STAT 315: Probability Basics (Section 2.2–2.5)

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What is probability?

- Probability describe an experiment whose outcome cannot be described with certainty. Keyword: random or stochastic
  - Roll of a dice
  - The price of GameStop stock or the price of Bitcoin tomorrow
  - The winner of Super Bowl LVIX: Chiefs of Eagles?
  - The amount of rain on Amherst due to Hurricane Ida

- Frequentist approach: deduce the probability by repeating the experiment N times (N very large) (Law of Large numbers)

$$P(\text{roll a 5}) \approx \frac{\text{number of 5 in N rolls}}{N}$$

$\longrightarrow$ Can be simulated on a computer!

- Subjective (Bayesian) approach: Probability is a measures of one's belief in the occurrence of a future event
  $\longrightarrow$ You can't repeat the Super Bowl but you can bet on it!

Review of set notation

- Denote sets of points by capital letters, $A, B_1, B_2, S, \cdots$, and points by lower case, $a_1, a_2, b, c, x, \cdots$
- If the elements ins A are a_1, a_2, a_3 we write $A = \{a_1, a_2, a_3\}$
- Denote by S the sets of all elements (the sample space) and by $\emptyset$ the set with no element.

Sets operations

- $A \subset B$ (A is contained in B): every element in A is also in B.
- $A \cup B$ (the union of A and B): the set elements which belong either to A or to B.
- $A \cap B$ (the intersection of A and B): the sets of elements which belong both to A and B.
- $\overline{A}$ (the complement of A): the set of element in S which do not belong to A.
- $B \setminus A = B \cap \overline{A}$: the elements in B which are not in A.

Laws of set algebra

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan's Law

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Use Venn's diagrams

Probabilistic experiment: discrete case

We use the language of set theory to describe an experiment with random outcomes:

Sample space and events

- S is called the sample space: the list of all possible outcomes of the experiment.

$$S = \{e_1, e_2, e_3, \cdots\} \quad \text{finite or countable set}$$

- A subset $A \subset S$ is called an event. Think of it as a question you ask about the experiment: does the event A occur?)

Example (Roll a pair of dice)

$$
\begin{aligned}
S &= \{(1,1),(1,2),(1,3),\cdots,(6,6)\} \quad S \text{ has 36 elements} \\
A &= \{\text{The sum of the dice is } 4\} = \{(1,3),(2,2)(3,1)\} \\
B &= \{\text{Exactly one } 6\} = \{(1,6),(6,1),(2,6),(6,2),\cdots(5,6)(6,5)\}
\end{aligned}
$$

Set operations in probabilistic language

Intuitive meaning of set operations

- $A \cap B = $ "A and B" $\longrightarrow$ both A and B occur.

- $A \cup B = $ "A or B" $\longrightarrow$ either A or B occur.

- $\overline{A} = $ "not A" $\longrightarrow$ A does not occur.

- $A \cap B = \emptyset \longrightarrow$ A and B are mutually exclusive, they cannot occur simultaneously.

- $B \setminus A = B \cap \overline{A} = $ "B but not A", B occurs but not A

Laws of probability I

Laws of Probability

S is the sample space. To every event A in S (i.e., $A \subset S$) we assign a number $P(A)$, called the probability of A, with the following properties

- Axiom 1: $0 \leq P(A) \leq 1$.
- Axiom 2: $P(S) = 1$.
- Axiom 3: If $A_1, A_2, A_3, \cdots$ are pairwise mutually exclusive $A_i \cap A_j = \emptyset$ if $i \neq j$ then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

For the discrete case $S = \{e_1, e_2, \cdots\}$ simply assign numbers $P(\{e_i\}) = p_i$ with $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. We have then
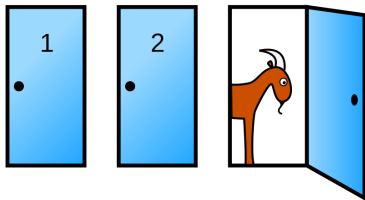
$$P(A) = \sum_{i:e_i \in A} p_i.$$

Laws of probability II

Simple consequence of the laws of probability

1. $P(\emptyset) = 0$

2. If $A \cap B = \emptyset$ (mutually exclusive) then $P(A \cup B) = P(A) + P(B)$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Inclusion-Exclusion)

4. If $A \subset B$ then $P(A) \leq P(B)$ (Monotonicity)

5. $P(\overline{A}) = 1 - P(A)$ (Complement)

The Monty Hall Problem

- There are 2 goats and $1 million hidden behind 3 doors.
- You pick a door (without opening it), say door 1.
- The game hosts open of the other two door (on the picture, door 3) and reveals a goat behind it.
- You are given the following xchoice
  - Keep your door?
  - Switch?
- What should you do to maximize your probability of winning?

Any questions?