Solutions of HWK #2

Prob 1 X_n = numbers of papers on the pile in the evening

(a) $S = \{0, 1, 2, 3, 4\}$

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) $P_{i0} \geq \frac{1}{3} \quad 0 \leq i \leq 4$. By Doeblin, there exists a unique stationary distribution $\pi(j)$

and $\lim_{n \to \infty} P^n_{ij} = \pi(j)$

Note that X_n is irreducible and aperiodic since $P^n_{00} > 0$ for all $n \geq 1$

c $\pi = \frac{3/2 - 1}{(3/2)^5 - 1} \left(\left(\frac{3}{2}\right)^4, \left(\frac{3}{2}\right)^3, \left(\frac{3}{2}\right)^2, \left(\frac{3}{2}\right)^1, 1 \right)$

( ) For large $E[X_n] \simeq \sum_{j=0}^{4} j\,\pi(j) = \cdots$

(d) $\sigma_0 = \inf\{n \geq 1, X_n = 0\}$ first return time

$$E[\sigma_0 \mid X_0 = 0] = \frac{1}{\pi(0)} = \frac{(3/2)^5 - 1}{3/2 - 1} \; \frac{1}{(3/2)^4}$$

Prob 2 (a) Since $\pi P = \pi$ we have

$$\pi(I - P + M) = \pi - \pi P + \pi M = \pi M$$

$$= (\pi(1), \ldots, \pi(N)) \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & & 1 \end{pmatrix} = (1, 1, 1, \ldots 1)$$

Since $\sum_i \pi(i) = 1$

If $(I - P + M)$ is invertible we have

$$\pi = (1, \ldots 1)(I - P + M)^{-1}$$

(b) • A left eigenvector for P is the same as a right eigenvector for P^* where $P^*_{ij} = P_{ji}$ (transposed matrix)

• The eigenvalues and their multiplicities are the same for P and P^* since $\det(\lambda - P) = \det(\lambda - P^*)$

• This implies that if π is the unique left eigenvector for P $\pi P = P$ then there exists a unique right eigenvector for P, $Px = x$. Since $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is always a right eigenvector it is the only one (up to a constant).

(c) To show that $(I - P + M)$ is invertible, we show that $(I - P + M)X = 0$ implies $X = 0$.

If $(I - P + M)X = 0$ then $\pi(I - P + M)X = 0$

$$\pi X - \pi P X + \pi M X = 0$$

$$\pi M X = 0$$

Since $\pi M = (1,\dots,1)$ we have $(1,1,1,\dots,1)X = 0$

But if $(1,\dots,1)X = 0$ then $MX = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} X = 0$

Then $0 = (I - P + M)X = (I - P)X = X - PX$
and so
$$PX = X.$$
By (b) $X = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ (up to a constant).

On the other hand $(1,\dots,1)X = 0$. But $(1,\dots,1)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \neq 0$

So the only option is $X = 0$ and thus $(I - P + M)$ is invertible.

<u>Prob 3</u> With $s=1$, $S=3$ X_n takes the value $\{0,1,2,3\}$

(a)
$$\begin{pmatrix} .1 & .2 & .3 & .4 \\ .1 & .2 & .3 & .4 \\ .3 & .3 & .4 & 0 \\ .1 & .2 & .3 & .4 \end{pmatrix} = P$$

(b) X_n is irreducible & aperiodic so $P_{ij}^n \to \pi(j)$
In the long run the expected amount of items in stock is
$$E[X_n] \underset{n \text{ large}}{\cong} \sum_j j\, \pi(j)$$

