosen in such a way that $\sum \mathcal{W}^\gamma(x-y) \approx \int \mathcal{J}(x)dx = 1$, so $\mathcal{W}^\gamma(x-y)$ indeed represents the intensity of interactions. Note that in (6) the interaction vanishes when $\|x-y\| \geq R\gamma^{-1}$ if $\mathcal{J}$ is supported on the ball of radius R. So as $\gamma \to 0$, an agent interacts very weakly but with a growing number of neighbors in the population. Frequently, in examples and simulations, we consider localized Gaussian-like kernels $\mathcal{J}(x) \propto \exp(-b\|x\|^2)$ for some $b > 0$.

The time evolution of the system is given by a continuous time Markov process $\{\sigma_t\}$ with state space S^Λ, in which each agent receives, independently of all the other agents, a strategy revision opportunity in response to his own exponential "alarm clock" with rate 1, and then updates his strategy according to a rate $c(x,\sigma,k)$ − the rate with which agent x switches to strategy k when the configuration is σ. This process is then characterized by a generator

$$(Lg)(\sigma) = \sum_{x\in\Lambda}\sum_{k\in S} c(x,\sigma,k)\left(g(\sigma^{x,k}) - g(\sigma)\right) \tag{7}$$

where g is a bounded function on S^Λ and

$$\sigma^{x,k}(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ k & \text{if } y = x \end{cases}$$

represents a configuration where the agent at site x switches from his current strategy $\sigma(x)$ to a new strategy k.

If $c(x,\sigma,k) > 0$ for all x, σ and k, the stochastic process can introduce any new strategy, even if it is not currently used in the population. We call this case *innovative* following Szabo and Fath (2007). If $c(x,\sigma,k) = 0$ for some x, σ, and k and hence a strategy which is not present in the population does not appear under the dynamics, we call the dynamics *non-innovative.* Furthermore if, upon switching, agents only consider the payoff of the new strategy we call the dynamics *targeting.* In contrast, when agents' decision depends on the payoff difference between the current strategy and the new strategy we call the dynamics *comparing.*

Precise technical assumptions for the strategy revision rates will be discussed later (Conditions **C1**−**C3** in Section 2.3); here, we give only a number of concrete examples commonly used in applications, and to which our results will apply. Several more examples of rates are discussed in the Appendix and the assumptions on our rates are satisfied by virtually all dynamics commonly used in evolutionary game theory (see Sandholm, 2010, for a more comprehensive discussion of rates and more examples). To define the rate we introduce

$$w(x,\sigma,k) := \sum_{y\in\Lambda} \mathcal{W}(x-y)\delta(\sigma(y),k)$$

where $\delta(i,j) = 1$ if $i = j$ and 0 otherwise; $w(x,\sigma,k)$ can be interpreted as the probability for an agent at site x to find a neighbor with strategy k, provided the neighbors are sampled with the probability distribution $\mathcal{W}(x-y)$. Let also F denote a non-negative function.

Examples of Rates

• **Comparing and Innovative**: The rate is $c(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$ and is comparing and innovative provided $F > 0$. When

$$c(x,\sigma,k) = \min\left\{1, \exp\left(u(x,\sigma,k) - u(x,\sigma,\sigma(x))\right)\right\},$$

the rate corresponds to a generalization of the well-known Metropolis algorithm. In particular, when the normal form game is a potential game, the corresponding Markov chain satisfies detailed balance and its invariant distribution can be explicitly expressed as a Gibbs distribution (Szabo and Fath, 2007).

• **Targeting and Innovative**: This case arises if $c(x,\sigma,k) = F(u(x,\sigma,k))$ and $F > 0$. If

$$c(x,\sigma,k) = \frac{\exp(u(x,\sigma,k))}{\sum_l \exp(u(x,\sigma,l))} \tag{8}$$

the rate is called "logit choice rule" in the game theory literature, and it is a generalization of the "Gibbs sampler" in statistics and of the "Glauber dynamics "of physics. The Markov process, in this case too, satisfies the detailed balance for potential games and has the same Gibbs invariant distribution as Metropolis dynamics.

• **Comparing and Non-innovative:** The rate has the form

$$c(x,\sigma,k) = w(x,\sigma,k)\left[F(u(x,\sigma,k) - u(x,\sigma,\sigma(x))\right]. \tag{9}$$

This specifies an imitation process: the first factor $w(x,\sigma,k)$ is the probability for an agent at x to choose an agent with strategy k and the second factor $F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$ gives the rate at which the new strategy k is adopted (Weibull, 1995; Benaim and Weibull, 2003; Hofbauer and Sigmund, 2003). The standard example is

$$c(x,\sigma,k) = w(x,\sigma,k)\left[u(x,\sigma,k) - u(x,\sigma,\sigma(x))\right]_+ \tag{10}$$

where $[s]_+ = \max\{s, 0\}$. The rate (10), in the mean-field case, gives rise to the famous replicator ODEs as the deterministic approximation. More generally if F in (9) satisfies

$$F(s) - F(-s) = s, \tag{11}$$

then the corresponding mean field ODE is the replicator dynamics. Note that $[s]_+$ satisfies condition (11). In the paper we frequently use

$$F_\kappa(s) := \frac{1}{\kappa}\log(\exp(\kappa s)+1) \tag{12}$$

and it is easily seen that the function (12) satisfies (11) and converges uniformly to $[s]_+$ as $\kappa \to \infty$; hence (12) can serve as a smooth regularization of (10), and we name a replicator equation using (12) by a regularized replicator equation.

2.2 Mesoscopic scaling and long-range interactions

We will consider the limit $\gamma \to 0$ in equation (6); i.e., the interaction range $\frac{1}{\gamma}$ becomes infinite and the agent at x interacts with a growing number of agents. In order to obtain a limiting equation, we rescale space and take a continuum limit. Let $\mathbb{A} \subset \mathbb{R}^d$ (mesoscopic domain) and $\mathbb{A}^\gamma := \gamma^{-1}\mathbb{A} \cap \mathbb{Z}^d$ (microscopic domain). If $\mathbb{A}$ is a smooth region in $\mathbb{R}^d$, then $\mathbb{A}^\gamma$ contains $\gamma^{-d}|\mathbb{A}|$ lattice sites and as $\gamma \to 0$, $\gamma\mathbb{A}^\gamma$ approximates $\mathbb{A}$.

At the mesoscopic scale the state of the system is described by the *strategy profile* function $f_t(u,i)$ – the density of agents with strategy i at u. The bridge between microscopic and mesoscopic scale is given by the *empirical measure* $\pi^\gamma(\sigma; du, di)$ defined as follows. For $(v,j) \in \mathbb{A} \times S$, let $\delta_{(v,j)}(du,di)$ denote the Dirac delta measure at (v,j).

Definition 1 (Empirical measure) *The empirical measure $\pi^\gamma : S^{\mathbb{A}^\gamma} \to \mathcal{P}(\mathbb{A} \times S)$ is the map given by*

$$\sigma \mapsto \pi^\gamma(\sigma; du, di) := \frac{1}{|\mathbb{A}^\gamma|}\sum_{x \in \mathbb{A}^\gamma} \delta_{(\gamma x, \sigma(x))}(du di) \tag{13}$$

where $\mathcal{P}(\mathbb{A} \times S)$ denotes the set of all probability measures on $\mathbb{A} \times S$.

Our main result is to show that, under suitable conditions,

$$\pi^\gamma(\sigma_t; du, di) \to f_t(u,i)du \qquad \text{in probability} \tag{14}$$

and $f_t(u,i)$ satisfies an integro-differential equation. Since σ_t is the state of the microscopic system at time t, $\pi^\gamma(\sigma_t; du, di)$ is a random measure, while $f_t(u,i)$ is a solution of a deterministic equation. So (14) is in a sense a form of a time-dependent law of large numbers. For this result to hold we need to assume that the initial distribution for σ_0 is sufficiently regular. For our purpose it will be enough to assume that the distribution of σ_0 is given by a *product measure with a slowly varying parameter.*

Definition 2 (Product measures with a slowly varying parameter) *The collection of measure $\{\mu^\gamma\}$ is called a family of product measures with a slowly varying parameter if $\mu_\gamma := \bigotimes_{x\in\mathbb{A}^\gamma} \rho_x$ on $S^{\mathbb{A}^\gamma}$ and there exists a profile $f(u,i)$ such that*

$$\rho_x(\{i\}) = f(\gamma x, i)$$

More general initial distributions can also be accommodated (See Kipnis and Landim, 1999), provided they can be associated to a mesoscopic strategy profile. Furthermore, below we will consider two types of boundary conditions:

(a) Periodic Boundary Conditions. Let $\mathbb{A} = [0,1]^d$. We assume that $\mathbb{A}^\gamma = \gamma^{-1}\mathbb{A} \cap \mathbb{Z}^d = [0, \frac{1}{\gamma}]^d \cap \mathbb{Z}^d$, and then extend the profile $f_t(u,i)$ and the configuration $\sigma_{\mathbb{A}^\gamma}$ periodically on $\mathbb{R}^d$ and $\mathbb{Z}^d$. Equivalently we can identify $\mathbb{A}$ with the torus $\mathbf{T}^d$ and similarly $\mathbb{A}^\gamma$ with the discrete torus $\mathbf{T}^{d,\gamma}$.

(b) Fixed Boundary Conditions. In applications it is also useful to consider the case where the configurations in some regions do not change with time. Let $\Lambda \subset \Gamma \subset \mathbb{R}^d$ be a region. We think of $\partial\Lambda := \Gamma\backslash\Lambda$ as the "boundary region" where agents do not revise their strategies. Since we consider compactly supported $\mathcal{J}$ we can take, for suitable $r > 0$

$$\Gamma := \bigcup_{u\in\Lambda} B(u,r),$$

where B denotes a ball centered at u with radius r. We define microscopic spaces, $\Lambda^\gamma := \gamma^{-1}\Lambda \cap \mathbb{Z}^d$ and $\Gamma^\gamma := \gamma^{-1}\Gamma \cap \mathbb{Z}^d$.

2.3 Main results

Let us consider first the case with periodic boundary conditions. Our assumptions on the interactions weights $\mathcal{W}^\gamma(x-y)$ are

(F) $\mathcal{W}^\gamma(x-y) = \gamma^d \mathcal{J}(\gamma(x-y))$ where $\mathcal{J}$ is nonnegative, continuous with compact support, and normalized, $\int \mathcal{J}(x)dx = 1$.

The mesoscopic strategy profiles are described by functions $f(u,i) \in \mathcal{M}(\mathbf{T}^d \times S)$ where

$$\mathcal{M}(\mathbf{T}^d \times S) := \left\{ f \in L^\infty(\mathbf{T}^d \times S) : 0 \le f(u,i) \le 1,\ \sum_i f(u,i) = 1 \text{ for all } u \in \mathbf{T}^d \right\} .$$

Let $\{\sigma_t^\gamma\}_{t \ge 0}$ be the stochastic process with generator L^γ given by

$$(L^\gamma g)(\sigma) = \sum_{x \in \mathbf{T}^{d,\gamma}} \sum_{k \in S} c^\gamma(x,\sigma,k) \left(g(\sigma^{x,k}) - g(\sigma) \right) \tag{15}$$

for $g \in L^\infty(S^{\mathbf{T}^{d,\gamma}})$. Our assumptions on the strategy revision rate $c^\gamma(x,\sigma,k)$ are that there exists a real-valued function

$$\mathbf{c}(u,i,k,\pi),\ u \in \mathbf{T}^d,\ i,k \in S,\ \pi \in \mathcal{P}(\mathbf{T}^d \times S)$$

such that

(C1) $\mathbf{c}(u,i,k,\pi)$ satisfies

$$\lim_{\gamma \to 0} \sup_{x \in \mathbf{T}^{d,\gamma}, \sigma \in S^{\mathbf{T}^{d,\gamma}}, k \in S} |c^\gamma(x,\sigma,k) - \mathbf{c}(\gamma x, \sigma(x), k, \pi^\gamma(\sigma))| = 0 \,,$$

(C2) $\mathbf{c}(u,i,k,\pi)$ is uniformly bounded: i.e., there exists M such that

$$\sup_{u \in \mathbf{T}^d, i,k \in S, \pi \in \mathcal{P}(\mathbf{T}^d \times S)} |\mathbf{c}(u,i,k,\pi)| \le M \,,$$

(C3) $\mathbf{c}(u,i,k,f du di)$ satisfies a Lipschitz condition with respect to f : i.e., there exists L such that for all $f_1,\ f_2 \in \mathcal{M}(\mathbf{T}^d \times S)$

$$\sup_{u \in \mathbf{T}^d, i,k \in S} |\mathbf{c}(u,i,k,f_1 du di) - \mathbf{c}(u,i,k,f_2 du di)| \le L \left\| f_1 - f_2 \right\|_{L^1(\mathbf{T}^d \times S)} \,.$$

In the appendix we show that all the classes of rates given in the examples in Section 2.1 and several others satisfy conditions **C1**$-$**C3**. For example, if $c^\gamma(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$, with $u(x,\sigma,i) = \sum_y \mathcal{W}^\gamma(x-y) a(i,\sigma(y))$ and the weights

$\mathcal{W}^\gamma(x-y)$ satisfy condition $\mathbf{F}$, then

$$\mathbf{c}(u,i,k,f) = F\left(\sum_{l\in S} a(k,l)\mathcal{J} * f(u,l) - a(i,l)\mathcal{J} * f(u,l)\right) \tag{16}$$

satisfies condition $\mathbf{C1}-\mathbf{C3}$ (recall that $\mathcal{J} * f(u,l) := \int_{\mathbf{T}^d} \mathcal{J}(u-v)f(v,l)dv$ is the convolution of $\mathcal{J}$ with f). A slight modification of (16) yields corresponding expressions for each choice of $c^\gamma(x,\sigma,k)$ in Section 2.1 (see the appendix for the complete list of these rates). In Section 2.4 we will explain how to obtain the function $\mathbf{c}(u,i,k,\pi)$ from the rates.

