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# Uncertainty quantification for generalized Langevin dynamics

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We present efficient finite difference estimators for goal-oriented sensitivity indices with applications to the generalized Langevin equation (GLE). In particular, we apply these estimators to analyze an extended variable formulation of the GLE where other well known sensitivity analysis techniques such as the likelihood ratio method are not applicable to key parameters of interest. These easily implemented estimators are formed by coupling the nominal and perturbed dynamics appearing in the finite difference through a common driving noise or common random path. After developing a general framework for variance reduction via coupling, we demonstrate the optimality of the common random path coupling in the sense that it produces a minimal variance surrogate for the difference estimator relative to sampling dynamics driven by independent paths. In order to build intuition for the common random path coupling, we evaluate the efficiency of the proposed estimators for a comprehensive set of examples of interest in particle dynamics. These reduced variance difference estimators are also a useful tool for performing global sensitivity analysis and for investigating non-local perturbations of parameters, such as increasing the number of Prony modes active in an extended variable GLE. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4971433>]

## I. INTRODUCTION

Sensitivity analysis (SA), understanding how changes in input parameters affect the output of a system, is a key component of uncertainty quantification (UQ), optimal experimental design, and analysis of model robustness, identifiability, and reliability.<sup>1,2</sup> The local sensitivity of a system can be analyzed by computing sensitivity indices that are formed by taking partial derivatives with respect to each of the input parameters. These indices quantify which parameter directions are most sensitive to perturbations.

The present article concerns SA techniques for the generalized Langevin equation (GLE) and other models of interest in particle dynamics. The Langevin equation (LE) models particle diffusion in the presence of a heat bath where the particle-bath interactions are reduced to an instantaneous drag force and a delta-correlated random force.<sup>3</sup> This approximation dramatically reduces the computational cost compared to explicitly resolving the particle-bath interactions. However, there are a number of compelling applications where the Langevin assumptions fail to produce a reliable model, such as anomalous diffusion. The GLE, a model that captures both diffusion and anomalous diffusion, incorporates “memory” into the drag force through the inclusion of a kernel depending on the history of the velocity. In many instances, this non-Markovian system can be mapped onto a Markovian system with additional degrees of freedom under physically reasonable assumptions, such as when the memory kernel can be approximated by a positive Prony series.<sup>4</sup> The resulting

extended variable formulation contains many parameters that must be tuned and is therefore an ideal candidate for SA and UQ. However, well known SA techniques such as likelihood ratio and pathwise methods are not applicable to analyze the sensitivity of key parameters of interest in the extended variable formulation. In particular, it is relevant to understand and compare discrete changes in the model such as modifying the number of extended variables. In contrast, Monte Carlo finite difference estimators of sensitivity indices are applicable to all parameters of interest in the extended variable GLE but introduce a bias error and typically have a large variance making them computationally expensive.

We give efficient Monte Carlo finite difference estimators via a coupling method for approximating goal-oriented sensitivity indices for a large class of stochastic differential equations (SDEs). In particular, we apply these estimators to an extended variable formulation of the GLE where the memory kernel can be approximated by a positive Prony series, a choice motivated by applications in anomalous diffusion in biological fluids.<sup>5,6</sup> Although uncertainties in drift, vibration, and tracking measurements may be present in microrheology, a central problem here is one of epistemic model form uncertainty. There is a wealth of data but few methods that allow one to compare and evaluate the models that are suggested by these data. Moreover, the extended variable GLE is typically fitted by matching summary statistics, such as the mean squared displacement (MSD) and velocity autocorrelation function (VACF), between microrheology observations and simulated data;<sup>5</sup> however, for many applications, matching summary statistics alone results in models that do not capture quantities of interest. For instance, the first passage time for a pathogen across a mucus barrier depends critically on model features that are not captured by summary statistics.<sup>7</sup>

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Further, “ergodicity breaking” is observed in biofluids, that is, the time averaged statistics within independent reaction paths do not converge to ensemble averages (the spreads in pathwise MSD are too large to be independently drawn from a uniform population).<sup>8</sup> Therefore, it is of fundamental importance to understand the GLE dynamics and the sensitivity of the dynamics to local and global perturbations.

In the context of anomalous diffusion for biological fluids, we mention that other authors have given a Bayesian methodology for comparing different models of anomalous diffusion that favor the extended variable GLE.<sup>7</sup> In addition to biological fluids, other recent interesting applications of the GLE include modeling nanoscale materials and solids,<sup>9–13</sup> thermostats for sampling classical and path integral molecular dynamics,<sup>14–19</sup> and, more generally, reduced order modeling.<sup>20–23</sup> This list of applications is far from exhaustive but nevertheless provides additional incentive for investigating SA and UQ techniques for the GLE and its extended variable formulations.

To provide further orientation consider, for simplicity, the sensitivity of the stochastic dynamics  $X_t(\theta)$ , depending on an input parameter  $\theta$ ,

$$\mathcal{S}(t, \theta; f) = \partial_\theta \mathbf{E}[f(X_t(\theta))],$$

for a given observable  $f$  where  $\partial_\theta$  is the derivative with respect to  $\theta$ . In general, the finite difference approach is to approximate the derivative above by a finite difference quotient and then obtain the required moments by Monte Carlo. For example, a forward difference with a bias parameter  $\varepsilon$  yields the sensitivity estimator,

$$\mathcal{S}_\varepsilon(t, \theta; f) = (\mathbf{E}[f(X_t(\theta + \varepsilon))] - \mathbf{E}[f(X_t(\theta))]) / \varepsilon,$$

where  $\mathcal{S}_\varepsilon \approx \mathcal{S}$  for  $\varepsilon$  small, and then the estimator is computed by approximating the expectations with sample averages. Similar expressions can be given for central differences and more general finite difference stencils. While this approach requires little analysis of the underlying model and is easily implemented, the introduction of the bias, and in particular its effect on the variance of  $\mathcal{S}_\varepsilon$ , a key quantity in evaluating the efficiency of the method, is often cited as a reason for pursuing alternative methods.<sup>24</sup> However, as we shall show, the variance of  $\mathcal{S}_\varepsilon$  can be reduced by choosing the right sampling strategy for the observable of the nominal and perturbed dynamics, respectively,  $f(X_t(\theta))$  and  $f(X_t(\theta + \varepsilon))$  in the expression above. For a comprehensive set of examples of interest in particle dynamics, we demonstrate that coupling the nominal and perturbed dynamics through a common driving noise, that is, a *common random path coupling*, reduces the variance of the finite difference estimator, often substantially. In particular, for the extended variable GLE with a *convex* potential, the reduction due to the common random path coupling is on the order of the bias squared—mitigating the effect of the systematic error. The common random path coupling also leads to reduced variance estimators for problems with *nonconvex* potentials, although the reduction is not expected to be on the order of the bias squared (cf. Figures 4 and 5). This is a topic that deserves further rigorous analysis that will be the subject of future work.

Other well known SA techniques for continuous time stochastic dynamics, including pathwise methods,<sup>25</sup> likelihood ratio methods,<sup>26,27</sup> and Malliavin methods,<sup>28,29</sup> produce

*unbiased* estimators of sensitivity indices by representing  $\mathcal{S}$  as the expectation of  $f$  (or its derivative) under a change of measure. A very good account of the interrelations among them has been given by other authors.<sup>24</sup> However, each of these methods is not suited to our application of interest, the GLE, for reasons that we detail below.

The pathwise and likelihood ratio methods are not applicable to key parameters of interest, those appearing simultaneously in the drift and diffusion terms, in the extended variable formulation of the GLE. The pathwise method views the dynamics at each fixed time as a density and takes a derivative of this density yielding the estimator,  $\mathcal{S}_P(t, \theta; f) = \partial_\theta \mathbf{E}[f(X_t(\theta))] = \mathbf{E}[f'(X_t(\theta))\partial_\theta X_t(\theta)]$ , requiring equality to hold when the order of differentiation and expectation is interchanged, a smooth observable  $f$ , and, of course, that the derivative of the process  $X_t(\theta)$  exists. In its most general form, if an expression  $\mathbf{E}[f(X_t(\theta))] = \int f(x_t)g(\theta, x_t)dx_t$  exists, then the likelihood ratio estimator,

$$\begin{aligned} \mathcal{S}_{LR}(t, \theta; f) &= \partial_\theta \mathbf{E}[f(X_t(\theta))] \\ &= \int f(x_t) [\partial_\theta \log g(\theta, x_t)] g(\theta, x_t) dx_t \\ &= \mathbf{E}[f(X_t(\theta))\partial_\theta \log g(\theta, X_t)], \end{aligned}$$

is obtained by bringing the derivative inside the integral and multiply and dividing by  $g$ . However, this formulation requires knowledge of the unknown density  $g$  and, in practice, pathwise information is substituted  $\mathcal{S}_{LR}(t, \theta; f) \approx \mathbf{E}[f(X_t(\theta))G(\{X_s\}_{0 \leq s \leq t})]$ . For both estimators, the application of these methods to key parameters of interest in the extended variable formulation of the GLE leads to perturbations in path-space that are not absolutely continuous, that is, the typical Girsanov change of measure does not hold in path-space.

An approach that circumvents this lack of a Girsanov transform for certain parameters, using the Malliavin derivative,<sup>30</sup> first appeared in finance applications for calculating sensitivities, known as Greeks, related to the pricing of securities.<sup>28,29,31</sup> Applied directly to continuous time dynamics, the Malliavin approach produces unbiased estimators  $\mathcal{S}_M = \mathbf{E}[f(X_T)h(\{X_s\}_{0 \leq s \leq T})]$  where  $h$  is a non-unique weight that involves a system of auxiliary processes obtained through Malliavin calculus but that does not depend on  $g$ . In particular, for overdamped Langevin dynamics, Malliavin weights are given for sensitivities with respect to parametric forces, that is, for parameters appearing in the drift term only.<sup>32</sup> While in principle the Malliavin method applies to other perturbations that cannot be handled by pathwise and likelihood ratio methods, it requires a number of auxiliary processes that may scale poorly with the system size and is not clearly computationally practical for the extended variable GLE.

We mention that, finite differences using common random numbers (CRNs) have been employed, based on empirical evidence, for SA with respect to parametric forces for the LE.<sup>33–35</sup> The sensitivity for invariant measures for parametric diffusions,  $\partial_\varepsilon(\int_{\mathbf{R}^d} f d\pi_\varepsilon)$ , has been considered,<sup>33</sup> and a mathematically rigorous justification of such objects has been given by other authors in relation to linear response theory.<sup>36</sup> In

the finance literature, it is of great interest to calculate sensitivities, or the Greeks, of option prices that are modeled by SDEs. In this setting, the sensitivity is usually calculated at a fixed terminal time, and then computing the sensitivity of this static distribution using a finite difference with CRN is a heuristic employed for certain Greeks, but not all applications allow such a nice coupling.<sup>37</sup> In the context of discrete state stochastic systems, the rates of convergence for finite difference estimators using CRN have been derived under various assumptions.<sup>38–42</sup> For physicochemical systems, several coupled finite difference methods have been demonstrated to reduce the variance of the estimator with respect to independent sampling,<sup>43–49</sup> and in particular examples are given in kinetic Monte Carlo (KMC) simulations where CRN couplings have minimal effect to the variance reduction of finite difference estimators for sensitivities and instead a nontrivial coupling is required due to the spatial effects of KMC and jump Markov dynamics.<sup>47</sup> Here, in contrast, we develop a general framework, at the level of the generators of the coupled SDEs, that allows us to formulate an optimization problem, locally in time, with minor assumptions to ensure the correct marginal statistics. That is, we formulate an associated maximization problem (see Optimization Problem 1 in Sec. II C), and we show that the problem is satisfied by the common random path coupling for a large subset of solutions. Intuition is developed in the examples of the Ornstein–Uhlenbeck (OU) process and LE dynamics (see the Appendix) for which the optimality of the common random path coupling can be derived directly without invoking a localization argument.

