Homework #4

Prob 1

To check for positive recurrence we try to solve $\pi P = \pi$. We have

$$(1-p)\,\pi(0) + (1-p)q\,\pi(1) = \pi(0) \qquad (1)$$

$$p\,\pi(0) + \big(pq + (1-p)(1-q)\big)\pi(1) + q(1-p)\,\pi(2) = \pi(1) \qquad (2)$$

$$p(1-q)\,\pi(j-1) + \big(qp + (1-p)(1-q)\big)\,\pi(j) + \big(q(1-p)\big)\,\pi(j+1) = \pi(j) \quad (*)$$

Since $1 - qp + (1-p)(1-q) = p(1-q) + q(1-p)$ $(*)$ is

$$p(1-q)\,\pi(j-1) + q(1-p)\,\pi(j+1) = \big(p(1-q) + q(1-p)\big)\,\pi(j)$$

It has the form

$$a\,\pi(j-1) + b\,\pi(j+1) = (a+b)\,\pi(j)$$

$\pi(j) = x^j$ gives the solutions $x = 1$

$$x = \frac{a}{b} = \frac{p(1-q)}{q(1-p)}$$

General solutions $\pi(j) = C_1 + C_2 \left(\frac{p(1-q)}{q(1-p)}\right)^j$

If you choose $\pi(0) = (1-q)$, (1) and (2) gives

$$\pi(0) = (1-q)\,, \quad \pi(j) = \left(\frac{p(1-q)}{q(1-p)}\right)^j$$

This can be normalized if $\left(\frac{p}{q}\frac{1-q}{1-p}\right) < 1 \iff \boxed{p<q}$

The normalization constant is

$$C = (1-q) + \sum_{j=1}^{\infty} \left(\frac{p}{q}\frac{1-q}{1-p}\right)^j = (1-q)\left[1 + \frac{p}{q-p}\right]$$

The long run average length of the queue is

$$C^{-1} \sum_{j=1}^{\infty} j \left(\frac{p}{q}\frac{1-q}{1-p}\right)^j = C^{-1} \frac{\frac{p}{q}\frac{1-q}{1-p}}{\left(1 - \frac{p}{q}\frac{1-q}{1-p}\right)^2}$$

To check for <u>transience</u> we solve the equation

- $0 \le \alpha(j) \le 1$
- $\alpha(0) = 1$, $\alpha(j) < 1$ for some j
- $\alpha(j) = \sum P_{ij}\, \alpha(i)$

We find $\quad \alpha(j) = q(1-p)\,\alpha(j-1) + (pq + (1-p)(1-q))\,\alpha(j) + p(1-q)\,\alpha(j+1)$

This is similar to the previous case

$$\alpha(j) = C_1 + C_2 \left(\frac{q}{p}\frac{1-p}{1-q}\right)^j$$

Choose $C_1 = 0$, we have a solution if $\frac{q}{p}\frac{1-p}{1-q} < 1$

i.e $\boxed{p > q.}$

If $p = q$ X_n is null recurrent.

Prob 2

Using the criterion for transience we have

$$\alpha(i) = \sum P_{ij}\, \alpha(j)$$

$$\alpha(i) = (1-p)\,\alpha(i-1) + p\,\alpha(i+2)$$

which leads, with $\alpha(i) = x^i$, to $p x^3 - x + (1-p) = 0$.

$x=1$ is a solution so $px^3 - x + (1-p) = (x-1)(px^2 + px - (1-p))$

The roots are thus $1, \dfrac{-1 \pm \sqrt{1 + \frac{4(1-p)}{p}}}{2}$.

We want a root which is positive and less than 1.
The only candidate

$$\frac{-1 + \sqrt{1 + 4\left(\frac{1-p}{p}\right)}}{2} < 1$$

$$\Rightarrow \quad 1 + 4\left(\frac{1-p}{p}\right) < 9$$

$$\frac{1-p}{p} < 2$$

$$p > \frac{1}{3}$$

$$\boxed{\text{Transient for } p > \tfrac{1}{3}}$$

Prob 3

The transition matrix has the form

$$P = \begin{pmatrix} 1-p_0 & p_0 & & & \\ 1-p_1 & 0 & p_1 & & \\ 1-p_2 & 0 & 0 & p_2 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Note that $P\{T_0 = n \mid X_0 = 0\} = P\{X_1 = 1, X_2 = 2, \dots, X_{n-1} = n-1, X_n = 0 \mid X_0 = 0\}$

