Math 597/697: Homework 5

1. The weather in Noho can be in the states 0 (sunny), 1 (foggy), 2 (gray and rainy) or 3 (cold and snowy). The transition between the these different states is governed by the transition matrix

$$\begin{pmatrix} 0 & .1 & .6 & .3 \\ .6 & 0 & .2 & .2 \\ .4 & .1 & 0 & .5 \\ .5 & .1 & .4 & 0 \end{pmatrix} \tag{1}$$

   The weather remains in state i with a mean time (in days) 4, 0.2, 3 and 5 respectively. Set up a continuous Markov chain which describes the weather in Noho and write down the generator.

2. A population of organisms consisting of both males and females. Any particular male is likely to mate with any particular female during the time interval h with probability $\lambda h + o(h)$. Each mating produces immediately an offspring which is male with probability p and female with probability $1-p$. Let M_t and F_t denote the numbers of males and females in the population. Derive the parameters of the Markov chain $\{F_t, M_t\}$, i.e. the v_i, P_{ij}.

3. (a) An airline reservation system has two computers, one on-line and one backup. The operating computer fails after an exponentially distributed time with parameter μ and is replaced by the backup. There is one repair facility and the repair time is exponentially distributed with parameter λ. Let X_t denote the number of computers in operating condition at time t. Write down the generator A of X_t and the backward and forward equations (no need to solve them). In the long run, what is the proportion of the time when the reservation system is on.

   (b) Answer the same questions in the case where the two machines are simultaneously online if they are in operating condition.

   (c) Starting with two machines in operation, compute in both cases, the expected value of the time T until both are in the repair facility. *Hint:* This is a hitting time. You can write $T = T_{21} + T_{10}$ where T_{21} is the first time the chain reach state 1 stating from state 2 and T_{10} is the first time the chain reach state 0 starting from 1. The expected value of T_{21} and T_{10} can be computed by conditioning on the first transition. If this hint does not please you, look at your textbook on pp.356.

4. A component is in two possible states 0=on or 1=off. A system consists of two components A and B which are independent of each other. Each component remains on for an exponential time with rate λ_i, $i = A, B$ and when it is off it remains off for an exponential time with rate $\mu_i, i = A, B$. Consider now a system which consists of two such components. Determine the long run probability that the system is operating if

   (a) They are working in parallel, i.e. at least one must be operating for the system to be operating.

   (b) They are working in series, i.e. both must work for the system to be operating.

   Hint: You might either construct a 4-state Markov chain or use the independence of both components.

5. Consider a continuous time branching process defined as follows. A organism lifetime is exponential with parameter λ and upon death, it leaves k offspring with probability p_k, $k \geq 0$. The organisms act independently of each other. For simplicity assume that $p_1 = 0$. Let X_t be the population at time t, find the generator and write down the forward and backward equations for the process. Specialize your results to the binary splitting case where the particle either splits in two or vanishes. Find the stationary distribution of $\{X_t\}$.

6. Consider the $M/M/\infty$ queue with arrival rate λ and service rate μ. Let X_t be the number of customer in the system at time t.

   (a) Write down the generator A of the Markov process

   (b) Show by solving $\pi A = 0$ that the stationary and limiting distribution is Poisson.

   (c) Derive and solve a differential equation for $E[X_t]$.

7. Consider a queuing system with one single server, arrival rate λ and serving rate μ.

   (a) When N customers are in the system, the arriving customer give up and do no enter the system. What is the state space and the generator of the system? What is the limiting distribution?

   (b) When n customers are in the system, an arriving customer will join the system with probability $1/(n+1)$. What is the state space and the generator of the system? What is the limiting distribution? Does it look familiar?

8. Consider a Yule process (pure birth with linear growth) and let T_i be the time it takes for a population of size i to reach $i+1$. In this problem we indicate a method to actually compute the transition probabilities $P_{mn}(t)$ which is not based on solving directly the backward of forward equation.

   (a) Argue that T_i is exponential with rate $i\lambda$.

   (b) Let $X_1, X_2, \ldots, X_n$ be n exponentials random variables with parameter λ. Show

   $$\max(X_1, \ldots, X_n) = T_n + T_{n-1} + \ldots T_1 \tag{2}$$

   Hint: Interpret X_i as time the failure of a component, T_n as the time of the first failure, T_{n-1} as the time between the first and second failure, etc....

   (c) Deduce from (b) that $P\{T_1 + \ldots + T_n < t\} = (1 - e^{-\lambda t})^n$.

   (d) Use (c) and (a) to obtain that

   $$P_{1n}(t) = (1 - e^{-\lambda t})^{n-1} - (1 - e^{-\lambda t})^n = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \tag{3}$$

   and hence that $P_{\{}X_t = n \,|\, X_0 = 1\}$ has a geometric distribution.

   (e) Deduce from (d) that for $n > m$

   $$P_{mn}(t) = \binom{n-1}{m-1}(e^{-\lambda t})^m (1 - e^{-\lambda t})^{n-m} \tag{4}$$

   Hint: Use that the sum of geometric R.V. is a negative binomial R.V. and that each individual reproduce independently of the others.