The stochastic process σ_t^γ induces a measure-valued stochastic process $\pi_t^\gamma := \pi^\gamma(\sigma_t^\gamma, dudi)$ for the empirical given in equation (13). Theorem 1 shows that the stochastic process π_t^γ has a deterministic limit.

Theorem 1 (Long Range Interaction and Periodic Boundary Condition) *Suppose the revision rate satisfies* $\mathbf{C1}-\mathbf{C3}$. *Let* $f \in \mathcal{M}(\mathbf{T}^d \times S)$ *and assume that the initial distribution* $\{\mu^\gamma\}_\gamma$ *is a family of measures with a slowly varying parameter associated to the profile of* f. *Then for every* $T>0$

$$\lim_{\gamma\to 0} \pi_t^\gamma(du,di) = f_t(u,i)\, dudi \textit{ in probability}$$

uniformly for $t \in [0,T]$ *and* f_t *satisfies the following differential equation: for* $u \in \mathbf{T}^d, i \in S$

$$\begin{aligned} \frac{\partial}{\partial t} f_t(u,i) &= \sum_{k\in S} \mathbf{c}(u,k,i,f) f_t(u,k) - f_t(u,i)\sum_{k\in S}\mathbf{c}(u,i,k,f) \\ f_0(u,i) &= f(u,i) \end{aligned} \tag{17}$$

Next let us consider fixed boundary conditions as in Section 2.2. In this case, the stochastic process, $\{\sigma_t^\gamma\}_{t\geq 0}$, is specified by the generator L^γ

$$(L^\gamma g)(\sigma_{\Gamma^\gamma}) = \sum_{x\in\Lambda^\gamma}\sum_{k\in S} c^\gamma(x,\sigma_{\Gamma^\gamma},k)(g(\sigma_{\Gamma^\gamma}^{x,k}) - g(\sigma_{\Gamma^\gamma})) \tag{18}$$

for $g \in L^\infty(S^{\Gamma^\gamma})$. Note that the summation in terms of x in (18) is taken over Λ^γ, which represents the fact that only individuals in Λ^γ revise their strategies, whereas the rate depends on the configuration in entire Γ^γ. For a given $f \in \mathcal{M}$, we define its restriction on Λ, $f_\Lambda(u,i) : f_\Lambda(u,i) = f(u,i)$ if $u \in \Lambda$ and $f_\Lambda(u,i) = 0$ if $u \in \Lambda^C$.

Theorem 2 (Long Range Interaction and Fixed Boundary Condition) *Suppose the revision rate satisfies* **C1** $-$ **C3**. *Let* $f \in \mathcal{M}(\Gamma^d \times S)$ *and assume that the initial distribution* $\{\mu^\gamma\}_\gamma$ *is a family of measures with a slowly varying parameter associated to the profile of* f. *Then for every* $T > 0$

$$\lim_{\gamma \to 0} \pi_t^\gamma(du, di) = \frac{1}{|\Gamma|} f_t(u, i)\, dudi \ \textit{in probability}$$

uniformly for $t \in [0, T]$ *and* $f_t = f_{\Lambda,t} + f_{\partial\Lambda,t}$ *satisfies the following differential equation: for* $u \in \Gamma, i \in S$

$$\begin{aligned} \frac{\partial}{\partial t} f_{\Lambda,t}(u,i) &= \sum_{k \in S} \mathbf{c}(u,k,i,f) f_{\Lambda,t}(u,k) - f_{\Lambda,t}(u,i) \sum_{k \in S} \mathbf{c}(u,i,k,f) \qquad (19) \\ f_0(u,i) &= f(u,i) \end{aligned}$$

Note that $\mathbf{c}(u,k,i,f) = c(u,k,i,f_\Lambda + f_{\partial\Lambda})$ is given by the similar formula to (16) with $\mathcal{J} * f(u) = \int_\Gamma \mathcal{J}(u-v) f(v) dv$ for $u \in \Lambda$; so the rates depend on $f_{\partial\Lambda}$ as well as f_Λ.

2.4 Heuristic derivation of the differential equations

In this section we justify, heuristically, the IDEs obtained in Theorems 1 and 2. For simplicity we assume periodic boundary conditions but the other case is similar. The differential equations (17) and (19) are examples of input-output equations. In particular, by summing over the strategy set, it is easy to see that $\sum_{i \in S} f_t(u,i)$ is independent of t and therefore if $f_0 \in \mathcal{M}$, then $f_t \in \mathcal{M}$ for all t. Also the space $\mathcal{M}$ can be thought of as a product over the space of the standard strategy simplex Δ of game theory, i.e., $\mathcal{M} = \prod_{u \in \mathbf{T}^d} \Delta$. As shown in evolutionary game theory textbooks (Weibull, 1995; Sandholm, 2010) one can derive heuristically the ODEs from corresponding stochastic processes. The main assumption used there is that the rates depend only on the average proportion of players with a given strategy. In this section we provide, for the convenience of a reader, a similar heuristic derivation from microscopic processes in the case of the spatial IDE (17); we replace global average by spatially localized averages as expressed in the limit of the empirical measure (13).

For microscopic sites x and y, let us denote by $u = \gamma x$ and $v = \gamma y$ the corresponding spatial positions at the mesoscopic level. For the sake of exposition let us suppose that $c^{\gamma}(x,\sigma,k)$ is given by

$$c^{\gamma}(x,\sigma,k)=F(u(x,\sigma,k)-u(x,\sigma,\sigma(x))\,.$$

For any continuous function g on $\mathbf{T}^d\times S$, by the definition of the empirical measure (13) we have the identity

$$\frac{1}{|\mathbf{T}^{d,\gamma}|}\sum_{x\in\mathbf{T}^{d,\gamma}}g(\gamma x,\sigma(x))=\int_{\mathbf{T}^d\times S}g(u,i)\pi^{\gamma}(\sigma,du,di)\,.$$

Since $\left|\mathbf{T}^{d,\gamma}\right|\approx\gamma^{-d}$ and if we *assume* that $\pi^{\gamma}\left(\sigma,du,di\right)\to f(u,i)dudi$, we obtain

$$\lim_{\gamma\to 0}\sum_{x\in\mathbf{T}^{d,\gamma}}\gamma^d g(\gamma x,\sigma\left(x\right))=\int_{\mathbf{T}^d\times S}g(u,i)f(u,i)dudi. \tag{20}$$

Using (20), we find

$$\lim_{\gamma\to 0}\sum_{x\in\mathbf{T}^{d,\gamma}}\gamma^d\mathcal{J}\;\left(\gamma(x-y)\right)a(k,\sigma(y))=\int_{\mathbf{T}^d\times S}a(k,l)\mathcal{J}(u-v)f(v,l)dvdl=\sum_{l\in S}a(k,l)\mathcal{J}*f(u,l).$$

Therefore if $\sigma(x)=i$ we then obtain

$$\begin{aligned}
c^{\gamma}(x,\sigma,k) &= F(u(x,\sigma,k)-u(x,\sigma,\sigma(x))\\
&= F\left(\sum_{y\in\mathbf{T}^{d,\gamma}}\gamma^d\mathcal{J}\;\left(\gamma x-\gamma y\right)a(k,\sigma(y))-\sum_{y\in\mathbf{T}^{d,\gamma}}\gamma^d\mathcal{J}\;\left(\gamma x-\gamma y\right)a(\sigma(x),\sigma(y))\right)\\
&\underset{\gamma\to 0}{\longrightarrow} F\left(\sum_{l\in S}a(k,l)\mathcal{J}*f(u,l)-\sum_{l\in S}a(i,l)\mathcal{J}*f(u,l)\right)=\mathbf{c}(u,i,k,f),
\end{aligned}$$

and this gives equation (16). After having identified rates, we can now explain how to derive the IDE (17). We write

$$\left\langle\pi^{\gamma},g\right\rangle(\sigma):=\int_{\mathbf{T}^d\times S}g(u,i)\pi^{\gamma}(\sigma,dudi),\qquad\langle f,g\rangle:=\int_{\mathbf{T}^d\times S}g(u,i)f(u,i)dudi,$$

where we view $\left\langle\pi^{\gamma},g\right\rangle(\sigma)$ as a function of the configuration σ. The action of the generator on this function is

$$L_\gamma \left\langle \pi^\gamma, g \right\rangle (\sigma) = \sum_{k \in S} \int_{\mathbf{T}^d \times S} \mathbf{c}(u, i, k, \pi^\gamma(\sigma)) \left(g(u, k) - g(u, i) \right) \pi^\gamma(\sigma, dudi) \,.$$

¿From the martingale representation theorem for Markov processes (for example see Ethier and Kurz, 1986) there exists a martingale $M_t^{g,\gamma}$ such that

$$\langle \pi_t^\gamma, g \rangle = \langle \pi_0^\gamma, g \rangle + \int_0^t ds \sum_{k \in S} \int_{\mathbf{T}^d \times S} \mathbf{c}(u, i, k, \pi_s^\gamma) \left(g(u, k) - g(u, i) \right) \pi_s^\gamma(dudi) + M_t^{g,\gamma}. \tag{21}$$

As $\gamma \to 0$, one proves that $M_t^{g,\gamma} \to 0$. Thus if $\pi_t^\gamma(dudi) \to f(t, u, i) dudi$ as $\gamma \to 0$, equation (21) becomes

$$\langle f_t, g \rangle = \langle f_0, g \rangle + \int_0^t ds \sum_{k \in S} \int_{\mathbf{T}^d \times S} \mathbf{c}(u, i, k, f_s) \left(g(u, k) - g(u, i) \right) f_s(u, i) dudi$$

and upon differentiating with respect to time, we find

$$< \frac{\partial f_t}{\partial t}, g >= \sum_{k \in S} \int_{\mathbf{T}^d \times S} \mathbf{c}(u, i, k, f_t) \left(g(u, k) - g(u, i) \right) f_t(u, i) dudi \tag{22}$$

which is the weak formulation of the IDE (17) obtained by integrating over u and i.

The proof of Theorem 1 and Theorem 2, which we present in the appendix, is a variation on the proof given in Comets (1987), Kipnis and Landim (1999), and Katsoulakis, Plechac, and Tsagkarogiannis (2005). Unlike these papers, in the case of non-innovative dynamics studied here there is no detailed balance condition, however the mesoscopic limit of the type (14) can still be carried out in the Kac scaling (6). Using the martingale representation (21), we show that $\{\mathbf{Q}^\gamma\}_\gamma$, a sequence of probability laws of $\{\pi_t^\gamma\}_\gamma$, is relatively compact. We then show that all the limit points are concentrated on the weak solutions of (19) and on measures absolutely continuous with respect to Lebesgue measure. Finally we demonstrate that the weak solutions of (19) are unique so that we conclude the convergence of $\mathbf{Q}^\gamma$ to the Dirac measure concentrated on the solution of (19).

$$\begin{tikzcd} \text{Macroscopic Proc. } \eta_t \arrow[r, "approximate", "Corollary4"'] & \text{Mean ODE } \rho_t(i) \\ \text{Microscopic Proc. } \sigma_t \arrow[u, "aggregate", "Theorem3"'] \arrow[r, "approximate", "Theorem1{,}2"'] & \text{Mesoscopic Equation } f_t(u,i) \arrow[u, "aggregate"'] \end{tikzcd}$$

Figure 3: **The relationships between the microscopic process and the macroscopic process and between the stochastic process and the deterministic approximation.**

2.5 Spatially uniform interactions: Mean-field Dynamics

The goal of this section is to show that under the assumption of uniform interactions the spatially aggregated process is still a Markov chain (such process is called lumpable). Furthermore our IDEs reduce then to the usual ODEs of evolutionary game theory, as it should be. The relationships between the various processes and differential equations is illustrated in Figure 3. Let us take periodic boundary conditions and uniform interactions, i.e., $\mathcal{J} \equiv 1$ on $\mathbf{T}^d$. Let us further define the aggregate variables

$$\eta^\gamma(i) := \frac{1}{|\mathbf{T}^{d,\gamma}|} \sum_{x \in \mathbf{T}^{d,\gamma}} \delta(\sigma(x), i)$$

which counts the proportion of agents with strategy i in the entire domain $\mathbf{T}^{d,\gamma}$. Note that this is obtained, equivalently, by integrating the empirical measure $\pi^\gamma(\sigma, dudi)$ over the spatial domain $\mathbf{T}^d$. We observe that η^γ depends on γ only through the size of the domain n^d i.e., $n^d = \frac{1}{\gamma^d}$ and $n^d \to \infty$ as $\gamma \to 0$. Furthermore since $\mathcal{J} \equiv 1$, the payoff $u(x, \sigma, k)$ depends on σ only through the aggregated variable $\eta^n(i)$. Indeed, we have

$$u(x, \sigma, k) := \frac{1}{n^d} \sum_{y \in \mathbf{T}^{d,n}} \sum_{l \in S} \delta(\sigma(y), l) a(k, l) = \sum_{i \in S} a(k, i) \eta^n(i)$$

Thus for the strategy revision rates, if $\sigma(x) = j$ we define

$$c^M(j, k, \eta^n) := c^\gamma(x, \sigma, k)\,,$$

since the right hand side is independent of x and depends only on σ through the corresponding aggregate variable η^n. Therefore η^n_t itself is a Markov process as we will show in Theorem 3 below, and the state space for η_t^n is the discrete simplex

$$\Delta^n = \left\{ \{\eta(i)\}_{i\in S} \, ; \, \sum_{i\in S} \eta(i) = 1 \; , n^d \eta(i) \in \mathbb{N}_+ \right\}$$

To capture the transition induced by an agent's strategy switching, we write

$$\eta^{j,k}(i) = \begin{cases} \eta(i) & \text{if } i \neq k, j \\ \eta(i) - \frac{1}{n^d} & \text{if } i = j \\ \eta(i) + \frac{1}{n^d} & \text{if } i = k \end{cases}$$

Thus $\eta^{j,k}$ is the state obtained from η if one agent switches his strategy from j to k.