In relation to SA, we also mention that information theoretic bounds can be used to screen parametric sensitivities.<sup>25,50–52</sup> In particular, information theoretic bounds involving the relative entropy<sup>53</sup> have been used to analyze the sensitivity of some parameters of interest in Langevin dynamics in equilibrium and non-equilibrium regimes.<sup>51</sup> These information theoretic methods are not goal oriented, that is, the dependence on the observable  $f$  is not explicit. Further they cannot be applied to key parameters of interest in the extended variable GLE as relative entropy calculations also require the absolute continuity of the measures arising from the nominal and perturbed dynamics.

In addition to local SA, the optimally coupled differences are a useful computational tool for *global* SA and for investigating *non-local perturbations* in parameters. In global SA, elementary effects are used to screen for sensitive parameters.<sup>54–56</sup> Calculating elementary effects involves sampling a number of finite difference estimators with various biases and stencils to survey the space of input parameters. The coupled finite differences might be used to efficiently build such global sensitivity indices. For exploring non-local perturbations, a key observation is that the finite difference estimators proposed are formed by coupling the nominal and perturbed dynamics, and there is no requirement that the perturbations be local or that the corresponding measures be absolutely continuous. In Sec. IV B, we demonstrate that the optimally coupled difference might be used to efficiently analyze, with respect to independent sampling, the effect of increasing the *number* of Prony modes active in an extended variable formulation of GLE dynamics.

The rest of this paper is organized as follows. To set the stage for our variance reduction technique, we review the errors committed in estimators for sensitivity indices for static distributions in Sec. II. Then we introduce a very general coupling framework and derive a maximization problem for the variance reduction. In Sec. III, we recall facts about the GLE and illustrate how the theory presented in Sec. II applies to the extended variable formulation, obtaining the optimality of the common random path coupling for a large subset of solutions. In Sec. IV, we provide numerical experiments involving SA for GLE that include (1) the sensitivity with respect to the coefficients of the Prony series approximation, for both convex and nonconvex potentials, and (2) the sensitivity with respect to the number of Prony modes, the latter not being formally a sensitivity index. Finally, in the Appendix, we provide supplemental examples that help build an intuition for the behavior of coupled finite difference estimators for other models of interest in the study of particle dynamics, namely, OU processes and the LE.

## II. EFFICIENT FINITE DIFFERENCE ESTIMATORS

In forming the Monte Carlo finite difference estimator for the sensitivity, the discretization of the derivative results in a systematic error or bias, while replacing the expected value with a sample average results in a statistical error. We denote the sample average of  $f$ , for a sample of size  $M$ , by  $\hat{f}(X_t) = M^{-1} \sum_{i=1}^M f(X_{i,t})$ , where the  $X_{i,t}$  are independent for each  $i \in \{1, \dots, M\}$ . A measure of the statistical error committed in computing  $S_\varepsilon$  is the variance, or more precisely, the standard deviation of the sample means which is proportional to the square root of the variance.

### A. Errors

To illustrate how these two errors behave, consider for simplicity the observable that depends on the process at the final time and define  $\hat{\phi}(\theta) = M^{-1} \sum_{i=1}^M X_{i,T}(\theta)$ , a random variable dependent on the parameter  $\theta$ . The forward difference estimator for this observable is

$$S_\varepsilon(T, \theta; \phi) \approx \hat{\Delta}(M, \varepsilon) = (\hat{\phi}(\theta + \varepsilon) - \hat{\phi}(\theta)) / \varepsilon,$$

where we write  $S_\varepsilon = \Delta(M, \varepsilon)$  to emphasize the dependence on  $M$  and  $\varepsilon$  and, in the sequel,  $\Delta_c$  for the central difference estimator. Note that under these assumptions, the target is a distribution, that is, there are no dynamics, and in this setting the following analysis, that gives the bias and variance of the estimator, is classical.<sup>57</sup> The expected value of the estimator is  $E[\hat{\Delta}] = (\varepsilon)^{-1}(\hat{\phi}(\theta + \varepsilon) - \hat{\phi}(\theta))$ , and if  $\hat{\phi}$  is (twice) differentiable in  $\theta$ , the bias is given by

$$\text{Bias}(\hat{\Delta}) = E[\hat{\Delta} - \hat{\phi}'(\theta)] = \hat{\phi}''(\theta)\varepsilon/2 + O(\varepsilon^2),$$

where the last equality can be seen by writing out the Taylor expansion for  $\hat{\phi}(\theta + \varepsilon)$ . The variance is

$$\text{Var}[\hat{\Delta}] = \varepsilon^{-2} \text{Var}[\hat{\phi}(\theta + \varepsilon) - \hat{\phi}(\theta)].$$

Assuming the pair  $(X_{i,T}(\theta + \varepsilon), X_{i,T}(\theta))$  is independent of other pairs for each  $i \leq M$ , then we have that

$$\text{Var}[\hat{\phi}(\theta + \varepsilon) - \hat{\phi}(\theta)] = M^{-1} \text{Var}[X^1 - X^2],$$

where we define  $(X^1, X^2) = (X_{1,T}(\theta + \varepsilon), X_{1,T}(\theta))$ . Thus, altogether we have

$$\text{Var}[\hat{\Delta}] = \varepsilon^{-2} M^{-1} \text{Var}[X^1 - X^2]. \quad (1)$$

An analysis of how the variance of this difference depends on  $\varepsilon$  provides insight into a strategy for efficiently computing the estimator.

From (1), we see that the  $\varepsilon$  dependence of  $\text{Var}[\hat{\Delta}]$  relies upon the  $\varepsilon$  dependence of  $\text{Var}[X^1 - X^2]$ . If  $X^1$  and  $X^2$  are independent, then  $\text{Var}[X^1 - X^2] = \text{Var}[X^1] + \text{Var}[X^2] \approx 2 \text{Var}[X]$ , where  $X$  is related to the distribution of the final time of the nominal dynamics. This implies  $\text{Var}[X^1 - X^2] = O(1)$  and hence  $\text{Var}[\hat{\Delta}] = O(\varepsilon^{-2} M^{-1})$ . In general for  $X^1$  and  $X^2$  that are not independent, we have that

$$\text{Var}[\hat{\Delta}] = \varepsilon^{-2} M^{-1} (\text{Var}[X^1] + \text{Var}[X^2] - 2 \text{Cov}[X^1, X^2]).$$

Thus, if  $X^1$  and  $X^2$  are positively correlated, then there is a net reduction to the variance of the estimator relative to independently sampling  $X^1$  and  $X^2$ . For instance, if the difference  $X^1 - X^2$  can be judiciously sampled so that  $\text{Var}[X^1 - X^2] = O(\varepsilon^2)$ , then  $\text{Var}[\hat{\Delta}] = O(M^{-1})$ , asymptotically eliminating the dependence of the estimator on the bias. For these static distributions, the well known technique of sampling using CRNs leads to reduced variance estimators.<sup>57</sup> Observe that all of the error estimates and relations above can be extended from this simple example with static distributions to the case of dynamics in a straight forward manner and, in particular, that (1) remains a quantity of interest for evaluating the efficiency of the finite difference estimator. Our goal will be to choose a sampling strategy for *dynamics* that will make the positive correlation between the distribution of the nominal and perturbed dynamics at each time step as large as possible while maintaining the correct marginal statistics.

We remark that at present we fix a bias and show that the common random path coupling produces a reduction to the variance *relative to independent sampling*. The mean squared error (MSE), formally

$$\text{MSE} = \text{Var} + \text{Bias} \cdot \text{Bias},$$

represents a balance between the statistical and systematic errors. While increasing the number of samples  $M$  decreases the variance with no effect on the bias, decreasing  $\varepsilon$  may increase the variance while decreasing the bias. For dynamics, different estimators, for example, central or forward differences, may have an optimal choice of bias that balances the two sources of error to achieve a minimal MSE, as is the case for static distributions.<sup>57</sup>

In Sec. II C, we demonstrate that coupling the nominal and perturbed dynamics using a common random path is an optimal strategy for sampling dynamics that reduces the variance of the estimator  $\mathcal{S}_\varepsilon$  relative to independent sampling. For SA of the extended variable GLE with convex potentials, the common random path coupling leads to substantial reductions, observed to be  $O(\varepsilon^2)$ , for sensitivities with respect to key parameters of interest. In these instances, since the statistical error scales like the square root of the variance, to reduce the error by a factor of 10 for independent sampling with a modest bias of  $\varepsilon = 0.1$  would require adding  $M = 10^4$  samples, in contrast to  $M = 10^2$  samples for the common random path coupling!

## B. Coupling dynamics

In what follows, we provide a very general framework that allows us to derive a coupling for dynamics that minimizes the variance of the difference between the nominal and perturbed processes appearing in Equation (1). We note that this difference need not be associated with a difference estimator, an aspect that we will exploit to analyze the sensitivity for non-local perturbations in Sec. IV B.

Consider the following pair of SDEs:

$$dX_t^k = b_k(X_t^k)dt + \sigma_k(X_t^k)dW_t^k, \quad (2)$$

subject to the initial condition  $X_0^k = x_0^k$ , for  $k = 1, 2$ , where  $X_t^k \in \mathbf{R}$ . We assume that for  $k \in \{1, 2\}$ ,  $(W_t^k)_{t \geq 0}$  are  $\mathbf{R}$ -valued independent Wiener processes, on a given stochastic basis, and that the coefficients  $b_k$  and  $\sigma_k$  satisfy the usual properties guaranteeing that each of the solutions is an Itô diffusion.<sup>58</sup> The infinitesimal generators of (2) are, respectively,

$$A_k f(x) = b_k(x)f'(x) + \frac{1}{2}\sigma_k^2(x)f''(x), \quad (3)$$

$f \in C_0^2(\mathbf{R})$ , where the prime indicates the derivative with respect to the argument. The generator encodes information about the statistics of the process.<sup>58,59</sup>

A coupling  $Z_t = (X_t^1, X_t^2)$  is produced by considering

$$dZ_t = B(Z_t)dt + \Sigma(Z_t)dW_t, \quad (4)$$

subject to initial conditions  $Z_0 = (x_0^1, x_0^2)$ , with given

$$W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \quad \text{and} \quad B(Z_t) = \begin{pmatrix} b_1(X_t^1) \\ b_2(X_t^2) \end{pmatrix}.$$

Here the diffusion matrix,

$$\Sigma(Z_t) = \begin{pmatrix} \Sigma_{11}(Z_t) & \Sigma_{12}(Z_t) \\ \Sigma_{21}(Z_t) & \Sigma_{22}(Z_t) \end{pmatrix},$$

depends on functions  $\Sigma_{ij}$ ,  $i, j \in \{1, 2\}$ , to be determined. Observe that (4) reduces to (2) by choosing  $\Sigma_{12} = \Sigma_{21} = 0$ ,  $\Sigma_{11}(Z_t) = \sigma_1(X_t^1)$ , and  $\Sigma_{22}(Z_t) = \sigma_2(X_t^2)$ . The generator for this extended system is given by

$$Af(z) = B(z) \cdot \nabla f(z) + \frac{1}{2}\Sigma(z)\Sigma^\top(z) : \nabla^2 f(z), \quad (5)$$

$f \in C_0^2(\mathbf{R}^2)$ , where  $z = (x_1, x_2)$ , and we use the notation  $M : N = \text{tr}(M^\top N)$  for the Frobenius product.

