Lecture 10: Evolutionary stable states: the sex ratio game

We discuss here a marvelous application of the concept of evolutionary stable states to explain why in most species, the number of male is roughly equal to the number of females. We will explain it in the following terms: *to maximize your number of descendants in the future generations (i.e. to propagate your genes as much as possible) you should have as many sons as daughters*

To express this game-theoretic terms we will think that the strategy of a players is his *sex ratio*:

$$\boxed{\text{Sex ratio } p = \text{ Expected proportion of males in your children}}$$

The sex ratio does not influence the number of children so we will consider *the number of grandchildren*. So we imagine we have three generations

$G0$: Generation of the parents, total population N_0
$G1$: Generation of the children, total population N_1
$G2$: Generation of the grandchildren, total population N_2

Let us assume that the sex ratio in the population is m and consider one particular individual in $G2$. This individual has one father and one mother in $G1$. If we assume that the population is well mixed so that we have random mating we obtain

$$\text{Probability that a randomly chosen individual in } G1 \text{ is the father is } \frac{1}{mN_1}$$

$$\text{Probability that a randomly chosen individual in } G1 \text{ is the mother is } \frac{1}{(1-m)N_1}$$

To compute the fitness we imagine that a distinguished individual in generation $G0$ has a sex ratio p in a population of sex ratio m. Its expected number of grandchildren is given by

$$\begin{aligned}\text{Expected number of grandchildren} &= p\,\frac{1}{mN_1}\,N_2 + (1-p)\,\frac{1}{(1-m)N_1}\,N_2 \\ &= \frac{N_2}{N_1}\left(\frac{p}{m} + \frac{1-p}{1-m}\right).\end{aligned}$$

The factor N_2/N_1 does not play any role and so we define

Fitness of sex ratio p in a population with sex ratio m

$$W(p,m) = \left(\frac{p}{m} + \frac{1-p}{1-m}\right) = \frac{1}{1-m} + p\frac{1-2m}{m(1-m)}$$

We note for future use that

- $W(p,m)$ is an increasing function of p for $m < 1/2$.
- $W(p,m)$ is a decreasing function of p for $m > t1/2$.
- $W(p,m)$ is constant in p for $m = 1/2$.

In order to determine if a certain sex ratio is *evolutionary stable* we imagine a population of sex ratio p which invaded by a small population of sex ratio q. After invasion the population has now sex ratio

$$m = (1-\epsilon)p + \epsilon q$$

for some small epsilon. To determine evolutionary stability we need to compare $W(p,m)$ (fitness of the original sex ratio) vs $W(p,m)$ (fitness of the mutant sex ratio).

1. If $p < 1/2$ then the new sex ratio $m = (1-\epsilon)p+\epsilon q$ is also $< 1/2$ (for sufficiently small ϵ). But for $m < 1/2$, $w(p,m)$ is increasing in p therefore we have $w(q,m) > w(p,m)$ for $q > p$ and thus a population of sex ratio $p < 1/2$ is not evolutionary stable since it can be invaded by a population with a larger sex ratio.

2. If $p > 1/2$ then the new sex ratio m is also $> 1/2$ (for sufficiently small ϵ). But for $m > 1/2$, $w(p,m)$ is decreasing in p therefore we have $w(q,m) > w(p,m)$ for $q < p$ and thus a population of sex ratio $p > 1/2$ is not evolutionary stable since it can be invaded by a population with a smaller sex ratio.

3. If $p = 1/2$ then we need to distinguish two subcases:

   (a) If $p = 1/2$ and $q > 1/2$ then $m = (1-\epsilon)1/2 + \epsilon q > 1/2$ and so we have $w(q,m) < w(1/2,m)$ for $q > 1/2$. This means that a population with sex ratio 1/2 cannot be invaded by mutants with sex ratio $q > 1/2$.

(b) If $p = 1/2$ and $q < 1/2$ then $m = (1-\epsilon)1/2 + \epsilon q < 1/2$ and so we have $w(q, m) < w(1/2, m)$ for $q < 1/2$. This means that a population with sex ratio $1/2$ cannot be invaded by mutants with sex ratio $q < 1/2$.

From this we conclude

The sex ratio $1/2$ is the only evolutionary stable sex ratio