Theorem 3 *Suppose the interaction is uniform, then η^n is a Markov chain with state space Δ^n and generator*

$$L^{M,n} g\left(\eta\right) = \sum_{k\in S} \sum_{j\in S} n^d \eta^n(j) c(j,k,\eta) (g(\eta^{n,j,k}) - g(\eta^n)) \, . \tag{23}$$

The factor n^d in (23) comes from the fact that in a time interval of size 1, on average n^d strategy switches take place, and among those, $n^d \eta^n(j)$ are switches from agents with type j. Theorem 3 shows that the stochastic process with uniform interactions coincides with multi-type birth and death process in population dynamics (Blume, 1998; Benaim and Weibull, 2003). In addition, following Kurtz (1970), Benaim and Weibull (2003), and Darling and Norris (2008), or as a special case of our result (Corollary 4 below) we can obtain mean field ODEs. Furthermore, at the mesoscopic level, the IDEs reduce to the usual ODEs of evolutionary game theory as follows (See Figure 3). We note that when $\mathcal{J} \equiv 1$, we can define

$$\rho(i) := \int f(u,i) du = \mathcal{J} * f(i)$$

so $\mathbf{c}(u,k,i,f)$ is independent of u and this again allows to define

$$\mathbf{c}^M(k,i,\rho) := \mathbf{c}(u,k,i,f) \tag{24}$$

Thus, from the IDE (17) we obtain

$$\frac{d\rho_t(i)}{dt} = \sum_{k\in S} \mathbf{c}^M(k,i,\rho)\rho_t(k) - \rho_t(i)\sum_{k\in S} \mathbf{c}^M(i,k,\rho). \tag{25}$$

For example, in the case of the comparing and imitative rate we have

$$\mathbf{c}^M(k,i,\rho) = \rho(i)F\left(\sum_{l\in S} a(i,l)\rho(l) - \sum_{l\in S} a(k,l)\rho(l)\right).$$

If $F(s) = \frac{1}{\kappa}\log\left(\exp\left(\kappa s\right)+1\right)$, then $F(s) - F(-s) = s$ and (25) becomes the (imitative) replicator dynamics. Other well-known mean field ODEs, such as logit dynamics and Smith dynamics, are similarly derived by choosing appropriate F. Finally, as a consequence of Theorem 1 we have the following corollary which is the *continuous-time* version of Benaim and Weibull (2003)'s result. To state the result, we write $\|\eta^n\|_u := \sup_{i\in S}|\eta^n(i)|$.

Corollary 4 (Uniform Interaction; Benaim and Weibull, 2003) *Suppose that the interaction is uniform and that the strategy revision rate satisfies* **C1 − C3**. *Suppose there exists $\rho \in \Delta$ such that the initial condition η_0^n satisfies*

$$\lim_{n\to\infty} \eta_0^n = \rho \text{ in probability}$$

Then for every $T > 0$

$$\lim_{n\to\infty} \eta_t^n(i) \longrightarrow \rho_t(i) \text{ in probability} \tag{26}$$

uniformly for $t \in [0,T]$ and $\rho_t(i)$ satisfies the following differential equation: for $i \in S$

$$\frac{d\rho_t(i)}{dt} = \sum_{k\in S} \mathbf{c}^M(k,i,\rho)\rho_t(k) - \rho_t(i)\sum_{k\in S} \mathbf{c}^M(i,k,\rho) \tag{27}$$

$$\rho_0(i) = \rho(i) \tag{28}$$

where $\mathbf{c}^M$ is given by (24). Moreover, there exist C and ϵ_0 such that for all $\epsilon \le \epsilon_0$, *there exists n_0 such that for all $n \geq n_0$*

$$P\left\{\sup_{t\leq T}\|\eta_t^n-\rho_t\|_u\geq\epsilon\right\}\leq 2\,|S|\,e^{-\frac{n^d\epsilon^2}{TC}}\,. \tag{29}$$

Estimates such as (29) describe the validity regimes of the approximation by mean field models (27) in terms both of agent number n and the time window $[0,T]$

3 Equilibrium Selection and Pattern Formation

In this section we illustrate the usefulness and the versatility of the IDE's derived in Section 2.3 by using a combination of linear analysis and numerical simulations. We will consider the following equations (see the rates in Section 2.1 and at the beginning of the Appendix)

(a) Logit/Glauber dynamics: If the rate is given by (8) we obtain the IDE

$$\frac{\partial}{\partial t}f_t(u,i) \;=\frac{\exp\left(\sum_{l\in S}a(i,l)\mathcal{J}*f_t(u,l)\right)}{\sum_{k\in S}\exp\left(\sum_{l\in S}a(k,l)\mathcal{J}*f_t(u,l)\right)}-f_t(u,i)$$

which generalizes the well-known logit ODE of game theory.

(b) Imitative replicator equation: Let us suppose that the rates are given by equation (9). Then we obtain

$$\begin{aligned}\frac{\partial}{\partial t}f_t(u,i) \;\;=\;\; \sum_{k\in S}\Bigg[&f(u,k)\mathcal{J}*f(u,i)F\left(\sum_{l\in S}(a(i,l)-a(k,l))\mathcal{J}*f_t(u,l)\right)\\ &-f(u,i)\mathcal{J}*f(u,k)F\left(\sum_{l\in S}(a(k,l)-a(i,l))\mathcal{J}*f_t(u,l)\right)\Bigg]\end{aligned} \tag{30}$$

Note that the equation depends explicitly on F. This is to be contrasted with the replicator ODE which is independent of F whenever F satisfies the relation $F(t)-F(-t)=t$. This is a purely spatial effect: indeed if we take $f(u,i)$ independent of u for all i then equation (30) reduces to the replicator ODE.

(c) Biological replicator equation: Note that one can also derive a "replicator IDE" using a "biological fitness"argument, i.e., the rate of change in the population of a given type is proportional to the difference between the fitness of this type and the average fitness in the population:

$$\frac{\partial}{\partial t} f_t(u,i) \;=\; f_t(u,i)\left[\sum_{l\in S} a(i,l)\mathcal{J}*f(u,l) - \sum_{k,l\in S} f(u,k)a(k,l)\mathcal{J}*f(u,l)\right]$$

This equation, while it still lacks a convincing derivation from a microscopic stochastic process is an interesting equation in itself and it shares many of the nice properties of the replicator ODEs.

3.1 Spatio-temporal Linear Stability

In this section we present the linear stability analysis of IDEs around stationary solutions as a first step to understand the generation and propagation of temporal and spatial morphologies; we refer to Murray (1989) for numerous examples and applications of linear stability analysis of partial differential equation models. Let us consider the following general type of integro-differential equations:

$$\begin{cases} \frac{\partial f}{\partial t} = F(\mathcal{J}*f, f) & \text{in } \Lambda\times(0,T] \\ f(0,x) = f^0(x) & \text{on } \Lambda\times\{0\} \end{cases}, \tag{31}$$

where $\Lambda \subseteq \mathbb{R}^d$ or $\Lambda = \mathbf{T}^d$, $f \in \mathcal{M}(\Lambda\times S)$, $\mathcal{J}*f := (\mathcal{J}*f_1, \mathcal{J}*f_2, \cdots, \mathcal{J}*f_n)^T$, and F is smooth in both arguments. First, observe that if f is spatially homogeneous, i.e., $f(u,t) = f(t)$, then $\mathcal{J}*f = f(\mathcal{J}*1) = f$, and thus the IDE (31) reduces to the ODE

$$\frac{\partial f}{\partial t} = F(f,f)\,.$$

This ODE, in turn, is exactly the ODE obtained if the interactions are uniform $\mathcal{J} \equiv$ const. This shows that the spatially homogenous solutions of (31) are exactly the stationary solutions of the corresponding mean-field ODE. In particular every spatially homogenous stationary solution f_0, satisfies $F(f_0,f_0) = 0$. We record this observation in Lemma 1.

Lemma 1 (Space Independent Stationary Solutions) *f_0 is a spatially independent stationary solution to (31) if and only if $F(f_0,f_0) = 0$.*

Next we study spatiotemporal perturbations of such constant states by linearizing around a spatially homogeneous stationary solution, f_0: let $f = f_0 + \epsilon D$ where

$D = D(u,t)$ and substituting into (31), we obtain

$$\epsilon \frac{\partial D}{\partial t} = F(f_0 + \epsilon \mathcal{J} * D, f_0 + \epsilon D). \tag{32}$$

For small ϵ we expand the right hand side of equation (32) around $\epsilon = 0$, ignore the terms of order ϵ^2 or smaller and obtain

$$\frac{\partial D}{\partial t} = M \mathcal{J} * D + ND \tag{33}$$

where $(M)_{i,j} := \frac{\partial F_i}{\partial r_j}$, $(N)_{i,j} := \frac{\partial F_i}{\partial s_j}$, and each derivative is evaluated at (f_0, f_0). We can solve (33) explicitly using Fourier transform (see the appendix for details) and obtain the following

Dispersion Relation: Eigenvalues for the solutions to (33) are given by

$$\lambda(k) = \text{eigenvalue}(M\hat{\mathcal{J}}(k) + N) \text{ for } k \in \mathbb{Z}^d \tag{34}$$

where $\hat{\mathcal{J}}(k) = \int_{\mathbf{T}^d} \mathcal{J}(u) e^{2\pi i k \cdot u} du$ are the Fourier coefficients of $\mathcal{J}$.

In general, dispersion relations are a useful tool to investigate the generation and early-stage propagation of spatial phenomena for nonlinear PDE or IDE, see for instance Murray (1989). In our case (34) provides the growth (or decay) rates of approximate solutions to equation (32). As we will see later, identifying the Fourier coefficients k which are linearly unstable (i.e., $\lambda(k) > 0$) allows us to identify the regions in phase space where instabilities occur and could lead, coupled with the nonlinear effects, to the formation of complex spatial structures. Finally, the linear stability analysis provides us computational benchmarks for our simulations.

3.2 Example: Two-strategy symmetric coordination games

We consider two-strategy symmetric coordination games with payoffs (5) being normalized in a way that $a(1,2) = a(2,1) = 0$ and $a(1,1) > 0$, $a(2,2) > 0$. If $p(u) \equiv f(u,1)$, using that $f(u,1) + f(u,2) = 1$ we can write a single equation for $p(u)$

and obtain an equation of the form (31) with

$$\textbf{Replicator IDE} \quad F_R(r,s) \;:\; = (1-s)rF_\kappa\left(\beta\left(r-\zeta\right)\right) - s(1-r)F_\kappa\left(\beta\left(\zeta-r\right)\right) \tag{35}$$

$$\textbf{Logit IDE} \quad F_L(r,s) \;:\; = l(\beta\left(r-\zeta\right)) - s \tag{36}$$

where $\zeta = \frac{a(2,2)}{a(1,1)+a(2,2)}$, $\beta = a(1,1)+a(2,2)$, $l\left(t\right) := \frac{1}{1+\exp(-t)}$, and $F_\kappa(t) := \frac{1}{\kappa}\log\left(\exp(\kappa t)+1\right)$ (recall equation (12)). In equations (35) and (36) r and s are variables representing $\mathcal{J}*p$ and p, respectively. Note that ζ is the mixed strategy Nash equilibrium and β is positive.

We refer to (35) at $\kappa = \infty$ as a replicator IDE, while we also consider the regularized replicator IDE (35) for $\kappa < \infty$, and refer to (36) as a logit IDE. In addition to the conditions for $\mathcal{J}$ stated in Section 2.3, we assume that $\mathcal{J}$ is symmetric: $\mathcal{J}(x) = \mathcal{J}(-x)$ for $x \in \Lambda$.

3.2.1 Stationary solutions and their linear stability

To find spatially homogenous stationary solutions, we need to set $F_R\left(p,p\right) = 0$ and $F_L\left(p,p\right) = 0$. Then, for the replicator case $p = 0, 1$, and ζ are three stationary solutions. In the case of logit dynamics, using $l(\kappa z) = \frac{1}{2} + \frac{1}{2}\tanh(\kappa\frac{z}{2})$ and changing the variable, $p \mapsto 2p-1 := u$, the differential equation becomes

$$\frac{\partial u}{\partial t} = -u + \tanh\left(\frac{\beta}{4}(\mathcal{J} * u + (1-2\zeta))\right) \tag{37}$$

which is the well-known Glauber mesoscopic equation (DeMasi, Orlandi, Presutti, and Triolo, 1994; Katsoulakis and Souganidis, 1997; Presutti, 2009) with β being the *inverse temperature*. All known results for (37), such as the existence of traveling wave solutions in one space dimension and the geometric evolution of interfaces between homogeneous stationary states in higher dimensions, are directly applicable to the logit dynamics. Because of this connection, we have the following characterization of stationary solutions to logit dynamics; the proof is the consequence of (37) and the analysis of Glauber dynamics (Presutti, 2009) or it can easily be done directly.

Lemma 2 *Suppose that the game is a coordination game. Then, there exists β_C such that for $\beta < \beta_C$ there exists one spatially homogenous stationary solution, p_1, and for $\beta > \beta_C$ there exist three spatially homogenous stationary solutions, p_1, p_2, and p_3.*

We note here the different role of β in each one of the IDEs (35) and (36). Since $\beta = a_{11} + a_{22}$, β measures the size of payoffs in coordination games, capturing the importance of the game to the players; as $\beta \to 0$, the payoffs become negligible. In replicator IDEs, a change in β merely corresponds to a time change in IDEs, as we can easily see from equations (35). As the game become less important, the replicator system evolves slowly; when the game is for high stakes, the individual's updating of strategy and, hence, the time evolution of the system is very fast. By contrast, in logit dynamics β becomes a parameter capturing the noise level. So when β is small (high noise), disorder pervades and the system converges everywhere to $\frac{1}{2}$; everyone randomizes between two strategies regardless of the payoffs. As β gets higher (less noise), logit dynamics approach best response dynamics, and the payoffs weigh more. In this situation, the solution converges everywhere to a Nash equilibrium.