With these ideas in mind, we view  $Z_t$  as a coupling of the nominal and perturbed dynamics in the sensitivity estimator and, as foreshadowed in (1) in Sec. II A, seek to minimize the variance of the difference

$$D(Z_t) = f(X_t^1) - f(X_t^2), \quad (6)$$

where  $X_t^1$  and  $X_t^2$  are the solutions of (2) for a given observable  $f$ . In general, this minimization can be achieved locally in time where the constraints are constructed using (3) and (5). For specific examples (see the Appendix), it is possible to obtain the optimal coupling directly without localizing in time.

A slight modification of the above setting is sufficiently general to consider the LE and the extended variable GLE, both models that we consider in the sequel. These two models can be cast as systems of Itô diffusions where some components might degenerate in that the noise term may vanish. Then

instead of the pair (2), which we view as representing the nominal and perturbed dynamics, we consider a larger system that decomposes into a system of the nominal dynamics and of the perturbed dynamics, where some equations are degenerate diffusions. These ideas will be explored in more detail in Sec. III after we derive a general formulation for the optimal coupling for (2).

### C. Optimal variance reduction

To obtain an optimal reduction to the variance of (6), we place the following constraints on the generator of the coupled system, namely,

$$\begin{aligned} Af(x_1, x_2) &= A_1 f_1(x_1), & \text{when } f(x_1, x_2) = f_1(x_1), \\ Af(x_1, x_2) &= A_2 f_2(x_2), & \text{when } f(x_1, x_2) = f_2(x_2), \end{aligned}$$

assuming  $f_1$  and  $f_2 \in C_0^2(\mathbf{R}^2)$ , but of course more general type class assumptions are available. These constraints ensure that the marginal statistics of the coupled system match the statistics of the appropriate diffusion solving (2). In particular, for  $g(z) = f(x_1)f(x_2)$  such that  $g \in C_0^2(\mathbf{R}^2)$ , after some manipulation, the generator  $A$  can be expressed, in terms of the generators (3),

$$\begin{aligned} Ag(z) &= (A_1 f(x_1))f(x_2) + (A_2 f(x_2))f(x_1) \\ &\quad + (\Sigma_{11}\Sigma_{21} + \Sigma_{12}\Sigma_{22})(z)f'(x_1)f'(x_2), \end{aligned} \quad (7)$$

provided that  $\Sigma_{11}^2(z) + \Sigma_{12}^2(z) = \sigma_1^2(x_1)$  and  $\Sigma_{21}^2(z) + \Sigma_{22}^2(z) = \sigma_2^2(x_2)$  hold for  $z = (x_1, x_2) \in \mathbf{R}^2$  and that the mixed partials of  $g$  are equal.

Next, we observe that the variance of (6) is equal to

$$\begin{aligned} \text{Var}[D(Z_t)] &= \text{Var}[f(X_t^1)] + \text{Var}[f(X_t^2)] \\ &\quad + 2\mathbf{E}[f(X_t^1)]\mathbf{E}[f(X_t^2)] - 2\mathbf{E}[f(X_t^1)f(X_t^2)]. \end{aligned}$$

In order to minimize the variance, we must maximize the last term in the above equation. Locally in time, that is, for small perturbations  $\delta t$ , we have that

$$\begin{aligned} \mathbf{E}[f(X_{\delta t}^1)f(X_{\delta t}^2)] &= \mathbf{E}[g(X_{\delta t}^1, X_{\delta t}^2)] \\ &= e^{\delta t A}g(X_0^1, X_0^2) = [I + \delta t A + O(\delta t^2)]g(X_0^1, X_0^2) \\ &= f(X_0^1)f(X_0^2) + \delta t(A_1 f(X_0^1))f(X_0^2) + \delta t(A_2 f(X_0^2))f(X_0^1) \\ &\quad + \delta t(\Sigma_{11}\Sigma_{21} + \Sigma_{12}\Sigma_{22})(X_0^1, X_0^2)f'(X_0^1)f'(X_0^2) + O(\delta t^2), \end{aligned} \quad (8)$$

where the last equality follows from (7). Using these facts, we now state the following formal optimization problem.

**Optimization Problem 1.** *The choice of the diffusion matrix  $\Sigma = \Sigma(z)$  in (4) that minimizes the variance of (6) is given by*

$$\max_{\Sigma} \mathcal{F}(\Sigma; f) = \max_{\Sigma} \{(\Sigma_{11}\Sigma_{21} + \Sigma_{12}\Sigma_{22})(z)f'(x_1)f'(x_2)\},$$

for  $z = (x_1, x_2) \in \mathbf{R}^2$ , under the constraints  $\Sigma^\top \Sigma \geq 0$  and

$$\begin{aligned} \Sigma_{11}^2(z) + \Sigma_{12}^2(z) &= \sigma_1^2(x_1), \\ \Sigma_{21}^2(z) + \Sigma_{22}^2(z) &= \sigma_2^2(x_2). \end{aligned} \quad (9)$$

To solve Optimization Problem 1, constraint (9) allows us to write  $\Sigma_{11}(z) = \sigma_1(x_1) \cos \theta_1$ ,  $\Sigma_{12}(z) = \sigma_1(x_1) \sin \theta_1$ ,  $\Sigma_{21}(z) = \sigma_2(x_2) \cos \theta_2$ , and  $\Sigma_{22}(z) = \sigma_2(x_2) \sin \theta_2$ . Then the objective function to maximize is simply

$$\begin{aligned} &(\Sigma_{11}\Sigma_{21} + \Sigma_{12}\Sigma_{22})(z)f'(x_1)f'(x_2) \\ &= \sigma_1(x_1)\sigma_2(x_2) \cos(\theta_1 - \theta_2)f'(x_1)f'(x_2). \end{aligned}$$

If  $f'(x_1)f'(x_2) \geq 0$ , then the maximum is achieved at  $\eta = \theta_1 = \theta_2$ . Thus we obtain a *family of couplings*

$$\begin{aligned} dX_t^1 &= b_1(X_t^1)dt + \sigma_1(X_t^1)(\cos(\eta)dW_t^1 + \sin(\eta)dW_t^2), \\ dX_t^2 &= b_2(X_t^2)dt + \sigma_2(X_t^2)(\cos(\eta)dW_t^1 + \sin(\eta)dW_t^2), \end{aligned}$$

for  $\eta \in [0, 2\pi]$ . This coupling is equivalent to generating approximations with a common Wiener process  $(\tilde{W}_t)_{t \geq 0}$ , since  $\tilde{W}_t \stackrel{d}{=} \cos(\eta)W_t^1 + \sin(\eta)W_t^2$ , that is, they are equal in distribution. Due to the localization argument in Equation (8), this coupling may be sub-optimal for observables computed over long time horizons. Indeed, for ergodic systems, observables of trajectories arising from perturbations in the force field become essentially uncorrelated since the trajectories depart exponentially as time increases.<sup>35</sup> For some explicit examples (see the Appendix), one obtains the optimality of the common random path coupling without requiring a localization argument. On the other hand, locally for the OU process, LE, and GLE, we observe that the reduction to the variance of the estimator for several parameters of interest is on the order of the bias squared; clearly this coupling must be optimal for the specific numerical experiments that follow because anything more would be miraculous—we would have produced a Monte Carlo estimator that could beat Monte Carlo.

We remark further that the restricted set of diffusion matrices does not include perturbations of the following form. Consider  $dX_t = \sqrt{T}dW_t$  and  $dY_t^\varepsilon = \sqrt{T}d\tilde{W}_t$  for independent Wiener processes  $(W_t)_{t \geq 0}$  and  $(\tilde{W}_t)_{t \geq 0}$ . Indeed,  $Y_t^\varepsilon$  does not define a local perturbation with respect to  $T$  in precisely the same manner as  $dX_t^\varepsilon = \sqrt{T + \varepsilon}dW_t$ . Such couplings arise in a different context and are natural when the driving noise is not Wiener but Poisson.<sup>60</sup> Nevertheless,  $\text{Cov}[Y_t^\varepsilon, X_t] < \text{Cov}[X_t^\varepsilon, X_t]$  and thus  $\text{Var}[X_t^\varepsilon - X_t] < \text{Var}[Y_t^\varepsilon - X_t]$ , so the diffusion that is a part of our solution set performs better than the alternative. We also remark that the above solution to the optimization problem does not depend on  $f$  due to the implicit assumption that  $f$  is monotonic throughout its domain. A more general solution to Optimization Problem 1 would incorporate the behavior of the function  $f$  into the coupling design.

In Sec. III, we introduce the GLE, a prototypical system with memory, and discuss an extended variable formulation which casts the problem into a form amenable to the preceding theory. We also introduce some notation and concepts germane to both examples in Sec. IV, including the technique used for fitting the Prony series, the normalized velocity autocorrelation function (VACF), and the integration scheme used.

## III. SYSTEMS WITH MEMORY

### A. Extended variable GLE

The GLE is a model of anomalous diffusion, that is, diffusion where the relationship between the mean squared displacement (MSD) of the particle and time is no longer linear, that occurs in complex or viscoelastic media typically found in biological applications. The GLE includes a temporally non-local drag force and a random force term with non-trivial correlations.<sup>61</sup> The position,  $X_t^i \in \mathbf{R}^d$ , and velocity,

$V_t^i \in \mathbf{R}^d$ , of particle  $i$  with mass  $m_i$  at time  $t$  are given by the GLE,

$$\begin{aligned} dX_t^i &= V_t^i dt, \\ m_i dV_t^i &= -\nabla U(X_t^i)dt - \int_0^t \kappa(t-s)V_s^i ds dt + F^i(t)dt, \end{aligned} \quad (10)$$

subject to initial conditions  $X_0^i = x_0$  and  $V_0^i = v_0$ , where  $-\nabla U$  is the conservative force and  $F^i$  is the random force. In the stochastic integro-differential equation for the velocity, the memory kernel  $\kappa$  characterizes the frictional force and, through the fluctuation-dissipation theorem,

$$E[F^i(t+s)F^j(t)] = k_B T \kappa(s) \delta_{ij}, \quad s \geq 0, \quad (11)$$

the random force, where  $k_B$  is Boltzmann's constant and  $T$  is the absolute (thermodynamic) temperature. This system is non-Markovian, that is, it has memory; the friction at time  $t$  may have a dependence on the velocity  $V(s)$ , for  $s < t$ .

