Math 645: Midterm solution

1. The third equation is $x_3' = 0$ with $x_3(0) = 0$ with solution $x_3(t) = 0$. The first equation is then $x_1' = -x_1$ with $x_1(0) = 1$ with solution $x_1(t) = e^t$. The second equation is then $x_2' = x_2 - e^t + t$ with $x_2(0) = 0$ whose solution is, by DuHamel formula

$$x_2(t) \;=\; \int_0^s e^{t-s}(s - e^s)\, ds \;=\; e^t \int_0^s (-1 + se^{-s}) \;=\; e^t(1-t) - (1+t)\,.$$

Remark: Computing the Jordan normal form is also an option, but the matrix can be easily put in triangular form by exchanging the first and second row.

2. (a) For the first problem using that $|x/(1+x^2)| \le 1/2$ we have

$$\begin{aligned}\left|2 - x - \frac{4xy}{1+x^2}\right| &\le 2 + |x| + 4\left|\frac{x}{1+x^2}\right||y| \le 2 + |x| + 2|y| \le 2 + \|(x,y)\|_\infty \\ \left|3x\left(1 - \frac{y}{1+x^2}\right)\right| &\le 3|x| + \frac{3}{2}|y| \le 3\|(x,y)\|_\infty \end{aligned} \qquad (1)$$

so that $\|f\|_\infty \le 2 + 3\|(x,y)\|_\infty$ and thus f is linearly bounded.

Remark: The result does not depend on the choice of norm (but the constants do). However the calculations are made much easier by choosing $\|\cdots\|_\infty$. An alternative is the Liapunov function $V(x,y) = x^2/2 + y^2/2$.

(b) For the second equation $x\,(3 - x^2 - 2y^2)$ has the same sign as x if (x,y) is outside the ellipse with equation $x^2 + 2y^2 = 3$. On the other hand $x^2 - y^3$ is positive if $y > x^{2/3}$ and negative if $y < x^{2/3}$.

Let us consider, for example, the rectangle $ABCD$ where $A = (-a, a^{2/3})$, $B = (a, a^{2/3})$, $C = (a, -a^{2/3})$ and $D = (-a, -a^{2/3})$. There exists a number a^* such that, for all $a > a^*$, the ellipse $x^2 + 2y^2 = 3$ is contained in the interior rectangle $ABCD$ and therefore on each side of the rectangles the vector field $(x(3 - x^2 - 2y^2)\,,\, x^2 - y^3)$ points inward. Given an initial condition (x_0, y_0) we choose a so large that (x_0, y_0) is contained in the rectangle $ABCD$ and $a > a^*$. The solution cannot escape out of the rectangle and thus stays bounded for all time $t > 0$.

3. (a) The matrix $A(t) = S(t)^{-1}BS(t)$ is similar (or conjugate) to B so that the eigenvalues of $A(t)$ are the same as those of B, i.e, -1 and -1.

(b) With transformation $y(t) = S(t)x(t)$ we have

$$y' \;=\; S'x + Sx' \;=\; S'S^1 y + SA(t)S^{-1}y \;=\; (S'S^{-1} + B)y\,, \qquad (2)$$

and $S'S^{-1}$ can be computed explicitly. One finds a system with constant coefficients for y

$$y' \;=\; \left(\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \right) y \qquad (3)$$

with eigenvalues $\lambda_{\pm} = -1 \pm \sqrt{-(4+a)a}$. If $-a(4+a) > 1$, i.e., $a \in (-2-\sqrt{3}, -2+\sqrt{3})$ then there is one negative and one positive eigenvalues. If we choose v to be an eigenvector for the positive eigenvalue λ_+ the solution $y(t) = e^{\lambda_+ t}$ satisfies $\lim_{t\to\infty} \|y(t)\| = \infty$. Since $S(t)$ and $S^{-1}(t)$ are bounded uniformly in t, $\lim_{t\to\infty} \|x(t)\| = \infty$ if and only if $\lim_{t\to\infty} \|y(t)\| \to \infty$.

