Math 697: Homework 4

Exercise 1 Problem 3.1, p. 82

Exercise 2 Machine 1 is currently working and machine 2 will be put in use at a time T from now. If the lifetimes of the machines 1 and 2 are exponential random variables with parameters λ_1 and λ_2, what is the probability that machine 1 is the first machine to fail?

Exercise 3 Consider a two-server system in which a customer is first served by server 1, then by server 2 and then departs. The service times at server i are exponential random variables with parameter μ_i. $i = 1, 2$. When you enter the system you find server 1 free and two customers at server 2, customer A in service and customer B waiting in line.

1. Find the probability P_A that A is still in service when you move over to server 2.
2. Find the probability P_B that B is still in service when you move over to server 2.
3. Compute $E[T]$, where T is the total time you spend in the system. *Hint:* Write $T = S_1 + S_2 + W_A + W_B$ where S_i is your service time at server i, W_A the amount of time you wait in queue when while A is being served, and W_B the amount of time you wait in queue when while B is being served.

Exercise 4

Let N_t be a Poisson process with rate λ and let $0 < s < t$. Compute

1. $P\{N_t = n + k | N_s = k\}$
2. $P\{N_s = k | N_t = n + k\}$
3. $E[N_t N_s]$

Exercise 5 Problem 3.2, p. 83

Exercise 6 A component is in two possible states 0=on or 1=off. A system consists of two components A and B which are independent of each other. Each component remains on for an exponential time with rate λ_i, $i = A, B$ and when it is off it remains off for an exponential time with rate $\mu_i, i = A, B$. Determine the long run probability that the system is operating if

1. They are working in parallel, i.e. at least one must be operating for the system to be operating.
2. They are working in series, i.e. both must work for the system to be operating.

Exercise 7 Problem 3.8, p. 84

Exercise 8 Problem 3.11, p. 84

Exercise 9 Problem 3.13, p. 85

Exercise 10 In a coffee shop, Anna is managing the single cash register. Customers enter the shop according to a Poisson process with parameter λ and the service time at the cash register is exponential with parameter μ. WE denote by X_t be the number of customers in the system, i.e. being served or waiting in line to be served. The customers in queue, but not the one in service, might get discouraged and decide to leave the line. Assume that each customer who joins the queue will leave after an exponential time with parameter γ if the the customer has not entered service before.

1. Suppose that customers enters the coffee shop and find one customer inside (being served at the register). Compute the expected amount of time he will spend in the system.
2. Describe X_t has birth and death process, i.e. give the parameter λ_n and μ_n and write down the generator of the process.
3. Determine for which parameter λ, μ, γ the process is positive recurrent.
4. Consider the special case $\mu = 2\gamma$. Compute the stationary distribution.

Exercise 11 For a general birth and death process, find a differential equation for the expected population at time t, $E[X_t]$. Solve this equation for the $M/M/1$ and $M/M/\infty$ queues.

Exercise 12 Consider a continuous time branching process defined as follows. A organism life-time is exponential with parameter λ and upon death, it leaves k offspring with probability p_k, $k \geq 0$. The organisms act independently of each other. For simplicity assume that $p_1 = 0$. Let X_t be the population at time t, find the generator and write down the forward and backward equations for the process. Specialize your results to the binary splitting case where the particle either splits in two or vanishes. Find the stationary distribution of $\{X_t\}$.

Exercise 13 An airline reservation system has two computers, one on-line and one backup. The operating computer fails after an exponentially distributed time with parameter μ and is replaced by the backup. There is one repair facility and the repair time is exponentially distributed with parameter λ. Let X_t denote the number of computers in operating condition at time t. Write down the generator A of X_t and the backward and forward equations (no need to solve them). In the long run, what is the proportion of the time when the reservation system is on.

Answer the same questions in the case where the two machines are simultaneously online if they are in operating condition.

Starting with two machines in operation, compute in both cases, the expected value of the time T until both are in the repair facility.