Math 645: Homework 1

1. Derive the following *error estimate* for the method of successive approximations. Let x be a fixed point given by this method. Show that

$$\|x - x_k\| \leq \frac{\alpha}{1-\alpha}\|x_k - x_{k-1}\|\,. \tag{1}$$

2. Consider the function $f(x) = e^x/4$ on the interval $[0,1]$. Show that f has a fixed point on $[0,1]$, compute the first five iterations and determine the error.

3. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) \;=\; \begin{cases} x + e^{-x/2} & \text{if } x \geq 0 \\ e^{x/2} & \text{if } x \leq 0 \end{cases}\,. \tag{2}$$

   (a) Show that $|f(x) - f(y)| < |x - y|$ for $x \neq y$.

   (b) Show that f does not have a fixed point.

   Explain why this does not contradict the Banach fixed point theorem.

4. Show that the assumption that "D is closed" cannot be omitted in general in the fixed point theorem. Find a set D which is not closed and a map $f : D \to E$ such that $f(D) \subset D$, f is a contraction, but f does not have a fixed point in D.

5. Show that $\|f\|_2$ is a norm on $\mathcal{C}([0,1]$.

6. Show that any norm $\|\cdot\|$ in $\mathbf{R}^n$ is equivalent to the euclidean norm $\|\cdot\|_2$. *Hint:* Write $x = x_1e_1 + \cdots + x_ne_n$ and use the triangle inequality and Cauchy-Schwartz to show that $\|x\| \leq C\|x\|_2$. Show that the map $|\|x\| - \|y\|| \leq C\|x - xy\|_2$ and deduce from this the equivalence of the two norms.

7. (a) Consider the norm of $\mathcal{C}([0,a])$ given by

$$\|f\|_{\mathrm{e}} \;=\; \max_{0\leq t\leq a} |f(t)|e^{-t^2}\,. \tag{3}$$

   (Why is it a norm?) Let

$$Tf(t) \;=\; \int_0^t sf(s)\,ds\,. \tag{4}$$

   Show that $\|Tf\|_\infty \leq \frac{a^2}{2}\|f\|_\infty$ and $\|Tf\|_{\mathrm{e}} \leq \frac{1}{2}\|f\|_{\mathrm{e}}$.

   (b) Show that the integral equation

$$x(t) \;=\; \frac{1}{2}t^2 + \int_0^t sx(s)\,ds\,, \quad t \in [0,a]\,, \tag{5}$$

   has exactly one solution. Determine the solution (i) by rewriting the equation as an initial value problem and solving it, (ii) by using the methods of successive approximations starting with $x_0 \equiv 0$.

8. Let us consider $\mathbf{R}^2$ with the norm $\|x\| = \max\{|x_1|, |x_2|\}$. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be given by
$$f(x_1, x_2) = \begin{pmatrix} x_1^2 + 2x_2^2 + 5 \\ 4x_1x_2 + 3 \end{pmatrix} \tag{6}$$
Let $K = \{(x_1, x_2)\,,\, |x_1| \le 1, |x_2| \le 2\}$. Find an explicit Lipschitz constant for f on K.

9. Apply the Picard-Lindelöf iteration to
$$x' = x^2\,, \quad x(0) = 1\,. \tag{7}$$
Compute the first three iterations $x_1(t)$, $x_2(t)$, $x_3(t)$ and show, by induction, that $x_n(t) = 1 + t + \cdots t^n + O(t^{n+1})$.

10. Apply the Picard-Lindelöf iteration to the Cauchy problem
$$x_1' = 2x_1 + x_2\,, \qquad x_1(0) = 0 \tag{8}$$
$$x_2' = t^2 + 2x_1\,, \qquad x_2(0) = 0 \tag{9}$$
Compute the first five terms in the taylor series of the solution.

11. Consider ODE $x' = f(t, x)$ where
$$f(t,x) = \begin{cases} \frac{6t^5x}{t^6+x^2} & (t,x) \neq (0,0) \\ 0 & (t,x) = (0,0) \end{cases} \tag{10}$$
Show that $f(t, x)$ is continuous but it does not satisfy a Lipschitz condition at the origin. Show that $x = \pm t^3$ and, more generally, any solution of $t^6 = x^2 + cx$ is a solution of the ODE. Deduce that the Cauchy problem $x' = f(t, x)$, $x(0) = 0$ has infinitely many solutions.

12. Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfy a *global Lipschitz condition*, i.e., there exists a positive $L > 0$ such that
$$\|f(x) - f(y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbf{R}^n\,. \tag{11}$$
Consider the Banach space $E = \{g : [t_0, \infty) \to \mathbf{R}^n\,,\, g(t) \text{ continuous }\}$ with the norm
$$\|g\|_\kappa = \sup_{0 \le t < \infty} \|g(t)\|e^{-\kappa t}\,. \tag{12}$$
Using the Banach fixed point theorem, show that the Cauchy problem $x' = f(x)$, $x_0 = x_0$ has a unique solution for $0 \le t < \infty$. *Hint*: Choose κ adequately.

13. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be of class $\mathcal{C}^1$ and satisfy $f(0, 0) = 0$. Consider the ODE
$$x'' = f(x, x')\,. \tag{13}$$
Show that every non-zero $x(t)$ solution of this ODE has simple zeros. Examples: the harmonic oscillator $x'' + x =$ or the mathematical pendulum $x'' + \sin(x) = 0$.