Math 623: Problem set 4

1. Problem 8, p. 92
2. Problem 18, p. 93
3. Problem 19, p.93.
4. Convolution product: Do Problem 21, Part (a)-(d) and do Problem 24.
5. Suppose $f(x)$ on $\mathbf{R}^{d_1}$ is measurable and $g(y)$ on $\mathbf{R}^{d_2}$ is measurable. Show that $F(x,y) = f(x)g(y)$ on $\mathbf{R}^{d_1+d_2}$ is measurable.
6. (a) In class we have shown that if E is a set of finite measure and f_n converges to f almost everywhere on E then f_n converges to f in measure on E. Show by a counterexample that the assumption that E has finite measure is necessary.

   (b) Show the following variant of Dominated Convergence Theorem. Assume that $|f_n| \le f$ for some $g \in L^1(\mathbf{R}^d)$ and f_n converges to f in measure. Prove

   (a) $\lim_n \int f_n dx = \int f dx$.

   (b) f_n converges to f in L^1.

   Hint: We now have 2 different proofs of DCT, find the right one to modify...)

   Hint: Sometimes the following (easy?) fact may come handy: A sequence of real numbers $\{a_n\}$ converges to A if and only every subsequence of $\{a_n\}$ has a convergent subsubsequence which converges to a.
7. Consider the function $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$ and for $0 \le x < \infty$ Show that

   (a) $\sum_{n=1}^{\infty} \int |f_n| dx = \infty$.

   (b) $\sum_{n=1}^{\infty} \int f_n dx = 0$.

   (c) $\sum_{n=1}^{\infty} f_n(x)$ is integrable on $[0,\infty)$ and $\int_{[0,\infty)} \sum_{n=1}^{\infty} f_n dx = \ln(b/a)$.
8. Let $T : \mathbf{R}^d \to \mathbf{R}^d$ be an invertible linear transformation. The goal of this problem is to show the change of variable formula $\int f(x)dx = |detT| \int f \circ T(x) dx$

   (a) Show that if E is compact the set $T(E)$ is also compact. Deduce from this that is E is a F_σ then $T(E)$ is a F_σ.

   (b) Show that if E has measure 0 then $T(E)$ has measure 0. Hint: You can use Problem 1 in Problem set #3.

   (c) Show that T maps measurable sets into measurable sets.

(d) Show that for any measurable set E we have $m(T(E)) = \det(T)m(E)$. *Hint:* You can use here the (elementary) fact from linear algebra that any invertible linear transformation can be written as a finite product of transformations of the types

$$
\begin{aligned}
T(x_1, \cdots, x_j, \cdots x_n) &= (x_1, \cdots, cx_j, \cdots x_n) \quad (c \neq 0) \\
T(x_1, \cdots, x_j, \cdots x_n) &= (x_1, \cdots, x_j + cx_k, \cdots x_n) \quad (k \neq j) \qquad (1) \\
T(x_1, \cdots, x_j, \cdots, x_k, \cdots x_n) &= (x_1, \cdots, x_k, \cdots, x_j, \cdots x_n) \qquad (2)
\end{aligned}
$$

Then use Fubini.

(e) Finally conclude that for any integrable f, $f \circ T$ is also integrable and $\int f(x)dx = |\det T| \int f \circ T(x)dx$

9. (a) Consider the function $f(x, y) = ye^{-(1+x^2)y^2}$ if $x \geq 0$ and $y \geq 0$ and 0 otherwise. Integrate this function over $\mathbf{R} \times \mathbf{R}$ to show that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$. Justify all your steps carefully!

   (b) Use part (a) to compute $\int_{\mathbf{R}} e^{-tx^2}\, dx = \sqrt{\pi/t}$ and to show that $\int x^{2n} e^{-x^2}\, dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$.

10. Problem 4 on page 146.