Math 645: Homework 4

1. Compute the general solution of
$$x' = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{pmatrix} x \,. \tag{1}$$

2. Transform the matrix
$$A = \frac{1}{9}\begin{pmatrix} 14 & 4 & 2 \\ -2 & 20 & 1 \\ -4 & 4 & 20 \end{pmatrix} \tag{2}$$
in Jordan normal form and compute the resolvent of $x' = Ax$. *Hint*: All eigenvalues are equal to 2.

3. The equation of motion of two coupled harmonic oscillators is
$$\begin{aligned} x_1'' &= -\alpha x_1 - \kappa(x_1 - x_2)\,, \\ x_2'' &= -\alpha x_2 - \kappa(x_2 - x_1)\,. \end{aligned} \tag{3}$$
Find a fundamental matrix for this system. You can either write it as a first order system and compute the chararcteristic polynomial or, better, stare at the equation long enough until you make a clever Ansatz. Discuss the solution in the case where $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = 0$.

4. Solve the Cauchy problem
$$\begin{aligned} x_1' &= x_1 - 3x_3^3 \\ x_2' &= 2x_1^2 + x_2 + 6x_3 + 1 \\ x_3' &= -3x_3 \end{aligned} \tag{4}$$
with $x(0) = (1, 0, 1)^T$.

5. Compute the resolvant $R(t, 0)$ (in real representation) for the ODE
$$\begin{aligned} x' &= \cos(t)x - \sin(t)y\,, \\ y' &= \sin(t)x + \cos(t)y\,. \end{aligned} \tag{5}$$
Hint: Find an equation for $z = x + iy$.

6. Consider the scalar equation (i.e. $n = 1$) $x' = f(t)x$ where $f(t)$ is continuous and periodic of period p.
   (a) Determine $P(t)$ and R in Floquet Theorem.
   (b) Give necessary and sufficient conditions for the solutions to be bounded as $t \to \pm\infty$ or to be periodic

7. Consider the differential equation $x'' + \epsilon f(t)x = 0$, where $f(t)$ is periodic of period 2π and
$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < \pi \\ 0 & \text{if} \quad \pi < t \le 2\pi \end{cases} \,. \tag{6}$$
For both $\epsilon = 1/4$ and $\epsilon = 4$

(a) Consider the fundamental solution $\Phi(t)$ which satisfies $\Phi(0) = \mathbf{I}$ and compute the corresponding transition matrix $C = e^{pR}$.

(b) Compute the Floquet multipliers (the eigenvalues of C).

(c) Describe the behavior of solution.

8. Let $A(t)$ be periodic of period p and consider ODE $x' = A(t)x$.

 (a) Show that the transition matrix C depends on the fundamental solution, but that the eigenvalues of $C = e^{pR}$ are independent of this choice.

 (b) Show that for each Floquet multiplier λ (the eigenvalue of C), there exists a solution of $x' = A(t)x$ such that $x(t+p) = \lambda x(t)$, for all t.

9. Consider the equation $x' = A(t)x$ where $A(t)$ is periodic of period p.

 (a) Let $\Phi(t)$ be the fundamental solution with $\Phi(0) = 1$. Use Floquet Theorem and Liouville Theorem to show that

$$\det(e^{pR}) \;=\; e^{\int_0^p \mathrm{Trace}(A(s))\,ds}\,. \tag{7}$$

 (b) Deduce from (a) that the characteristic exponents μ_i satisfy

$$\mu_1 + \cdots + \mu_n = \frac{1}{p}\int_0^p \mathrm{Trace}(A(s))\,ds \tag{8}$$