Next we examine the linear stability of these stationary solutions. By differentiating F_R, F_L, we find similarly to (34) the dispersion relations for the replicator IDE:

$$\begin{array}{ll} p = 0 & \lambda_R(k) = F_\kappa\left(-\beta\zeta\right)\hat{\mathcal{J}}(k) - F_\kappa\left(\beta\zeta\right) \\ p = 1 & \lambda_R(k) = F_\kappa\left(\beta\left(\zeta - 1\right)\right)\hat{\mathcal{J}}(k) - F_\kappa\left(\beta\left(1 - \zeta\right)\right) \\ p = \zeta & \lambda_R\left(k\right) = \left(\frac{\log(2)}{\kappa} + \beta\zeta\left(1 - \zeta\right)\right)\hat{\mathcal{J}}(k) - \frac{\log(2)}{\kappa} \end{array}$$

Table 1: **Dispersion Relations for the Replicator IDE**

Note that by our assumptions of $\mathcal{J}$, $\hat{\mathcal{J}}(k)$ is real-valued and $|\hat{\mathcal{J}}(k)| < 1$ for all k. Using this fact, we obtain Proposition 1.

Proposition 1 (Linear Stability for the Replicator IDE) $p = 0, 1$ *are linearly stable for the replicator dynamics for coordination games.*

Figure 4 shows one example of the dispersion relations for $p = \zeta$. Observe that $\lambda\left(k\right) > 0$ for $k = 0$, ± 1, ± 2 and the solutions to linear equation (33) is of the form, $e^{2\pi ik\cdot x}$ (see appendix). So, when $k = 0$, the corresponding solution is constant along space and the eigenvalue $\lambda(0)$ is the eigenvalue for the linearized equation of the mean-field ODE (27). Thus $\lambda(0) > 0$ merely shows that ζ is unstable in mean-field ODE, and when $k = 0$ we do not expect to observe any non-trivial spatial morphologies. At $k = \pm 1$, the corresponding solution has a period 1, involving $\cos(x)$, $\sin(x)$ or both and this solution may grow fast, dominating other solutions with different frequencies.

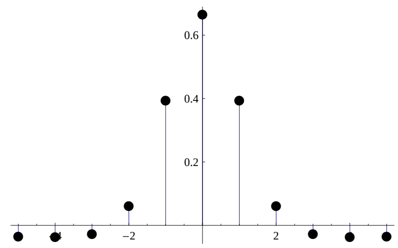


Figure 4: **Dispersion Relation for the mixed strategy equilibrium in regularized replicator dynamics.** The figure shows the dispersion relation $\lambda_R(k)$ at $p = \zeta$. $\mathcal{J}(x) = \exp\left(-bx^2\right) / \int \exp\left(-bx^2\right) dx$, $b = 20$, $\kappa = 20$, $\beta = 3, \zeta = \frac{1}{3}$.

Note that the nonlinearity of the replicator IDE implies a bound on the solutions, so that they remain in the simplex, at each spatial location. An initially fast growing solution may be bounded due to the nonlinearity effects and, hence, may develop to a spatially heterogeneous solution. This is how we obtain the pattern formation in Figure 2 (upper panels). For $k = \pm 2$, we expect a similar spatial phenomenon, but now the solution involves $\cos(2x)$ or $\sin(2x)$. Hence, we anticipate a finer pattern and, indeed, observe this in the numerical simulation of Figure 2 (lower panels).

In the case of logit dynamics, we note that $l'(t) = l(t)\,(1 - l(t))$, hence we easily obtain the dispersion relation for any stationary solution, p_0:

$$\lambda_L(k) = \beta(1 - p_0)p_0\hat{\mathcal{J}}(k) - 1, \;\; k \in \mathbb{Z}^d \tag{38}$$

Proposition 2 (Linear Stability for the logit IDEs) *Suppose that* $0 < \hat{\mathcal{J}}(k)$ *for all* k. *When* $\beta < \beta_C$, *the unique stationary solution* p_0 *is linearly stable. When* $\beta > \beta_C$, *two stationary solutions,* p_1, p_3, *are linearly stable where three stationary solutions* p_1, p_2, *and* p_3 *are arranged in* $p_3 < p_2 < p_1$.

We note that the Gaussian kernel satisfies the hypothesis, $0 < \hat{\mathcal{J}}(k)$ for all k. From table 1, we see that the dispersion relation for $p = \zeta$ in the replicator IDEs approaches

$$\lambda_R(k) = \beta\zeta(1 - \zeta)\hat{\mathcal{J}}(k) \text{ as } \kappa \to \infty$$

and, as a result, $\lambda_L(k)$ is less than $\lambda_R(k)$ at $\kappa = \infty$. Thus, we expect that the unstable steady solutions of the replicator IDE are 'more unstable' than the corresponding ones for logit dynamics, and developed patterns in the replicator case may persist longer; this conjecture is numerically confirmed in Figure 8 below.

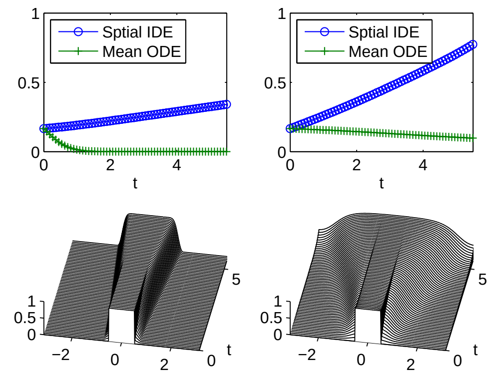


Figure 5: **Comparison of equilibrium selections in mean-field ODEs and IDEs (Periodic BC).** The upper left panel shows population densities of strategy 1 for the mean replicator ODE and IDE. The upper right panel depicts the case of logit rule. The bottom panels show the actual solutions of IDEs which were used for the comparison. $N = 512$. $\Lambda = [-\pi, \pi]$. $dt = 0.001/(0.25N^2)$, $a_{11} = \frac{20}{3}$, $a_{22} = \frac{10}{3}$, $a_{12} = a_{21} = 0$. $b = 2$ for the Gaussian kernel. The initial density in the upper panel is $\frac{1}{6}$ and the initial datum for IDEs is $\mathbf{1}\{-\frac{\pi}{6} < x < \frac{\pi}{6}\}$.

Remark: Overall, the linearized analysis depicted in the dispersion relations in Table 1 or in (38) for the deterministic mesoscopic IDE, allows us to easily create a *phase diagram* for pattern generation and strategies segregation, i.e. a systematic representation of the parameter regimes of the microscopic models for which we expect to have nontrivial spatial structures. On the other hand such calculations, typically referred to in the engineering literature as 'systems tasks', are prohibitive using conventional Monte Carlo simulations such as the ones in Szabo and Fath (2007), for the complex microscopic processes in Section 2.1; this is due not only to the expense of spatial Monte Carlo simulations with many agents and strategies, but even more importantly to the large number of parameters involved in the microscopic models.

3.2.2 Equilibrium selection: Mean-Field ODE versus Mesoscopic IDE

To understand the importance of spatial interactions in our model compared to existing mean-field models, we investigate the problem of equilibrium selection for mean-field ODEs and spatial, mesoscopic IDEs. We ran numerical simulations of the solutions to the spatial IDEs and the corresponding mean-field ODEs whose initial values are chosen to be the spatial averages of the initial data to the IDEs.

In the upper panels in Figure 5, we present the comparisons of population densities of strategy 1 in the coordination game with payoffs, $a(1,1) = \frac{20}{3}$, $a(2,2) = \frac{10}{3}$, $a(1,2) = a(2,2) = 0$. The values for IDE are computed by integrating the corresponding spatial solutions over space. The bottom panels show the evolutions of the spatial solutions which were used to generate the upper panels. We used both the replicator and logit equations; the left panels correspond to the replicator equation, and the right panels describe the logit dynamics. As we see from the upper panels, the solutions to mean-field ODEs converge to the equilibrium where everyone in the population coordinates to strategy 2, since the initial density $\frac{1}{6}$ belongs to the basin of the attraction of this equilibrium. However, a small island of 1-strategists in the spatial domain induces a transition toward an equilibrium of coordinating to strategy 1, even though the total population density using strategy 1 is still $\frac{1}{6}$. In the replicator IDE case, the system reaches a *metastable state* − a state where both strategies coexist for a very long time − and form a 1-dimensional pattern similar to Figure 2. In case of the logit IDE, the propagation of strategy 1 is much faster than the replicator IDE, and typically the system converges to the equilibrium of coordination to strategy 1. Heuristically, this is because agents located near the island of 1-strategists, but are playing strategy 2, face a roughly 50% chance of interacting with 1-stategists and 50% chance of interacting with 2-strategists and, since strategy 1 yields a higher payoff than strategy 2, these agents are better off by adopting strategy 1. This mechanism propagates strategy 1 to the whole spatial domain in IDEs.

In similar simulations, though not reported in the paper, we have observed that mean-field ODEs tend to overestimate the speed of convergence to equilibrium. The mean-field ODE systems converge to equilibrium exponentially fast, while in the IDEs the convergence is much slower, which represents the fact that patterns are metastable; for related metastable behavior for scalar reaction-diffusion equations we refer for instance to Carr and Pego (1989).Thus, one needs to exercise caution in studying equilibrium selection and the convergence of the system using mean-field equations, especially when the spatial consideration of system is important.

3.2.3 Traveling front solution as a way of equilibrium selection: Imitation versus Perturbed Best Responses

Suppose that the domain is a subset of $\mathbb{R}$ with the fixed boundary conditions or the whole real line $\mathbb{R}$. Then, this provides a natural setting to study traveling front solutions, see for instance Figure 1. A solution is called a traveling front or wave solution if it moves at a constant speed: i.e., a traveling front solution $p(x,t)$ can be written as $P(x-ct)$ for some constant c and some function P. The existence of traveling front solutions for the logit dynamics is the direct consequences of known results for the Glauber equations. When $\zeta = \frac{1}{2}$, the existence of a unique standing wave (i.e. $c = 0$) was proved and when there are three equilibrium states, the existence of traveling waves was established (DalPasso and DeMottoni, 1991; DeMasi, T.Gobron, and Presutti, 1995; Orlandi and Triolo, 1997). Particularly, if $\zeta < \frac{1}{2}$ one can find a solution that satisfies $P(-\infty) = 0$ and $P(\infty) = 1$, and travels at a negative speed. Thus the value of $P(\infty)$ propagates to the whole real line and as $t \to \infty$, the solution becomes 1 everywhere; coordination to a state with the higher payoffs becomes a dominating behavior. However, there is no existing rigorous result, so far, on the replicator IDE, though we have observed this solution in numerical simulations.

To compare the traveling wave solutions for each mesoscopic dynamics, we first study the shapes of the standing waves. This is because the shapes of the standing waves may depend on how "diffusive" the system is and the diffusiveness of the system may, in turn, determine the speed of traveling waves. As in the usual analysis of Allen-Cahn type PDE and Glauber IDE, we believe that the sharpness of the standing wave varies with the diffusion effect of the equations and the more "diffusive" the system is, the faster interfaces move (Carr and Pego, 1989; Katsoulakis and Souganidis, 1997).

As Figure 6 shows, the shape of the standing wave in the replicator dynamics with $\kappa = \infty$ is much sharper than that of the logit dynamics. In other numerical simulations, we have observed that the shape of the regularized replicator dynamics depend on κ; as κ become larger, the shape is getting sharper. Since $F_\kappa(t) \to [t]_+$ as $\kappa \to \infty$, as κ increases marginal gains from switching to a different strategy become higher in response to increases in the payoff of that strategy; in particular, at $\kappa = \infty$, this marginal gain becomes infinity. Thus in the replicator IDEs of high payoffs, there is a zero probability for actions against the optimal choice, hence the interface is very sharp. However, the players in the logit dynamics do not have zero probabilities for doing such an action when an agent is right on the "interface"; i.e., there is a nonzero probability to select something not optimal. That creates the "mushy" mixed region of a transition, see the schematic in Figure 1(a) (b). From this observation we infer that the logit dynamic is more "diffusive" than the replicator dynamic with $\kappa = \infty$; hence the interfaces in the logit IDEs would move faster than those in the replicator IDEs. This is numerically exhibited in Figure 7.

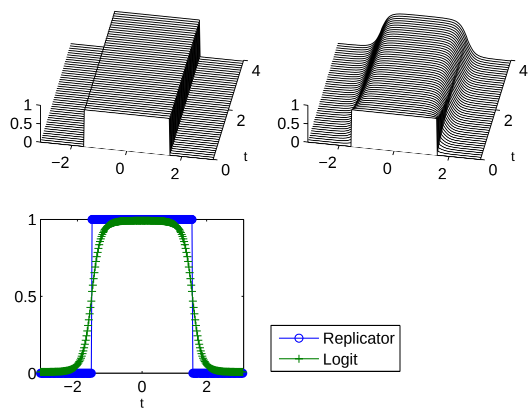


Figure 6: **Comparison of standing waves between the replicator and the logit dynamic ($a_{11} = a_{22}$, Periodic BC).** The upper left panel shows the time evolution of the population density of strategy 1 in the replicator dynamic. The upper right panel describes the case of the logit dynamic. The bottom panel shows the shapes of standing waves in both cases at time 4. We consider the replicator with $\kappa = \infty$. $N = 256$. $\Lambda = [-\pi, \pi]$. $dt = 0.001/(0.25N^2)$, $a_{11} = 5$, $a_{22} = 5$, $a_{12} = a_{21} = 0$. $b = 2$. The initial datum is $\mathbf{1}_{[-\frac{1}{2}\pi, \frac{1}{2}\pi]}$

We note that in the coordination game used for Figure 7, the equilibrium of coordination to strategy 1 is the one predicted by the existing equilibrium selection theories (Harsanyi and Selton, 1988; Young, 1998; Hofbauer, Hutson, and Vickers, 1997; Hofbauer, 1997). Particularly Hofbauer (1997) shows, under the best response dynamics, the existence of a traveling wave solution which drives out the equilibrium of strategy 2, and at the same time propagates the equilibrium of strategy 1. Although we observe the existence of similar traveling wave solutions under various dynamics, the speed of traveling varies dramatically. As Figure 9 shows the transition is extremely slow in the replicator equation with $\kappa = \infty$. So, when the society is characterized by imitative behaviors and marginal gains from switching is high, our model predicts that the transition to a "better equilibrium" is very slow and it takes a long time for equilibrium selection to occur.

