Math 597/697: Solutions for Homework 1

1. (a) We have

$$\begin{aligned} E[X] &= P\{X=1\} + 2P\{X=2\} + 3P\{X=3\} + \ldots \\ &= P\{X=1\} + P\{X=2\} + P\{X=3\} + \ldots \\ &\qquad + P\{X=2\} + P\{X=3\} + \ldots \\ &\qquad\qquad + P\{X=3\} + \ldots \\ &\qquad\qquad \vdots \\ &= P\{X \geq 1\} + P\{X \geq 2\} + P\{X \geq 3\} + \ldots \\ &= P\{X > 0\} + P\{X > 1\} + P\{X > 2\} + \ldots \end{aligned} \tag{1}$$

(b) We have $P\{X \geq x\} = \int_x^\infty f(x)\,dx$ and so by the fundamental theorem of calculus

$$\frac{d}{dx} P\{X \geq x\} = -f(x)\,. \tag{2}$$

Integrating by parts

$$E[X] = -xP\{X \geq x\}|_0^\infty + \int_0^\infty P\{X \geq x\}\,dx \tag{3}$$

To see that the first term vanishes we note that

$$xP\{X \geq x\} = x\int_x^\infty f(y)\,dy \leq \int_x^\infty yf(y)\,dy \tag{4}$$

Since $E[X] = \int_0^\infty yf(y)\,dy$ is finite $\int_x^\infty yf(y)\,dy$ goes to 0 as x goes to infinity, by the bounded convergence Theorem for example.

2. (a)

$$E[X^n] = f^{(n)}(0)\,, \quad Var(X) = f''(0) - f'(0)^2 \tag{5}$$

(b)

$$E[e^{tX}] = \sum_{n=1}^\infty e^{tn}(1-p)^{n-1}p = \frac{e^t p}{1 - e^t(1-p)} \tag{6}$$

and $E[X] = 1/p$, $Var(x) = (1-p)/p^2$.

(c)

$$E[e^{tX}] = \frac{1}{b-a}\int_a^b e^{tx}\,dx = \frac{e^{tb} - e^{ta}}{t(b-a)} \tag{7}$$

and $E[X] = (a+b)/2$, $Var(X) = (b-a)^2/12$.

3. $p_Y(y) = \frac{e^{-y}}{y} \int_0^\infty e^{-x/y} = e^{-y}$ so

$$E[X|Y=y] \;=\; \int_0^\infty \frac{x}{y} e^{-\frac{x}{y}}\, dx \;=\; y\,. \tag{8}$$

4. We have $P\{Y=1\} = 5/9$ and

$$p_{X|Y}(1|1) \;=\; \frac{1}{5}\,, \quad p_{X|Y}(2|1) \;=\; \frac{3}{5}\,, \quad p_{X|Y}(3|1) \;=\; \frac{1}{5}\,. \tag{9}$$

So

$$E[X|Y=1] = \frac{1}{5} + \frac{6}{5} + \frac{3}{5} \;=\; 2\,. \tag{10}$$

The random variables X and Y are not independent, since for example

$$P\{X=1\} = \frac{2}{9} \neq P\{X=1|Y=1\} = \frac{1}{5} \tag{11}$$

5. (a)

$$P\{Y=1\} \;=\; 3\frac{5}{11}\frac{6}{10}\frac{5}{9} \tag{12}$$

(b) If $Y=1$, then X can be 0, 1, or 2. For example if $X=0$ we must draw two red balls and 1 black balls, this occurs with probability $3\frac{5}{11}\frac{2}{10}\frac{1}{9}$, etc... We have

$$\begin{aligned} p_{X|Y}(0|1) &= \frac{3\frac{5}{11}\frac{2}{10}\frac{1}{9}}{3\frac{5}{11}\frac{6}{10}\frac{5}{9}} = \frac{2}{30} \\ p_{X|Y}(1|1) &= \frac{6\frac{5}{11}\frac{4}{10}\frac{2}{9}}{3\frac{5}{11}\frac{6}{10}\frac{5}{9}} = \frac{16}{30} \\ p_{X|Y}(2|1) &= \frac{3\frac{5}{11}\frac{4}{10}\frac{3}{9}}{3\frac{5}{11}\frac{6}{10}\frac{5}{9}} = \frac{12}{30} \end{aligned} \tag{13}$$

so that

$$E[X|Y=1] \;=\; \frac{4}{3} \tag{14}$$

6. With replacement we have

(a)

$$P\{Y=1\} \;=\; 3\frac{5}{11}\frac{6}{11}\frac{6}{11} \tag{15}$$

(b) We have

$$
\begin{aligned}
p_{X|Y}(0|1) &= \frac{3\frac{5}{11}\frac{2}{11}\frac{2}{11}}{3\frac{5}{11}\frac{6}{11}\frac{6}{11}} = \frac{1}{9} \\
p_{X|Y}(1|1) &= \frac{6\frac{5}{11}\frac{4}{11}\frac{2}{11}}{3\frac{5}{11}\frac{6}{11}\frac{6}{11}} = \frac{4}{9} \\
p_{X|Y}(2|1) &= \frac{3\frac{5}{11}\frac{4}{11}\frac{4}{11}}{3\frac{5}{11}\frac{6}{11}\frac{6}{11}} = \frac{4}{9}
\end{aligned}
\tag{16}
$$

so that

$$E[X|Y=1] = \frac{4}{3} \tag{17}$$

7. Let us condition on the first door the prisoner is using. We find

$$E[X] = 1\frac{1}{3} + (2+E[X])\frac{1}{3} + (4+E[X])\frac{1}{3} \tag{18}$$

This gives $E[X] = 7$. If the prisoner remembers the doors he has chosen before then

$$E[X] = 1\frac{1}{3} + 3\frac{1}{6} + 5\frac{1}{6} + 7\frac{1}{6} + 7\frac{1}{6} = 4 \tag{19}$$

