Dissipative Transport

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Nonequilibrium stationary states (NESS) describe the state of a mechanical system driven and maintained out of equilibrium by external forces. The main characteristic of a NESS is that it sustains sustain steady flows or equivalently that it exhibits positive entropy production. We discuss general features of the *fluctuations* of the entropy production and corresponding flows.

The NESS and the entropy production have been formally defined in the entry "Nonequilibrium Stationary States" to which we refer for details. A probability measure for ν is a NESS for the dynamical system (M, Φ_t) with a given reference measure μ if ν is ergodic and is an SRB-measure with respect to μ. The corresponding entropy production σ is defined by $\sigma(x) = \frac{d}{dt} \log \frac{d\mu_t}{d\mu}|_{t=0}$.

In comparison to the theory of equilibrium states in statistical mechanics little is known about the general properties of NESS. A notable exception is a general and striking universal property of the fluctuations of the entropy production in the NESS which occur if the system is *time-reversible*, i.e. if there exists an involution i of the phase space M such that, for all t,

$$\Phi_t \circ i \, = \, i \circ \Phi_{-t} \, .$$

Let us consider the ergodic average of the entropy production

$$\frac{1}{t} \int_0^t \sigma \circ \Phi_s(x) \tag{1}$$

and let us assume that x is distributed according to the NESS ν. This defines a stationary sequence of real-valued random variable and we say that this sequence satisfies a *Large Deviation Principle* if there exists a convex rate function $I(z)$ such that

$$\lim_{t \to \infty} \frac{1}{t} \log \mu \left\{ x \, ; \, \frac{1}{t} \int_0^t \sigma \circ \Phi_s(x) \in [a, b] \right\} \, = \, - \inf_{z \in [a,b]} I(z) \, .$$

Fluctuation Theorem: *Let ν be a NESS and let σ be the entropy production. Then the ergodic average $t^{-1}\int \sigma \circ \Phi^s ds$ satisfies a large deviation principle with a rate function $I(x)$ which has the symmetry*

$$I(x) - I(-x) \;=\; -x\,,$$

i.e. the odd part of $I(x)$ is linear with slope $-1/2$.

The symmetry of the rate function means that

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\log\frac{\mu\left\{x\,;\,\frac{1}{t}\int \sigma\circ\Phi^s ds \in [z-\epsilon, z+\epsilon]\right\}}{\mu\left\{x\,;\,\frac{1}{t}\int \sigma\circ\Phi^s ds \in [-z-\epsilon, -z+\epsilon]\right\}} \;=\; z\,,$$

i.e. there is a *universal* relation between for the ration of the probabilities to observe, in the NESS, fluctuations of the entropy production to be equal to z and its negative value $-z$.

This symmetry can be also expressed in terms of the logarithmic moment generating function

$$e(\alpha) \;=\; \lim_{t\to\infty}\frac{1}{t}\log\int e^{-\alpha\int_0^t \sigma\circ\Phi_s ds} d\nu\,.$$

The rate function $I(z)$ is the Legendre transform of $e(\alpha)$ and the symmetry of the rate function is equivalent to the relation

$$e(\alpha) = e(1-\alpha)\,.$$

Entropy production, Flows, and Linear Response Theory: In situation of physical interests the entropy production can be written in the form

$$\sigma \;=\; \sum_{i=1}^{N} F_i J_i \;=\; F\cdot J.$$

where $F = (F_1, \cdots, F_N)$ are the *thermodynamical forces* and $J = (J_1, \cdots, J_N)$ are the conjugated *flows* or *currents*.

The logarithmic moment generating function for the vector valued random variable $t^{-1}\int J\circ\Phi_s ds$ is given by

$$e(\lambda, E) \;=\; \lim_{t\to\infty}\frac{1}{t}\log\int e^{-\int_0^t \gamma\cdot J\circ\Phi_s ds} d\nu_F\,,$$

where the second variable indicated the dependence of the NESS ν_F (and possibly also the dynamics Φ_t) on the thermodynamical forces F. This rate function inherits the symmetry

$$e(\lambda, F) = e(F-\lambda, F)\,.$$

Note that

$$\frac{\partial e}{\partial \lambda_i}(0,F) \;=\; -\int J_i d\nu_F$$

is the expectation of the flow in the stationary state ν_F. Therefore the *linear response coefficients* L_{ij} satisfy

$$L_{ij} \;=\; \frac{\partial}{\partial F_j}\int J_i d\nu_F|_{F=0} \;=\; -\frac{\partial e^2}{\partial \lambda_i \partial F_j}(0,0)\,.$$

The symmetry of $e(\lambda, F)$, on the other hand, implies that

$$\frac{\partial e^2}{\partial \lambda_i \partial F_j}(0,0) \;=\; -\frac{1}{2}\frac{\partial e^2}{\partial \lambda_i \partial \lambda_j}(0,0)$$

and thus we obtain

$$L_{ij} = \frac{1}{2}\frac{\partial e^2}{\partial \lambda_i \partial \lambda_j}(0,0) \;=\; \frac{1}{2}\int_{-\infty}^{\infty}\left(\int F_i F_j \circ \phi_t \, d\nu_0\right) dt\,,$$

i.e., *Kubo formula* and *Onsager relations* $L_{ij} = L_{ji}$.

We have given here a *formal* derivation, freely assuming that the rate function are differentiable and exchanging derivative with limits. This can be justified, for example, under the assumption that

(a) The expectation values of the flows in the NESS are differentiable.

(b) The current-current correlation functions are integrable, i.e. sufficiently fast mixing.

(c) The finite-time moment generating function $t^{-1}\log\int e^{-\int_0^t \gamma\cdot J\circ\Phi_s ds} d\nu_F$, . are uniformly bounded in a *complex* neighborhood of the origin (allowing the use of Vitali convergence Theorem).

A mathematical proof of these properties requires therefore very strong ergodic properties of the system under consideration. Some example of physical systems where the fluctuation theorem and the corresponding linear response theory have been established are

(i) Time-reversible Anosov discrete and continuous time dynamical systems on compact manifolds

(ii) Certain examples of open classical systems consisting of a collection of oscillators (harmonic or anharmonic) and the reservoirs are free phonons fields or infinite collection of harmonic oscillators.

(iii) A large class of random dynamical systems admitting a Markovian or Gibbsian description.

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