$$= p_0 p_1 \cdots p_{n-2}(1 - p_{n-1})$$

$$= \underbrace{p_0 p_1 \cdots p_{n-2}}_{\equiv U_{n-2}} - \underbrace{p_0 p_1 \cdots p_{n-1}}_{U_{n-1}}$$

$$P\{T_0 \le N \mid X_0 = 0\} = (1-p_0) + (U_0 - U_1) + (U_1 - U_2) + \dots + (U_{N-2} - U_{N-1})$$

$$= (1 - p_0) + U_0 - U_{N-1} = 1 - U_{N-1}$$

So we have recurrence if $P\{T_0 < \infty \mid X_0 = 0\} = 1$

OR $U_N \xrightarrow[N \to \infty]{} 0$

(a) $U_n = \frac{1}{\cancel{2}} \frac{\cancel{2}}{\cancel{3}} \frac{\cancel{3}}{4} \cdots \frac{n}{n+1} \frac{n+1}{n+2} = \frac{1}{n+2} \xrightarrow[n \to \infty]{} 0$ $\boxed{\text{recurrent}}$

(b) $U_n = \frac{1}{2} \frac{1}{3} \cdots \frac{1}{n+2} = \frac{1}{(n+2)!} \xrightarrow[n \to \infty]{} 0$ $\boxed{\text{recurrent}}$

(c) ~~~~ $U_n = \frac{1}{2}\,\frac{2}{3}\,\frac{5}{6}\cdots\frac{(n-1)^2+1}{(n-1)^2+2}\,\frac{n^2+1}{n^2+2}$

In order to determine if $U_n \to 0$ we note that

$$\prod_{i=0}^{\infty} p_i > 0 \quad \text{iff} \quad -\log\left(\prod_{i=0}^{\infty} p_i\right) < \infty$$
$$= -\sum_{i=0}^{\infty} \log(p_i)$$
$$= -\sum_{i=0}^{\infty} \log\left(1-(1-p_i)\right)$$

Note that $\lim_{x\to 0} \frac{-\log(1-x)}{x} = 1$ by L'Hospital

So $\sum_{i=0}^{\infty} \log(1-(1-p_i)) < \infty \iff \sum_{i=0}^{\infty}(1-p_i) < \infty$

Here we have $\sum_{i=0}^{\infty}(1-p_i) = \sum_{i=0}^{\infty}\frac{1}{i^2+2} < \infty$

and so $\lim_{n\to\infty} U_n > 0$

$\Rightarrow$ $\boxed{X_n \text{ is transient}}$

<u>Positive recurrence</u>

<u>For (a)</u> $P\{\tau_0 = n \mid X_0 = 0\} = U_{n-2} - U_{n-1} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$\Rightarrow \sum n P\{\tau_0 = n \mid X_0 = 0\} = \sum \frac{1}{n+1} = \infty$ (not positive recurrent)

For (b) $P\{\tau_0 = n \mid X_0 = 0\} = \frac{1}{n!} - \frac{1}{(n+1)!} = \frac{1}{n!}\,\frac{n}{n+1}$ (positive recurrent)

Prob 4

$$P = \begin{pmatrix} 1-p & p & & & \\ 1-p & 0 & p & & \\ 1-p & 0 & 0 & p & \\ \vdots & \vdots & & & \ddots \end{pmatrix}$$

As in prob 3 $P\{\tau_0 = n \mid X_0 = 0\} = p^{n-1}(1-p)$ $n = 1, 2, 3, \dots$

geometric R.V.

$$\Rightarrow E[\tau_0 \mid X_0 = 0] = \frac{1}{(1-p)} < \infty$$

positive recurrent.

