Math 523H–Homework 5

1. Use an $\epsilon - \delta$ argument to show that the following functions are continuous:

   (a) $f(x) = \sqrt{x}$ for any point $x_0 \geq 0$,

   (b) $f(x) = x^3$ for any point x_0. *Hint:* $(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$.

2. Consider the function defined by

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Show by an $\epsilon - \delta$ argument that this function is not continuous at 0.

3. Suppose $f : A \to \mathbb{R}$ is continuous at $x_0 \in A$ and that $g : B \to \mathbb{R}$ is continuous at $y_0 = f(x_0) \in B$. Show that the composition $g \circ f$ is continuous at x_0.

4. (a) Let f and g be continuous functions on $[a, b]$ and assume that $f(a) \leq g(a)$ and $g(b) \leq f(b)$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

   (b) Show that the equation $x = \cos(x)$ has at least one solution in $(0, \pi/2)$.

5. Prove that a polynomial of odd degree has at least one real root.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous which satisfies $f(a)f(b) < 0$ for some $a \neq b$. Show that there exists and x such that $f(x) = 0$.

7. (a) Consider the function $f(x) = \frac{x^3 - 3x^2 - 13x + 15}{x^2 - 1}$. At which point is this function continuous? Describe the discontinuities of the function.

   (b) Consider the function $f(x) = \frac{\sqrt{1 + 3x^2 + 2x^4} - 1}{x^2}$ for $x \neq 0$. Can you extend f to a continuous function on $\mathbb{R}$?

8. (a) Suppose that $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Show that f is discontinuous at every x.

   (b) Suppose that $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational. Show that f is continous at 0 but discontinuous at every other point.

   (c) If $x = p/q$ where p and q have no common factor define $f(x) = 1/q$ and if x is irrational define $f(x) = 0$. Show that f is discontinuous at every rational point but continuous at every irrational point.