Math 645: Homework 3

1. Consider the Cauchy problem for the Ricatti equation $x' = t^2 + x^2$, $x(0) = 1$. Show that the solution $x(t)$ satisfies the bounds

$$\frac{1}{1-t} \leq x(t) \leq \tan(t + \pi/4) \tag{1}$$

Hint: Consider the Cauchy problems $u' = u^2$ and $v' = 1 + v^2$ and compare $x(t)$ with $u(t)$ and $v(t)$.

2. Let $U \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$ be an open set and let $f : U \to \mathbf{R}^n$ be a continuous functions. Suppose that $f(t, x, c)$ satisfies a local Lipschitz condition, in the sense that for given c, $f(t, x, c)$ satisfies a local Lipschitz condition. Prove that the solution of the Cauchy problem $x' = f(t, x, c)$, $x(t_0) = x_0$, depends continuously on the parameter c.

3. The spectral radius $\rho(A)$ of a $n \times n$ matrix A is defined as

$$\rho(A) = \max\{|\lambda|\,;\, \lambda \text{ eigenvalue of } A\}\,. \tag{2}$$

Let $A \in \mathcal{L}(\mathbf{R}^n)$. Show that for any norm on $\mathbf{R}^n$ we have the inequality $\rho(A) \leq \|A\|$, and that if A is symmetric we have the equality $\|A\|_2 = \rho(A)$.

4. Let

$$A = \begin{pmatrix} 0.999 & 1000 \\ 0 & 0.999 \end{pmatrix} \tag{3}$$

   (a) Compute the spectral radius of A as well as $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$.

   (b) Find a norm on $\mathbf{R}^n$ such that $\|A\| \leq 1$.

5. Show that for any matrix A, there exists a matrix D such that $D^{-1}AD$ is upper triangular. *Hint:* Let λ and v be such that $Av = \lambda v$. Define $Q = (v, y_1, \cdots, y_n)$ where the y_j and v form a basis of $\mathbf{R}^n$ 0r $\mathbf{C}^n$. Compute $Q^{-1}AQ$.

6. Show that for any $n \times n$ matrix A and any *epsilon* > 0, there exists a norm such that $\|A\| \leq \rho(A) + \epsilon$. *Hint:* There exists a matrix D such that DAD^{-1} is upper triangular (or maybe even in Jordan normal form). Consider the diagonal matrix S with entries $1, \mu^{-1}, \cdots, \mu^{1-n}$. Set $\|x\|_\mu = \|SDx\|$ where $\|\cdot\|$ is any norm on $\mathbf{R}^n$.

7. Find matrices A and B such that $e^{A+B} \neq e^A e^B$.

8. Show that if $A(t)$ is antisymmetric, i.e., $A^T = -A$, then the resolvant of $x' = A(t)x$ is orthogonal. *Hint:* Show that the scalar product of two solutions is constant.

9. Using the definition of the exponential matrix, compute the e^{tA} for the following matrices

   (a) $A = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix}$ with $\kappa > 0$.

   (b) $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

10. **(D'Alembert reduction method)**. Consider the ODE $x' = A(t)x$ where $A(t)$ is a $n \times n$ matrix and assume that we know one non-trivial solution $x(t)$. Show that one can reduce the equation $x' = A(t)x$ to the problem $z' = B(t)z$ where $z \in \mathbf{R}^{n-1}$ and $B(t)$ is a $(n-1) \times (n-1)$ matrix. *Hint:* Without loss of generality you may assume that $x_n(t) \neq 0$. Look for solutions of the form $y(t) = \phi(t)x(t) + z(t)$, where $\phi(t)$ is a scalar function and z has the form $z = (z_1, \cdots, z_{n-1}, 0)^T$.

11. (a) Using the previous problem, compute the resolvent $R(t, 1)$ of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x \,, \tag{4}$$

using the fact that $x(t) = (t^2, -t)^T$ is a solution. *Hint:* The solution is

$$\begin{pmatrix} t^2(1 - \log t) & -t^2 \log t \\ t \log t & t(1 + \log t) \end{pmatrix} \tag{5}$$

(b) Compute the solution of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x + \begin{pmatrix} t \\ -t^2 \end{pmatrix} \,, \tag{6}$$

with initial condition $x(1) = (0, 0)^T$.

12. Consider the system

$$x' = A(t)x \,, \qquad A(t) = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix} \,. \tag{7}$$

(a) Compute the resolvent of (7).

(b) Show that $R(t, t_0) \neq \exp\left(\int_{t_0}^t A(s)\, ds\right)$.

(c) Show that $A(t)$ does not commute with $\int_{t_0}^t A(s)\, ds$.

(d) Show that if $A(t)$ does commute with $\int_{t_0}^t A(s)\, ds$ then the resolvent for $x' = A(t)x$ is $R(t, t_0) = \exp\left(\int_{t_0}^t A(s)\, ds\right)$