Finally we present another comparison between the imitative behavior with the perturbed best response rule using unequal payoff coordination games ($a_{11} > a_{22}$) with the periodic boundary condition (Figure 8). Observe that the time evolution of the replicator dynamic IDE in the left panel of Figure 8 corresponds to the 1-dimensional snap shot of the pattern formation in two dimensional replicator systems in Figure 2. In Figure 8, the replicator system developed a spatial pattern; in the

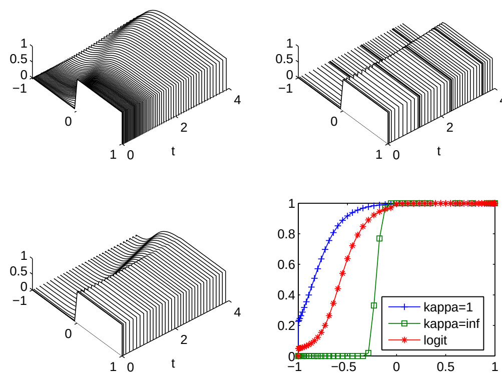


Figure 7: **Comparison of traveling waves between the replicator and the logit IDEs (Fixed BC).** The upper panels show the time paths of the population densities for strategy 1 in the replicator with $\kappa = 1$ (left) and the one with $\kappa = \infty$ (right). The lower left panel shows the case of the logit dynamic. In the bottom right panel we show the shapes of traveling waves at time 4. $N = 256$. $\Lambda = [-1, 1]$, $\partial\Lambda = [-3, -1] \cup [1, 3]$ with the fixed boundary condition $p(x) = 0$ for $x \in [-3, -1]$ and $p(x) = 1$ for $x \in [1, 3]$, $dt = 0.001/(0.05N^2)$, $a_{11} = 20/3, a_{12} = a_{21} = 0$. $b = 2$ for the Gaussian kernel. The initial datum is $\mathbf{1}_{[0,1]}$.

logit dynamic all population coordinate to an equilibrium of strategy 1 exponentially fast. Thus, in a society where agents adopt strategies by imitating their neighbors, the significant proportion of the population may spend a long time in an inefficient equilibrium, whereas agents with perturbed best response rules coordinate "better" to an efficient outcome.

Throughout numerical simulations, we frequently observed the development of patterns in the replicator IDEs, while this is not the case for logit IDEs, except for the equal payoff coordination games. We have also observed the similar pattern formations in the regularized replicator IDEs for a reasonable range of κ; the regularized replicator IDEs with $\kappa = 10$ showed similar patterns to the replicator IDEs.

3.3 PDE approximations of IDEs

If the interaction kernel $\mathcal{J}$ is highly concentrated at the origin, or equivalently, the density f varies slowly with respect to space, we can consider $\mathcal{J}_\epsilon(x) = \epsilon^{-d}\mathcal{J}(x/\epsilon)$ as an interaction kernel for small ϵ. Then by a change of variables and a Taylor

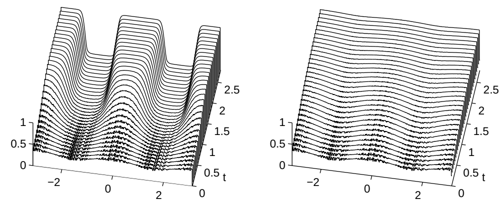


Figure 8: **Replicator versus Logit (Periodic BC).** The left panel shows population density of strategy 1 for the replicator IDE with $\kappa = \infty$, and the right panel depicts the population density in the logit dynamics. $N = 512$. $\Lambda = [-\pi, \pi]$ with the periodic condition. $dt = 0.001/(0.05N^2)$, $a_{11} = 20/3$, $a_{22} = 10/3$, $a_{12} = a_{21} = 0$. $b = 10$. for the Gaussian kernel. The initial datum is $\frac{1}{2} + \frac{1}{10}\,\mathrm{rand}\cos(2x)$,where rand denotes a realization of the uniform random variable at each node.

expansion, we find

$$\mathcal{J}_\epsilon * f \approx f + \frac{\epsilon^2}{2} J_2 \Delta f$$

where we ignore smaller order terms like ϵ^3 and $\Delta f = (\Delta f_1, \Delta f_2, \cdots, \Delta f_n)^T$, $\Delta f_1 = \frac{\partial^2 f_1}{\partial r_1^2} + \cdots + \frac{\partial^2 f_1}{\partial r_d^2}$, and $J_2 = \int_\Lambda |w|^2 \mathcal{J}(w) dw$. Thus, by expanding $F(f + \frac{\epsilon^2}{2} J_2 \Delta f, f)$ in equation 31 around $\epsilon \approx 0$ again, we find the PDE approximations of IDEs:

$$\frac{\partial f}{\partial t} = F(f, f) + \frac{1}{2}\epsilon^2 J_2 M \Delta f \tag{39}$$

where $(M)_{i,j} := \frac{\partial F_i}{\partial r_j}$, and the derivatives are evaluated at (f, f). Intuitively, the coordinating behaviors imply that agents try to choose the same strategy as their neighbors, and this, in turn, means that the density of a given strategy tends to diffuse toward locations where the coordination of that strategy is more likely. This is how our original IDEs are related to the reaction diffusion equations in (39). For specific PDE expressions, we find

$$\begin{aligned} &\textbf{Replicator} \quad \frac{\partial f}{\partial t} = \beta f(1-f)(f-\zeta) \\ &\qquad\qquad + \left[\beta f(1-f) + (1-f)F_\kappa\left(\beta\left(f-\zeta\right)\right) + fF_\kappa\left(\beta\left(\zeta-f\right)\right)\right]\frac{\epsilon^2}{2}J_2\Delta f \\ \\ &\textbf{Logit} \qquad \frac{\partial f}{\partial t} = l(\beta\left(f-\zeta\right)) - f + \beta l(\beta(f-\zeta))(1-l(\beta(f-\zeta)))\frac{\epsilon^2}{2}J_2\Delta f \end{aligned} \tag{40}$$

Both PDEs in (40) are reaction diffusion equations, whose reaction terms are of the same functional form as the mean field reactions (term $\beta f(1-f)(f-\zeta)$ in the replicator and $l(\beta\left(f-\zeta\right)) - f$ in the logit). The diffusion terms are *nonlinear* as the coefficients of the terms Δf depend on the strategy density f. In PDE that Hutson and Vickers (1992), Vickers, Hutson, and Budd (1993), Hofbauer, Hutson, and Vickers (1997), and Hofbauer (1997) have studied for the existence of traveling wave solutions and pattern formation, the diffusion coefficients are constant, implicitly modeling 'fast' diffusion of strategies between players at different lattice sites in space at the microscopic level, in contrast to the 'slow' strategy updating dynamics; such derivations of reaction-diffusion PDE from interacting particle systems with combined fast/slow mechanisms are discussed in Durrett (1999) and references therein. However, in our long-range interaction models the diffusion terms are concentration-dependent, induced by the nonlinearities in the logit and replicator microscopic stochastic dynamics, which in turn are heuristically discussed in Figure 1(a). In biology models, when the population pressure tends to enhance dispersal as the population density increases, the density dependent reaction diffusion models have been used (Murray, 1989; Morishita, 1971; Shigesada, 1980).

Overall the PDEs in (40) provide additional insights for the IDEs in (35) and (36), and their corresponding microscopic stochastic dynamics. For example, in the case of (35), when p is close to either 0 or 1 the diffusion term is weakest and when p lies in the intermediate range, the effect becomes strong. This means that the individuals playing strategy 1 diffuse fast, as p reaches $\frac{1}{2}$, because it is more likely for them to play with 2-strategists, so more likely to be uncoordinated. When it is highly likely to be coordinated, as in $p = 0$ or 1, the individuals with the corresponding strategy do not diffuse at all.

A Appendix

A.1 Various strategy revision rates and proof of Theorem 2

Strategy Revision Rates

We show that condition **C1**−**C3** are satisfied for the following strategy revision rates:

- $c^{\gamma}(x,\sigma,k) = F(u(x,\sigma,k))$: $\mathbf{c}(u,i,k,f) = F\left(\sum_l a(i,l)\,\mathcal{J} * f(u,l)\right)$
- $c^{\gamma}(x,\sigma,k) = F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$: $\mathbf{c}(u,i,k,f) = F\left(\sum_l [a(k,l) - a(i,l)]\mathcal{J} * f(u,l)\right)$
- $c^{\gamma}(x,\sigma,k) = \sum_y w(x,y,\sigma,k) F(u(x,\sigma,k))$: $\mathbf{c}(u,i,k,f) = \mathcal{J} * f(u,k)\, F\left(\sum_l a(k,l)\,\mathcal{J} * f(u,l)\right)$
- $c^{\gamma}(x,\sigma,k) = \sum_y w(x,y,\sigma,k) F(u(x,\sigma,k) - u(x,\sigma,\sigma(x)))$: $\mathbf{c}(u,i,k,f) = \mathcal{J} * f(u,k)\, F\left(\sum_l [a(k,l) - a(i,l)]\mathcal{J} * f(u,l)\right)$
- $c^{\gamma}(x,\sigma,k) = \frac{\exp(u(x,\sigma,k))}{\sum_l \exp(u(x,\sigma,l))}$: $\mathbf{c}(u,i,k,f) = \frac{\exp(\mathcal{J} * f(u,k))}{\sum_l \exp(\mathcal{J} * f(u,l))}$

if F satisfies the global Lipschitz condition:i.e., for all $x, y \in Dom(F)$, there exists $L > 0$ such that $|F(x) - F(y)| \leq L|x - y|$. Note that the list above is far from being exhaustive; one can easily invent various other rates which satisfy **C1**−**C3**. Since the verifications of the conditions are similar, we will check the conditions for the following rate (41) in the periodic domain.

$$c^{\gamma}(x,\sigma,k) = F\Big(\sum_{y\in\Lambda^{\gamma}} \gamma^d a(k,\sigma(y))\mathcal{J}(\gamma(y-x)) - \sum_{y\in\Lambda^{\gamma}} \gamma^d a(\sigma(x),\sigma(y))\mathcal{J}(\gamma(x-y))\Big) \tag{41}$$

Lemma 3 *The rate given by (41) satisfies* **C1** − **C3**.

Proof. Let

$$\begin{aligned}
\bar{c}^{\gamma}(u,i,k,\sigma) \quad &: \quad = F\Big(\sum_{y\in\Lambda^{\gamma}} \gamma^d a(k,\sigma(y))\mathcal{J}(u-\gamma y) - \sum_{y\in\Lambda^{\gamma}} \gamma^d a(i,\sigma(y))\mathcal{J}(u-\gamma y)\Big) \\
\mathbf{c}(u,i,k,\pi) \quad &: \quad = F\Big(|\Gamma|\int^{\gamma\times S} a(k,l)\mathcal{J}(u-v)\pi(dvdl) - |\Gamma|\int^{\gamma\times S} a(i,l)\mathcal{J}(u-v)\pi(dvdl)\Big)
\end{aligned}$$

where we associate u to γx and v to γy. First we note that

$$\begin{aligned}
&\left|\sum_{y\in\Lambda^{\gamma}} \gamma^d a(k,\sigma(y))\mathcal{J}(\gamma x - \gamma y) - |\Gamma|\int_{\Lambda\times S} a(k,l)\mathcal{J}(rx - v)\pi^{\gamma}(\sigma, dvdl)\right| \leq \\
&\left|\gamma^d - \frac{|\Lambda|}{|\Lambda^{\gamma}|}\right| \sum_{y\in\Lambda^{\gamma}} a(k,\sigma(y))\mathcal{J}(\gamma x - \gamma y) \leq \left|\gamma^d\,|\Lambda^{\gamma}| - |\Lambda|\right| M \to 0 \text{ uniformly in } x,\sigma,k
\end{aligned} \tag{42}$$

where $M := \sup_{i,k,u,v} a(i,j)\mathcal{J}(u,v)$. So by using the Lipschitz condition of F, we have

$$\begin{aligned}
&|c^{\gamma}(x,\sigma,k) - c(\gamma x, \sigma(x), k, \pi^{\gamma}(\sigma))| \leq |\bar{c}^{\gamma}(\gamma x, \sigma(x), k, \sigma) - \mathbf{c}(\gamma x, v, \sigma(x), k, \pi^{\gamma}(\sigma))| \\
\leq\ & L \sup_{\substack{x\in\Lambda^{\gamma} \\ \sigma\in S^{\Lambda^{\gamma}} \\ k\in S}} \left|\sum_{y\in\Gamma^{\gamma}} \gamma^d a(k,\sigma(y))\mathcal{J}(\gamma x - \gamma y) - |\Gamma|\int_{\Gamma\times S} a(k,l)\mathcal{J}(\gamma x - v)\pi(\sigma, dvdl)\right| + \\
& L \sup_{\substack{x\in\Lambda^{\gamma} \\ \sigma\in S^{\Lambda^{\gamma}} \\ k\in S}} \left|\sum_{y\in\Gamma^{\gamma}} \gamma^d a(\sigma(x),\sigma(y))\mathcal{J}(\gamma x - \gamma y) - |\Gamma|\int_{\Gamma\times S} a(\sigma(x),l)\mathcal{J}(\gamma x - v)\pi(\sigma, dvdl)\right| \to 0 \text{ uniformly in } x,\sigma,k
\end{aligned}$$

Hence **C1** is satisfied. Since $\mathbf{c}(u,i,k,\pi)$ is uniformly bounded, **C2** is satisfied. Again **C3** follows from the fact that $\mathbf{c}\left(u,i,k,\pi\right)$ is uniformly bounded and F satisfies the Lipschitz condition. ■

Notations

We use the following notations in the proof of Theorem 1 and Theorem 2.