A general strategy for analyzing (10) involves mapping the non-Markovian system onto a Markovian system with suitably many additional degrees of freedom.<sup>62</sup> An extended variable formulation can often be obtained through physically realistic assumptions on  $\kappa$  that suggest a particular representation for the memory kernel. For example, when the memory kernel is posited to have the form of a power law then a positive Prony series has been identified as a good representation although more general formulations exist.<sup>6,62</sup> In general, observe from (11) that  $\kappa$  is the covariance function for the driving noise. Then a sufficient condition on  $\kappa$  for an extended variable formulation to hold is when the driving noise has a spectral density  $|p(k)|^{-2}$ , where  $p(k) = \sum_m c_m (-ik)^m$  is a polynomial with real coefficients and roots in the upper half plane.<sup>63</sup> A separate topic, not addressed in this work, is at what level of generality to represent the kernel or subsequently how to fit the parameters to experimental data. Indeed, much work has been done in the harmonic analysis and signal processing literature on fitting exponential functions to data since de Prony's classical work.<sup>64–67</sup> The important observation here is that the mapping onto Markovian dynamics yields a system of (degenerate) Itô diffusions with a large number of parameters. This results in systems for which local and global SA are highly relevant and for which finite difference estimators are useful for SA of all parameters of interest.

The issue of which representation to use aside, when the memory kernel can be represented by a positive Prony series,

$$\kappa(t) = \sum_{k=1}^{N_k} \frac{c_k}{\tau_k} e^{-t/\tau_k}, \quad t \geq 0,$$

then the non-Markovian GLE can be mapped onto a higher dimensional Markovian problem in  $dN_k$ -extended variables. This extended variable GLE is given by

$$\begin{aligned} mdV_t &= -\nabla U(X_t)dt + \sum_{k=1}^{N_k} S_t^k dt, \\ dX_t &= V_t dt, \\ dS_t^k &= -\frac{1}{\tau_k} S_t^k dt - \frac{c_k}{\tau_k} V_t dt + \frac{1}{\tau_k} \sqrt{2k_B T c_k} dW_t^k, \end{aligned} \quad (12)$$

subject to  $X_0 = x_0$ ,  $V_0 = v_0$ , and  $S_0^k = s_0^k$ , for independent Wiener processes  $(W_t^k)_{t \geq 0}$ . Here we omit the obvious extension to a system of many particles in the interest of brevity. In the absence of a conservative force and for the harmonic potential,  $U(X_t) = \omega^2 X_t^2 / 2$ , analytic expressions can be given for the statistics of the dynamics and for certain observables of interest including the VACF and MSD.<sup>68–70</sup> For other potentials, numerical integrators for this system that are stable for a wide range of parameter values are available and implemented in the LAMMPS software package.<sup>4</sup> Moreover, these schemes exactly conserve the first and second moments of the integrated velocity distribution in certain limits and stably approach the LE in the limit of small  $\tau_k$ , the latter of which is a property proven to hold for the extended variable GLE by other authors.<sup>71</sup>

As formulated, (12) can be viewed as a system of (degenerate) Itô diffusions. Thus, we can form a system of nominal and perturbed dynamics in the spirit of (2), for  $k \geq 2$ . In addition to any parameter appearing in the potential and  $T$ , we are interested in analyzing the sensitivity with respect to  $\tau_k$  and  $c_k$ ,  $k \in \{1, \dots, N_k\}$ . The pathwise and likelihood ratio methods outlined in the Introduction are not applicable to these latter parameters of interest. Since in general the  $c_k$  and  $\tau_k$  are obtained from experimentally observed data, it is desirable to analyze the sensitivity of the model with respect to uncertainties arising from the fitting procedure, for example, due to errors in the measurement or lack of data.

## B. Optimal coupling for extended variable GLE

Presently, we apply the most basic aspects of the theory presented in Sec. II to the simple example of an extended variable GLE with one extended degree of freedom, i.e., one Prony mode, where the dynamics is subject to a harmonic confining potential with frequency parameter  $\omega$ . That is, we consider the system  $dZ_t = BZ_t dt + \Sigma dW_t$  for the coupling  $Z = (X, V, S, \tilde{X}, \tilde{V}, \tilde{S})$ , where  $B$  and  $(\Sigma)_{ij} = \sigma_{ij}$  are  $6 \times 6$  coefficient matrices to be determined. Here and below we suppress the extended variable index and denote the perturbed system variables with tildes for the ease of notation.

An optimal coupling is found by matching the statistics of the marginals of the coupled process to the statistics of the nominal and perturbed processes. By writing out the infinitesimal generators of the corresponding SDEs, this requirement immediately characterizes  $B$  and implies that the only nonzero elements of  $\Sigma$  are  $\sigma_{33}$ ,  $\sigma_{63}$ ,  $\sigma_{36}$ , and  $\sigma_{66}$ . Formally, the optimization problem can be stated as follows.

**Optimization Problem 2 (1-mode GLE with harmonic potential).** The choice of diffusion matrix  $(\Sigma)_{ij} = \sigma_{ij}$  that minimizes the variance of  $D(Z_t)$  is given by

$$\max_{\Sigma} \mathcal{F}(\Sigma; f) = \max_{\sigma} (\sigma_{33}\sigma_{63} + \sigma_{36}\sigma_{66}) \frac{\partial}{\partial x_3 \partial x_6} f(z),$$

for  $\sigma = (\sigma_{11}, \dots, \sigma_{66})$  for all  $z \in \mathbf{R}^4$ , under the constraints  $\Sigma^\top \Sigma \geq 0$  and

$$\begin{aligned} \sigma_{33}^2 + \sigma_{36}^2 &= \gamma \sqrt{c} / \tau, \\ \sigma_{63}^2 + \sigma_{66}^2 &= \gamma \sqrt{c} / \tilde{\tau}, \end{aligned}$$

where  $\gamma = \sqrt{2k_B T}$ .

Thus, for this problem, an optimal family of couplings  $Z(\eta)$ , indexed by  $\eta \in [0, 2\pi]$ , is given by

$$B = \begin{pmatrix} L & \mathbf{0} \\ \mathbf{0} & \tilde{L} \end{pmatrix},$$

with

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -\omega^2 & 0 & 1 \\ 0 & -\frac{c}{\tau} & -\frac{1}{\tau} \end{pmatrix},$$

and  $\Sigma$  with only nonzero elements  $\sigma_{33} = \gamma \sin(\eta) \sqrt{c}/\tau$ ,  $\sigma_{63} = \gamma \sin(\eta) \sqrt{c}/\tau$ ,  $\sigma_{36} = \gamma \cos(\eta) \sqrt{c}/\tau$ , and  $\sigma_{66} = \gamma \cos(\eta) \sqrt{c}/\tau$ , where  $W = (-, -, W^3, -, -, W^6)$  for independent Wiener processes  $(W_t^3)_{t \geq 0}$  and  $(W_t^6)_{t \geq 0}$  (here several components of  $W$  are irrelevant due to the zero rows and columns in  $\Sigma$ ). For each fixed  $\eta$ , this coupling is equivalent to choosing a common random path for generating the dynamics of  $S$  and  $\tilde{S}$ . Extending this optimization problem to an  $N_k$ -mode GLE leads to the expected strategy, namely, the common random path coupling for generating  $S^k$  and  $\tilde{S}^k$ , for each  $k \in \{1, \dots, N_k\}$ . Each extended variable requires an independent common random path for  $N_k$  independent Wiener processes in total, as dictated by (12).

In the remainder of this section, we introduce the notation and concepts that are relevant for the numerical experiments in Sec. IV where we test the variance reduction obtained by the common random path coupling suggested by the theory above.

### C. Fitting Prony series

In the numerical experiments that follow, we consider (10) with a power law memory kernel given by

$$\kappa(t-s) = \frac{\gamma_\lambda}{\Gamma(1-\lambda)}(t-s)^{-\lambda}, \quad (13)$$

for  $\lambda \in (0, 1)$  where  $\Gamma$  is the gamma function. For (13), one can obtain an approximate Prony series with  $N_k$  modes by assuming logarithmically spaced  $\tau_k$  and then fitting the  $c_k$  using a least squares method over an interval two decades longer than the simulation length.<sup>4</sup> This simplification retains a rich enough family of parameters  $c_k$  to illustrate the variance reduction achieved by the common random path coupling. In Figure 1, we illustrate this fitting procedure for Prony series with  $N_k$  modes compared to measurements of (13) with  $\gamma_\lambda=1.0$  and  $\lambda=0.5$ . We choose sufficiently many data points to ensure a stable least squares approximation.

### D. Integration scheme

We integrate the system using a modified Verlet method proposed by other authors, ensuring that the choice of method parameters satisfies the consistency condition and preserves the Langevin limit.<sup>4</sup> In many molecular dynamics simulations, the initial velocity  $v_0$ , and hence  $s_{k,0}$ , is chosen from a thermal distribution. In the numerical experiments below, the initial conditions for the particle position and velocity are taken to be definite and  $s_{k,0} = 0$  for all  $k$ . This is done to minimize the sources of the statistical error thus, clarifying the reporting of deviations in the numerical results. The inclusion

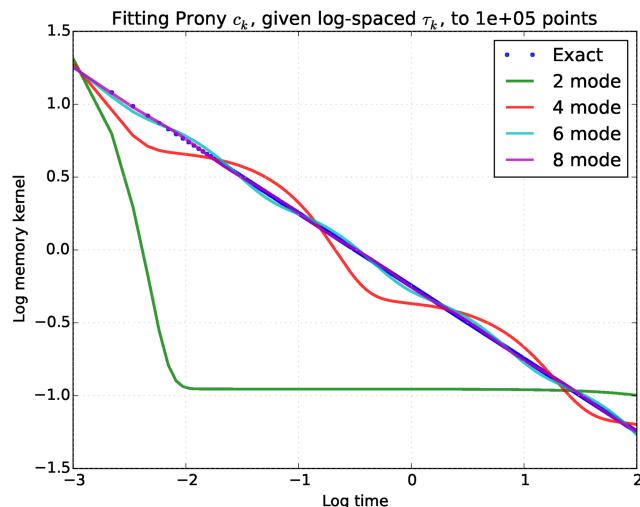


FIG. 1. A least squares fit of the Prony coefficients  $c_k$ ,  $k \in \{1, \dots, N_k\}$ , assuming log-spaced  $\tau_k$ , for each of  $N_k$  modes. This fit is sufficient to illustrate the variance reduction achieved by the common random path coupling.

of thermal initial conditions does not pose a challenge to the method.

### E. Normalized autocorrelation functions

The results of our numerical experiments are given primarily in terms of normalized autocorrelation functions. Formally, the normalized VACF is given by

$$\overline{\text{VACF}(t)} = \langle V_t V_0 \rangle / \langle V_0 V_0 \rangle = \langle V_t \rangle / v_0,$$

where the second equality holds when the initial velocity is definite. A similar definition is assigned to the normalized position autocorrelation function (PACF). For the GLE with a harmonic confining potential and a power law memory kernel, expressions for the autocorrelation functions can be given in terms of Mittag-Leffler functions and their derivatives.<sup>68</sup> We compute the normalized VACF and PACF using the integrated velocity and position distributions and the fast Fourier transform method.<sup>72</sup>

As discussed in the Introduction, it is of fundamental importance to understand the sensitivity of the GLE dynamics to local and global perturbations. Fitting parameters based on

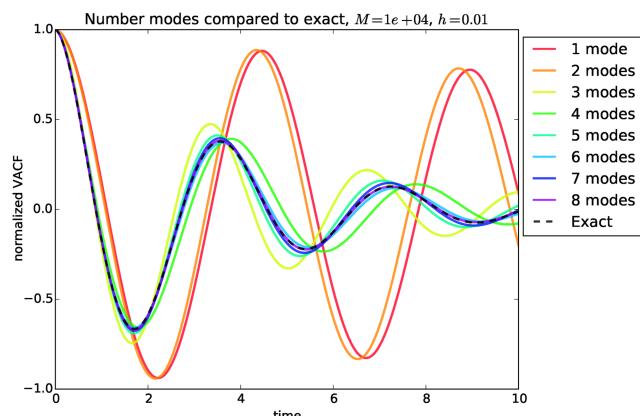


FIG. 2. Small changes to the number of modes leads to qualitatively different behavior of the  $\overline{\text{VACF}}$  for the GLE with a harmonic confining potential.

summary statistics alone can lead to models that do not capture quantities of interest. Nevertheless, we shall demonstrate the optimal coupling method for summary statistic observables for the simple models discussed above where analytic representations are available. Even for these simple models, the VACF exhibits striking qualitative differences to global model perturbations such as decreasing the number of Prony modes. Figure 2 illustrates the VACF for models with a varying number of Prony modes, i.e., extended variables, compared to an asymptotically exact quantity for the normalized VACF for the GLE.

