Math 597/697: Homework 6

1. (a) Let X be normal random variable with mean μ and σ^2. Show that $Y = aX + b$ is a normal random variable. What are the mean and variance of Y?

   (b) Let X_1 and X_2 be independent normal random variable with mean 0 and variance σ_i^2, $i = 1, 2$. Show that $X_1 + X_2 = X_1 - X_2$

2. Suppose X_t is a standard Brownian motion and let $Y_t = a^{-1/2} X_{at}$ with $a > 0$. Show that Y_t is a standard Brownian motion. *Hint:* Prob. (1)

3. Let X_t and Y_t be independent standard one-dimensional Brownian motion.

   (a) Show that $Z_t = X_t - Y_t$ is a Brownian motion. What is the variance parameter?

   (b) True or false: With probability 1 $X_t = Y_t$ infinitely often?

4. Let X_t be a Brownian motion with variance parameter σ^2. Compute $E[X_s X_t]$. *Hint:* Treat $s < t$ and $s > t$ separately.

5. Let X_t be a standard Brownian motion and let us define, for $\alpha > 0$,

$$V_t = e^{-\alpha t/2} X(e^{\alpha t}) . \tag{1}$$

   The process V_t is called an Ornstein-Uhlenbeck process. Compute the mean and the variance of V_t.

6. The motion of an oil spill (assume it is pointwise) on the surface of the ocean is a 2-dimensional Brownian motion with variance parameter $\sigma^2 = 1/2$. Let C be a square with side lengths 2 and assume that the oil spill, at time 0, is at the center of the square C. Estimate the probability that the oil spill exits the square before time 1? *Hint:* Use the reflection principle for each of the component of X_t.

7. Let X_t be a standard Brownian motion. Let M be the random variable given by

$$M = \sup_{0 \le t \le 1} X_t \tag{2}$$

   It is the maximum of X_t on the time interval $[0, 1]$. Compute the density of the RV M. *Hint:* Compute $P\{M \ge a\}$ using the Reflection Principle.

8. Let X_t be a Brownian motion with drift parameter μ and variance parameter σ^2.

   (a) Compute $E[X_s X_t]$.

   (b) Let $g(x)$ be a smooth bounded function with bounded derivatives and let $g(t,x) = E[g(X_t)|X_0 = x]$. Show that $g(t,x)$ satisfy the equation

$$\frac{\partial g}{\partial t}(t,x) \; = \; \mu \frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} \tag{3}$$