• $\{\Sigma_t^\gamma\}$ is the stochastic process taking values σ_t with generator L^γ given in equation (18) and the sample space $D\left([0,T],S^{\Gamma^\gamma}\right)$.

• $\{\Pi_t^\gamma\}$ is the stochastic process for the empirical measure taking values π_t with the sample space $D([0,T],\mathcal{P}(\Lambda\times S))$ and we denote by $\mathbf{Q}^\gamma$ the law of the process $\{\Pi_t^\gamma\}$ and by $\mathbf{P}$ the probability measure in the underlying probability space.

The proof of Theorems 1 and 2 are so similar that we only prove Theorem 2 and leave the modifications needed to prove Theorem 1 to the readers.

Martingale Estimates

For $g\in C\left(\Gamma\times S\right)$ we set

$$h\left(\sigma\right):=\left\langle\pi^\gamma,g\right\rangle=\frac{1}{|\Gamma^\gamma|}\sum_{y\in\Gamma^\gamma}g(\gamma y,\sigma(y)) \tag{43}$$

We define $M_t^{g,\gamma},\langle M_t^{g,\gamma}\rangle$ as follows: for $g\in C(\Gamma\times S)$

$$M_t^{g,\gamma}=\langle\Pi_t^\gamma,g\rangle-\langle\Pi_0^\gamma,g\rangle-\int_0^t L^\gamma\left\langle\Pi_s^\gamma,g\right\rangle ds,\langle M_t^{g,\gamma}\rangle=\int_0^t\left[L^\gamma\left\langle\Pi_s^\gamma,g\right\rangle^2-2\left\langle\Pi_s^\gamma,g\right\rangle L^\gamma\left\langle\Pi_s^\gamma,g\right\rangle\right]ds \tag{44}$$

Since h is measurable, so $M_t^{g,\gamma}$ and $\langle M_t^{g,\gamma}\rangle$ are $\mathcal{F}_t-$martingale with respect to $\mathbf{P}$, where $\mathcal{F}_t$ is the filtration generated by $\{\Sigma_t\}$ (Ethier and Kurz, 1986; Darling and Norris, 2008).

Lemma 4 *For $g\in C(\Gamma\times S)$ there exist C such that*

$$\left|L^\gamma\left\langle\pi^\gamma,g\right\rangle\right|\leq C,\quad\left|L^\gamma\left\langle\pi^\gamma,g\right\rangle^2-2\left\langle\pi^\gamma,g\right\rangle L^\gamma\left\langle\pi^\gamma,g\right\rangle\right|\leq\gamma^d C$$

Proof. For h in (43),we have

$$h(\sigma^{x,k})-h(\sigma)=\frac{1}{|\Gamma^\gamma|}\left(g(\gamma x,k)-g(\gamma x,\sigma(x))\right)$$

and so we have equation (45) below. Now let $q(\sigma):=\left\langle\pi^\gamma,g\right\rangle^2$. Then

$$\begin{aligned}q(\sigma^{x,k})-q(\sigma)&=\frac{1}{|\Gamma^\gamma|^2}(\sum_{y\in\Lambda^\gamma}g(\gamma y,\sigma^{x,k}(y)))^2-\frac{1}{|\Gamma^\gamma|^2}(\sum_{y\in\Lambda^\gamma}g(\gamma y,\sigma(y)))^2\\&=\frac{1}{|\Gamma^\gamma|^2}\left(g(\gamma x,k)-g(\gamma x,\sigma(x))\right)^2+\frac{2}{|\Gamma^\gamma|^2}\left(g(\gamma x,k)-g(\gamma x,\sigma(x))\right)\sum_{y\in\Lambda^\gamma}g(\gamma y,\sigma(y))\end{aligned}$$

Thus we have

$$L^\gamma\left\langle\pi^\gamma,g\right\rangle=\frac{1}{|\Gamma^\gamma|}\sum_{k\in S}\sum_{x\in\Lambda^\gamma}c^\gamma(x,\sigma(x),k)\left(g(\gamma x,k)-g(\gamma x,\sigma(x)\right) \tag{45}$$

$$L^\gamma\left\langle\pi^\gamma,g\right\rangle^2-2\left\langle\pi^\gamma,g\right\rangle L^\gamma\left\langle\pi^\gamma,g\right\rangle=\frac{1}{|\Gamma^\gamma|^2}\sum_{k\in S}\sum_{x\in\Lambda^\gamma}c^\gamma(x,\sigma(x),k)\left(g(\gamma x,k)-g(\gamma x,\sigma(x)\right)^2 \tag{46}$$

Therefore from **C1** $-$ **C2**, $|\Gamma^\gamma|\approx|\Gamma|\,\gamma^{-d}$,and $|\Lambda^\gamma|\approx|\Lambda|\,\gamma^{-d}$, the results follow. ■

Proposition 3 *Let $g \in C(\Gamma \times S)$ and τ^γ and δ^γ such that*
(1) τ^γ is a stopping time on the process $\{\Pi_t^\gamma : 0 \le t \le T\}$ with respect to the filtration $\mathcal{F}_t$.
(2) δ^γ is a constant, $0 \le \delta^\gamma \le T$ and $\delta^\gamma \to 0$ as $\gamma \to 0$.
Then for $\epsilon > 0$, there exists C such that

$$\text{(i)}\ \mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| \ge \epsilon\right\} \le \frac{\gamma^d CT}{\epsilon^2} \quad \text{and (ii)}\ \mathbf{P}\left\{\omega : \left|M_{\tau^\gamma+\delta^\gamma}^{g,\gamma} - M_{\tau^\gamma}^{g,\gamma}\right| \ge \epsilon\right\} \le \frac{\gamma^d C\delta^\gamma}{\epsilon^2}$$

and there exists γ_0 such that for $\gamma < \gamma_0$

$$\text{(iii)}\ \mathbf{P}\left\{\omega : \left|\int_{\tau^\gamma}^{\tau^\gamma+\delta^\gamma} L^\gamma \langle \Pi_s^\gamma, g\rangle\, ds\right| \ge \epsilon\right\} = 0$$

Proof. We first show (iii). Let C as in Lemma 4. Since $\delta^\gamma \to 0$, there exists γ_0 such that $\delta^\gamma < \frac{\epsilon}{2C}$ for $\gamma \le \gamma_0$. Then by Lemma 4

$$\left|\int_{\tau^\gamma}^{\tau^\gamma+\delta^\gamma} L^\gamma \langle \Pi_s^\gamma, g\rangle\, ds\right| \le \delta^\gamma C < \frac{\epsilon}{2}, \text{ for } \gamma \le \gamma_0.$$

For (i), let γ be fixed first. Since $(M_0^{g,\gamma})^2 - \langle M_0^{g,\gamma}\rangle = 0, \mathbf{P}$ a.e. and $(M_t^{g,\gamma})^2 - \langle M_t^{g,\gamma}\rangle$ is $\mathcal{F}_t-$martingale, by martingale inequality and Lemma 4, we have,

$$\mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| > \epsilon\right\} \le \frac{1}{\epsilon^2}\mathbf{E}\left[(M_T^{g,\gamma})^2\right] = \frac{1}{\epsilon^2}\mathbf{E}\left[\langle M_T^{g,\gamma}\rangle\right] \le \frac{\gamma^d CT}{\epsilon^2}$$

For (ii), by Lemma 4, Chevyshev inequality, and Doob's optional stopping, we have

$$\mathbf{P}\left\{\omega : \left|M_{\tau^\gamma+\delta^\gamma}^{g,\gamma} - M_{\tau^\gamma}^{g,\gamma}\right| > \epsilon\right\} \le \frac{1}{\epsilon^2}\mathbf{E}\left[(M_{\tau^\gamma+\delta^\gamma}^{g,\gamma} - M_{\tau^\gamma}^{g,\gamma})^2\right] = \frac{1}{\epsilon^2}\mathbf{E}\left[\langle M_{\tau^\gamma+\delta^\gamma}^{g,\gamma}\rangle - \langle M_{\tau^\gamma}^{g,\gamma}\rangle\right] \le \frac{\gamma^d C\delta^\gamma}{\epsilon^2}$$

■

Next we prove an exponential estimate. We let $r_\theta(x) = e^{\theta|x|} - 1 - \theta|x|$ and $s_\theta(x) = e^{\theta x} - 1 - \theta x$ for x, $\theta \in \mathbb{R}$. We define

$$\phi(\sigma,\theta) := \sum_{k\in S}\sum_{x\in\Lambda^\gamma} c^\gamma(x,\sigma,k) r_\theta(h(\sigma^{x,k}) - h(\sigma)),\ \ \psi(\sigma,\theta) := \sum_{k\in S}\sum_{x\in\Lambda^\gamma} c^\gamma(x,\sigma,k) s_\theta(h(\sigma^{x,k}) - h(\sigma))$$

Then, from Proposition 8.8 in Darling and Norris (2008), we have for $M_T^{g,\gamma}$ in (44)

$$Z_t^{g,\gamma} := \exp\left\{\theta M_t^{g,\gamma} - \int_0^t \psi(\Sigma_s^\gamma,\theta) ds\right\}$$

is a supermartingale for $\theta \in \mathbb{R}$. Now we let $C_g := 2\sup|g(u,i)|$, $C_c := \sup|c^\gamma(x,\sigma,k)|$.

Lemma 5 (Exponential Estimate) *There exist C depends on C_g, C_c, S and ϵ_0 such that for all $\epsilon \le \epsilon_0$ we have*

$$\mathbf{P}\left\{\sup_{t\le T} |M_t^{g,\gamma}| \ge \epsilon\right\} \le 2e^{-\frac{|\Lambda^\gamma|\,\epsilon^2}{TC}}$$

Proof. We choose $\epsilon_0 \le \frac{1}{2}|S| C_g C_c T$ and let $A = \frac{1}{|\Lambda^\gamma|}|S| C_g^2 C_c e$, $\theta = \frac{\epsilon}{AT}$. Then since r_θ is increasing in $\mathbb{R}_+$,

$$r_\theta\left(h(\sigma^{x,k}) - h(\sigma)\right) \le r_\theta\left(\frac{1}{|\Lambda^\gamma|}C_g\right) \le \frac{1}{2}\left(\frac{1}{|\Lambda^\gamma|}C_g\theta\right)^2 e^{\frac{1}{|\Lambda^\gamma|}\theta C_g} \text{ for all } \sigma \in S^{\Lambda^\gamma}$$

where in the last line we used $e^x - 1 - x \le \frac{1}{2}x^2 e^x$ for all $x > 0$. Also for $\epsilon \le \epsilon_0$,

$$\frac{1}{|\Lambda^\gamma|}\theta C_g = \frac{1}{|\Lambda^\gamma|}\frac{\epsilon}{AT}C_g \le \frac{1}{|\Lambda^\gamma|}\frac{1}{2}\frac{|S|\,C_g^2 C_c}{A} \le \frac{1}{2e} < 1$$

Thus

$$\int_0^T \phi(\Sigma_t^\gamma,\theta)dt \le |S|\,|\Lambda^\gamma|\,\frac{1}{|\Lambda^\gamma|^2}\frac{1}{2}C_g^2\theta^2 e^{\frac{1}{|\Lambda^\gamma|}\theta C_g} C_c T \le \frac{1}{2}\frac{1}{|\Lambda^\gamma|}\,|S|\,C_g^2 C_c e\theta^2 T = \frac{1}{2}A\theta^2 T \text{ for all } \omega \in \Omega$$

So, since $\psi(\sigma,\theta) \le \phi(\sigma,\theta)$,

$$\begin{aligned}\mathbf{P}\left\{\sup_{t\le T} M_t^{g,\gamma} > \epsilon\right\} &= \mathbf{P}\left\{\sup_{t\le T} Z_t^{g,\gamma} > \exp[\theta\epsilon - \int_0^T \psi(\Sigma_t^\gamma,\theta)dt]\right\} \le \mathbf{P}\left\{\sup_{t\le T} Z_t^{g,\gamma} > \exp[\theta\epsilon - \frac{1}{2}A\theta^2 T]\right\} \\ &\le e^{\frac{1}{2}A\theta^2 T - \theta\epsilon} = e^{-\frac{|\Lambda^\gamma|\,\epsilon^2}{TC}}\end{aligned}$$

where we choose $C := 2\,|S|\,C_g^2 C_c e$. Since the same inequality holds for $-M_t^{g,\gamma}$, we obtain the desired result.

