Math 597/697: Homework 3

1. Let $D_1, D_2, \cdots$ be the successive values from indepenedent rolls of a standard six-sided die. Let $S_n = D_1 + D_2 + \cdots + D_n$. Let

$$T_1 = \min\{n \geq 1 : S_n \text{ is divisible by } 8\},$$

$$T_2 = \min\{n \geq 1 : S_n - 1 \text{ is divisible by } 8\}.$$

Find $E[T_1]$ and $E[T_2]$.

Hint: Write S_n modulo 8 as a Markov chain

2. Let X_n and Y_n be two independent Markov chains with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}. \tag{1}$$

Suppose $X_0 = 0$ and $Y_0 = 2$ and let

$$T = \inf\{n : X_n = Y_n\}.$$

   (a) Find $E[T]$.

   (b) What is $P\{X_T = 2\}$?

   (c) In the long run what percentage of time are both chains in the same state?

Hint: Consider the nine-state Markov chain $Z_n = (X_n, Y_n)$.

3. You start to play backgammon with a friend. You have \$5 and your friend \$20. If you win a game your friend gives you \$1 and if you loose you give him \$1. Being a better player, the probability that you win any single game is .6. What is the probability that you wipe out your friend. What if you start with \$10?

4. Let X_n be an irreducible Markov chain and let

$$M_{ij} = E[\sigma_j | x_0 = i] = \sum_{n \geq 1} P\{\sigma_j \geq n | X_0 = i\}$$

denote the expected first return time to the state j, starting from i.

(a) Analyzing the first step show that

$$M_{ij} = 1 + \sum_{\substack{k \\ k \neq j}} P_{ik} M_{kj}$$

(b) Let π_i be the stationary distribution. Multiplying both sides by π_i and summing over i show that

$$\pi_j = \frac{1}{M_{jj}} \,.$$

Remark: This provides another derivation than the one in class for the formula $\pi_j = E[\sigma_j | x_0 = j]^{-1}$.

5. Let X_n be a Markov chain with state space $\{1, 2, \cdots, 6\}$ and transition matrix

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/10 & 0 & 0 & 9/10 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 & 0 \\ 1/10 & 3/10 & 1/5 & 1/5 & 1/10 & 1/10 \\ 0 & 3/10 & 0 & 0 & 7/10 & 0 \\ 1/4 & 1/4 & 1/8 & 1/8 & 1/8 & 1/8 \end{pmatrix} . \tag{2}$$

(a) Determine the communication classes and whether they are recurrent or transient.

(b) Reorder the states and put the matrix in normal form.

(c) Compute the stationary distribution associated with each recurrent classes.

(d) Determine the mean time spent in transient states.

(e) Starting in a transient state compute the probabilities to be absorbed in the various recurrent classes.

(f) Using (a)-(e) determine

$$\lim_{n \to \infty} P_{ij}^n \,,$$

for all i, j.

6. Mr Smith is growing grapes for his winery. The fall weather in his area follows a daily pattern which can be modeled as a 4-state Markov chain with states 0 (sunny and warm), 1 (cool), 2 (gray and dreary), 3 (raining). The transition matrix for this chain is given by

$$P = \begin{pmatrix} .4 & .2 & .1 & .3 \\ .4 & .3 & ..2 & .1 \\ .6 & .1 & .1 & .2 \\ .2 & .4 & .3 & .1 \end{pmatrix} . \tag{3}$$

Today it is now gray and dreary. Another sunny and warm day would bring his grapes to perfection, but rain would badly damage his harvest. Mr Smith must decide whether to pick his grapes today or risk waiting for another sunny day. To help him decide compute the probability that a sunny day will occur before rain.

Hint: To do this transform 0 and 3 into absorbing states.

7. Consider the following Markov chain with state space $S = \{0, 1, 2, 3, \cdots\}$. A sequence of positive number $p_1, p_2, \cdots$ is given with $\sum_{n=1}^{\infty} p_i = 1$. When the Markov chain reaches 0 it chooses a new state with probability p_i. When the state is in another state than 0 it moves deterministically toward 0. In other words

$$P_{0,i} = p_i \,, \quad i \geq 1$$

and

$$P_{ii-1} = 1 \,, \quad i \geq 1$$

Determine when the Markov chain is positive recurrent.

Hint: Compute $E[\sigma_0 \,|\, X_0 = 0]$.

8. Consider the Markov chain with state space $S = \{0, 1, 2, 3, \cdots\}$ and transition probabilities

$$P_{ii+1} = 2/3 \,, \quad P_{i0} = 1/3 \,.$$

Show that the chain is positive recurrent and give the stationary distribution π.

9. Let P be a transition matrix and define

$$A^{(n)} \;=\; \frac{1}{n} \sum_{j=0}^{n-1} P^k \,.$$

Suppose that there exists numbers $\pi(j)$ with $\sum_j \pi(j) = 1$ such that

$$\lim_{n \to \infty} A_{ij}^{(n)} \;=\; \pi(j) \,.$$

Show that π is a stationary distribution.

10. Let $\{a_n\}$ and $\{b_n\}$ be sequences of numbers. We say that $a_n \sim b_n$ if $\lim_{n \to \infty} a_n / b_n = 1$. Show that if $a_n \sim b_n$ then

$$\sum_n a_n < \infty \quad \text{ if and only if } \quad \sum_n b_n < \infty \,.$$