You can compute π with Prob 2

(c) To take into account unfulfilled demand we construct a new Markov chain with state space

$$\ldots -2, -1, 0, 1, \ldots, S-1, S$$

We say $Y_n = -1$ if one more item was requested than the amount of items in stock.
For $s=1$, $S=3$ we have unfulfilled demand only if we start with 2 items and 3 are requested

$$S = \{-1, 0, 1, 2, 3\}$$

$$P = \begin{array}{c} \\ \left(\begin{array}{ccccc} 0 & .1 & .2 & .3 & .4 \\ 0 & .1 & .2 & .3 & .4 \\ 0 & .1 & .2 & .3 & .4 \\ .1 & .2 & .3 & .4 & 0 \\ 0 & .1 & .2 & .3 & .4 \end{array}\right) \end{array} \begin{array}{c} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$

Average number of unfulfilled demands is the proportion of days when $Y_n = -1$.
In the long run it is $\pi(-1)$. You can compute π with Prob 2.

Prob 4 (a) 2 classes $\{0,1\}$ $\underbrace{\{2,3,4,5\}}$

recurrent transient

aperiodic aperiodic

($P_{00}^n > 0$ for all n) $\to P_{22}^n > 0$ for all n

(b) Use Doeblin and verify that

$$P_{j0}^4 > 0 \quad \text{and} \quad P_{j1}^3 > 0 \quad \text{for all } 0 \le j \le 5.$$

So $P_{ij}^n \to \pi(j)$ and $\pi(0) > 0$ $\pi(1) > 0$

$\pi(2) = \pi(3) = \pi(4) = \pi(5) = 0$

Prob 5 • 3 classes $\{0,1\}$ $\{2,4\}$ $\{3,5\}$

recurrent recurrent transient

aperiodic aperiodic aperiodic

• We can construct one stationary distribution for each recurrent class

$$\begin{array}{c} 0 \\ 1 \end{array}\begin{pmatrix} .5 & .5 \\ .3 & .7 \end{pmatrix} \qquad \begin{array}{c} 2 \\ 4 \end{array}\begin{pmatrix} .1 & .9 \\ .7 & .3 \end{pmatrix}$$

$$\pi_1 = (* \; * \; 0 \; 0 \; 0 \; 0) \qquad \pi_2 = (0 \; 0 \; * \; 0 \; * \; 0)$$

$\alpha \pi_1 + (1-\alpha)\pi_2$ $0 \le \alpha \le 1$ is also a stationary distribution.

Prob6 (a) $S = \{0, 1, 2, \dots, 2d\}$

$$P_{i,i+1} = \frac{2d-i}{2d}$$

$$P_{i,i-1} = \frac{i}{2d}$$

(b) irreducable and thus recurrent.
periodic with period 2

(c) (b) implies that $\frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{k} \to \pi(j)$
and π is unique.

(d) • $\sum_{j=0}^{2d} \pi(j) = 2^{-2d} \sum_{j=0}^{2d} \binom{2d}{j} = 2^{-2d}\, 2^{2d} = 1$

(Binomial Theorem $2^n = \sum_{j=0}^{n} \binom{n}{j}$)

• Stationarity $\pi P(j) = \sum \pi(\ell) P_{\ell j} = \pi(j-1) P_{j-1,j} + \pi(j+1) P_{j+1,j}$

We have

$$\binom{2d}{j-1} \cdot \frac{2d-(j-1)}{2d} + \binom{2d}{j+1} \frac{j+1}{2d}$$

$$= \binom{2d-1}{j-1} + \binom{2d-1}{j} = \binom{2d}{j}$$

and so $\pi(j)$ is stationary

(e) We are looking for $E[T_0 \mid X_0 = 0]$

$$= \frac{1}{\pi(0)}$$

$$= 2^{2d}$$

For $d = 50$ we get 2^{100} steps $\cong 4 \cdot 10^{19}$ years which is 1 billion times the known lifetime of the universe ($\sim 10^{10}$ years). So be very patient. :)

<u>Prob 7</u> Note that 1 is an absorbing state

<u>$i=1$</u> $P\{T_3 = \infty \mid X_0 = 1\} = 1$

<u>$i=2$</u> $P\{T_3 = 1 \mid X_0 = 2\} = 1/3$

$$P\{T_3 = 2 \mid X_0 = 2\} = P\{X_2 = 3, X_1 = 2 \mid X_0 = 1\}$$
$$= 1/3\,(1/6)$$

