Math 523H–Homework 3

1. For each of the following sequences, compute $\sup\{s_n\}$, $\inf\{s_n\}$, $\limsup\{s_n\}$ and $\liminf\{s_n\}$ and determine all the accumulation points.
   (a) $s_n = 5^{(-1)^n}$
   (b) $s_n = (-1)^n + \sin(\frac{n\pi}{2})$.
   (c) $s_n = (-1)^n \frac{n+5}{n}$
   (d) $s_n = n\cos(\frac{n\pi}{4})$
2. Construct a sequence whose accumulation points are all the non-negative integers $0, 1, 2, \cdots$.
3. Show the following facts:
   (a) If the sequence $\{s_n\}$ converges then every subsequence of $\{s_n\}$ converges to the same limit.
   (b) A sequence $\{s_n\}$ converges if and only if $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.
4. **Equivalent definitions of** $\limsup$**:** Suppose $\{s_n\}$ is a bounded sequence. In class we have defined $\limsup s_n$ as

$$\xi = \limsup_{n\to\infty} s_n = \sup\{x \mid s_n > x \text{ for infinitely many } n\}$$

   and have established in the Bolzano-Weierstrass theorem that

$$\xi = \limsup_{n\to\infty} s_n \text{ is the largest accumulation point of the sequence } \{s_n\}$$

   which gives another characterization of $\limsup$. Here is a third one: prove the formula

$$\xi = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} t_n \text{ where } t_n = \sup_{k\geq n}\{s_k\}\,.$$

   Hint: Show that t_n is a monotone sequence and that its limit t is greater than ξ. Then show that t is an accumulation point.
5. Write down the three characterizations of $\liminf$ similarly to Problem 4. (You do not need to prove it.) Show also that

$$\liminf s_n = -\limsup(-s_n)\,.$$

6. Let $\{s_n\}$ and $\{v_n\}$ be bounded sequences. Show that

$$\begin{aligned} \limsup(s_n + v_n) &\leq \limsup(s_n) + \limsup(v_n) \qquad (1) \\ \liminf(s_n + v_n) &\geq \liminf(s_n) + \liminf(v_n) \qquad (2) \end{aligned}$$

Provide examples that show that the equalities may be strict.
Hint: Choose the right definition!

7. Consider the series $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$. Give at least four different proofs that the series converges.

8. Determine which of the following series converge, and which ones converges absolutely. Justify your answer by stating the appropriate criterion.

   (a) $\sum \dfrac{n^4}{2^n}$.

   (b) $\sum \dfrac{100^n}{\sqrt{n!}}$

   (c) $\sum \dfrac{\cos^2(n^2)}{n^2}$

   (d) $\sum \dfrac{(-1)^n}{n^{1/3}}$