# Math 597/697: Solutions for Homework 2

1. We have

$$P = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 11/18 & 7/18 \\ 7/16 & 9/16 \end{pmatrix} \tag{1}$$

$$P^3 = \begin{pmatrix} 107/216 & 109/216 \\ 327/576 & 249/576 \end{pmatrix}. \tag{2}$$

   (a) This is $P^3_{01} = 109/216$

   (b) This is $P_{01}P^2_{11} = 1/3 \times 9/16 = 9/48$

   (c) This is $(1/4, 3/4)P^2 = (277/576, 299/576)$

2.

$$P = \begin{pmatrix} .4 & .2 & .4 \\ .6 & 0 & .4 \\ .2 & .5 & .3 \end{pmatrix}. \tag{3}$$

Solving the equation $\pi P = \pi$ gives $\pi = (75/198, 51/198, 72/198)$

3.

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix}. \tag{4}$$

You can draw a communication diagram or put it in normal form. Ordering the states as $0, 1, 2, 4, 3, 5$ we find

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix}. \tag{5}$$

and one sees immediately that there are two recurrent classes $\{0, 1\}$ and $\{2, 4\}$ and one transient class $\{3, 5\}$.

4. *The Ehrenfest urn model.* $X_n$ can take any of the $2a+1$ values $0, 1, \cdots, 2a$ and so $Y_n$ can take the value $-a, -a+1, \cdots, a$. The only non vanishing matrix element of $P$ are $P_{ii+1}$ and $P_{ii-1}$ since at each step the number of balls in urn $A$ either increases or decreases by one unit. If $Y_n = i$ there are $X_n = a+i$ balls in urn $A$ and so $a-i$ balls in urn $B$. The number of balls in urn $A$ will increase by one unit if we pick a ball from urn $B$ and this occurs with probability $(a-i)/2a$. Hence we obtain $P_{ii+1} = (a-i)/2a$ and similarly $P_{ii-1} = (a+i)/2a$.

   One sees easily that there is only one class which is recurrent.

5. *An inventory model.* $X_n$ is the number of items in stock at the end of period $n$ (before restocking). If $X_n > s$ then no restocking occurs and the number of items in stock at the end of the next period will $X_n - \xi_{n+1}$. If $X_n \leq s$ then restocking occurs and the number of items in stock at the end of the next period will be $S - \xi_{n+1}$.

   If we order the states as $\{S, S-1, \ldots, s, s-1, \ldots\}$ the transition matrix is given by

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{6}$$

   For the example consider here $s = 0$ and $S = 2$, one does not sell more that 2 items per period so the state space is $\{2, 1, 0, -1\}$ and the transition matrix is thus

$$P = \begin{pmatrix} .5 & .4 & .1 & 0 \\ 0 & .5 & .4 & .1 \\ .5 & .4 & .1 & 0 \\ .5 & .4 & .1 & 0 \end{pmatrix}. \tag{7}$$

6. The weather one one day depends on the weather of two previous day, it depends not only on the past, so it is not a Markov chain. If we take the weather of two consecutive day as a state, there are four states { 0=RR, 1=NR, 2=RN, 3=NN } where $R$ indicates rain and $N$ indicates no rain, the first letter the weather on the first day and the second the

weather on the second day. We find

$$P = \begin{pmatrix} .8 & 0 & .2 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{pmatrix}. \tag{8}$$

If it rained yesterday but not the day before yesterday the initial state is $1 = NR$ and if it rains tomorrow the final state is either $1 = NR$ or $0 = RR$ depending on the weather today. So the desired probability is $P^2_{10} + P^2_{11} = P_{10}P_{00} + P_{12}P_{21} = 3/5$

7. (a) Note that we have

$$f_i \; = \; P\{x_n = 1 \text{ for some n}|x_0 = i\} \; = \; \sum_{n=1}^{\infty} P\{\tau_i = n|X_0 = i\}\,. \tag{9}$$

Recurrence means $f_i = 1$ and so $P\{\tau_i = \infty|X_0 = i\} = 0$, transience means $f_i < 1$ and so $P\{\tau_i = \infty|X_0 = i\} = 1 - f_i > 0$.

