Math 645: Homework 5

1. Consider the equation for the mathematical pendulum
$$x'' + \sin(x) = 0\,, \quad x(0) = \epsilon\,, x'(0) = 0\,, \tag{1}$$
where ϵ is supposed to be small. Show that the solution can be written in the form
$$x(t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + O(\epsilon^4)\,. \tag{2}$$
Compute $x_1(t)$, $x_2(t)$, and $x_3(t)$. *Hint:* Taylor expansion.

2. Determine the critical points and study their stability for the equation $x' = \alpha x + \beta x^3$ for all values of α and β.

3. Consider the equation $x'' + bx' + kx = 0$. Construct a stability diagram for this system in the (b, k) plane, i.e., indicate the stability of the equation and describe orbits (source, sink, saddle, etc...) as a function of b and k.

4. Study the stability of the critical points of the equation
$$\begin{aligned} x_1' &= (x_1 - x_2)(1 - x_1 - x_2)/3\,, \\ x_2' &= x_1(2 - x_2)\,. \end{aligned} \tag{3}$$

5. Consider the FitzHugh-Nagumo equation
$$\begin{aligned} x_1' &= f_1(x_1, x_2) = g(x_1) - x_2\,, \\ x_2' &= f_2(x_1, x_2) = \sigma x_1 - \gamma x_2\,, \end{aligned} \tag{4}$$
where σ and γ are positive constants and the function g is given by $g(x) = -x(x - 1/2)(x - 1)$. Show that as the ratio σ/γ increases the systems undergoes a bifurcation one equilibrium state to three equilibrium states. Compute the critical points and determine their stability properties. Make a graph of the orbits before and after the bifurcation.

6. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be of class $\mathcal{C}^1$. Assume that the solutions of $x' = f(x)$ exists for all $t \in \mathbf{R}$ and denote by ϕ^t the corresponding flow $\phi^t(x) = x(t, 0, x)$.

   (a) Prove Liouville Theorem
   $$\det\left(\frac{\partial \phi^t}{\partial x}\right) = \exp\left(\int_0^t \mathrm{div} f(\phi^s(x))\, ds\right)\,. \tag{5}$$
   where $\mathrm{div} f(x) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x)$. Hint: Use Liouville theorem for linear ODE and the variational equation.

(b) Show that ϕ^t is volume preserving if and only if $\mathrm{div} f = 0$. *Hint*: A map $T : \mathbf{R}^n \to \mathbf{R}^n$ is volume preserving if $\mathrm{vol}(T(A)) = \mathrm{vol}(A)$ for all sets A compact for which ∂A is negligible (for Rieman integral) or Lebesgue measurable (if you prefer Lebesgue integral).

7. Consider the equation

$$x' = A(t)x + f(t,x)\,, \tag{6}$$

where

(a) $A(t)$ is continuous and periodic of period p.

(b) g is continuous and locally Lipschitz and

$$\lim_{\|x\|\to 0} \sup_{t>t_0} \frac{\|g(t,x)\|}{\|x\|} = 0 \tag{7}$$

Let R be the matrix given in Floquet Theorem. Show that 0 solution of (6) is stable if all the negative eigenvalues of R have negative real part and is unstable if at least one eigenvalue of R has positive real part. *Hint:* Consider the change of variables $x = P(t)y$ where $P(t)$ is the periodic matrix given in Floquet Theorem.