#### IV. NUMERICAL EXPERIMENTS

The numerical experiments below focus on SA for the extended variable GLE for one particle in one dimension with a power law memory kernel. The first experiment, in Sec. IV A, concerns the sensitivity with respect to the Prony coefficients  $c_k$  where the coefficients are fit using the method described in Sec. III C. We observe that the reduction to the variance of the difference (6) for the optimally coupled dynamics is on the order of the bias squared for convex potentials.

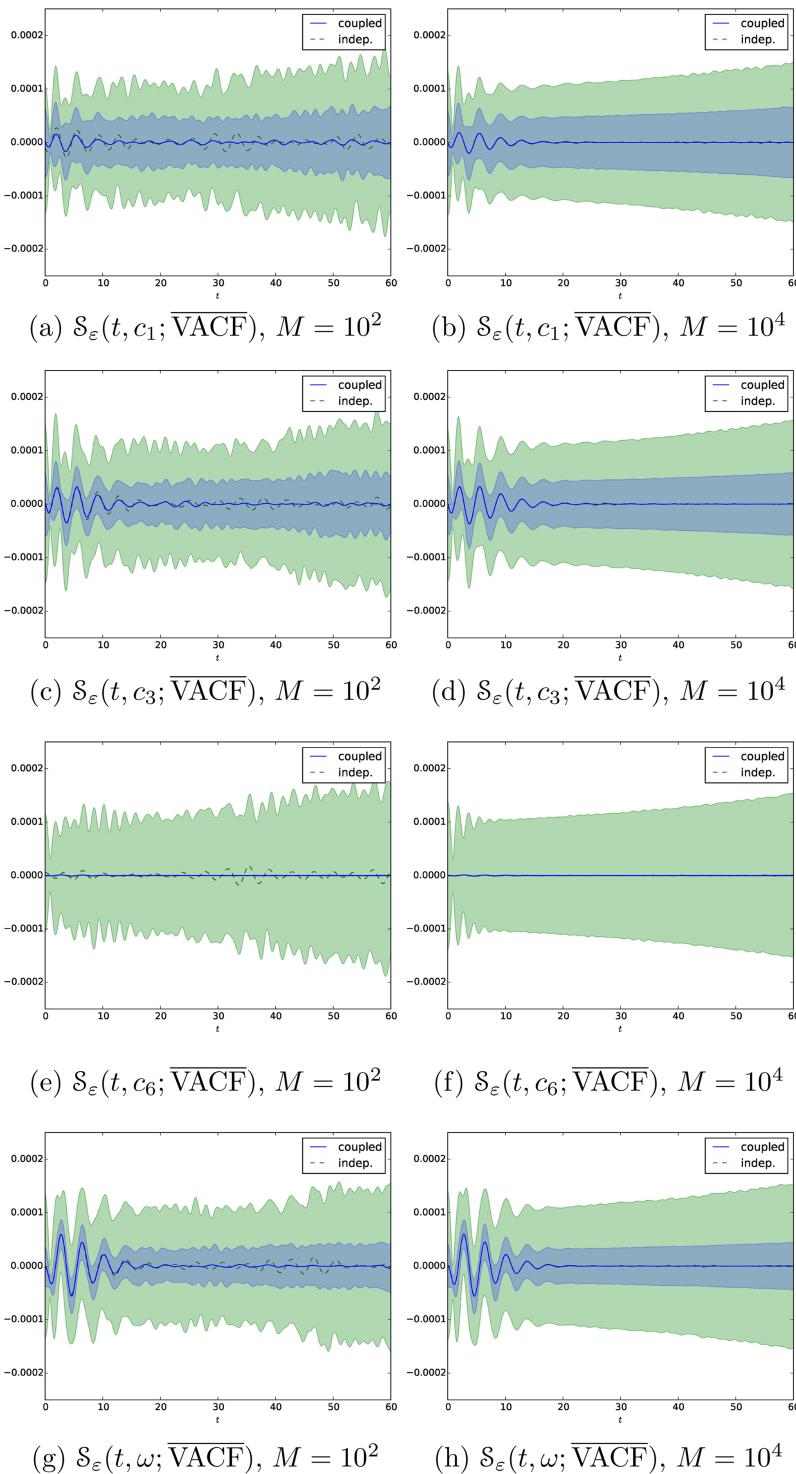


FIG. 3. The computational advantage of the common random path coupling is illustrated by the reduced sample size required to obtain meaningful estimates for sensitivities. Here we plot the sample mean with error bars denoting two standard deviations, based on  $M = 10^2$  samples (left column) and  $M = 10^4$  samples (right column), for various parameters.

We remark that the experiments that follow do not use physically significant units. The difference estimator and the variance of  $D(Z_t)$  depend on the scale of the parameters of interest and this should be taken into account when understanding the results, for example, the limits on the  $y$ -axis in Figures 4 and 5 have no relation to experimental resolution. That is, for  $u(\theta) = \mathbf{E}[f(X_t^\theta)]$ , the finite difference estimator of  $\partial_\theta u$  depends on the scale of  $\theta$ , and therefore it may be of interest in certain applications to compute re-scaled sensitivities, such as

$$\begin{aligned}\partial_{\log \theta} u &= \theta \partial_\theta u, \\ \partial_\theta \log u &= (\partial_\theta u)/u, \text{ and} \\ \partial_{\log \theta} \log u &= \theta(\partial_\theta u)/u,\end{aligned}$$

where  $\partial_{\log \theta}$  is the derivative with respect to  $\log \theta$ .<sup>73</sup>

### A. Sensitivity with respect to Prony coefficients

We begin by computing local sensitivities for the proposed model with a harmonic confining potential. In particular, we investigate the sensitivity with respect to the Prony coefficients  $c_k$  for  $k \in \{1, \dots, N_k\}$ , the harmonic potential frequency  $\omega$ , and the temperature  $T$ , that is, for a set of parameters  $\theta = (\omega, T, c_1, \dots, c_{N_k})$ . For the observable  $\overline{\text{VACF}}(t) = f(V_t^\theta)$ , the Monte Carlo finite difference estimator  $\mathcal{S}_\varepsilon(t, \theta; \overline{\text{VACF}}) = \Delta_c(M, \varepsilon)$  based on the central difference is given by

$$\hat{\Delta}_c(M, \varepsilon) = (\hat{f}(V_t^{\theta_i+\varepsilon}) - \hat{f}(V_t^{\theta_i-\varepsilon}))/2\varepsilon,$$

where  $V_t^{\theta_i \pm \varepsilon}$  denotes a small  $\varepsilon$  perturbation with respect to the parameter  $\theta_i$  leaving all other  $\theta_j, j \neq k$ , fixed. We compute  $\hat{\Delta}_c$  with a bias  $\varepsilon = 0.01$  for dynamics that are driven by a common random path and that are driven by independent paths. In Figure 3, we compare the sample mean of estimators  $\mathcal{S}_\varepsilon$ , along with two standard deviations, for various parameters. The key observation here is that the optimal coupling dramatically reduces the variances of the difference estimator, relative

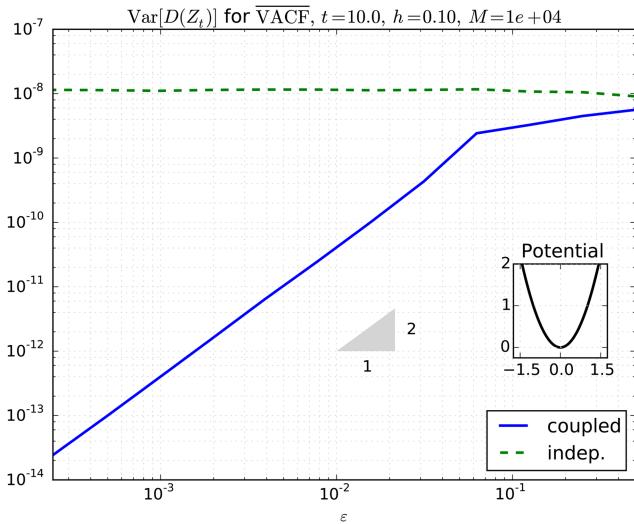


FIG. 4. For  $\mathcal{S}_\varepsilon(t = 10, c_1; \overline{\text{VACF}})$  for an  $N_k = 8$  mode formulation of GLE,  $\text{Var}[D(Z_t)] = O(\varepsilon^2)$  for the common random path coupling in contrast to  $\text{Var}[D(Z_t)] = O(1)$  for the naively sampled independent difference.

to the independently sampled dynamics, even for a modestly sized sample.

The precise nature of the reduction can be deduced by varying  $\varepsilon$  for a fixed index  $\mathcal{S}_\varepsilon(t, \theta_i; \overline{\text{VACF}})$ . In Figure 4, the variance of the difference (6) is compared for dynamics coupled with a common random path and independent dynamics for  $\mathcal{S}_\varepsilon(t, c_1; \overline{\text{VACF}})$ . For the optimally coupled dynamics, the reduction is  $\text{Var}[D(Z_t)] = O(\varepsilon^2)$ , that is, on the order of the bias squared and, in contrast,  $\text{Var}[D(Z_t)] = O(1)$  for the difference of the independent dynamics. Recalling the discussion of errors in Sec. II A, we see that for this example,  $\text{Var}[\hat{\Delta}_c] = O(M^{-1})$  in the case of the optimally coupled dynamics. That is, the optimal coupling eliminates the dependence of the variance of the estimator on the bias, asymptotically, in the case of a convex potential.

