Math 623: Homework 5

1. Exercise 4, p. 146
2. Exercise 5, p.146
3. Exercise 7, p. 147
4. Suppose f and g are of bounded variation on $[a, b]$.
   (a) Show $af = +bg$ is of bounded variation for abitrary $a, b \in \mathbf{R}$.
   (b) Show that fg, $|f|^p$ for any $1 \leq |p| < \infty$, and, if $\inf_{x \in [a,b]} |g(x)| \geq c > 0$, f/g are of bounded variation.
5. Suppose $a, b > 0$. Let $f(0) = 0$ and $f(x) = x^a \sin(x^{-b})$ for $0 < x \leq 1$. Show that f is of bounded variation if and only if $a > b$.
6. We say that a function $f : [a, b] \to \mathbf{R}$ satisfies a H ölder condition with exponent α if there exists a constant $L > 0$

   $$|f(x) - f(y)| \leq L|x - y|^\alpha$$

   for all $x, y \in [a, b]$. Note that a Hölder condition with exponent $\alpha = 1$ is also called a Lipschitz condition.

   (a) Show that if f satisfies a H ölder condition with exponent α with $\alpha > 1$ then f must be a constant. So only the case $0 < \alpha \leq 1$ is non-trivial.
   (b) Show that $f(x) = \sqrt{x}$ on $[0, 1]$ satisfies a H ölder condition with exponent $1/2$ but not a H ölder condition with exponent $\alpha > 1/2$. Is $f = \sqrt{x}$ of bounded variation?
   (c) Show that $f(x) = x^a \sin(x^{-a})$ satisfies a a H ölder condition with exponent $\alpha < 1$. This shows that, unless $\alpha = 1$, a Hölder condition is not sufficient to ensure that f is of bounded variation.
   Hint: Show using the mean value theorem that $|f(x + h) - f(x)| < Ch/x$ and also that $|f(h) - f(0)| \leq C'h^a$.
7. Compute the positive and negative variation of $f(x) = x^3 - |x|$, $-1 \leq x \leq 1$ and $f(x) = \cos(x)$ for $0 \leq x \leq 2\pi$.
8. Exercise 13, p. 147
9. Exercise 15, p. 148
10. Exercise 19, p.148
11. Exercise 22 (a), p.149
12. Exercise 24, p. 150