Lecture 11: Probability models

Probability is the mathematical toolbox to describe phenomena or experiments where randomness occur. To have a probability model we need the following ingredients

- A *sample space* S which is the collection of all possible outcomes of the (random) experiment. We shall consider mostly finite sample space S.
- A *probability distribution*. To each element $i \in S$ we assign a probability $p(i) \in S$.

  $$p(i) \;=\; \text{Probability that the outcome } i \text{ occurs}$$

  We have
  $$0 \le p(i) \le 1 \qquad \sum_{i \in S} p(i) = 1\,.$$

- An *event* A is a subset of the sample space S. It describes an experiment or an observation that is compatible with the outcomes $i \in A$. The probability that A occurs, $P(A)$, is given by
  $$P(A) \;=\; \sum_{i \in A} p(i)\,.$$

Example: Thoss three (fair) coins and record if the coin lands on tail (T) or head (H). The sample space is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and has $8 = 2^3$ elements. For fair coins it is natural assign the probability 1/8 to each outcome.

$$P(HHH) = P(HHT) = \cdots = P(TTT) \;=\; 1/8$$

An example of an event A is that at least two of the coins land on head. Then

$$A = \{HHH, HHT, THH, HHT\} \qquad P(A) = 1/2\,.$$

■

The *basic operations of set theory* have a direct probabilistic interpretation:

- The event $A \cup B$ is the set of outcomes wich belong either to A or to B. We say that $P(A \cup B)$ is the probability that either A or B occurs.
- The event $A \cap B$ is the set of outcomes which belong to A and to B. We say that $P(A \cup B)$ is the probability that A and B occurs.

- The event $A \setminus B$ is the set of outcomes which belong to A but not to B. We say that $P(A \setminus B)$ is the probability that A occurs but B does not occurs.

- The event $\overline{A} = S \setminus A$ is the set of outcomes which do not belong to A. We say that $P(\overline{A})$ is the probability that A does not occur.

We have the following simple rules to compute probability of events. Check them!

Theorem 1. *Suppose A and B are events. Then we have*

1. $0 \leq P(A) \leq 1$ *for any event* $A \subset S$.
2. $P(A) \leq P(B)$ *if* $A \subset B$.
3. $P(\overline{A}) = 1 - P(A)$
4. $P(A \cup B) = P(A) + P(B)$ *if A and B are disjoint.*
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *for general A and B.*

Example: Tossing three coins again let A be the event that the first toss is head while B is the event that the second toss is tail. Then $A = \{HHH, HHT, HTH, HTT\}$, and $B = \{HTH, HTT, TTH, TTT\}$, $A \cap B = \{HTH, HTT\}$. We have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/2 + 1/2 - 1/4 = 3/4$. ■

Odds vs probabilities. Often, especially in gambling situations the randomness of the experiment is expressed in terms of *odds* rather than probabilities. For we make a bet at 5 to 1 odds that U of X will beat U of Z in next week basketball game. What is meant is that the probability that X wins is thought to be 5 times greater than the probability that Y wins. That is we have

$$p = P(\text{X wins}) = 5P(\text{Y wins}) = 5P(\text{X looses}) = 5(1-p)$$

and thus we have $p = 5(p-1)$ of $p = 5/6$. More generally we have

$$\boxed{\text{The odds of an event } A \text{ are } r \text{ to } s \iff \frac{P(A)}{1-P(A)} = \frac{r}{s} \iff P(A) = \frac{r/s}{r/s+1}}$$

Uniform distribution. In many examples it is natural to assign the same probability to each event in the sample space. If the sample space is S we denote by the cardinality of S by

$$\#S = \text{ number of elements in } S$$

Then for every event $i \in S$ we set

$$p(i) = \frac{1}{\#S},$$

and for any event A we have

$$p(A) = \frac{\#A}{\#S}$$

Example. Throw two fair dice. The sample space is the set of pairs (i, j) with i and j an integer between 1 and 6 and has cardinality 36. We then obtain for example

$$P(\text{Sum of the dice is 2}) = \frac{1}{36}, \quad P(\text{Sum of the dice is 9}) = \frac{4}{36}, \cdots$$