For nonlinear and non-convex potentials, the common random path coupling reduces the variance of the estimator, although the rate is not expected to be  $O(\varepsilon^2)$ . In Figure 5, the  $N_k = 8$  mode formulation of GLE is considered with

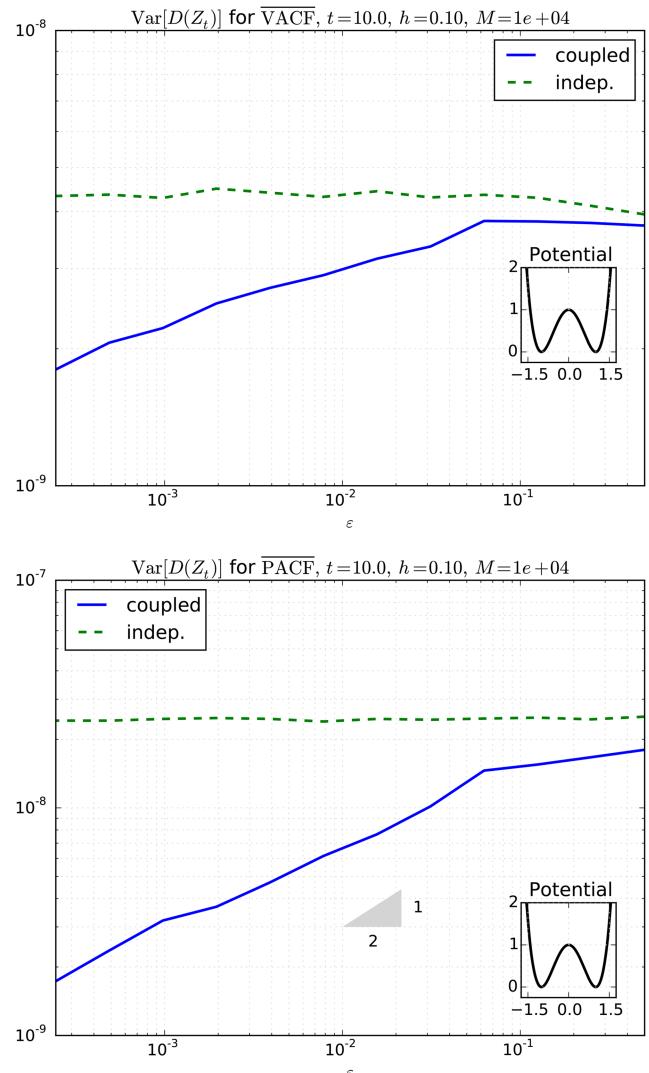


FIG. 5. For simple nonlinear non-convex potentials, there is a net reduction to the variance from the common random path coupling. Here, for the double well potential,  $U(X_t) = (1 - X_t^2)^2$ , with  $k_B T = 0.5$ , the reduction is less than  $O(\sqrt{\varepsilon})$  for both the PACF and VACF.

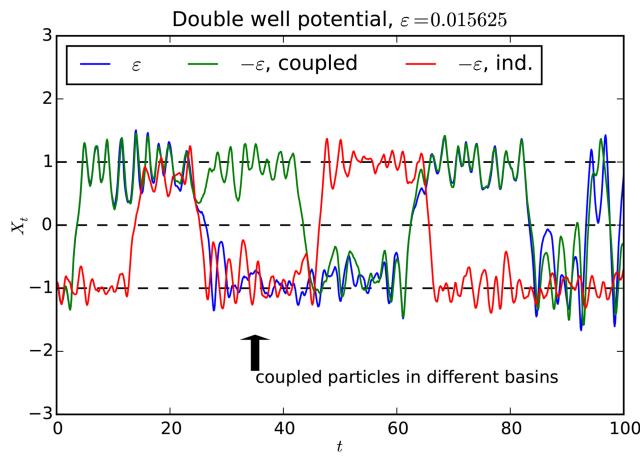


FIG. 6. For nonlinear non-convex potentials, less substantial reductions to the variance are observed. For the double well potential, the coupled dynamics can be “pushed” into separate basins decreasing the covariance between the coupled paths.

a simple double-well potential,  $U(X_t) = (1 - X_t^2)^2$ , and  $k_B T = 0.5$  for the sensitivities  $S_\varepsilon(t = 10, c_1; \overline{\text{VACF}})$  and  $S_\varepsilon(t = 10, c_1; \overline{\text{PACF}})$ . In this setting, we observe a decay of less than  $O(\sqrt{\varepsilon})$  for both observables. In particular, for the double well potential, the position time series, see Figure 6, indicates that the perturbed dynamics can be pushed into a basin different from the location of the nominal dynamics for some time. This decreases the correlation between the two paths and lessens the reduction to  $\text{Var}[D(Z_t)]$ .

For the extended variable GLE with a harmonic potential and a power law memory kernel, since analytical expressions exist for several observables of interest including the VACF,<sup>68</sup> the maximum relative error for approximating the power law memory kernel with a given number of Prony modes can be computed *a priori*.<sup>4</sup> For more complicated potentials, exact expressions for observables and statistics of the dynamics are not available. Further, in reality one would like to fit the Prony modes to experimentally obtained data. Such a procedure would likely involve complex inference methods and a nonlinear fitting to obtain the  $\tau_k$  and  $c_k$ . In such instances, it would be highly relevant to test the sensitivity of the fitted parameters.

## B. Sensitivity with respect to number of Prony modes

From Figure 2, we see that changing the number of Prony modes has a qualitative impact on the  $\overline{\text{VACF}}$ . This motivates the numerical experiment that follows, where we analyze the effect of increasing the *number* of Prony modes. That is, for  $N_1 < N_2$  consider two systems with  $N_1$  and  $N_2$  extended variables, respectively. Given the difference  $D(Z_t) = f(V_t^{N_1}) - f(V_t^{N_2})$ , define a sensitivity

$$\mathcal{S}^* = \sum_t |\widehat{D(Z_t)}|^2 / \sigma_{Z_t},$$

where the carat denotes the sample mean,  $\sigma_{Z_t}$  is the standard deviation of the associated sample mean  $\widehat{D(Z_t)}$ , and the sum is over the space of discrete time points up to a fixed time  $t < T$ . Although this sensitivity is not a sensitivity in the sense of the gradients introduced previously,  $\mathcal{S}^*$  gives a quantitative

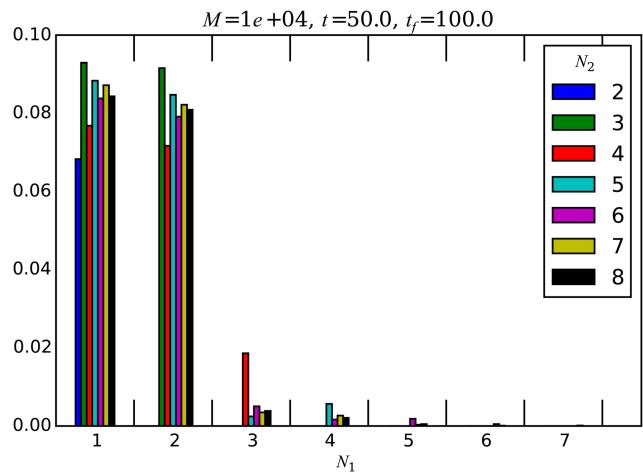


FIG. 7. The non-local sensitivity  $\mathcal{S}^*$  gives a quantitative characterization of the difference between the observed  $\overline{\text{VACF}}$  for models with different numbers of modes (cf. Figure 2).

characterization of the difference between the two systems, see Figure 7.

The optimal coupling can be used to reduce the variance of such non-local sensitivities. Here we investigate the difference between a nominal model with  $N_1 = n$ , for  $n = \{1, \dots, 8\}$ , and a perturbed model with one additional mode  $N_2 = N_1 + 1$ . In Figure 8, we plot the variance of the difference generated by these nominal and perturbed dynamics for both the optimally coupled and independent cases, illustrating the reduced computational cost in sampling the optimally coupled dynamics in comparison to independent sampling. Here the Prony series are fit separately for the nominal and perturbed dynamics using the method outlined in Sec. III C. Auxiliary variables  $c_{N_1+1} = 0$  and  $\tau_{N_1+1} = 1$  are added to the nominal system so that the vectors for the nominal and perturbed dynamics have the same size, and then the common random path coupling is naively carried out for each of the components.

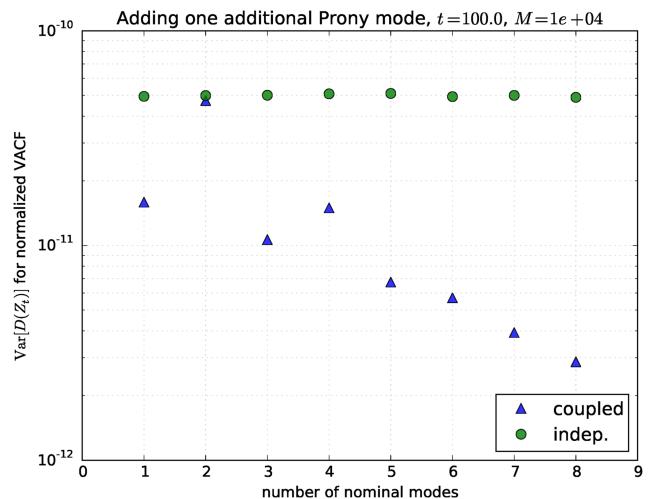


FIG. 8. The common random path coupling is a valid tool for global SA, as illustrated by the reduced computational cost in computing  $\text{Var}[D(Z_t)]$  where the difference is between a nominal model with a fixed number of Prony modes and an alternative model with one additional Prony mode. This difference, although it cannot be expressed as a derivative, provides a characterization of the sensitivity to a discrete change to the system.

## V. CONCLUSIONS

We develop a general framework for the variance reduction via coupling for goal-oriented SA of continuous time stochastic dynamics. This theory yields efficient Monte Carlo finite difference estimators for sensitivity indices that apply to all parameters of interest in an extended variable formulation of the GLE. Other well known SA techniques, such as likelihood ratio and pathwise methods, are not applicable to key parameters of interest for this model. These estimators are obtained by coupling the nominal and perturbed dynamics appearing in the difference estimator through a common random path and are thus easy to implement. Strong heuristics are provided to demonstrate the optimality of the common random path coupling in this setting. In particular, for the extended variable GLE with a convex potential, the reduction to the variance of the estimator is on the order of the bias squared, mitigating the effect of the bias error on the computational cost. Moreover, the common random path coupling is a valid computation tool for non-local perturbations and finite difference estimators for the global SA with applications to model form uncertainty, such as comparing different models obtained from truncating the Prony series in GLE, and other aspects of UQ.

## ACKNOWLEDGMENTS

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## APPENDIX: OTHER EXAMPLES OF INTEREST IN PARTICLE DYNAMICS

### 1. OU processes

OU processes are simple and easily analyzed yet are insightful as they possess several important features: the processes are Markovian, Gaussian, and stationary under the appropriate choice of initial conditions. Further, we note that the evolution of the extended variables in (12) is described by an OU process.

Consider the SDE,

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad (\text{A1})$$

subject to the initial condition  $X_t = x_0 \sim h$ , for a given distribution  $h$ , with scalar coefficients  $\theta, \sigma > 0$ , and  $\mu \in \mathbf{R}$ . Here  $(W_t)_{t \geq 0}$  is a Wiener process on a given stochastic basis. The solution to (A1), given by

$$X_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dW_s,$$

for  $t \in [0, T]$ , is the OU process. This process depends on parameters  $\theta, \mu, \sigma, x_0$ , and  $h$ .