4. By differentiating with respect to t one verifies that $e^{At}x_0 + \int_0^t e^{A(t-s)}B(s)x(s)\,ds$ is a solution. By uniqueness it is the only solution and so the solution must have this form.

   If all eigenvalues of A have negative real part, then there exists constants $K > 0$ and $\mu > 0$ such that $\|e^{At}\| \le Ke^{\mu t}$ so that $\sup_{t>0} \|e^{At}\| \le K$. Therefore for any $t > 0$ we have

$$\|x(t)\| \le \|e^{At}x_0\| + \int_0^t \|e^{A(t-s)}\|\|B(s)\|\|x(s)\|\,ds \;\le\; K\|x_0\| + K\int_0^t \|B(s)\|\|x(s)\|\,. \tag{4}$$

   By Gronwall Lemma with $f(t) = \|x(t)\|$ and $h(s) = K\|B(s)\|$ we obtain

$$\|x(t)\| \;\le\; Ke^{K\int_0^t \|B(s)\|\,ds} \;\le\; Ke^{K\int_0^\infty \|B(s)\|\,ds} \;<\; \infty\,. \tag{5}$$

   Taking the supremum on the left side shows that $\sup_{t>0} \|x(t)\| < C < \infty$.

5.
   - Claim 1: For any $t \in [0,\alpha]$, the sequence $\{x_n(t)\}_{n\ge 0}$ is bounded.
     Proof: : $x_0(t) = 0$ is bounded by b. Assume that $x_{n-1}(s) \le b$ for $t \in [0,\alpha]$ then $x_n(t) = \int_0^t f(s, x_{n-1}(s)\,ds \le Mt \le b$ for $t \in [0,\alpha]$. ■

   - Claim 2: For any $t \in [0,\alpha]$, the sequence $\{x_n(t)\}_{n\ge 0}$ is monotone increasing.
     Proof: Since $x_0(t) = 0$ and f is positive we have $x_1(t) = x_1(t) - x_0(t) = \int_0^t f(s, x_0(s))\,ds \ge 0$ for all $t \in [0,\alpha]$. Let us assume that $x_n(t) > x_{n-1}(t)$ for all $t \in [0,\alpha]$, then we have $x_{n+1}(t) - x_n(t) = \int_0^t f(s, x_n(s)) - f(s, x_{n-1}(s))\,ds \ge 0$ since f is increasing. ■

   - Claim 3: The sequence of function $\{x_n(t)\}$ is equicontinuous.
     Proof: For $0 \le s < t \le \alpha$ we have, using Claim 1, $|x_n(t) - x_n(s)| = x_n(t) - x_n(s) = \int_s^t f(u, x_{n-1}(u))\,du \le M(t-s) = M|t-s|$. This shows that the function $t \mapsto x_n(t)$ uniformly Lispchitz continuous on with a Lipschitz constant which is independent of M. This implies equicontinuity. ■

   Claims 1 and 2 implies that, for fixed t, the sequence $\{x_n(t)\}$ is bounded and monotone increasing. Therefore it converges pointwise to a function $x(t)$. Claim 3 together with Arzela-Ascoli Theorem implies that there exists a convergent subsequence $\{x_{n_k}(t)\}_{k\ge 1}$ which converges *uniformly* to $x(t)$. But for all $n \ge n_k$, Claim 2 implies that $x_{n_k}(t) \le x_n(t) \le x(t)$ and thus $x_n(t)$ converges uniformly to $x(t)$. Therefore $x(t)$ is continuous. Using the uniform convergence and the integral equation $x_n(t) = \int_0^t f(s, x_{n-1}(s))\, ds$ we can interchange limit and integral and show that the continuous fucntion $x(t)$ satisfies the integral equation and therefore is differentiable and satisfies $x' = f(t, x)$.