Stationary distribution

$(1-p)(\pi_0 + \pi_1 + \cdots) = \pi_0$

$\pi_0 p = \pi_1$

$\pi_1 p = \pi_2$

$\vdots$

$\pi_n p = \pi_{n+1}$

$$\pi_0 = \frac{1}{E[\tau_0 \mid X_0 = 0]} = (1-p) \quad \text{and so} \quad \underline{\pi_n = p^n(1-p)} \quad n = 0, 1, 2, 3 \dots$$

Prob 5

(a) is easy

(b) We have

$$P\{Z_A = n,\ Z_B = m\} = P\{Z_A = n, Z_B = m \mid Z_A + Z_B = m+n\}\ P\{Z_A + Z_B = m+n\}$$

$$= \binom{m+n}{n} P_A^{\,n} P_B^{\,m} \; e^{-\lambda} \frac{\lambda^{m+n}}{(m+n)!}$$

$$= \frac{\cancel{(m+n)!}}{m!\,n!} P_A^{\,n} P_B^{\,m} \lambda^m \lambda^n e^{-\lambda P_A} e^{-\lambda P_B} \frac{1}{\cancel{(m+n)!}}$$

$$= \frac{(\lambda P_A)^n}{n!} e^{-\lambda P_A} \times \frac{(\lambda P_B)^m}{m!} e^{-\lambda P_B}$$

This shows that Z_A and Z_B are independent Poisson R.V. with parameters λP_A and λP_B

(c) The sum of 2 independent Poisson R.V with parameters μ_1 and μ_2 is again a Poisson R.V with parameter $\mu_1 + \mu_2$ (This is a similar argument as in (b)).

- If X_n is Poisson with rate μ_n then

$$X_{n+1} = \xi_{n+1} + R(X_n)$$

is Poisson with rate $p\mu_n + \lambda$.

- If X_0 is Poisson with rate μ_0, then

X_n is Poisson with rate

$$\mu_n = \lambda(1 + p + p^2 + \cdots + p^{n-1}) + p^n \mu_0$$

As $n \to \infty$ $\mu_n \to \frac{\lambda}{1-p}$

X_n has a limiting distribution which also stationary.

Since X_n is irreducible and aperiodic this is the unique stationary distribution.

$$\pi(j) = \frac{\left(\frac{\lambda}{1-p}\right)^j}{j!} e^{-\frac{\lambda}{(1-p)}}$$

Prob 6 $a = P\{\text{dies out}\}$ is the smallest root of

$$a = \phi(a)$$

where $\phi(s) = E[s^X] = \sum_{j=0}^{\infty} p_j s^j$

(a) $\phi(s) = \frac{1}{4} + \frac{3}{4}s^2$

$\phi(s) = s \qquad \frac{3}{4}s^2 - s + \frac{1}{4} = 0$

$(s-1)\left(\frac{3}{4}s - \frac{1}{4}\right) \qquad s = 1, \frac{1}{3}$

$a = \frac{1}{3}$

(b) $s = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2$ or $s^2 - 2s + 1 = 0 \quad s = 1$

$a = 1$

(c) $s = \frac{1}{6} + \frac{1}{2}s + \frac{1}{3}s^3$. Since $s=1$ is a root we

$\frac{1}{3}s^3 - \frac{1}{2}s + \frac{1}{6} = (s-1)\left(\frac{1}{3}s^2 + \frac{1}{3}s - \frac{1}{6}\right)$

$s = 1, -\frac{1}{2} \pm \sqrt{\frac{3}{2}}$

$a = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

1d) $E[s^Y] = \sum_{j=0} s^j (1-q) q^j = (1-q) \sum_{j=0}^{\infty} (sq)^j$

$$= \begin{cases} \dfrac{1-q}{1-sq} & \text{if } s < \dfrac{1}{q} \\ \infty & \text{otherwise} \end{cases}$$

$$s = \frac{1-q}{1-sq} \Rightarrow \begin{array}{l} s = 1 \\ s = \dfrac{1-q}{q} \end{array}$$

So $a = \begin{cases} 1 & q \le \frac{1}{2} \\ \dfrac{1-q}{q}, & q > \frac{1}{2} \end{cases}$

Prob 9 This is nothing but a branching process!

During each service time of a customer, this customer reproduce and dies out!

P{ coffee break } = P{ dies out }

$$s = \frac{2}{10} + \frac{2}{10} s + \frac{6}{10} s^2 \Rightarrow \begin{array}{l} s = 1 \\ s = \frac{1}{3} \end{array}$$

P{ coffee break } = $\frac{1}{3}$

Prob 7

- If $\mu = \sum n p_n > 1$ then there is a positive probability $a < 1$ that the population dies out.

So there is a positive probability $b = 1-a$ that the population never dies out, i.e. never returns to 0

So $\mu < 1 \Rightarrow$ transient.