8. (a) $X = \sum_{i=1}^{N} T_i$.

(b) N has a geometric distribution with $p = 1/3$ since $P\{N = n\}$ is the probability to choose the "right" door in the n-th trial. Hence $E[N] = 3$.

(c) $E[T_N] = 1$ since the last door leads outside.

(d) To compute $E[\sum_{i=1}^{N} T_i | N = n]$, note that n-th door is the first one and the the $n-1$ first doors were either the second or the third one. If the prisoner chooses k times the second and $n-1-k$ times the third then the travel times is $1 + k2 + (n-k-1)4$. The probability to choose k times the second door is a binomial with parameters $n-1$ and $1/2$. Hence

$$
\begin{aligned}
E[\sum_{i=1}^{N} T_i | N = n] &= \sum_{k=0}^{n-1} (1 + k2 + (n-k-1)4) \binom{n}{k} \frac{1}{2}^{n-1} \\
&= 1 + 4(n-1) - 2(n-1)\frac{1}{2} = 3n - 2
\end{aligned}
$$

(e) We find $E[X] = E[E[\sum_{i=1}^{N} T_i|N]] = E[3N-2] = 7$.

9. (a) Let us condition on X, which is a binomial random variable with parameters n and p. If the gamblers wins $X = k$ of his first n games he will play k more and the probability to win l more games has a binomial distribution with paramters k and p. Hence

$$E[N|X=k] \;=\; k + kp\,. \tag{20}$$

So

$$E[N] \;=\; E[E[N|X]] \;=\; E[X+Xp] \;=\; (1+p)E[X] \;=\; (1+p)pn \tag{21}$$

(b) Let us condition on Y, which is a geometric random variable with parameter p. If the gamblers wins for the first time at his n-th trial, $Y = n$ and he will play n more and the probability to win k more games has a binomial distribution with parameters n and p. Hence

$$E[N|Y=n] \;=\; 1 + np\,. \tag{22}$$

So

$$E[N] \;=\; E[E[N|Y]] \;=\; E[1+Yp] \;=\; 1+pE[Y] \;=\; 1+p\frac{1}{p} \;=\; 2 \tag{23}$$

10. Condition on the time T at which you arrive. If $T = t$ then every other guest has, independentlyof each other, a probability t to arrive before you and a probability $1-t$ to arrive after you. So the number of people who arrive before you is a binomial with parameters 12 and t. So

$$E[N|T=t] \;=\; 12t \tag{24}$$

and

$$E[N] \;=\; E[E[N|T]] \;=\; 12E[T] \;=\; 12\frac{1}{2} \;=\; 6 \tag{25}$$

11. Conditioning on Λ we find

$$\begin{aligned} P\{N \geq 3\} &= \int_0^{1/2} P\{N \geq 3|\Lambda = \lambda\} 2d\lambda \\ &= 2\int_0^{1/2} \left(1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2}{2}e^{-\lambda}\right) d\lambda \\ &= \frac{33}{4}e^{-1/2} - 5\,. \end{aligned} \tag{26}$$

12. In both cases we will condition on the outcomes of the first two rolls. There are three possible events: the first one called $A6$ in which A rolls a 6 and wins (probability 5/36), the second called $B7$ in which A does not roll a 6 and B rolls a 7 and wins (probability $31/36 \times 1/6$), and the third one called NO in which A does not roll a 6 and B does not roll a 7 (probability $31/36 \times 5/6$) and the games starts again fresh.

(a) We have

$$\begin{aligned}
& P(A\ wins) \\
&= P(A\ wins|A6)P(A6) + P(A\ wins|B7)P(B7) \\
&\quad + P(A\ wins|NO)P(NO) \\
&= 1\frac{5}{36} + 0\frac{31}{36}\frac{1}{6} + P(Awins)\frac{31}{36}\frac{5}{6}\,. \qquad (27)
\end{aligned}$$

and one finds $P(A\ wins) = \frac{30}{61}$.

(b) Let N the number of rolls until somebody wins. We have

$$\begin{aligned}
E[N] &= E[N|A6]P(A6) + E[N|B7]P(B7) + E[N|NO]P(NO) \\
&= 1\frac{5}{36} + 2\frac{31}{36}\frac{1}{6} + (2 + E[N])\frac{31}{36}\frac{5}{6}\,. \qquad (28)
\end{aligned}$$

and one finds $E[N] = \frac{402}{61}$.