Math 523H–Homework 10

1. Compute the convergence radius of the following series

   (a) $\sum_{n=0}^{\infty} n! x^n$

   (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$

   (c) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} x^n$

   (d) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

   (e) $\sum_{n=0}^{\infty} 2^{-n} x^{3n}$ *Hint:* The answer is not 2.

   (f) $\sum_{n=0}^{\infty} x^{n!}$

2. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Justifying all your steps show that $f'(x) = f(x)$. *Hint:* You are not allowed to use that $f(x) = e^x$ and your calculus.....

3. Let $c_n = \left(\frac{4+2(-1)^n}{5}\right)^n$.

   (a) Compute $\limsup_n |c_n|^{1/n}$, $\liminf_n |c_n|^{1/n}$, $\limsup_n \left|\frac{c_{n+1}}{c_n}\right|$, $\liminf_n \left|\frac{c_{n+1}}{c_n}\right|$

   (b) Find the radius of convergence for the series $\sum_n c_n x^n$.

   (c) Determine the exact interval in which the series $\sum_n c_n x^n$ converge.

4. Consider the binomial series $(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x + \frac{a(a-1)(a-2)}{3!}x^2 + \cdots$ The convergence radius of the series is 1. Examine the convergence of the series at $x = \pm 1$.

5. Consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{n!} = \frac{\cos(2x)}{1!} + \frac{\cos(4x)}{2!} + \frac{\cos(8x)}{3!}$$

   (a) Show that the series and all its derivatives converges uniformly on $\mathbb{R}$.

   (b) Compute the Taylor series of the function as the origin and shows that it diverges at all $x \neq 0$.

6. Compute the number e using the Taylor series for e^x around $x = 0$ with remainder term (in whichever form you want). How many terms do you need to consider to compute accurately e with an error or no more than 10^{-5}. Compute such an approximation.

7. (a) Using your calculus and your knowledge of derivatives, use Taylor theorem to derive the power series for $\sin(x)$. (Bound the remainder term!). What is the convergence radius?

(b) Using part (a), compute the power series for the Fresnel integral $\int_0^x \sin(t^2)dt$, and determine its convergence radius.

8. Show
$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$
Hint: Use the series for the exponential function for $x^x = e^{x\ln(x)}$ and then compute $\int_0^1 x^n \ln(x)^n dx$ by integrating by parts.