Solutions of HWK #1

#1 $P = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix}$ $P^2 = \begin{pmatrix} 11/16 & 7/12 \\ 7/16 & 9/16 \end{pmatrix}$ $P^3 = \begin{pmatrix} \frac{107}{216} & \frac{109}{216} \\ \frac{109}{192} & \frac{83}{192} \end{pmatrix}$

(a) $P^3_{01} = \frac{109}{216}$ (b) $P_{11} P^2_{10} = \frac{1}{4}\frac{7}{16}$

(c) $P_{00} \cdot P_{00} P_{01} = \left(\frac{1}{3}\right)^2 \frac{2}{3}$ (d) $\left(\frac{1}{4}, \frac{3}{4}\right) P^3 = \left(\frac{1}{4}\frac{107}{216} + \frac{3}{4}\frac{109}{192}, \frac{1}{4}\frac{109}{216} + \frac{3}{4}\frac{83}{192}\right)$

#2 If X_n is the weather on day n, $X_n = 0$ (rainy) or 1 (not rainy)

(a) $P(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}) \neq P(X_n = i_n \mid X_{n-1} = i_{n-1})$

so X_n is not a Markov chain.

(b) $Y_n = (i, j)$ where i weather at day $\underline{n-1}$, j weather at day $\underline{n}$

then Y_n depends only on Y_{n-1} so Markov.

We have $P_{(i,j)(k,\ell)} \neq 0$ only if $j = k$.

$$\begin{array}{c|c} & \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} & \begin{pmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .4 & .6 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & .2 & .8 \end{pmatrix} \end{array}$$

For example $P_{(0,0),(0,0)} = P\left(\text{n-th rainy (n-1)-th rainy} \,\middle|\, \begin{array}{c}\text{n-1 th rainy} \\ \text{n-2 th rainy}\end{array}\right)$

$= P(X_n = 0 \mid X_{n-1} = 0 \; X_{n-2} = 0)$

$= 0.8$

etc....

3.) States of the system is $j = 0, 1, 2, 3$

If $j = 0$ A: R RR B: W W W

so $P_{01} = P\{\text{pick a red from A and pick a white from B}\}$
$= 1$

RR W WW R

$P_{10} = P\{\text{pick white from A, pick red from B}\}$
$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P_{11} = P\{\text{pick white from A, pick white from B}\}$
$+ P\{\text{pick red from A, pick red from B}\}$
$= \frac{1}{3}\frac{2}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}$

etc....

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The stationary distribution is unique, by Doeblin Thm, e.g. $P^2_{i1} > 0 \quad 0 \le i \le 3$

Computing $\pi P = \pi$ gives

$\frac{1}{9}\pi(2) = \pi(1)$, $\pi(1) + 4/9\,\pi(2) + 4/9\,\pi(3) = \pi(2)$

$\frac{4}{9}\pi(2) + 4/9\,\pi(3) + \frac{1}{9}\pi(4) = \pi(3)$ $\quad \frac{1}{9}\pi(3) = \pi(4)$

$\Rightarrow \frac{1}{9}\pi(2) = \pi(1)$, $\pi(2) = \pi(3)$, $\pi(4) = \frac{1}{9}\pi(3)$ $\Rightarrow \boxed{\pi = \left(\frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}\right)}$

#4 $\cdot\ Z_n = (X_n, Y_n)$: State space : $(i,j),\ i \in S,\ j \in S$.

$\cdot$ Z_n is a Markov chain since Z_n depends only on Z_{n-1}.

$\cdot\ P\{Z_n = (i_n, j_n) \mid Z_{n-1} = (i_{n-1}, j_{n-1})\}$

$= P\{X_n = i_n, Y_n = j_n \mid X_{n-1} = i_{n-1}, Y_{n-1} = j_{n-1}\}$

$= P\{X_n = i_n \mid X_{n-1} = i_{n-1}\}\ P\{Y_n = j_n \mid Y_{n-1} = j_{n-1}\}$

X_n and Y_n are independent

$= P_{i_{n-1} i_n}\ P_{j_{n-1} j_n}$

So $\boxed{P_{(i,j),(k,\ell)} = P_{ik}\, P_{j\ell}}$

#5 $P = \begin{pmatrix} .4 & .4 & .2 \\ .3 & .4 & .3 \\ .2 & .4 & .4 \end{pmatrix}$ All ~~rows~~ columns are strictly positive, so by Doeblin Theorem there is a unique stationary distribution $\pi(j)$, $\pi(j) > 0$.