■

Convergence

Lemma 6 (Relative Compactness) *The sequence $\{\mathbf{Q}^\gamma\}$ in $\mathcal{P}(D([0,T];\mathcal{P}(\Lambda\times S)))$ is relatively compact.*

Proof. By Proposition 1.7 in Kipnis and Landim (1999, p.54), we show that $\{\mathbf{Q}^\gamma g^{-1}\}$ is relatively compact in $\mathcal{P}(D([0,T];\mathbb{R}))$ for each $g \in C(\Lambda\times S)$, where the definition of $\mathbf{Q}^\gamma g^{-1}$ is as follows: for any Borel set A in $D([0,T];\mathbb{R})$

$$\mathbf{Q}^\gamma g^{-1}(A) := \mathbf{Q}^\gamma\{\pi.\in D([0,T];\mathcal{P}(\Lambda\times S)) : \langle \pi., g\rangle \in A\}$$

So, from Theorem 1 in Aldous (1978) and Prohorov Theorem in Billingsley (1968, p.125), it is enough to show that
(i) for $\eta > 0$, there exists a such that

$$\mathbf{Q}^\gamma g^{-1}\left\{x \in D([0,T];\mathbb{R}) : \sup_t |x(t)| > a\right\} \le \eta \text{ for } \gamma \le 1$$

(ii)

$$\mathbf{P}\left\{\omega : \left|\left\langle \Pi^\gamma_{\tau^\gamma+\delta^\gamma}, g\right\rangle - \langle \Pi^\gamma_{\tau^\gamma}, g\rangle\right| > \epsilon\right\} \to 0$$

for all $\epsilon > 0$, for $(\tau^\gamma, \delta^\gamma)$ satisfying the condition (1) and (2) of Proposition 3. For (i), since g is bounded, it is enough to choose $a = 2\sup|g(u,i)|$; i.e., $\mathbf{Q}^\gamma g^{-1}\{x \in D([0,T];\mathbb{R}) : \sup_t |x(t)| > a\} = \mathbf{Q}^\gamma\{\pi. : \sup_t |\langle \pi_t, g\rangle| > a\} = 0$ since $|\langle \pi., g\rangle| < a$ for all π. For (ii)

$$\begin{aligned}&\mathbf{P}\left\{\omega : \left|\left\langle \Pi^\gamma_{\tau^\gamma+\delta^\gamma}, g\right\rangle - \langle \Pi^\gamma_{\tau^\gamma}, g\rangle\right| > \epsilon\right\} \le \mathbf{P}\left\{\omega : \left|M^{g,\gamma}_{\tau^\gamma+\delta^\gamma} - M^{g,\gamma}_{\tau^\gamma}\right| > \frac{\epsilon}{2}\right\} + \mathbf{P}\left\{\omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| > \frac{\epsilon}{2}\right\} \\ \le\ & \frac{\gamma^d C\delta^\gamma}{\epsilon^2} \quad \text{for } \gamma \le \gamma_0 \text{ chosen in Proposition 3}\end{aligned}$$

■

Let $\mathbf{Q}^*$ be a limit point of $\{\mathbf{Q}^\gamma\}$ and choose a subsequence $\{\mathbf{Q}^{\gamma_k}\}$ converging weakly to $\mathbf{Q}^*$. Hereafter we denote the stochastic process defined on Λ^γ by $\left\{\Sigma^{\Lambda^\gamma}\right\}$ and its restriction on Γ^γ by $\left\{\Sigma^{\Gamma^\gamma}\right\}$. With these notations, equation (44) becomes

$$\left\langle \Pi_t^{\Gamma^\gamma}, g\right\rangle = \left\langle \Pi_0^{\Gamma^\gamma}, g\right\rangle + \frac{|\Lambda^\gamma|}{|\Gamma^\gamma|}\int_0^t ds \sum_{k\in S}\int_{\mathbf{\Lambda}\times S} c(u,i,k,\Pi_s^{\Gamma^\gamma})\,(g(u,k) - g(u,i))\,\Pi_s^{\Lambda^\gamma}(dudi) + M_t^{g,\gamma} \tag{47}$$

Let $\pi \in \mathcal{P}(\Gamma \times S)$ and we define $\pi_\Lambda(du, di) := \pi(du \cap \Lambda, di)$.

Lemma 7 (Characterization of Limit Points) *For all $\epsilon > 0$,*

$$\mathbf{Q}^* \left\{ \pi. : \sup_{t\in[0,T]} \langle \pi_t, g\rangle - \langle \pi_0, g\rangle - \int_0^t ds \sum_{k\in S} \left[\int_{\Lambda\times S} c(u,i,k,\pi_s)(g(u,k) - g(u,i))\pi_{\Lambda,s}(dudi) \right] > \epsilon \right\} = 0,$$

i.e. the limiting process is concentrated on the weak solutions of the IDE (19).

Proof. First we define $\Phi : D([0,T], \mathcal{P}(\Lambda \times S)) \to \mathbb{R}$

$$\pi. \mapsto \left| \sup_{t\in[0,T]} \langle \pi_t, g\rangle - \langle \pi_0, g\rangle - \int_0^t ds \sum_{k\in S} \left[\int_{\Lambda\times S} c(u,i,k,\pi_s)(g(u,k) - g(u,i))\pi_{\Lambda,s}(dudi) \right] \right|$$

Then Φ is continuous, hence $\Phi^{-1}((\epsilon,\infty))$ is open. From the weak convergence of $\{\mathbf{Q}^{\gamma_k}\}$ to $\mathbf{Q}^*$,

$$\mathbf{Q}^* \{\pi. : \Phi(\pi.) > \epsilon\} \le \liminf_{l\to\infty} \mathbf{Q}^{\gamma_l} \{\pi. : \Phi(\pi.) > \epsilon\}$$

Also,

$$\mathbf{Q}^\gamma \{\pi. : \Phi(\pi.) > \epsilon\} = \mathbf{P} \left\{ \omega : \sup_{t\in[0,T]} |M_t^{g,\gamma}| > \epsilon \right\} \le \frac{\gamma^d CT}{\epsilon^2} \text{ (by Proposition 3) for } \gamma < \gamma_0$$

The first equality follows from (47) and the following equality:

$$\Pi_{\Lambda,s}(dudi) = \frac{1}{|\Gamma^\gamma|} \sum_{x\in\Gamma^\gamma\cap\Lambda} \delta_{(\gamma x, \Sigma_s^{\Gamma^\gamma}(x))}(dudi) = \frac{1}{|\Gamma^\gamma|} \sum_{x\in\Lambda^\gamma} \delta_{(\gamma x, \Sigma_s^{\Lambda^\gamma}(x))}(dudi) = \frac{|\Lambda^\gamma|}{|\Gamma^\gamma|} \Pi_s^{\Lambda^\gamma}(dudi).$$

■

We denote explicitly by $dm \otimes dv$ as a product measure of Lebesgue measure on Λ and the counting measure on S.

Lemma 8 (Absolutely Continuity) *We have*

$$\mathbf{Q}^*\{\pi. : \pi_t(dudi) \text{ is absolutely continuous with respect to } dm \otimes dv \text{ for all } t \in [0,T]\} = 1\,.$$

Proof. We define $\Phi : D([0,T]; \mathcal{P}(\Gamma \times S)) \to \mathbb{R}, \pi. \mapsto \sup_{t\in[0,T]} |\langle \pi_t, g\rangle|$. Then Φ is continuous. Also

$$|\langle \pi^\gamma, g\rangle| \le \frac{1}{|\Gamma^\gamma|} \sum_{x\in\Gamma^\gamma} |g(\gamma x, \sigma(x))| \le \sum_{l\in S} \frac{1}{|\Gamma^\gamma|} \sum_{x\in\Gamma^\gamma} |g(\gamma x, l)|$$

Thus

$$\sup_{t\in[0,T]} |\langle \pi_t^\gamma, g\rangle| \le \sum_{l\in S} \frac{1}{|\Gamma^\gamma|} \sum_{x\in\Gamma^\gamma} |g(\gamma x, l)|$$

We write $\pi_\cdot^*$ be a trajectory on which all $\mathbf{Q}^*$'s are concentrated. Then $\Pi_\cdot^\gamma \xrightarrow{\mathcal{D}} \pi_\cdot^*$ (convergence in distribution), so $\mathbf{E}(\Phi(\Pi_\cdot^\gamma)) \to \mathbf{E}(\Phi(\pi_\cdot^*))$. Also $\frac{1}{|\Gamma^\gamma|} \sum_{x\in\Gamma^\gamma} |g(\gamma x, l)| \to \int_\Lambda |g(u,l)|\, du$ for all l by the Riemann sum approximation. Thus,

$$\sup_{t\in[0,T]} |\langle \pi_t^*, g\rangle| = \Phi(\pi_\cdot^*) = \lim_{\gamma\to 0} \mathbf{E}(\Phi(\Pi_\cdot^\gamma)) \le \lim_{\gamma\to 0} \sum_{l\in S} \frac{1}{|\Gamma^\gamma|} \sum_{x\in\Gamma^\gamma} |g(\gamma x, l)| = \int_{\mathbf{\Gamma}\times S} |g(u,l)|\, dm \otimes dv$$

Therefore, for all $t \in [0,T]$, for all $g \in C(\Gamma \times S)$,

$$\left| \int_{\mathbf{\Gamma}\times S} g(u,l)\pi_t^*(dudl) \right| \le \int_{\mathbf{\Gamma}\times S} |g(u,l)|\, dm \otimes dv$$

so for all $t \in [0,T]$ π_t^* is absolutely continuous with respect to $dm \otimes dv$. ■

We also see that all limit points of the sequence $\{\mathbf{Q}^\gamma\}$ are concentrated on the trajectories that equal to $f^0 dudi$ at time 0, since

$$\begin{aligned} & \mathbf{Q}^* \left\{ \pi_\cdot : \left| \int g(u,i)\pi_0(dudi) - \frac{1}{|\Gamma|} \int g(u,i) f^0(u,i) dudi \right| > \epsilon \right\} \\ \leq \quad & \liminf_{k\to\infty} \mathbf{Q}^{\gamma_k} \left\{ \pi_\cdot : \left| \int g(u,i)\pi_0(dudi) - \frac{1}{|\Gamma|} \int g(u,i) f^0(u,i) dudi \right| > \epsilon \right\} = 0, \end{aligned}$$

where the definition of sequence of product measures with a slowly varying parameter implies the last equality by Proposition 0.4 Kipnis and Landim (1999, p.44).

So far we have shown that $\mathbf{Q}^*$'s are concentrated on the trajectories that are the weak solutions of the integro-differential equations. Next we show the uniqueness of weak solutions defined in the following way. Let $\mathcal{A}(f)(u,i) := \sum_{k\in S} \mathbf{c}(u,k,i,f) f_\Lambda(t,u,k) - f_\Lambda(t,u,i) \sum_{k\in S} \mathbf{c}(u,i,k,f)$. For an initial profile $f^0 \in \mathcal{M}$, $f \in \mathcal{M}$ is a weak solution of the Cauchy problem:

$$\frac{\partial f_t}{\partial t} = \mathcal{A}(f_t), \;\; f_0 = f^0 \tag{48}$$

if for every function $g \in C(\Gamma \times S)$, for all $t < T$, $\langle f_t, g\rangle = \int_0^t \langle \mathcal{A}(f_s), g\rangle \, ds$.Observe that from **C3** $\mathcal{A}$ satisfies the Lipschitz condition: there exists C such that for all $f, \tilde{f} \in L^\infty([0,T];\; L^\infty(\Gamma\times S))$, $\left\| \mathcal{A}(f) - \mathcal{A}(\tilde{f}) \right\|_{L^2(\Gamma\times S)} \leq C \left\| f - \tilde{f} \right\|_{L^2(\Gamma\times S)}$.

Lemma 9 (Uniqueness of Weak Solutions) *Weak solutions of the Cauchy problem (48) which belong to*
$L^\infty\left([0,T]; L^2(\Gamma\times S)\right)$ *are unique.*

Proof. Let f_t, $\tilde{f}_t$ be two weak solutions and $\bar{f}_t := f_t - \tilde{f}_t$. Then, we have

$$\left\langle \bar{f}_t, g \right\rangle = \int_0^t \left\langle \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s), g \right\rangle ds \text{ for all } g \in C(\Gamma\times S)$$

We show that $t \mapsto \left\| \bar{f}_t \right\|^2_{L^2(\Gamma\times S)}$ is differentiable. Define a mollifier $\eta(x) := C \exp\left(\frac{1}{|x|-1}\right)$ if $|x| < 1$, $:= 0$ if $|x| \geq 1$, $C > 0$ is a constant such that $\int_{\mathbb{R}^d} \eta(x)dx = 1$. For $\epsilon > 0$, set $\eta_\epsilon(x) := \epsilon^{-d}\eta(\epsilon^{-1}x)$. For each $u \in \Gamma$, $i \in S$, define $h^\epsilon_{u,i}(v,k) = \eta_\epsilon(u-v)\, \mathbf{1}_{\{i=k\}}$ and

$$\bar{f}^\epsilon_t(u,i) := \int_{\Gamma\times S} \left(f_t(v,k) - \tilde{f}_t(v,k) \right) h^\epsilon_{u,i}(v,k) dvdk$$

Then,

$$\left| \left\langle \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s), h^\epsilon_{u,i} \right\rangle \right| \leq \left\| \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s) \right\|_{L^2} \left\| h^\epsilon_{u,i} \right\|_{L^2} \leq C \left\| f_s - \tilde{f}_s \right\|_{L^2} \left\| h^\epsilon_{u,i} \right\|_{L^2} \leq C \sup_{s\in[0,T]} \left\| f_s - \tilde{f}_s \right\|_{L^2} \left\| h^\epsilon_{u,i} \right\|_{L^2}.$$

Since $f_s - \tilde{f}_s \in L^\infty\left([0,T]; L^2(\Gamma\times S)\right)$ and $h^\epsilon_{u,i} \in C(\Gamma\times S)$ for each u, i, $t \mapsto \bar{f}^\epsilon_t(u,i)$ is differentiable and its derivative is $\bar{f}^{\epsilon\prime}_t(u,i) = \left\langle \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s), h^\epsilon_{u,i} \right\rangle$. Also, it follows that $\left\| \bar{f}^\epsilon_t \right\|^2_{L^2}$ is differentiable with respect to t and

$$\frac{d}{dt} \left\| \bar{f}^\epsilon_t \right\|^2_{L^2} = \int_{\Gamma\times S} 2 \left\langle \mathcal{A}(f_t) - \mathcal{A}(\tilde{f}_t), h^\epsilon_{u,i} \right\rangle \bar{f}^\epsilon_t(u,i) dudi, \text{ so } \left\| \bar{f}^\epsilon_t \right\|^2_{L^2} = \int_0^t \left[\int_{\Gamma\times S} 2 \left\langle \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s), h^\epsilon_{u,i} \right\rangle \bar{f}^\epsilon_t(u,i) dudi \right] ds$$