$\blacksquare$

The birthday problem. A classical problem in probability is the following. What is the probability that among N people at least 2 have the same birthday. As it turns out, and at first sight maybe surprisingly, one needs few people to have a high probability of matching birthdays. For example for $N = 23$ there is a probability greater than $1/2$ than at least two people have the same birthday. To compute this we will make the simplifying but reasonable assumptions that there is no leap year and that every birthday is equally likely.

If there are N people present, the sample space S is the set of all birthdays of the everyone. Since there is 365 choice for everyone we have

$$\#S = 365^N$$

We consider the event

$$A = \text{ at least two people have the same birthday}$$

It is easier to consider instead

$$B = \overline{A} = \text{ no pair have the same birthday}$$

To compute the cardinality of B we make a list of the N people (the order does not matter). There is 365 choice of birthday for the first one on the list, for the second one on the list, there is only 364 choice of birthday if they do not have the same birthday. Continuing in the same way we find

$$\#B = 365 \times 364 \times 363 \times \cdots \times (365 - N + 1)$$

and so

$$\begin{aligned} P(B) &= \frac{365 \times 364 \times 363 \times \cdots \times (365 - (N-1)}{365^N} \\ &= 1 \times \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \times \left(1 - \frac{N-1}{365}\right) \end{aligned}$$

To compute this efficiently we recall from calculus (use L'Hospital rule) that for number x we have

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

We take $n = 365$ which is a reasonably large number and take $x = -1, -2, \cdots$. We have

$$\left(1 - \frac{j}{365}\right) \approx e^{-j/365}.$$

We find then

$$\begin{aligned} P(B) &\approx 1 \times e^{-1/365} \times e^{-2/365} \times e^{-(N-1)/365} \\ &= e^{-(1+2+\cdots N-1)/365} \\ &= e^{-N(N-1)/730} \end{aligned}$$

by using the well-known identity $1 + 2 + \cdots N = \frac{N(N+1)}{2}$.

How many people are needed to have a probability of 1/2 of having 2 same birthday. We have

$$P(B) = \frac{1}{2} \approx e^{-N(N-1)/730} \iff N(N-1) = 730 \ln(2)$$

Even for moderately small N, $N(N-1) \approx N^2$ and so we find the approximate answer

$$N \approx \sqrt{730 \ln(2)} = 22.49$$

That is if there are 23 people in a room, the probability that two have the same birthday is greater than 1/2. Similarly we find that if there are

$$N \approx \sqrt{730 \ln(10)} = 40.99$$

people in the room there is a probability greater than .9 than two people have the same birthday.

Exercise 1: Bob and Maria are taking a math class with final grades A, B or C. The probability that Bob gets a B is .3 and the probability that Maria gets a B is .4. The probability that neither gets an A but at least one gets a B is .1. What is the probability that at least one gets a B but neither gets a C?

Exercise 2:

1. What odds should you give in favor of the following event?
   (a) A card chosen at random from a 52-card deck is an ace?
   (b) Exactly two heads will turn up when three coins are tossed?
2. In a horse race the odds that Romance wins are given 2 to 3 while the odds that Downhill wins are 1 to 2. Give the odds that either Romance or Downhill wins?

Exercise 3:

1. Suppose you roll three dice repeatedly say m times. Show the probability that he three dice will turn up sixes *at least once* is $1 - (1 - \frac{1}{216})^m$.
2. Let us compute the number of times m^* that we need to roll three dices to see at least once three sixes with probability at least 1/2. Find an approximation for m^* using the formula

   $$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n\alpha} = e^{-\alpha}$$

   for suitable n and α.