# Math 597/697: Homework 4

1. Consider the simple queuing model introduced in class. During each time period exactly a new customer arrives with probability $p$ and no customer arrives with probability $1-p$. During each time period exactly one customer is served with probability $q$ and zero is served with probability $(1-q)$. The transition probabilites are

$$P_{00} = (1-p)\,, \quad P_{01} = p$$
$$P_{ii-1} = (1-p)q\,, \ P_{ii} = pq + (1-p)(1-q)\,, \ P_{ii+1} = p(1-q)(1)$$

   Determine for which $p$ and $q$ the Markov chain is positive recurrent, recurrent and transient. For the positive recurrent case determine the stationary distribution and the long run average length of the queue.

2. Consider the Markov chain with state space $S = \{0, 1, 2, 3, \cdots\}$ and transition probabilities

$$P_{0,2} = p\,, \quad P_{00} = 1-p \quad P_{ii+2} = p\,, \quad P_{i,i-1} = 1-p\,, \quad i \geq 1\,.$$

   For which values of $p$ is the chain transient?

3. Consider Markov chains with state space $S = \{0, 1, 2, 3, \cdots\}$ and transition probabilities

   (a)
$$P_{i0} = \frac{1}{i+2}\,, \quad P_{i,i+1} = \frac{i+1}{i+2}\,, \quad i \geq 0\,.$$

   (b)
$$P_{i0} = \frac{i+1}{i+2}\,, \quad P_{i,i+1} = \frac{1}{i+2}\,, \quad i \geq 0\,.$$

   (c)
$$P_{i0} = \frac{1}{i^2+2}\,, \quad P_{i,i+1} = \frac{i^2+1}{i^2+2}\,, \quad i \geq 0\,.$$

   For each case determine if the chain is transient, null recurrent or positive recurrent.

4. Consider independent trials which result in successs S with probability $p$ and failure F with probability $q = 1-p$. We say that a success run of length $r$ happened at trial $n$ if the outcomes in the preceeding $r+1$ trials, including the the present trial as the last, were F, S, S, ... S. Now let us denote by $X_n$ the length of the trial run at trial $n$. This is a Markov chain with state space $\{0, 1, 2, 3, \ldots\}$.

(a) Verify that the transition probabilities are $p_{i,0} = (1-p)$ and $p_{ii+1} = p$

(b) Show that 0 is positive recurrent by computing $E[\tau_0 | x_0 = 0]$.

(c) To compute the stationary distribution, first use (b) to determine $\pi_0$ and then use the equation $\pi P = \pi$ to determine $\pi_1, \pi_2, \ldots$.

5. Consider the following Markov chain. At times $n = 1, 2, 3, \cdots$ $\xi_n$ particles are added in a box where $\xi_n$ are i.i.d. random variables with a Poisson distribution with parameter $\lambda$, i.e.,

$$P\{\xi_n = k\} = \frac{\lambda^k}{k!} e^{-\lambda} .$$

Suppose that any of the particles in the box at time $n$ independently of all other particles and of how the particles are added to the box has probability $p$ of remaining in the box at time $n+1$ and probability $q = 1 - p$ of being removed from the box. The number of particles in the box at time $n$, $X_n$ is a Markov chain which can be expressed as

$$X_{n+1} = \xi_{n+1} + R(X_n) ,$$

where $R(X_n)$ denotes the number of particles present at time $n$ and which remain at time $n+1$.

(a) Show that the Markov chain is irreducible and aperiodic.

(b) As a preparation prove the following fact. Let $Z$ be a Poisson random variable with parameter $\mu$ describing the number of some items. The items occur in two types, type $A$ with probability $p_A$ and type $B$ with probability $p_B = 1 - p_A$ and Let $Z = Z_A + Z_B$ where $Z_A$ are the number of items of type $A$ and $Z_B$ are the number of items of type $B$. Show that $Z_A$ and $Z_B$ are Poisson random variables with parameter $\mu p_A$ and $\mu p_B$.

(c) Use $A$ to show that if the initial distribution of $X_0$ is Poisson with with some parameter $\nu$ then $X_1$ has also a Poisson distribution. Compute the probability distribution of $X_n$ and determine the limiting and stationary distribution.

6. For a branching process, calculate the probability $a$ that the population eventually dies out starting with one individual when

(a) $p_0 = \frac{1}{4}$, $p_2 = \frac{3}{4}$

(b) $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$

(c) $p_0 = \frac{1}{6}$, $p_1 = \frac{1}{2}$, $p_3 = \frac{1}{3}$

(d) $p_i = (1-q)q^i$ for some $0 < q < 1$.

7. Consider a branching process with offspring distribution given by $p_n$. One makes this process irreducible by asserting that if the the population ever dies out, then in the next generation one new individual appears (i.e. $P_{01} = 1$). Determine for which values of $p_n$ the chain is positive recurrent, null recurrent, transient.

8. An electric light that has survived $n$ seconds fails during the $(n+1)st$ second with probability $q$ (with $0 < q < 1$).

   (a) Let $X_n = 1$ if the light is functioning at time $n$ seconds, and $X_n = 0$ otherwise. Let $T$ be the time of failure of the light (in seconds), i.e.,
   $$T = \inf\{n\,;\, X_n = 0\}\,. \tag{2}$$
   Determine $E[T]$.

   (b) A building contains $m$ lights of the type described above, which behave independently of each other. At time 0 they are all functioning. Let $Y_n$ denote the number of lights functioning at time $n$. Specify the transition matrix of $Y_n$.

   (c) Find the moment generating function
   $$\phi_n(s) = E[s^{Y_n}] \tag{3}$$
   of $Y_n$. *Hint*: Express $\phi_n$ in terms of $\phi_{n-1}$ and solve the recursion relation.

   (d) Use the moment generating function to find $P\{Y_n = 0\}$ and $E[Y_n]$.

9. Jamie is working in a bookstore, ordering books that are not in store and that the customers request. Each order takes 5 minutes to complete. While each order is being filled there is a probability $p_j$ that $j$ more customers arrive with $p_0 = .2$, $p_1 = .2$, $p_2 = .6$. Jamie cannot take a coffee break until a service is completed and no one is waiting in line to order a book. When Jamie starts her shift there is one customer waiting. What is the probability that she ever will take a coffee break.