Math 597/697: Homework 2

1. The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3 someone takes all papers in the pile and put them in the recycling bin. Also if ever there at least five papers in the pile, Mr Smith, with probability one, take the papers to the bin in the afternoon.

   (a) Consider the number of papers in the pile in the evening and describe it with a Markov chain. What are the state space and transition probabilities?

   (b) Show that there is a unique limiting distribution. Compute it.

   (c) After a long time what would be the expected number of papers in the pile?

   (d) Assume that the piles starts with 0 papers. What is the expected time until the pile will have again 0 papers.

2. If the state space is large it can be difficult and tedious to compute the stationary distribution π for the Markov chain with transition probability P, i.e., to solve $\pi P = \pi$. The purpose of this exercise is to derive a simple, easily implementable, algorithm to compute stationary distributions.

   We define the following matrices: I is the $N \times N$ identity matrix and M is the $N \times N$ matrix whose entries are all 1, i.e.,

$$I = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

   Algorithm for the stationary distribution π: Suppose π is the unique stationary distribution for the transition matrix P, then

$$\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}.$$

   It is easy to invert a matrix on a computer and this makes the computation of π on a computer straightforward.

   In order to justify this algorithm you need to prove the following statement

(a) Assume that the matrix $(I - P + M)$ is invertible. Show then $\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}$.

(b) Assume that π is the unique stationary distribution for P. Show then that the unique non-zero solution of $Px = x$ is the column vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

(c) By definition to show that $(I - P + M)$ is invertible you need to show that the only solution of $(I - P + M)x = 0$ is $x = 0$. To do this multiply $(I - P + M)x = 0$ on the left by π and deduce from this that $(1, \cdots, 1)x = 0$. Show then that $Mx = 0$ and $Px = x$. Conclude.

3. We consider the following inventory model for a certain item. Stock levels are inspected at fixed time intervals, say every evening. The following restocking policy is followed. Fix two number $0 \le s \le S$, if the inventory level is $\le s$ then the stock is replenished up to level S. If the inventory level is $> s$ and $\le S$, then no action is taken. Assume furthermore that the demand for the item during successive inspections and replenishments of the stocks is given by i.i.d random variables D_n. If the demand during a single day exceeds the number of items in stock, it is assumed that this demand remains unfullfilled. Let X_n to be the stock level at the inspection time just before restocking.

   (a) Describe this model using the state space $\{0, 1, \cdots S\}$ and determine the transition matrix P if $s = 1$, $S = 3$ and

$$P\{D_i = 0\} = .4\,, \quad P\{D_i = 1\} = .3\,,$$
$$P\{D_i = 2\} = .2\,, \quad P\{D_i = 3\} = .1$$

   (b) Compute the expected amount of items in stock at the moment of inspection in the long run.

   (c) If you want to determine if your restocking policy is efficient an important quantity to compute for this model is the average amount of command per day which goes unfullfilled. To do this consider the Markov chain Y_n where Y_n is defined similarly as in X_n but is also allowed to take negative value. For example $Y_n = -1$ means that 1 command was unfilled that day, this happens for example if $X_n = 2$ and $D_n = 3$. Determine the state space for Y_n and the transition matrix with the same distribution for D_n. Compute the average number of unfullfilled commnads per day in the long run.

4. Consider a Markov chain with state space $\{0, \cdots, 5\}$ and transition

matrix

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .8 & .1 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix}. \tag{1}$$

(a) Determine the communication classes and whether they are recurrent, transient, periodic, aperiodic?

(b) Show that there exists a unique stationary distibution $\pi(j)$ and that $\lim_{n\to\infty} P_{ij}^n = \pi(j)$ for all $1 \le i \le n$. For which j is $\pi(j) > 0$?

(c) Use the algorithm in problem 2. to compute π on your computer.

5. Consider a Markov chain with state space $\{0, \cdots, 5\}$ and transition matrix

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix}. \tag{2}$$

Determine the communication classes. Are they recurrent, transient, periodic, aperiodic? How many stationary distribution are there?

6. *The Ehrenfest urn model.* This is a model of a diffusion of molecules through a membrane. Imagine two containers, labeled A and B, contain a total of $2d$ balls. At each time unit a ball is selected at random from the totality of $2d$ balls and moved to the other container (think a molecule passing through the membrane). Each such selection generates a transition of the process. Clearly one would expect the balls to fluctuate with a average drift from the urn with more balls towards the urn with less balls. Let X_n denote the number of balls in urn A at time n. .

(a) Describe the state space and the transition matrix for X_n?

(b) Give the communication classes of the process. Are they recurrent? transient? periodic? aperiodic?

(c) Show that there exists a unique stationary distribution $\pi(j)$ with $\pi(j) > 0$ for all j. Show that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P_{ij}^k = \pi(j)$.

(d) Show that

$$\pi(j) = \binom{2d}{j} 2^{-2d}$$

is the stationary distribution for X_n.

(e) Compute the expected value and the variance of the number of balls in urn A in the long run.

(f) Suppose that $d = 50$ and that the transitions occurs at the rate of 1000 per second. You decide to put all the balls in urn B so that urn A is empty. You decide to sit down and watch until you see the urn A emtpy again. Compute the expected value of the time you have to wait. Please express your answers in adequate units, minutes?, day?, years?, centuries? What do you think about that?

7. Consider the Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 3/5 & 1/15 \end{pmatrix}.$$

Let τ_3 to be the *first return time* to the state 3, i.e.,

$$\tau_3 = \inf\{n \geq 1\,;\, X_n = 3\}.$$

Compute the probabilities $P\{\tau_3 = n | X_0 = i\}$ for $i = 1, 2, 3$. Don't forget $n = \infty$.

8. The Markov property means that the future depends on the present but not on the past, i.e.,

$$P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}, \cdots X_0 = i_0\} = P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}\}.$$

(a) Show that the Markov property implies that the past depends only on the present but not on the future, i.e.,

$$P\{X_n = i_0 \,|\, X_1 = i_1, \cdots X_n = i_n\} = P\{X_0 = i_0 \,|\, X_1 = i_1\}.$$

(b) Show that the Markov property also implies that, given the present, the past and the future are independent, i.e.,

$$\begin{aligned} & P\{X_{n+1} = i_{n+1}\,,\, X_{n-1} = i_{n-1} \,|\, X_n = i_n\} \\ = \; & P\{X_{n+1} = i_{n+1} \,|\, X_n = i_n\} P\{X_{n-1} = i_{n-1} \,|\, X_n = i_n\}. \end{aligned}$$

9. Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers. Define the sequence $\{b_n\}_{n\geq 1}$ by

$$b_n = \frac{a_0 + \cdots + a_{n-1}}{n} = \frac{1}{n} \sum_{j=0}^{n-1} a_j.$$

Show that if $\lim_{n\to\infty} = a$ then $\lim_{n\to\infty} b_n = a$.