Math 697 Fall 2014: Week 3

Exercise 1 ***(More algorithms to compute π)***

1. The estimator for π constructed in class is based on the indicator RV

$$I = 1 \quad \text{if} \quad V_1^2 + V_2^2 \le 1 \tag{1}$$

   where V_1 and V_2 are uniform random variable on $[-1, 1]$. As an alternative show that you can use instead the estimator $J = (1 - U^2)^{1/2}$ where U is a random number. Determine which one of the estimator has smaller variance.

2. Show also that you can apply the method of antithetic variables to the algorithm in part 1. and compute the corresponding variance reduction (you may want to compute the necessary integral for the covariance numerically...)

Exercise 2 ***(Computer exercise)***

Write a program to compute the number π using one of the algorithm described in Exercise 1. Your program should produce the following output: if you fix a number of iteration n the output should be an estimator for π as well as a confidence interval for your estimator.

Exercise 3 ***(Hit-or-miss method)***

1. Suppose that you wish to estimate the volume of a set B contained in the Euclidean space $\mathbf{R}^k$. You know that B is a subset of A and you know the volume of A. The "hit-or-miss" method consists in choosing n independent points uniformly at random in A and use the fraction of points which lands in B to get an estimate of the volume of B. (We used this method to compute the number π in class.) Write down the estimate I_n obtained with this method and compute $\text{var}(I_n)$. (This will be expressed in terms of the volume of A and B.)

2. Suppose now that D is a subset of A and that we know the volume of D and the volume of $D \cap B$. You decide to estimate the volume of B by choosing n points at random from $A \setminus D$ and counting how many land in B. What is the corresponding estimator I_n' of the volume of B for this second method? Show that this second method is better than the first one in the sense that $\text{var}(I_n') \le \text{var}(I_n)$.

3. How would you use this method concretely to improve the estimation of the number π? Compute the corresponding variances.

Exercise 4 ***(Computing integrals)***

Suppose f is a function on the interval $[0, 1]$ with $0 < f(x) < 1$. Here are two ways to estimate $I = \int_0^1 f(x)dx$.
(a) Use the "hit-or-miss" from the previous problem with $A = [0, 1] \times [0, 1]$ and $B = \{(x, y) \,:\, 0 \le x \le 1\,,\, 0 \le y \le f(x)\}$.

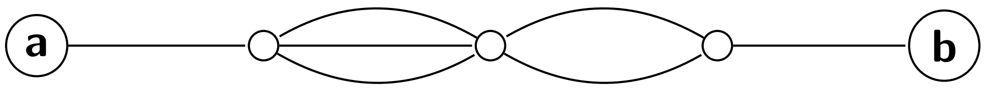

Figure 1: Network 1

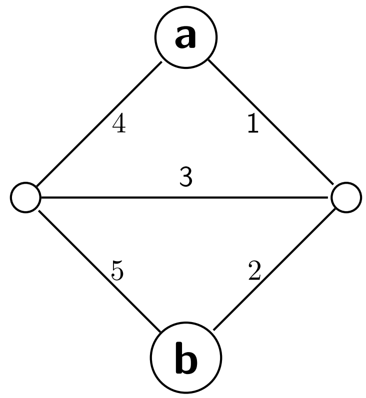

Figure 2: Network 1

(b) Use the simple sampling algorithm with $U_1, U_2, \cdots$ be i.i.d. uniform random variables on $[0,1]$ and

$$\hat{I}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i)\,.$$

Determine which one of theses two methods is the most efficient by comparing the variance.

Exercise 5 Consider the networks in Figures 1 and 2. Assume that for each edge the probability that the edge is operating is p, independently of the other edges. We say that the network is operating if there is a path from (a) to (b) along operating edges.

1. Compute that the probability that the network in Figure 1 is operating.
2. Compute that the probability that the network in Figure 2 is operating using inclusion-exclusion.
3. Compute that the probability that the network in Figure 2 is operating by conditioning first whether the edge 3 is operating or not.

Exercise 6 ***(Variance reduction by conditioning)*** Suppose X and Y are RV with joint pdf $f_{XY}(x,y)$. Define for all y such that $f_Y(y) = \int f_{XY}(x,y)dx \neq 0$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

For each fixed y this is a pdf of a RV which we write $X|Y = y$ with expectation $E[X|Y = y]$ and variance $\mathrm{Var}(X|Y = y)$. Further we can define now the random variable $E[X|Y]$ as a continuous random variable which takes values $E[X|Y = y]$ with probability density $f_Y(y)$, and similarly for $\mathrm{Var}(X|Y)$.

1. Show the formulas $E[X] = E[E[X|Y]]$ and $\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$.

2. Suppose that in a Monte-Carlo simulation we use the RV to estimate θ (i.e. $E[X] = \theta$). Show that for any other random variable Y, the random variable $E[X|Y]$ is a better (in the sense of variance) estimator for θ.

3. Consider the estimator I for π given in (1). Compute the estimator $E[X|V_1]$ and compare the variances (see problem 1).