(b)

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ .25 & .25 & .25 & .25 \end{pmatrix}. \tag{10}$$

0 and 1 are recurrent states and 2 and 3 are transient states.
Starting from 3, $X_1 = 3$ with probability 1/4 and hence $P\{\tau_3 = 1|X_0 = 3\} = 1/4$. With probability 3/4 $X_1$ is not 3 in which case the chain never returns to 3, hence $P\{\tau_3 = \infty|X_0 = 3\} = 3/4$.
Similar argument shows that $P\{\tau_2 = 1|X_0 = 2\} = 1/2$ and $P\{\tau_2 = \infty|X_0 = 2\} = 1/2$.
Start in state 1, $X_1 = 1$ with probability 1/2 and hence $\tau_1 = 1$ with probability 1/2, $X_2 = 1$ with probability 1/2 so $\tau_1 = 2$ with probability $1/2 \times 1/2 = 1/4$, etc... so we have

$$P\{\tau_1 = n|X_0 = 1\} = (1/2)^n \quad n = 1, 2, 3, \ldots \tag{11}$$

and similarly if we start from state 2.

8. (a) For large $n$

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/8 & 2/3 & 5/24 \\ 0 & 1/6 & 5/6 \end{pmatrix}, \quad P^n \approx \begin{pmatrix} .182 & .364 & .455 \\ .182 & .364 & .455 \\ .182 & .364 & .455 \end{pmatrix}. \tag{12}$$

Clearly $\lim_{n\to\infty} P_{ij}^n = \pi_j$, independent of the initial state $i$. The limiting distribution is $\pi \approx (.182, .364, .455)$. The Markov chain is irreducible, recurrent and aperiodic. We obtain that $P\{X_n = j\} \approx \pi_j$ for large $n$, independently of the initial state.

(b) Random walk on $\{0, 1, 2, 3, 4\}$ with reflecting boundary conditions.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

We have

$$P^n \approx \begin{pmatrix} 0 & .50 & 0 & .50 & 0 \\ .25 & 0 & .50 & 0 & .25 \\ 0 & .50 & 0 & .50 & 0 \\ .25 & 0 & .50 & 0 & .25 \\ 0 & .50 & 0 & .50 & 0 \end{pmatrix}. \tag{14}$$

for $n$ odd and

$$P^n \approx \begin{pmatrix} .25 & 0 & .50 & 0 & .25 \\ 0 & .50 & 0 & .50 & 0 \\ .25 & 0 & .50 & 0 & .25 \\ 0 & .50 & 0 & .50 & 0 \\ .25 & 0 & .50 & 0 & .25 \end{pmatrix}. \tag{15}$$

for $n$ even. Clearly $\lim_{n\to\infty} P_{ij}^n$ does not exist, for example $P_{i1}$ oscillates between 0 and 0.25, $P_{i2}$ oscillates between 0 and 0.50, etc... One checks easily that every state has period 2. The invariant distribution is $(1/8, 1/4, 1/4, /14, 1/8)$ (why is that so?). The Markov chain is irreducible recurrent and with period 2.

(c) Random walk on $\{0, 1, 2, 3, 4\}$ with absorbing boundary conditions

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$P^n \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .75 & 0 & 0 & 0 & .25 \\ .50 & 0 & 0 & 0 & .50 \\ .25 & 0 & 0 & 0 & .75 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

$\lim_{n\to\infty} P_{ij}^n$ does exist, but depends on $i$, for example starting from state 1, for large $n$ $P\{X_n = 0|X_0 = 1\} \approx 0.75$ and $P\{X_n = 4|X_0 = 1\} \approx 0.25$. Here 0 and 4 are absorbing states and $1, 2, 3$ are transient. For large $n$, $P\{X_n \in \{1, 2, 3\}\} \approx 0$. The stationary distributions are $\pi_1 = (1, 0, 0, 0, 0)$ and $\pi_2 = (0, 0, 0, 0, 1)$ as well as all linear combination $\alpha\pi_1 + (1 - \alpha)\pi_2$, $0 < \alpha < 1$.

(d)

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 1/8 & 2/3 & 5/24 \\ 0 & 0 & 0 & 1/6 & 5/6 \end{pmatrix}, \tag{18}$$

$$P^n \approx \begin{pmatrix} .25 & .75 & 0 & 0 & 0 \\ .25 & .75 & 0 & 0 & 0 \\ 0 & 0 & .182 & .364 & .455 \\ 0 & 0 & .182 & .364 & .455 \\ 0 & 0 & .182 & .364 & .455 \end{pmatrix}. \tag{19}$$

$\lim_{n\to\infty} P_{ij}^n$ does exist, but depends on $i$. The chain splits into two smaller noninteracting chains, one with state space $\{1, 2\}$ and the other with state space $\{3, 4, 5\}$. Each "subchain" converges to its own limiting distribution. You can verify that all stationary distributions are linear combinations of $\pi_1 = (.25, .75, 0, 0, 0)$ and $\pi_2 = (0, 0, .182, .364, .455)$