Math 523H–Homework 9

1. Show that if $g(x)$ is differentiable at x_0 and $g(x_0) \neq 0$ then $1/g(x)$ is differentiable at x_0 in two ways:
   (a) Using the definition of the derivative as a limit.
   (b) Using our third formulation of the derivative (Caratheodory formulation) (g is differentiable if there exists a function $\phi(x)$ continuous at x_0 such that $g(x) = g(x_0) + \phi(x)(x - x_0)$).

2. Using one of the equivalent definition of the derivative show that $f(x) = \sqrt{x}$ is differentiable for $x > 0$ and that $g(x) = x^{1/3}$ is differentiable at $x \neq 0$. Show also that $x^{1/3}$ is not differentiable at $x = 0$.

3. Consider the function $f_n = x^n \sin(1/x^3)$ for $x \neq 0$ and $f(0) = 0$ where n is a non-negative integer. For which values of n is $f(x)$ (a) continuous? (b) differentiable? (c) twice differentiable? when are the derivatives continuous? *Hint: You may use that* $\sin(x)$ *and* $\cos(x)$ *are differentiable and their derivatives without proving it.*

4. Prove that if f is differentiable at x_0 then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

5. The function $\sin(x)$ is bijective on $[-\pi/2, \pi/2]$ and $\arcsin(x)$ is its inverse function. Compute the derivative of *arcsin*(x). *Hint: You may use the derivative of* $\sin(x)$ *without proving it.*

6. Let $f : [a, b] \to \mathbb{R}$ be continuous and n times differentiable on (a, b). Show that $f(x)$ has $n+1$ zeros in $[a, b]$ then there exists $\xi \in (a, b)$ such that $f^{(n)}(\xi) = 0$. *Hint: Apply Rolle's theorem several times.*

7. Let f be a continuous function on $\mathbb{R}$ and define

$$G(x) = \int_0^{\sin(x)} f(t)dt$$

   Show that $G(x)$ is differentiable and and compute $G'(x)$.

8. Compute
   (a) $\lim_{x \to 0} (1 + 2x)^{1/x}$
   (b) $\lim_{x \to 0} \frac{e^{2x} - \cos(x)}{x}$
   (c) $\lim_{x \to 0} \left[\frac{1}{\sin(x)} - \frac{1}{x} \right]$

(d) $\lim_{x\to 0} \cos(x)^{1/x^2}$

(e) $\lim_{x\to 0} \frac{1-\cos(2x)-2x^2}{x^4}$.