$\vdots$

$$P\{T_3 = n \mid X_0 = 2\} = (1/3)(1/6)^{n-1}$$

$$P\{T_3 = \infty \mid X_0 = 2\} = 1 - \frac{1}{3}\left(\sum_{n=0}^{\infty}\left(\frac{1}{6}\right)^n\right)$$
$$= 1 - \frac{1}{3}\,\frac{1}{5/6} = \frac{3}{5}$$

<u>$i=3$</u> Similar to <u>$i=2$</u>

Prob 8 · Let $P_{ij} \doteq P\{X_n = j \mid X_{n-1} = i\}$ and let ϕ_0 be (any) initial distribution and $\phi_n = \phi_0 P^n$

· The Markov property is equivalent to

$$P\{X_n = i_n, \ldots, X_0 = i_0\} = \phi_0(i_0) P_{i_0 i_1} \cdots P_{i_{n-1} i_n}$$

(a) $P\{X_0 = i_0 \mid X_1 = i_1, \ldots, X_n = i_n\}$

$$= \frac{P\{X_0 = i_0, \ldots, X_n = i_n\}}{P\{X_1 = i_1, \ldots, X_n = i_n\}} = \frac{\phi_0(i_0) P_{i_0 i_1} \cancel{\cdots P_{i_{n-1} i_n}}}{\phi_1(i_1) \cancel{P_{i_1 i_2} \cdots P_{i_{n-1} i_n}}}$$

$$= \frac{\phi_0(i_0) P_{i_0 i_1}}{\phi_1(i_1)} = \frac{P\{X_0 = i_0, X_1 = i_1\}}{P\{X_1 = i_1\}}$$

$$= P\{X_0 = i_0 \mid X_1 = i_1\} \quad \blacksquare$$

(b). $P\{X_{n+1} = i_{n+1}, X_{n-1} = i_{n-1} \mid X_n = i_n\}$

$$= \frac{\phi_{n-1}(i_{n-1}) P_{i_{n-1} i_n} P_{i_n i_{n+1}}}{\phi_n(i_n)}$$

· $P\{X_{n+1} = i_{n+1} \mid X_n = i_n\} \, P\{X_{n-1} = i_{n-1} \mid X_n = i_n\}$

$$= \frac{\cancel{\phi_n(i_n)} P_{i_n i_{n+1}}}{\cancel{\phi_n(i_n)}} \cdot \frac{\phi_{n-1}(i_{n-1}) P_{i_{n-1} i_n}}{\phi_n(i_n)} = \frac{\phi_{n-1}(i_{n-1}) P_{i_{n-1} i_n} P_{i_n i_{n+1}}}{\phi_n(i_n)}$$

$\Rightarrow$ we obtain (b).

Prob 9 If $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = a$ then

(i) For any $\varepsilon > 0$ there exists N s.t. for $n > N$
$|a_n - a| < \varepsilon$.

(ii) The sequence $\{a_n\}$ is bounded: there exists $M > 0$ s.t. $|a_n| \le M$ for any $n \ge 0$.

Choose $n > N$ and write

$$b_n - a = \frac{(a_0 - a) + \cdots + (a_{n-1} - a)}{n}$$

$$= \underbrace{\frac{(a_0 - a) + \cdots + (a_{N-1} - a)}{n}}_{N \text{ terms}} + \underbrace{\frac{(a_{N} - a) + \cdots (a_{n-1} - a)}{n}}_{n-N \text{ terms}}$$

$$|b_n - a| \le \frac{2NM}{n} + \frac{n-N}{n}\varepsilon$$

$$\le \frac{2NM}{n} + \varepsilon$$

If we choose n large enough $\left(n > \frac{2NM}{2\varepsilon} \right)$

then $|b_n - a| \le 2\varepsilon$ and thus $b_n \to a$