As discussed in Sec. II, we are interested in minimizing the variance of (6), where  $(X_t^1, X_t^2)$  is given by the system

$$\begin{aligned} dX_t^1 &= \theta_1(\mu_1 - X_t^1)dt + \sigma_1 dW_t^1, & X_0^1 &= x_0^1 \sim h_1 \\ dX_t^2 &= \theta_2(\mu_2 - X_t^2)dt + \sigma_2 dW_t^2, & X_0^2 &= x_0^2 \sim h_2. \end{aligned}$$

Then

$$\begin{aligned} \text{Var}[D(Z_t)] &= \text{Var}[f(X_t^1)] + \text{Var}[f(X_t^2)] \\ &\quad + 2 \mathbf{E}[f(X_t^1)] \mathbf{E}[f(X_t^2)] - 2 \mathbf{E}[f(X_t^1)f(X_t^2)] \\ &= \text{Var}[f(X_t^1)] + \text{Var}[f(X_t^2)] \\ &\quad - 2 \text{Cov}[f(X_t^1), f(X_t^2)], \end{aligned}$$

and hence to minimize the variance of the difference we seek to maximize the covariance appearing in the expression above. If  $X_t^1$  and  $X_t^2$  are independent, that is, they are generated with independent processes  $W^1$  and  $W^2$ , then the covariance in question will vanish. If we inject some dependence between  $X^1$  and  $X^2$  so that the covariance is nonzero, we find, after cancellation (for linear  $f$ ), that the covariance is given by

$$\mathbf{E} \left[ f \left( \sigma_1 e^{-\theta_1 t} \int_0^t e^{\theta_1 s} dW_s \right) f \left( \sigma_2 e^{-\theta_2 t} \int_0^t e^{\theta_2 s} dW_s \right) \right].$$

This covariance is maximized when the stochastic integral processes above are dependent, which occurs when the driving processes  $W^1$  and  $W^2$  are assumed to be linearly dependent.

We shall look at two concrete observables, to gain intuition on the variance reduction introduced by the common random path coupling for the sensitivity with respect to different parameters. For simplicity, we shall further assume that  $x_0^1 = x_0^2$  and that  $\mu_1 = \mu_2$  are definite. Then these terms do not play a role since cancellations occur, for example, when  $\mathbf{E}[x_0]^2 = \mathbf{E}[x_0^2]$ . In these examples, the coupling with a common random path reduces the variance in the computation of the central difference estimator by a factor  $O(\varepsilon^2)$  for the sensitivity with respect to  $\theta$  and  $\sigma$ .

For both observables and for the sensitivity with respect to  $\theta$  and  $\sigma$ , we find that  $\text{Var}[D(Z_t)] = O(\varepsilon^2)$  when sampling coupled paths and  $\text{Var}[D(Z_t)] = O(1)$  when sampling independent paths. Therefore, for standard first order difference estimators of the sensitivity indices, we have  $\text{Var}[\hat{\Delta}_c] = O(M^{-1})$ , when sampling optimally coupled paths but  $\text{Var}[\hat{\Delta}_c] = O(\varepsilon^{-2} M^{-1})$ , for independently sampled paths. For the OU process, the optimal coupling eliminates the asymptotic dependence of the variance of the estimator on  $\varepsilon$ , in contrast to the case of sampling independent paths.

#### a. Finite time observable

Consider the finite time observable,  $f(X_t) = X_T$  for  $T < \infty$ . The expression for the covariance simplifies to

$$\begin{aligned} \text{Cov}[X_T^1, X_T^2] &= \sigma_1 \sigma_2 e^{-(\theta_1 + \theta_2)T} \mathbf{E} \left[ \int_0^T e^{\theta_1 u} dW_u \int_0^T e^{\theta_2 v} dW_v \right] \\ &= \sigma_1 \sigma_2 e^{-(\theta_1 + \theta_2)T} \int_0^T e^{(\theta_1 + \theta_2)s} ds \\ &= \sigma_1 \sigma_2 e^{-(\theta_1 + \theta_2)T} (e^{(\theta_1 + \theta_2)T} - 1) / (\theta_1 + \theta_2) \\ &= \sigma_1 \sigma_2 (1 - e^{-(\theta_1 + \theta_2)T}) / (\theta_1 + \theta_2). \end{aligned}$$

Thus the variance of the difference  $D(Z_t)$  converges to a constant, depending on  $\varepsilon$ , as  $T \rightarrow \infty$ . As the variance of the

difference does not vanish, the coupling with a common random path is a useful computational technique for all finite times.

Consider now the sensitivity with respect to  $\theta$ . Then  $\theta_1(\theta, \varepsilon)$  and  $\theta_2(\theta, \varepsilon)$  can be viewed as functions of  $\theta$  and  $\varepsilon$ , i.e.,  $\theta_1 = \theta + \varepsilon$  and  $\theta_2 = \theta - \varepsilon$  for the central difference. To determine the asymptotic dependence of  $\text{Var}[D(Z_t)]$  on  $\varepsilon$ , we expand the variance of the difference in a series in  $\varepsilon$  at zero. For standard first order differences (central, forward, and backward), in the case of independent sampling, one finds

$$\text{Var}[X_T^1] + \text{Var}[X_T^2] = \sigma^2 \theta^{-1} - \sigma^2 \theta^{-1} e^{-2T\theta} + O(\varepsilon),$$

since  $\theta_1(0) = \theta_2(0) = \theta$ . That is, the variance of the difference is  $O(1)$ . In contrast, for sampling with common random paths,

one finds

$$\text{Var}[X_T^1] + \text{Var}[X_T^2] - 2 \text{Cov}[X_T^1, X_T^2] = O(\varepsilon^2).$$

A similar story holds for the sensitivity with respect to  $\sigma$ ,  $\text{Var}[D(Z_t)] = O(1)$  for independent sampling and  $\text{Var}[D(Z_t)] = O(\varepsilon^2)$  for sampling with common random paths, when using standard first order differences.

### b. Time average observable

Next we consider the time average observable defined by  $\bar{X} = T^{-1} \int_0^T X_s ds$ . Once again, we wish to investigate the dependence of  $\text{Var}[D(Z_t)]$  on  $\varepsilon$  for the case of coupled paths and independent sampling. The expression for the covariance in this instance is

$$\begin{aligned} \text{Cov}[\bar{X}^1, \bar{X}^2] &= \mathbf{E} \left[ T^{-1} \int_0^T \sigma_1 e^{-\theta_1 s} \int_0^s e^{\theta_1 u} dW_u ds T^{-1} \int_0^T \sigma_2 e^{-\theta_2 t} \int_0^t e^{\theta_2 v} dW_v dt \right] \\ &= \sigma_1 \sigma_2 T^{-2} \int_0^T \int_0^T e^{-\theta_1 s - \theta_2 t} \mathbf{E} \left[ \int_0^t e^{\theta_1 u} dW_u \int_0^s e^{\theta_2 v} dW_v \right] ds dt \\ &= \sigma_1 \sigma_2 T^{-2} \int_0^T \int_0^T e^{-\theta_1 s - \theta_2 t} \int_0^{s \wedge t} e^{(\theta_1 + \theta_2)r} dr ds dt \\ &= \sigma_1 \sigma_2 T^{-1} \left( (\theta_1 \theta_2 + \theta_2^2)^{-1} + (\theta_1^2 + \theta_1 \theta_2)^{-1} \right) + O(T^{-2}). \end{aligned}$$

First, we look at the sensitivity with respect to the parameter  $\theta$ . As in the case for the finite time observable, we expand  $\text{Var}[D(Z_t)]$  in a series in  $\varepsilon$  at zero. For standard first order differences, this yields

$$\begin{aligned} \text{Var}[D(Z_t)] &= 2\sigma^2 T^{-1} \theta^{-2} - 3\sigma^2 T^{-1} \theta^{-3} \\ &\quad + 4\sigma^2 T^{-2} \theta^{-3} e^{-T\theta} - \sigma^2 T^{-2} \theta^{-3} e^{-2T\theta} + O(\varepsilon), \end{aligned}$$

for independently sampled paths. Working in a similar fashion, we find in contrast that

$$\text{Var}[D(Z_t)] = \varepsilon^2 (4\sigma^2 T^{-1} \theta^{-4} + O(T^{-2})) + O(\varepsilon^4),$$

for the coupled paths. For the sensitivity with respect to  $\sigma$ , the story is the same. The independently sampled paths behave like

$$\text{Var}[D(Z_t)] = 2\sigma^2 T^{-1} \theta^{-2} + \varepsilon^2 2T^{-1} \theta^{-2} + O(T^{-2}) (1 + \varepsilon^2),$$

and the coupled paths behave like

$$\text{Var}[D(Z_t)] = \varepsilon^2 (cT^{-1} \theta^{-2} + O(T^{-2})),$$

for a constant  $c$ . In Figure 9, we observe the theoretically obtained values for the reduction to the variance for the sensitivity with respect to  $\sigma$  of an OU process with parameters  $\theta = 1$ ,  $\mu = 1.2$ ,  $\sigma = 0.3$ , and  $x_0 = 2$ . The time average is computed up to time  $T = 10$  and each variance is computed using  $M = 10^3$  independent samples of an optimally coupled AR(1) process and an independent AR(1) process.

## 2. Langevin dynamics

We consider the LE with particle mass  $m = 1$ ,

$$\begin{aligned} dX_t &= V_t dt, \\ dV_t &= -\omega^2 X_t dt - \beta V_t dt + \sqrt{2\beta k_B T} dW_t^1, \end{aligned}$$

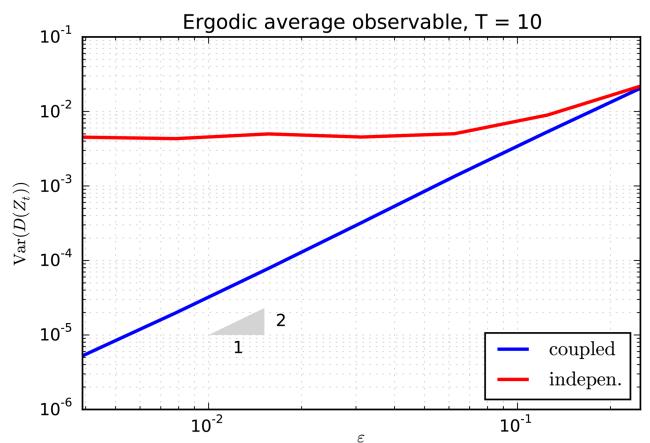


FIG. 9. For the OU process, the variance of the estimator for the sensitivity with respect to  $\sigma$  for the time averaged observable is  $O(1)$  for the independently sampled difference and  $O(\varepsilon^2)$  for the optimally coupled difference. Here consider an OU process with parameters  $\theta = 1$ ,  $\mu = 1.2$ ,  $\sigma = 0.3$ , and  $x_0 = 2$  and compute the average up to time  $T = 10$ . Each variance is based on  $M = 10^3$  independent samples of optimally coupled AR(1) processes and independent AR(1) processes.

for  $t \in [0, T]$ , subject to  $X_0 = x_0$  and  $V_0 = v_0$ , where  $(W_t^1)_{t \geq 0}$  is the Wiener process. This system can be written as a two-dimensional OU process  $Y_t = (X_t, V_t)$ , given by

$$dY_t = -BY_t dt + \Sigma dW_t, \quad Y_0 = (x_0, v_0), \quad (\text{A2})$$

for  $W_t = (0, W_t^1)$  with coefficient matrices

$$B = \begin{pmatrix} 0 & -1 \\ \omega^2 & \beta \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2\beta k_B T} \end{pmatrix}.$$

The general solution to (A2) is given by

$$Y_t = e^{-Bt} Y_0 + \int_0^t e^{-B(t-s)} \Sigma dW_s,$$

for  $t \in [0, T]$ , where, for this example,  $e^{-Bt}$  can be written as (except in the critically damped case) in a closed form in terms of the eigenvalues of  $B$ ,  $\mu_1 = \beta/2 + \sqrt{\beta^2/4 - \omega^2}$  and  $\mu_2 = \beta/2 - \sqrt{\beta^2/4 - \omega^2}$ .<sup>74</sup> That is, the position and velocity are given component-wise by

$$\begin{aligned} X_t &= \mu^{-1} \left( x_0 (\mu_1 e^{-\mu_2 t} - \mu_2 e^{-\mu_1 t}) + v_0 (e^{-\mu_2 t} - e^{-\mu_1 t}) \right. \\ &\quad \left. + \sqrt{\gamma(\mu_1 + \mu_2)} \int_0^t (e^{-\mu_2(t-s)} - e^{-\mu_1(t-s)}) dW_s \right), \\ V_t &= \mu^{-1} \left( x_0 \omega^2 (e^{-\mu_1 t} - e^{-\mu_2 t}) + v_0 (\mu_1 e^{-\mu_1 t} - \mu_2 e^{-\mu_2 t}) \right. \\ &\quad \left. + \sqrt{\gamma(\mu_1 + \mu_2)} \int_0^t (\mu_1 e^{-\mu_1(t-s)} - \mu_2 e^{-\mu_2(t-s)}) dW_s \right), \end{aligned}$$

for  $\mu^{-1} = (\mu_1 - \mu_2)$  and  $\gamma = 2k_B T$ . We shall further assume, for simplicity, that both  $x_0$  and  $v_0$  are definite.