Then since $f^\epsilon_t \to f_t$ in $\| \, \|_{L^2}$ and $\bar{f}_t \in L^\infty\left([0,T]; L^2(\Gamma\times S)\right)$ for a given t, we have $\left|\, \left\| \bar{f}^\epsilon_t \right\|^2_{L^2} - \left\| \bar{f}_t \right\|^2_{L^2} \right|$ $\to 0$. Also because $\left\langle \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s), h^\epsilon_{u,i} \right\rangle \to \mathcal{A}(f_t)(u,i) - \mathcal{A}(\tilde{f}_t)(u,i)$ for $a.e.u$, and all i, t, by the dominant convergence theorem we have

$$\left\|\bar{f}_t\right\|_{L^2}^2 = \int_0^t 2\left\langle \mathcal{A}(f_s) - \mathcal{A}(\tilde{f}_s), \bar{f}_s\right\rangle ds,$$

so $\left\|\bar{f}_t\right\|_{L^2}^2$ is differentiable and

$$\frac{d}{dt}\left\|\bar{f}_t\right\|_{L^2}^2 = \left\langle \mathcal{A}(f_t) - \mathcal{A}(\tilde{f}_t), \bar{f}_t\right\rangle \leq 2\left\|\mathcal{A}(f_t) - \mathcal{A}(\tilde{f}_t)\right\|_{L^2}\left\|\bar{f}_t\right\|_{L^2} \leq C\left\|\bar{f}_t\right\|_{L^2}^2$$

Hence from Gronwell lemma, the uniqueness of the solutions follows. ■

Lemma 10 (Convergence in Probability) *We have*

$$\Pi_t^\gamma(du, di) \longrightarrow \frac{1}{|\Gamma|} f(t, u, i)\, du di \text{ in probability } .$$

Proof. So far we established $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^*$ (converge weakly) and equivalently $\Pi_\cdot^\gamma \to \pi_\cdot^*$ in Skorohod topology (topology on $D([0,T], \mathcal{P}(\mathbf{T}^d \times S))$). If we show that $\Pi_t^\gamma \to \pi_t^*$ weakly in $\mathcal{P}\left(\Gamma^d \times S\right)$ or equivalently $\Pi_t^\gamma \xrightarrow{\mathcal{D}} \pi_t^*$ in distribution for fixed time $t < T$, then we have

$$\Pi_t^\gamma \xrightarrow{\mathbf{P}} \pi_t^* \text{ in probability.} \tag{49}$$

Since $\Pi_\cdot^\gamma \to \pi_\cdot^*$ in Skorohod topology implies $\Pi_t^\gamma \to \pi_t^*$ weakly for continuity points of $\pi_\cdot^*$ (p.112 Billingsley, 1968), it is enough to show that $\pi_\cdot^* : t \mapsto \pi_t^*$ is continuous for all $t \in [0,T]$ to obtain (49). Let $t_0 < T$ and $\{g_k\}$ a dense family in $C(\Gamma \times S)$. Since

$$\left|\int_{t_0}^t \langle A(\pi_s^*), g_k\rangle\, ds\right| \leq (t - t_0) \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle$$

we choose $\delta \leq \min\{1, \epsilon\}$. Then for $|t - t_0| \leq \delta$,

$$\frac{\left|\int_{t_0}^t \langle A(\pi_s^*), g_k\rangle\, ds\right|}{1 + \left|\int_{t_0}^t \langle A(\pi_s^*), g_k\rangle\, ds\right|} \leq \frac{\delta \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle}{1 + \delta \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle} \leq \delta \frac{\sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle}{1 + \sup_{s\in[0,T]} \langle \mathcal{A}(\pi_s^*), g_k\rangle} \leq \delta$$

so $\|\pi_t - \pi_{t_0}\|_{\mathcal{P}(\Gamma\times S)} \leq \epsilon$ and $\pi_\cdot^* : t \mapsto \pi_t^*$ is continuous, all $t \in [0,T]$, thus all $t \in [0,T]$ are continuity point of $\pi_\cdot^*$ ■

Proof of Theorem 2

From Lemma 10 we have, for $t < T$

$$\Pi_t^{\Lambda^\gamma}(du, di) = \frac{|\Gamma^\gamma|}{|\Lambda^\gamma|}\Pi_t^{\Gamma^\gamma}(du \cap A, di) \xrightarrow{\mathbf{P}} \frac{1}{|\Lambda|} f_t(du \cap A, di)$$

Since $f_t(u,i)(du \cap \Lambda)di = f_{\Lambda,t}(u,i)du di$, from (21) we obtain

$$\langle f_t, g\rangle = \langle f_0, g\rangle + \int_0^t ds \sum_{k\in S}\int_{\Lambda\times S} c(u, i, k, \frac{1}{|\Gamma|} f_s du di)\,(g(u,k) - g(u,i))\, f_{\Lambda,s}(du di)$$

Since $|\Gamma^\gamma|\,\Pi_t^{\Gamma^\gamma} = |\Lambda^\gamma|\,\Pi_t^{\Lambda^\gamma} + |\Lambda^{\gamma c}|\,\Pi_0^{\Lambda^{\gamma c}}$, $|\Gamma|\,\Pi_t^{\Gamma^\gamma} \xrightarrow{\mathbf{P}} f_t(u,i)du di$, $|\Lambda|\,\Pi_t^{\Lambda^\gamma} \xrightarrow{\mathbf{P}} f_{\Lambda,t}(u,i)du di$, and $|\Lambda^c|\,\Pi_0^{\Lambda^{\gamma c}} \xrightarrow{\mathbf{P}} f_{\Lambda^c}(u,i)du di$, we have $f_t = f_{\Lambda,t} + f_{\Lambda^c}$ for all t.

A.2 Proof of Theorem 3

To do this first we define a reduction mapping, $\phi : S^{\Lambda^n} \to \Delta^n$,

$$\sigma \mapsto \phi(\sigma), \quad \phi(\sigma)(i) := \frac{1}{|\Lambda^n|} \sum_{y \in \Lambda^n} \delta_{\sigma(y)}(\{i\})$$

For $g \in L^\infty(\Delta^n; \mathbb{R})$ we let $f := g \circ \phi \in L^\infty(S^{\Lambda^n}; \mathbb{R})$, where $f(\sigma) = g(\eta)$. Then for $\eta = \phi(\sigma)$, we have $f(\sigma^{x,k}) - f(\sigma) = g(\eta^{\sigma(x),k}) - g(\eta)$ since

$$\phi(\sigma^{x,k})(i) = \frac{1}{n^d} \sum_{y \in \Lambda_n} \delta_{\sigma(y)}(\{i\}) + \frac{1}{n^d}\delta_k(\{i\}) - \frac{1}{n^d}\delta_{\sigma(x)}(\{i\}) = \eta^{\sigma(x),k}(i)$$

Proof of Theorm 3. We check the case of imitative and comparing rates. Other cases can be treated as a special case. By writing $m^n(k) := \sum_l a(k,l)\eta^n(l)$, we find

$$\begin{aligned}
L_n f(\sigma) &= \sum_{k \in S} \sum_{x \in \Lambda_n} \eta(k) F(m^n(k) - m^n(\sigma(x)))(g(\eta^{\sigma(x),k}) - g(\eta)) \\
&= \sum_{k \in S} \sum_{j \in S} \left[\sum_{x \in \Lambda_n} \delta_{\sigma(x)}(\{j\}) \right] \eta(k) F(m^n(k) - m^n(j))(g(\eta^{j,k}) - g(\eta)) \\
&= \sum_{k \in S} \sum_{j \in S} n^d \eta(j)\eta(k) F(m^n(k) - m^n(j))(g(\eta^{j,k}) - g(\eta)) := \sum_{k \in S} \sum_{j \in S} n^d c^M(\eta, j, k)(g(\eta^{j,k}) - g(\eta))
\end{aligned}$$

Thus we obtain

$$L^n g(\eta) = \sum_{k \in S} \sum_{j \in S} n^d c^M(\eta, j, k)(g(\eta^{j,k}) - g(\eta))$$

and this makes $\{\eta_t\}$ a Markov chain and the rate is given by $c^M(\eta, j, k)$. ■

A.3 Proof of Corollary 4

Proof. It is enough to prove the exponential estimate. From (21) we recall that

$$\left\langle \Pi_t^\gamma, g \right\rangle = \left\langle \Pi_0^\gamma, g \right\rangle + \int_0^t \sum_{k \in S} \int_{\mathbf{T}^d \times S} c(u, i, k, \Pi_s^\gamma) \left(g(u,k) - g(u,i) \right) \Pi_s^\gamma(dudi) ds + M_t^{g,\gamma}$$

for $g \in C(\mathbf{T}^d \times S)$. By taking $g(u,i) = 1$ if $i = l$, $g(u,i) = 0$ otherwise, we find

$$\eta_{t,l}^n = \eta_{0,l}^n + n^d \int_0^t \left[\sum_{i \in S} c^M(i, l, \eta_s^n)\eta_{s,l}^n - \sum_{k \in S} c^M(l, k, \eta_s^n)\eta_{s,l}^n \right] ds + M_t^{l,n}$$

We define $\beta_l(x) := \sum_{i \in S} \mathbf{c}^M(i, l, x)x_l - \sum_{k \in S} \mathbf{c}^M(l, k, x)x_l$. Thus we have

$$\eta_{t,l}^n = \eta_{0,l}^n + n^d \int_0^t \beta_l(\eta_s^n) ds + M_t^{l,n}, \quad \rho_{t,l} = \rho_{0,l} + \int_0^t \beta_l(\rho_s) ds$$

From Lemma 5, we have $\mathbf{P}\left\{ \sup_{t \le T} \left| M_t^{l,n} \right| \ge \delta \right\} \le 2e^{-n^d \frac{\delta^2}{TC_0}}$ for each l and for $\delta \le \delta_0$, where we note that the choices of C_0 and δ_0 does not depend on g since $|g(u,i)| \le 1$ for all u, i. Thus, $\mathbf{P}\left\{ \sup_{t \le T} \|M_t^n\|_u \ge \delta \right\} \le$

$2\,|S|\,e^{-\frac{n^d\delta^2}{TC_0}}$. Therefore for $t \le T$, using the Lipschitz condition of β we obtain

$$\sup_{\tau\le t}\|\eta_\tau^n - \rho_\tau\|_u \le \|\eta_0^n - \rho_0\|_u + L\int_0^t \sup_{\tau\le s}\|\eta_\tau^n - \rho_\tau\|_u\,ds + \sup_{t\le T}\|M_t^n\|_u$$

For ϵ_0 in Lemma 5, we let $\delta = \frac{1}{3}e^{-LT}\epsilon$ for $\epsilon < \epsilon_0$ and define

$$\Omega_0 = \left\{\omega : \|\eta_0^n - \rho_0\|_u \le \delta\right\},\ \Omega_1 = \left\{\omega : \sup_{t\le T}\|M_t^n\|_u \le \delta\right\}$$

Then when $\omega \in \Omega_0 \cap \Omega_1$, we have $\sup_{\tau\le T}\|\eta_\tau^n - \rho_\tau\|_u \le 2\delta e^{LT}$ by Gronwell lemma. Choose n_0 such that $\|\eta_0^n - \rho_0\|_u \le \delta$ for *a.e.* ω for $n \ge n_0$. Then for $\epsilon \le \epsilon_0$ and $n \ge n_0$,

$$\begin{aligned}
P\left\{\sup_{\tau\le T}\|\eta_\tau^n - \rho_n\| \ge \epsilon\right\} &\le P\left(\Omega_0^c\right) + P\left(\Omega_1^c\right) \le P\left\{\omega : \|\eta_0^n - \rho_0\|_u \ge \delta\right\} + P\left\{\omega : \sup_{t\le T}\|M_t^n\|_u \ge \delta\right\} \\
&\le 2\,|S|\,e^{-\frac{n^d\delta^2}{TC_0}} = 2\,|S|\,e^{-\frac{n^d\epsilon^2}{TC}}
\end{aligned}$$

where $C := 9C_0e^{2LT}$. ■

A.4 Solutions of Linear IDE

Applying Fourier transform to (33) element by element, we obtain

$$\frac{\partial \hat{D}(k)}{\partial t} = (M\hat{\mathcal{J}}(k) + N)\hat{D}(k) \tag{50}$$

for each $k \in \mathbb{Z}^d$ and $\hat{D}(k) \in \mathbb{C}^{|S|}$. By solving the ODE system (50) for each k and using the inverse formula, we obtain

$$D(x,t) = \sum_{k\in\mathbb{Z}} e^{(M\hat{\mathcal{J}}(k)+N)t}\hat{g}(k)e^{2\pi i x\cdot k}$$

where $e^{(M\hat{\mathcal{J}}(k)+N)t}$ is $|S| \times |S|$ matrix, $\hat{g}(k)$ is $|S| \times 1$ vector.

A.5 Proof of Proposition 2

Proof. First we note that $p_1 > \zeta$, $p_2, p_3 < \zeta$, $\lim_{\kappa\to\infty} p_2 = \zeta$,

$$\beta(1 - l(\beta(p_i - \zeta)))l(\beta(p_i - \zeta)) < 1 \text{ for } i = 1,3,\ \ \beta(1 - l(\beta(p_i - \zeta)))l(\beta(p_i - \zeta)) > 1 \text{ for } i = 2.$$

Suppose that $\beta > \beta_C$ and consider p_1. Since $l(\beta(p_1 - \zeta)) = p_1$, we have $\beta(1-p_1)p_1 < 1$. Then since $\hat{\mathcal{J}}(k) \le 1$ for all k, we have

$$\lambda(k) = \beta(1-p_1)p_1\hat{\mathcal{J}}(k) - 1 < \beta(1-p_1)p_1 - 1 < 0$$

Thus p_1 is linearly stable. Similar argument shows that p_3 is linearly stable. ■

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