# Math 697 Fall 2014: Week 9

**Exercise 1** Given a branching process with the following offspring distributions determine the extinction probability $a$.

**(a)** $p(0) = .25, p(1) = .4, p(2) = .35$

**(b)** $p(0) = 5, p(1) = .1, p(3) = .4$

**(c)** $p(0) = .62, p(1) = .30, p(2) = .02, p(3) = .02, p(6) = .02, p(13) = .02$

**(d)** $p(i) = (1-q)q^i$

**Exercise 2** ***(Computer exercise)*** Write down a computer program which compute the extinction probability given a distribution of offsprings $p(k)$ by solving the equation $a = \phi(a)$.

- Use it to compute the extinction probability if

  $$p(0) = 1/10, p(1) = 3/10, p(2) = 2/10, p(4) = 1/20, p(5) = 1/20, p(8) = 1/10, p(12) = 2/10$$

- Use the previous algorithm to compute the probability that the population dies out after $n = 20, 100, 200, 1000, 1500, 2000, 5000$ generations if $p(0) = p(1) = p(2) = 1/3$.

**Exercise 3** Consider a population with the following rules of (asexual) reproduction. An individual has probability $q$ to leave long enough to reproduce and if it does, it produces 1 or 2 offsprings with equal probability and after this no longer reproduces and eventually dies. Suppose that the population starts with 4 individuals.

- For which values of $q$ is it guaranteed that the population will eventually die out?

- If $q = 0.9$ what is the probability that the population will survive forever?

**Exercise 4** Consider a branching process with offspring distribution given by $p_n$. One makes this process irreducible by asserting that if the the population ever dies out, then in the next generation one new individual appears (i.e. $P_{01} = 1$). Determine for which values of $p_n$ the chain is positive recurrent, null recurrent, transient.

**Exercise 5** Jamie is working in a bookstore, ordering books that are not in store and that the customers request. Each order takes 5 minutes to complete. While each order is being filled there is a probability $p_j$ that $j$ more customers arrive with $p_0 = .2$, $p_1 = .2$, $p_2 = .6$. Jamie cannot take a coffee break until a service is completed and no one is waiting in line to order a book. When Jamie starts her shift there is one customer waiting. What is the probability that she ever will take a coffee break.

**Exercise 6** An electric light that has survived $n$ seconds fails during the $(n+1)st$ second with probability $q$ (with $0 < q < 1$).

1. Let $X_n = 1$ if the light is functioning at time $n$ seconds, and $X_n = 0$ otherwise. Let $T$ be the time of failure of the light (in seconds), i.e.,

$$T \; = \; \inf\{n\,;\, X_n = 0\}\,. \tag{1}$$

   Determine $E[T]$.

2. A building contains $m$ lights of the type described above, which behave independently of each other. At time 0 they are all functioning. Let $Y_n$ denote the number of lights functioning at time $n$. Specify the transition matrix of $Y_n$.

3. Find the moment generating function

$$\phi_n(s) \; = \; E[s^{Y_n}] \tag{2}$$

   of $Y_n$. *Hint*: Express $\phi_n$ in terms of $\phi_{n-1}$ and solve the recursion relation.

4. Use the moment generating function to find $P\{Y_n = 0\}$ and $E[Y_n]$.