Computing $\pi = \left(\frac{3}{10}, \frac{4}{10}, \frac{3}{10}\right)$

#6 (a) $\pi_1 = (\pi_1(1), \pi_1(2), \dots, \pi_1(N))$ $\quad \sum_j \pi_1(j) = 1 \quad \pi_1(j) \geq 0$

$\pi_2 = (\pi_2(1), \dots, \pi_2(N))$ $\quad \sum_j \pi_2(j) = 1 \quad \pi_2(j) \geq 0$

If $0 < \alpha < 1$ then $\alpha \pi_1(j) + (1-\alpha)\pi_2(j) \geq 0$

and $\sum_j \alpha\pi_1(j) + (1-\alpha)\pi_2(j) = \alpha + 1 - \alpha = 1$.

(b) $\pi P = \pi$ gives

$$\left.\begin{array}{l} \frac{2}{5}\pi_1 + \frac{3}{10}\pi_2 = \pi_1 \\ \frac{3}{5}\pi_1 + \frac{7}{10}\pi_2 = \pi_2 \end{array}\right\} \Rightarrow \pi_1 = \frac{1}{2}\pi_2$$

$$\left.\begin{array}{l} \frac{1}{3}\pi_3 + \frac{1}{6}\pi_4 = \pi_3 \\ \frac{2}{3}\pi_3 + \frac{5}{6}\pi_4 = \pi_4 \end{array}\right\} \Rightarrow \pi_3 = \frac{1}{4}\pi_4$$

2 independent equations for π_1, π_2 and π_3, π_4

$$\pi_1 = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)$$

2 solutions

$$\pi_2 = \left(0, 0, \frac{1}{5}, \frac{4}{5}\right)$$

$\Rightarrow$ Using (a)

$$\pi = \left(\frac{\alpha}{3}, \frac{2\alpha}{3}, \frac{1-\alpha}{5}, \frac{4(1-\alpha)}{5}\right)$$

is a stationary distribution for $0 \le \alpha \le 1$.

#7 Suppose $\sum_{i=1}^{N} P_{ij} = 1$ and choose

$\pi = \left(\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}\right)$ then $\sum_{i=1}^{N} \pi(i) = \sum_{i=1}^{N} \frac{1}{N} = N \cdot \frac{1}{N} = 1$

and $\pi P(j) = \sum_i \pi(i) P_{ij} = \sum_i \frac{1}{N} P_{ij} = \frac{1}{N} \sum_i P_{ij}$

$$= \frac{1}{N} = \pi(j)$$

So π is stationary.

#8 The state space S consists of the 52! permutations of 52 cards.

To compute P_{ij}, pick any permutation i of the 52 cards. The states j which are accessible from i, ($i \to j$) are the ones obtained by picking one card out of 52, with probability $\frac{1}{52}$ and putting it on top.

Each row has 52 non zero entries out of 52!, each one equal to $\frac{1}{52}$

To see what are the columns, note that given a state j, the states i such that $i \to j$, are obtained by picking the first card on top of the deck and putting in any of 52 possible spots.

So $\sum_{i=1}^{52!} P_{ij} = 1$.

By #6, the uniform distribution $\pi(j) = \frac{1}{52!}$ is stationary.

#9 $P^T_{ij} = \frac{\pi(j) P_{ji}}{\pi(i)}$, $\pi P = \pi$

(a) • P^T_{ij} is stochastic

$$\sum_j P^T_{ij} = \sum_j \frac{\pi(j) P_{ji}}{\pi(i)} = \frac{1}{\pi(i)} \sum_j \pi(j) P_{ji}$$

$$= \frac{\pi(i)}{\pi(i)} = 1$$

• π is stationary for P^T since

$$\pi P^T(j) = \sum_i \pi(i) P^T_{ij} = \sum_i \cancel{\pi(i)} \frac{\pi(j)}{\cancel{\pi(i)}} P_{ji}$$

$$= \pi(j) \sum_i P_{ji} = \pi(j)$$

(b) $P\{\bar{X}_0 = i_0, \ldots, \bar{X}_n = i_n\}$

$$= \pi(i_0) P^T_{i_0 i_1} P^T_{i_1 i_2} \cdots P^T_{i_{n-1} i_n}$$

$$= \cancel{\pi(i_0)} \frac{\cancel{\pi(i_1)}}{\cancel{\pi(i_0)}} P_{i_1 i_0} \frac{\pi(i_2)}{\cancel{\pi(i_1)}} P_{i_2 i_1} \cdots \frac{\pi(i_n)}{\cancel{\pi(i_{n-1})}} P_{i_n i_{n-1}}$$

$$= \pi(i_n) P_{i_n i_{n-1}} \cdots P_{i_2 i_1} P_{i_1 i_0}$$

$$= P(X_0 = i_n, \ldots, X_{n-1} = i_1, X_n = i_0)$$

So $\bar{X}$ is X backward in time