For the Langevin dynamics, we form the coupled system  $Z_t = (Y_t, \tilde{Y}_t)$ , where  $\tilde{Y}$  solves (A2) for  $\tilde{B}$  and  $\tilde{\Sigma}$  depending upon perturbed parameters (also denoted with tildes in the sequel) and with an independent Wiener process  $\tilde{W}$ . Once again, we are interested in minimizing the variance of the difference  $D(Z_t) = f(Y_t) - f(\tilde{Y}_t)$ , for linear observables  $f$ . Note that  $D(Z_t)$  is a vector quantity (i.e.,  $\text{Var}[D(Z_t)]$  is the variance-covariance matrix),

$$\text{Var}[D(Z_t)] = \text{Var}[f(Y_t)] - 2 \text{Cov}[f(Y_t), f(\tilde{Y}_t)] + \text{Var}[f(\tilde{Y}_t)],$$

for  $f(Y_t) = (f(X_t), f(V_t))$ , where  $\text{Cov}[f(Y_t), f(\tilde{Y}_t)]$  has components  $\text{Cov}[f(X_t), f(\tilde{X}_t)]$ ,  $\text{Cov}[f(V_t), f(\tilde{V}_t)]$ , and cross terms  $\frac{1}{2}(\text{Cov}[f(V_t), f(\tilde{X}_t)] + \text{Cov}[f(X_t), f(\tilde{V}_t)])$ . This covariance is zero when  $Y_t$  and  $\tilde{Y}_t$  are independent and can be maximized when  $Y_t$  and  $\tilde{Y}_t$  are linearly dependent, which is equivalent to generating  $Y_t$  and  $\tilde{Y}_t$  using common random paths  $W_t = \tilde{W}_t$ . Next, we investigate the asymptotic dependence of  $\text{Var}[D(Z_t)]$  on  $\varepsilon$  for two observables, related to a finite time horizon and a time average, for sensitivities with respect to  $\beta$ .

### a. Finite time observable

Consider the finite time observable  $f(Y_t) = Y_T$ . Using the component wise expression above, the covariance term related to the positions can be expressed in terms of the eigenvalues of the drift matrices for the nominal and perturbed systems.

That is, we let  $\text{Cov}[X_T, \tilde{X}_T] = \phi(\mu_1, \mu_2, \tilde{\mu}_1, \tilde{\mu}_2)$ , where

$$\begin{aligned} \phi(\mu_1, \mu_2, \tilde{\mu}_1, \tilde{\mu}_2) &= \frac{\gamma \sqrt{(\mu_1 + \mu_2)(\tilde{\mu}_1 + \tilde{\mu}_2)}}{(\mu_1 - \mu_2)(\tilde{\mu}_1 - \tilde{\mu}_2)} \left( \frac{1 - e^{-(\mu_1 + \tilde{\mu}_1)T}}{\mu_1 + \tilde{\mu}_1} \right. \\ &\quad \left. - \frac{1 - e^{-(\mu_1 + \tilde{\mu}_2)T}}{\mu_1 + \tilde{\mu}_2} - \frac{1 - e^{-(\mu_2 + \tilde{\mu}_1)T}}{\mu_2 + \tilde{\mu}_1} + \frac{1 - e^{-(\mu_2 + \tilde{\mu}_2)T}}{\mu_2 + \tilde{\mu}_2} \right). \end{aligned}$$

Similar expressions can be given for the covariances related to the velocity and the cross terms. Here the eigenvalues of the nominal and perturbed systems are (linear) functions of  $\varepsilon$  (and  $\beta$ ) that are related by the type of difference quotient chosen to approximate the sensitivity.

In the case of a centered difference,  $\mu_1 = \mu_1(\varepsilon)$  and  $\mu_2 = \mu_2(\varepsilon)$  are defined, in the obvious way, as  $\mu_1(\varepsilon) = (\beta + \varepsilon)/2 + \sqrt{(\beta + \varepsilon)^2/4 - \omega^2}$  and  $\mu_2(\varepsilon) = (\beta + \varepsilon)/2 - \sqrt{(\beta + \varepsilon)^2/4 - \omega^2}$ , and hence  $\tilde{\mu}_1 = \mu_1(-\varepsilon)$  and  $\tilde{\mu}_2 = \mu_2(-\varepsilon)$ . In this case, we can write  $\text{Var}[X_T] = \psi(\mu_1(\varepsilon), \mu_2(\varepsilon))$  and  $\text{Var}[\tilde{X}_T] = \psi(\mu_1(-\varepsilon), \mu_2(-\varepsilon))$  where we define  $\psi(\mu_1, \mu_2) = \phi(\mu_1, \mu_2, \mu_1, \mu_2)$ .

The asymptotic dependence of  $\text{Var}[D(Z_t)] = \text{Var}[X_t] + \text{Var}[\tilde{X}_t] - 2 \text{Cov}[X_t, \tilde{X}_t]$  on  $\varepsilon$  can now be obtained by expanding the quantity of interest in a series in  $\varepsilon = 0$ , using the representations above. That is, for each term appearing above, we have

$$\text{Var}[X_t] = \psi|_{\varepsilon=0} + \partial_\varepsilon \psi|_{\varepsilon=0} \varepsilon + \partial_\varepsilon^2 \psi|_{\varepsilon=0} \varepsilon^2 + O(\varepsilon^3),$$

$$\text{Var}[\tilde{X}_t] = \psi|_{\varepsilon=0} - \partial_\varepsilon \psi|_{\varepsilon=0} \varepsilon + \partial_\varepsilon^2 \psi|_{\varepsilon=0} \varepsilon^2 + O(\varepsilon^3),$$

$$\text{Cov}[X_t, \tilde{X}_t] = \phi|_{\varepsilon=0} + \partial_\varepsilon \phi|_{\varepsilon=0} \varepsilon + \partial_\varepsilon^2 \phi|_{\varepsilon=0} \varepsilon^2 + O(\varepsilon^3),$$

where  $\partial_\varepsilon^k$  denotes the  $k$ th derivative with respect to  $\varepsilon$ . Noting that  $\psi|_{\varepsilon=0}$  is non-zero, it follows that  $\text{Var}[X_T - \tilde{X}_T] = O(1)$  for independently sampled paths. For the common random path coupling, the zeroth order term in the expansion for  $\text{Var}[D(Z_t)]$  vanishes, since  $\psi|_{\varepsilon=0} = \phi|_{\varepsilon=0}$ . In this particular case, the first order term,  $\partial_\varepsilon \phi|_{\varepsilon=0} = 0$ , also vanishes since  $\partial_{\mu_j} \phi = \partial_{\tilde{\mu}_j} \phi$  and  $\tilde{\mu}'_j(0) = -\mu'_j(0)$ , for  $j = 1, 2$ . Finally, noting that since  $\partial_{\mu_j} \phi$  is not symmetric in  $(\mu_j, \tilde{\mu}_j)$ , the second order term in the expansion for  $\text{Var}[D(Z_t)]$  does not vanish, yielding  $\text{Var}[D(Z_t)] = O(\varepsilon^2)$ . Explicit expansions can also be calculated for other standard first order differences and for the other covariance terms with similar asymptotic rates observed, namely,  $O(\varepsilon^2)$  for the common random path coupling and  $O(1)$  for independently sampled paths.

### b. Time average observable

Let  $\bar{X} = T^{-1} \int_0^T X_t dt$  and consider the time average observable  $f(Y_t) = (\bar{X}, \bar{V})$ . As in the case of the time average observable for the OU process, the expectation can be exchanged with the integral in time, yielding explicit expressions for the covariances as in the Appendix, Subsection 2 a. Investigations into the asymptotic dependence of  $\text{Var}[D(Z_t)]$  yield  $O(\varepsilon^2)$  in the optimally coupled case and  $O(1)$  in the independent case. These rates are observed experimentally in Figure 10 where we consider  $\text{Var}[D(Z_t)] = (\varphi)_{ij}$  (i.e.,  $\varphi_{11}$

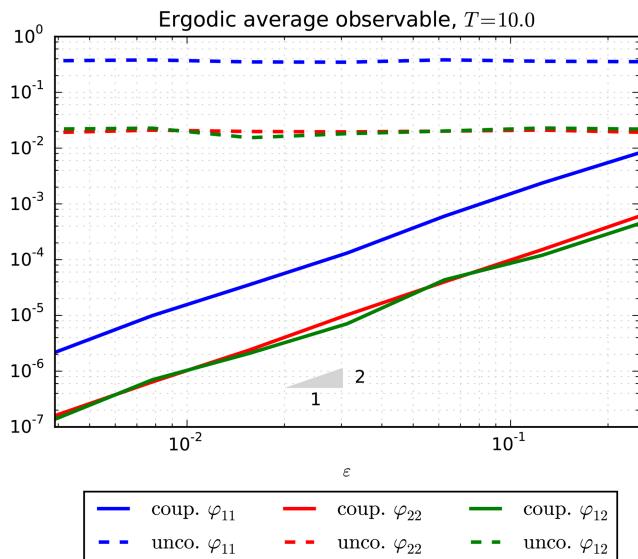


FIG. 10. The sample variance of the components of  $\text{Var}[D(Z_t)]$ , for the time average observable  $f(Z_t) = \bar{Z}$ , is on the order of  $O(\epsilon^2)$  for optimally coupled paths, in contrast to  $O(1)$  for independent paths. Here we consider the sample variance of  $\varphi_{11}$ ,  $\varphi_{22}$ , and  $\varphi_{12}$ , based on  $M = 10^3$  samples, for the sensitivity in  $\beta$ , at  $\beta = 1$ , for time averages up to  $T = 10$ .

$= \text{Var}[\bar{X}^1 - \bar{X}^2]$ ), based on  $M = 10^3$  samples, for a central difference perturbation in  $\beta$ , at  $\beta = 1$  (the underdamped case  $\beta < 2\omega$ ). The time averages are computed up to a final time  $T = 10$  for sample paths from Langevin dynamics, with fixed parameters  $x_0 = -1$ ,  $v_0 = -0.1$ ,  $\omega = 1$ ,  $m = 1$ , and  $\gamma = 1$ , integrated using the BAOAB method<sup>75</sup> with  $\Delta t = 10^{-3}$ .

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