Math 623: Problem set 1

1. Exercise 1, p. 37
2. Exercise 2, p. 37
3. Exercise 3, p. 38
4. Exercise 4, p. 38
5. Exercise 9, p. 40
6. Exercise 11, p. 41
7. Show that a countable union of set of measure 0 has measure 0 directly from the definition:

$$m(E) = 0 \quad \text{iff} \quad 0 \,=\, \inf\{\sum_{j=1}^{\infty} |Q_j|\,;\, Q_j \text{ closed cubes}\,, E \subset \bigcup_{j=1}^{\infty} Q_j\}\,.$$

   In particular any countable set has measure 0, e.g. the rational numbers in $[0,1]$. What is the Lebesgue measure of the set of irrational numbers in $[0,1]$?

8. Let $f : [a,b] \to \mathbf{R}$ be a *bounded* function. We denote by P a finite partition of $[a,b]$, i.e., $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. We say that the function f is Riemann integrable over the interval $[a,b]$ and that its integral is equal to V if for any $\epsilon > 0$ there exists $\delta > 0$ such that if P is any partition with $|t_i - t_{i-1}| \le \delta$ for $i = 1, \cdots, n$ and t_i^* are any points in $[t_{i-1}, t_{i-1}]$ we have

$$\left| \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}) - V \right| \le \epsilon\,.$$

   This definition is not very convenient to use in practice and the following equivalent criterion for Riemann integrability (due to Darboux) is more useful: Given a partition P we let $F_i = \sup_{x\in[t_{i-1},t_{i-1}]} f(x_i)$ and $f_i = \inf_{x\in[t_{i-1},t_{i-1}]} f(x_i)$ and define the lower and upper Darboux sums, $L(f,P)$ and $U(f,P)$ by

$$U(f,P) = \sum_{i=1}^{n} F_i(t_i - t_{i-1})\,, \quad L(f,P) = \sum_{i=1}^{n} f_i(t_i - t_{i-1})\,.$$

   One can show that a bounded function f is Riemann integrable iff

$$\sup_P L(f,P) = \inf_P U(f,P)$$

   where the inf and sup are taken over all finite partition P. In other words, f is integrable if for any $\epsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) \le \epsilon$.

If f is Riemann integrable then the Riemann integral of f is denoted by $V = \int_a^b f(x)dx$ and is $\inf U(f,P)$ (or $\sup L(f,P)$).

Using this criterion construct a sequence of functions $f_n : [0,1] \to \mathbf{R}$ such that

(i) For all x the sequence $\{f_n(x)\}$ is decreasing to $f(x)$.

(ii) f_n is Riemann integrable and $\int_0^1 f_n dx = 1$ for all n.

(iii) f is not Riemann integrable.

Hint: Consider an enumeration $r_1, r_2, r_3 \cdots$ of the rationals in $[0,1]$ and define f_n by $f_n(r_1) = f_n(r_2) = \cdots f_n(r_n) = 0$ and $f_n(x) = 1$ otherwise.

9. Let $h : \mathbf{R} \to \mathbf{R}$ be the function such that h is periodic of period 1 and $h(x) = x$ for $-1/2 < x < 1/2$ and $h(1/2) = 0$. In particular h is discontinuous whenever x is half of an odd integer. Consider now

$$f(x) = \sum_{n=1}^{\infty} \frac{h(nx)}{n^2} \,. \tag{1}$$

   (a) Prove that the convergence in Eq. (1) is uniform.

   (b) Prove that if $x = a/2b$ where a is odd and a and b are relatively prime the function f is disconinuous at x and satisfies

$$\lim_{\epsilon \to 0} \left[f\left(\frac{a}{2b} - \epsilon\right) - f\left(\frac{a}{2b} + \epsilon\right) \right] = \frac{\pi^2}{8b^2}$$

   Hint: You may use that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{4}$.

   (c) Prove that f is Riemann integrable on any interval $[a,b]$ *Hint:* Note that $h(nx)$ is Rieman integrable for any n.

   (d) Show that F is not differentiable at any $x = a/2b$.

$$F(x) = \int_0^x f(t)dt \,.$$

10. Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers (i.e., there exists $M > 0$ such that $|x_n| \leq M$ for all $n \geq 1$. Recall that b is an **accumulation point** for $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}_{j\geq 1}$ such that $\lim_{j\to\infty} x_{n_j} = b$

    Consider the sets

$$X = \{x\,;\ \text{infinitely many } x_n \text{ are } > x\}\,, \quad Y = \{x\,;\ \text{infinitely many } x_n \text{ are } < x\}\,.$$

    and define

$$\xi := \sup X\,, \quad \eta := \inf Y\,.$$

(a) Prove that ξ is the largest accumulation point of $\{x_n\}$ and that η is the smallest accumulation point of $\{x_n\}$. We write

$$\xi = \limsup_{n\to\infty} x_n \qquad \text{the limit superior of the sequence } \{x_n\}\,.$$

$$\eta = \liminf_{n\to\infty} x_n \qquad \text{the limit inferior of the sequence } \{x_n\}\,.$$

(b) Show the formulas

$$\limsup_{n\to\infty} x_n \;=\; \lim_{n\to\infty} \sup_{k\geq n} x_k \;=\; \inf_{n\geq 1} \sup_{k\geq n} x_k\,.$$

$$\liminf_{n\to\infty} x_n \;=\; \lim_{n\to\infty} \inf_{k\geq n} x_k \;=\; \sup_{n\geq 1} \inf_{k\geq n} x_k\,.$$

(c) Prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$$

and show that the inequalities can be strict (find such examples).

(d) Exhibit a sequence $\{x_n\}$ with $0 \leq x_n \leq 1$ such that any number in $[0,1]$ is an accumulation point of $\{x_n\}$. *Hint:* The rational are dense in $[0,1]$.

11. Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of measurable subsets of $\mathbf{R}^n$. We define

$$\begin{aligned}
\limsup_{n\to\infty} E_n &= \Big\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for infinitely many } n\Big\} \\
\liminf_{n\to\infty} E_n &= \Big\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for all but finitely many } n\Big\}\,.
\end{aligned}$$

(a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\,, \qquad \text{and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\,.$$

(b) Show that

$$\begin{aligned}
m\left(\liminf_{n\to\infty} E_n\right) &\leq \liminf_{n\to\infty} m\left(E_n\right) \\
\limsup_{n\to\infty} m\left(E_n\right) &\leq m\left(\limsup_{n\to\infty} E_n\right) \qquad \text{provided } m\Big(\bigcup_{n=1}^{\infty} E_j\Big) < \infty\,.
\end{aligned} \tag{2}$$

(c) Exercise 16, p.42. (Borel-Cantelli Lemma).