Math 645: Homework 6

1. Determine the stability of the $(0,0)$ solution of

$$\begin{aligned} x_1' &= x_1 x_2^2 - 2x_2\,, \\ x_2' &= x_1 - x_1^2 x_2\,. \end{aligned}$$

2. Determine the stability of the $(0,0)$ solution of

$$\begin{aligned} x' &= 2xy + x^3\,, \\ y' &= x^2 - y^5\,. \end{aligned}$$

3. Consider the equation

$$x' = Ax + f(x)\,, \tag{1}$$

where f is locally Lipschitz and satisfy the condition

$$\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0\,. \tag{2}$$

Assume that A is diagonalizable and that all its eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ are real. Show directly, using Liapunov functions, that

   (a) If all the eigenvalues of A are negative, then 0 is asymptotically stable.

   (b) If, for $0 < p < n$, $\lambda_1 \le \cdots \lambda_p < 0$ and $0 < \lambda_{p+1}, \cdots < \lambda_n$ then 0 is unstable.

4. Consider the system

$$\begin{aligned} x' &= 2y(z-1)\,, \\ y' &= -x(z-1) \\ z' &= xy\,. \end{aligned} \tag{3}$$

   (a) Show that $(0,0,1)$ is stable.

   (b) Is it asymptotically stable?

5. (a) For the following two equations

$$q' + q^3 + \frac{1}{2}q^2 - \frac{1}{2}q = 0\,, \tag{4}$$

$$q'' + q^3 + \frac{1}{2}q^2 - \frac{1}{2}q = 0\,. \tag{5}$$

$$\tag{6}$$

sketch the phase portrait of the system (on the same phase plane). How are they related? Determine the critical points, their stability properties (including their basin of attractions).

(b) Consider the equation

$$q'' + q' + q^3 + \frac{1}{2}q^2 - \frac{1}{2}q = 0 \tag{7}$$

Determine the critical points and their stability properties (including their basin of attractions). *Hint:* Use Lasalle stability theorem.

6. Let $p,\ q \in \mathbf{R}^n$ and consider the Hamiltonian $H(p,q) = \sum_{i=1}^n \frac{p_i^2}{2} + W(q)$ where $W(q)$ is of class $\mathcal{C}^2$. Assume that a is a nondegenerate critical point of W, i.e.

$$\nabla W(a) = 0 \quad \text{and} \quad \det\left(\frac{\partial^2 W}{\partial q_i \partial q_j}(a)\right) \neq 0\,. \tag{8}$$

Show that if a is a local minimum of W then $(p,q) = (0,a)$ is a stable solution for the Hamiltonian equation $q'' = -\nabla W(q)$ and show that $(p,q) = (0,a)$ is a unstable solution if a is not local minimum (i.e. a local maximum or a saddle point for W). *Hint:* Linearize around the critical point and study the eigenvalues.

7. Consider the Lorentz system

$$\begin{aligned} x' &= 10(y-x)\,, \\ y' &= rx - y - xz\,, \\ z' &= xy - \frac{8z}{3}\,. \end{aligned} \tag{9}$$

For $r = 28$ these are the values of the parameters that Lorentz originally considered. In this exercise you will establish some basic properties of these equations.

(a) Show that if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t)$.

(b) The positive and negative z-axes are invariant sets.

(c) The origin is a critical point for all values of r. Show that $0 < r < 1$ the origin is a stable critical point and its basin of attraction is $\mathbf{R}^3$. *Hint:* $V(x,y,z) = x^2 + 10y^2 + 10z^2$.

(d) Show that the origin is unstable for $r > 1$.

(e) Show that for any values of r, there exists an ellipsoid such that every solution enters the ellispsoid and then remains trapped there forever. *Hint:* See example 1.6.9. in the classnotes.

(f) Show that for any bounded region $D \subset \mathbf{R}^3$ with smooth boundary, we have $\lim_{t\to\infty} \mathrm{vol}(\phi^t(D)) = 0$ where ϕ^t is flow defined by (9). *Hint:* Use Liouville Theorem from Homework 5.

(g) Show that for $r > 1$ there are two more critical points at

$$Q_{\pm} = \left(\pm\sqrt{\frac{8(r-1)}{3}}\,,\ \pm\sqrt{\frac{8(r-1)}{3}}\,,\ r-1\right) \tag{10}$$

The eigenvalues for the linearization at $Q_\pm$ satisfy

$$P_r(z) \;=\; z^3 + (1 + \frac{8}{3} + 10)z^2 + \frac{8}{3}(10 + r)z + 2\frac{8}{3}10(r - 1) \;=\; 0 \qquad (11)$$

(you don't need to compute this...). Show that for $r > 1$, r sufficiently close to 1, $P_r(z)$ has three distincts negative real roots. *Hint*: Compute the roots for $r = 1$.