Math 597/697: Spring 06 Final Exam

You can use your class-notes and the textbook but nothing else. Don't talk to each other, talk to me.

1. **A discrete-time queueing model.** We consider a Markov chain describing, for example, the following car inspection station. The inspector enters the inspection station hourly (for example at 8:00, 9:00, and so on). If there are some cars waiting he takes the first one and inspects it, the inspection lasting exactly one hour. If there are no cars in line the inspector leaves and comes back one hour later. The inflow of cars into the inspection station during a one-hour period is given by a random variable ξ. We have

$$\begin{aligned} & P\{ \text{ j cars enter the inspection station during an hour } \} \\ & = P\{\xi = j\} \equiv p_j\,, \quad \text{for } j = 0, 1, 2, \cdots \end{aligned}$$

with $p_j \geq 0$ and $\sum_{j=0}^{\infty} p_j = 1$

(a) Let ξ_n, $n = 1, 2, \cdots$ be i.i.d. random variables which have the same distribution as ξ. Show that the length of the line at the inspection station, X_n, measured hourly is a Markov chain given by $X_{n+1} = \xi_{n+1}$ if $X_n = 0$ and $X_{n+1} = X_n + \xi_{n+1} - 1$ if $X_n \geq 1$. Show also that the transition probabilities are given by

$$P_{0j} = p_j \quad \text{and} \quad P_{ij} = p_{j-i+1} \ \text{ for } i \geq 1\ , j \geq i-1\,. \tag{1}$$

(b) Show that if either $p_0 = 0$ or $p_0 + p_1 = 1$ the chain is not irreducible.

(c) Show that if $p_0 > 0$ and $p_0 + p_1 < 1$ the chain is irreducible.

(d) In the case where the Markov chain is not irreducible determine which states are absorbing, recurrent, transient. Distinguish the cases (i) $p_1 = 1$, (ii) $p_0 > 0$, $p_1 > 0$, and $p_0 + p_1 = 1$, (iii) $p_0 = 1$, (iv) $p_0 = 0$, $p_1 < 1$.

(e) We denote by $\sigma_j = \inf\{n > 0, x_n = j\}$ be the first return time to the state j and we set $\rho_{ij} = P(\sigma_j < \infty \,|\, X_0 = i)$. Show the following identities which hold for general Markov chains:

$$\begin{aligned} P\{\sigma_j = n+1 \,|\, X_0 = i\} &= \sum_{k\neq j} P_{ik} P\{\sigma_j = n \,|\, X_0 = k\}\,, \quad n \geq 1\,, \\ P\{\sigma_j \leq n+1 \,|\, X_0 = i\} &= P_{ij} + \sum_{k\neq j} P_{ik} P\{\sigma_j \leq n \,|\, X_0 = k\}\,, \quad n \geq 0\,, \\ \rho_{ij} &= P_{ij} + \sum_{k\neq j} P_{ik}\rho_{kj} \end{aligned}$$

(f) For the queueing Markov chain with transition probabilities (1) show that the identity

$$\rho_{00} = \rho_{10}$$

holds and show also that

$$\rho_{ii-1} = \rho_{i-1i-2} = \cdots = \rho_{10}\,,$$

i.e. that ρ_{ii-1} is independent of i.

(g) Show that for the queueing chain

$$P\{\sigma_0 = m \,|\, X_0 = i\} = \sum_{k=1}^{m-1} P\{\sigma_{i-1} = k \,|\, X_0 = i\} P\{\sigma_0 = m - k \,|\, X_0 = i - 1\}\,,$$

and deduce from this that

$$\rho_{i0} = \rho_{ii-1}\rho_{i-10}\,, \quad i \geq 2\,.$$

(h) Conclude that ρ_{00} is a solution of the equation

$$\rho = \phi(\rho)$$

where $\phi(s)$ is the moment generating function of ξ, i.e $\phi(\xi) = E[s^\xi] = \sum_{n \geq 0} p_n s^n$.

As we have seen in the chapter on branching processes the equation $\rho = \phi(\rho)$ has always $\rho = 1$ as a solution. If $\mu = E[\xi] \leq 1$ then 1 is the only solution while if $\mu > 1$ then there exists another solution $0 < \rho < 1$ if $p_0 > 0$. As for branching processes (with the a very similar proof which you **do not** need to repeat here) one shows that the desired probability ρ_{00} is the smallest solution of $\rho = \Phi(\rho)$ and thus

$$X_n \text{ recurrent} \qquad \text{iff} \quad \mu \leq 1\,.$$

2. **Transition probabilities for the $M/M/\infty$ queue.**

(a) Let $Y_1, \cdots, Y_k$ be k independent uniform random variable on $[0,t]$. Let $0 = s_0 < s_1 < \cdots s_m = t$ be an arbitrary partition of $[0,t]$ into m subintervals and denote by Z_k the number of $Y_i's$ in the interval $[s_{i-1}\,,\, s_i]$, $i = 1, \cdots, m$. What is the joint distribution $P\{Z_1 = n_1, \cdots, Z_m = n_m\}$ of $Z_1, \cdots, Z_m$?

