Name:_____

Solve the first 5 problems and only **one** out of problems 6 and 7. (If you solve both 6 and 7, then I will not grade 7).

1. (20 points)

   Compute the contour integral

   $$\int_C e^{\bar{z}} dz,$$

   where C is the boundary of the rectangle with vertices at the points 0, 2, $2 + i$, and i, oriented counterclockwise. *Caution: the exponent of the integrand is the complex conjugate $\bar{z}$ of z.*

2. (18 points) Let C be the circle of radius 5 centered at the origin and transversed counterclockwise. Compute

$$\int_C \frac{e^z}{z^2 + 1} dz.$$

3. (18 points) Let C be the circle $\{z \text{ such that } |z| = 10\}$, transversed counterclock-wise. Evaluate

$$\int_C \frac{\sin(z)}{(z - \frac{\pi}{2})^n} dz,$$

for all integers n (positive, negative, or zero).

4. (16 points) Prove the equality

$$\left| \int_C \frac{z+1}{z-1} dz \right| \leq 6\pi,$$

where C is the semi-circle, given by the parametrization $z(t) = 2e^{it}$, $0 \leq t \leq \pi$.

5. (16 points) Let C_1 be the curve, consisting of the piece of the graph of $y = \sin(x)$, given by the parametrization

$$z(x) \;=\; (x + i\sin(x)), \quad 0 \le x \le 2\pi.$$

Let C_2 be the piece of the graph of $y = -\sin(x)$, given by the parametrization

$$z(x) \;=\; (x - i\sin(x)), \quad 0 \le x \le 2\pi.$$

Compute the difference

$$\int_{C_1} \frac{dz}{z - \frac{\pi}{2}} - \int_{C_2} \frac{dz}{z - \frac{\pi}{2}}$$

6. (12 points) Determine whether the following statements are true or false. **Justify your answers**.

a) Let C be any contour from 1 to $8i$, which does not pass through 0. Then the following equality holds

$$\int_C \frac{dz}{z} = \ln(8) + \frac{\pi}{2}i.$$

b) Let $P(z) = (z - 1)(z - 2i)(z + 4 + 5i)(z - 9i)$ and C_R the circle of radius R, centered at the origin, transversed counterclockwise. Let I_R be the value of the integral

$$\int_{C_R} \frac{dz}{P(z)}.$$

Then $I_R = I_{100}$, for all radii R satisfying $R \geq 100$.

7. (12 points) a) (4 points) Let $f(z)$ and $g(z)$ be entire functions, and set $P(z) :=$ $f(z)g(z)$. Prove the equality

$$\frac{P'(z)}{P(z)} = \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}.$$

b) (8 points) Let $P(z)$ be a polynomial of degree n. Let C_R the circle of radius $R > 0$, centered at the origin, transversed counterclockwise. Prove, the equality

$$\int_{C_R} \frac{P'(z)}{P(z)} dz = 2n\pi i,$$

provided R is sufficiently large. *Hint: Do the case $n = 1$ first.*