- If $\mu \leq 1$ then the populations dies out with probability 1. So $\mu \geq 1 \Rightarrow$ recurrent.

- If $\mu < 1$ then we have (see class)

$$\mu^n = E[X_n] = \sum_{k=1}^{\infty} k P\{X_n = k\} \geq \sum_{k=1}^{\infty} P\{X_n = k\} = P\{X_n \geq 1\}$$

So $P\{X_n \leq 1\} \leq \mu^n$

If R denotes the return time to 0

$$P\{R \geq k\} = P\{X_1 \geq 1, X_2 \geq 1, \dots, X_k \geq 1\}$$
$$\leq P\{X_k \geq 1\} \leq \mu^k$$

$$\Rightarrow E[R] = \sum_k P\{R \geq k\} < \infty$$

so positive recurrent.

Prob 8 (a) $X_0 = 1$, $P\{X_1=1\} = q$, $P\{X_2=1\} = q^2$, ... $P\{X_n=1\}=q^n$

$P\{T=1\} = (1-q)$, $P\{T=2\} = (1-q)q$, ..., $P\{T=n\} = (1-q)q^{n-1}$

$$\boxed{E[T] = \frac{1}{1-q}}$$

(b) The number of working lights decrease, so

If $k \le \ell$ $\quad P\{X_{n+1}=k \mid X_n=\ell\} = P_{k\ell}$

$= P\{k \text{ out of } \ell \text{ lamps survive}\}$

$= \binom{\ell}{k} q^k (1-q)^{\ell-k}$

(c) Moment generating function

Solution 1 $\quad \phi_n(s) = E[s^{Y_n}]$

$$\phi_n(s) = E[s^{Y_n}] = \sum_{k=0}^{m} P\{Y_n=k\} s^k$$

$$= \sum_{k=0}^{m} \sum_{\ell=k}^{m} P\{Y_n=k \mid Y_{n-1}=\ell\} P\{Y_{n-1}=\ell\} s^k$$

interchange summations $\quad = \sum_{\ell=0}^{m} \sum_{k=0}^{\ell} P\{Y_{n-1}=\ell\} \binom{\ell}{k} q^k (1-q)^{\ell-k} s^k$

binomial theorem $\quad = \sum_{\ell=0}^{m} P\{Y_{n-1}=\ell\} \left(sq + (1-q) \right)^k$

Therefore

$$\phi_n(s) = \phi_{n-1}\left(sq + (1-q) \right)$$

Solve the recursion

$$\phi_n(s) = \phi_{n-1}\left(sq + (1-q) \right)$$

$$= \phi_{n-2}\left((1-q) + \left((1-q) + sq \right) q \right)$$

$$= \phi_{n-2}\left((1-q) + q(1-q) + sq^2 \right)$$

$$= \phi_{n-3}\left((1-q) + q(1-q) + q^2(1-q) + sq^3 \right)$$

$$= \cdots = \phi_0\left((1-q)\left[1 + q + q^2 + \cdots q^{n-1} \right] + sq^n \right)$$

$$= \phi_0\left((1-q) \frac{1-q^n}{1-q} + s\, q^n \right)$$

$$= \phi_0\left((1-q^n) + s\, q^n \right)$$

and $\phi_0(s) = s^m$ since $Y_0 = m$

$$\Rightarrow \quad \phi_n(s) = \left((1-q^n) + s\, q^n \right)^m$$

Solution 2 It is simpler to note that

$$Y_n = X_n^{(1)} + \cdots + X_n^{(m)}$$

where $X_n^{(k)} = \begin{cases} 1 & \text{if the } k\text{-th lamp work at time } n \\ 0 & \text{otherwise} \end{cases}$

$X_n^{(k)}$ are independent identically distributed

$$\Rightarrow E[s^{Y_n}] = \left(E[s^{X_n^{(1)}}]\right)^m$$

$$E[s^{X_n}] = 1\, P\{X_n = 0\} + s\, P\{X_n = 1\}$$

$$= (1-q^n) + q^n s$$

$$\text{So } E[s^{Y_n}] = \left((1-q^n) + q^n s\right)^m$$

(d) $P\{Y_n = 0\} = \phi_n(0) = 1 - q^n$

$$E[Y_n] = \phi_n'(1) = m\, q^n \left[(1-q^n) + q^n\right]^{m-1}$$

$$= m\, q^n$$