(b) Let N_t be a Poisson process with rate λ and let $0 = s_0 < s_1 < \cdots s_m = t$ be an arbitrary partition of $[0,t]$ into m subintervals. Let X_i be the number of events occurring between time s_{i-1} and time s_i, $i = 1, \cdots, m$. Show that for integers $n_1, \cdots, n_m$ adding up to k

$$P\{X_1 = n_1, \cdots, X_m = n_m \,|\, N_t = k\}$$

has the same distribution as $Z_1, \cdots, Z_m$ found in (a).

Remark: This means that the arrival times of a Poisson process up to times conditioned on $\{N_t = k\}$ are independent and uniformly distributed on $[0,t]$.

(c) Assume that a customer enters a queue at a time which is uniformly distributed on $(0,t]$ and upon arrival he is immediately served at one service station. The service time is assumed to be exponential with parameter μ. Show that the probability p_t that he is still in service at time t is given by

$$p_t = \frac{1 - e^{-\mu t}}{\mu t} \tag{2}$$

(d) Consider an $M/M/\infty$ queue where the customers arrive following a Poisson process with rate λ and upon arrival are immediately served at one of infinitely many service counter, the service time being exponentially distributed with parameter μ. We denote by X_t the number of people in the system, i.e the number of people who are still in service at time t. We write

$$X_t = X_t^{(1)} + X_t^{(2)}\,,$$

where $X_t^{(2)}$ denote the number of customers present initially (at time $t = 0$) which are still in service at time t and $X_t^{(1)}$ denotes the number of customers entering the system in $(0,t]$ which are still in service at time t. Note that $X_0^{(1)} = 0$ and $X_0^{(2)} = X_0$.

Show the following

i. $X_t^{(1)}$ and $X_t^{(2)}$ are independent.

ii. $X_t^{(1)}$ is Poisson with parameter

$$\frac{\lambda}{\mu}(1 - e^{-\mu t})$$

iii. Assume that $X_0^{(2)} = j$ then show that $X_t^{(2)}$ is binomial with parameter j and $e^{-\mu t}$.

iv. Describe, *in words*, what are the transition probabilities $P_{jn}(t)$.

v. Assume the initial state of the queue X_0 has a Poisson distribution with parameter ν . Show then that X_t has a Poisson distribution with parameter

$$\frac{\lambda}{\mu} + \left(\nu - \frac{\lambda}{\mu}\right) e^{-\mu t}\,.$$

(e) Use (d) to determine $\lim_t \to \infty P\{X_t = k\}$.

3. **Generating a uniform distribution on the permutations of** $\{1, 2, \cdots, n\}$. In this problem we will use the following notation. If x is positive real number we denote by $[x]$ the integer part of x, i.e. $[x]$ is the greatest integer less than or equal x. For example $[2.37] = 2$.

   Consider the following algorithm to generate a random permutation of the elements $1, 2, 3, \cdots, n$. We will denote by S(i) the element in position i. For example for the permutation $(2, 4, 3, 1, 5)$ of 5 elements we have $S(1) = 2$, $S(2) = 4$, and so on.

   (a) Set $k = 1$

   (b) Set $S(1) = 1$

   (c) If $k = n$ stop. Otherwise let $k = k + 1$.

   (d) Generate a random number U, and let

   $$S(k) = S([kU] + 1)\,,$$

   $$S([kU] + 1) = k\,.$$

   Go to step 3.

   Explain, in words, what the algorithm is doing. Show that at iteration k, – i.e. when the value of $S(k)$ is initially set– $S(1), S(2), \cdots, S(k)$ is a random permutation of $1, 2, \cdots, k$.

   Hint: Use induction and prove that

   $$P_k\{i_1, \cdots, i_{j-1}, k, i_j, \cdots i_{k-2}, i\} \;=\; P_{k-1}\{i_1, \cdots, i_{j-1}, i, i_j, \cdots, i_{k-2}\}\frac{1}{k}\,.$$

   where P_k is the probability distribution obtained at iteration k.