Math 697: Homework 3

Exercise 1 A mouse is performing a symmetric random walk on the positive integer $\{0, 1, 2, 3, \cdots\}$: if it is in state i it is equally likely to move to state $i-1$ or $i+1$. The state 0 is the mouse's home filled with lots of tasty cheese. If the mouse ever reaches its home it will stay there forever. On the other hand there is a bad cat who tries to catch the mouse and each time the mouse moves there is a probability $1/5$ that the cat will kill the mouse.

To describe this process as a Markov chain consider an extra state $*$ which corresponds to the mouse being dead. The state space is $S = \{*, 0, 1, 2, \cdots\}$ and X_n denotes the position of the mouse at time n. Compute the corresponding transition matrix.

Compute the probability that the mouse reaches safety if it starts in state i, i.e.,

$$p_i \equiv P\{X_n = 0 \text{ for some n} \mid X_0 = i\}\,, i = 0, 1, 2, \cdots$$

Exercise 2 Problem 2.1, p. 57

Exercise 3 Problem 2.2, p. 57

Exercise 4 Problem 2.4, p. 58

Exercise 5 Problem 2.6, p. 58

Exercise 6 Problem 2.7, p. 58

Exercise 7

Consider the following Markov chain. At times $n = 1, 2, 3, \cdots$ ξ_n particles are added in a box where ξ_n are i.i.d. random variables with a Poisson distribution with parameter λ, i.e.,

$$P\{\xi_n = k\} = \frac{\lambda^k}{k!} e^{-\lambda}\,.$$

Suppose that any of the particles in the box at time n independently of all other particles and of how the particles are added to the box has probability p of remaining in the box at time $n+1$ and probability $q = 1 - p$ of being removed from the box. The number of particles in the box at time n, X_n is a Markov chain which can be expressed as

$$X_{n+1} = \xi_{n+1} + R(X_n)\,,$$

where $R(X_n)$ denotes the number of particles present at time n and which remain at time $n+1$.

1. Show that the Markov chain is irreducible and aperiodic.

2. As a preparation prove the following fact. Let Z be a Poisson random variable with parameter μ describing the number of some items. The items occur in two types, type A with probability p_A and type B with probability $p_B = 1 - p_A$ and Let $Z = Z_A + Z_B$ where Z_A are the number of items of type A and Z_B are the number of items of type B. Show that Z_A and Z_B are Poisson random variables with parameter μp_A and μp_B.

3. Use A to show that if the initial distribution of X_0 is Poisson with with some parameter ν then X_1 has also a Poisson distribution. Compute the probability distribution of X_n and determine the limiting and stationary distribution.

Exercise 8 Problem 2.8, p. 58

Exercise 9 Consider a branching process with offspring distribution given by p_n. One makes this process irreducible by asserting that if the the population ever dies out, then in the next generation one new individual appears (i.e. $P_{01} = 1$). Determine for which values of p_n the chain is positive recurrent, null recurrent, transient.

Exercise 10 An electric light that has survived n seconds fails during the $(n+1)st$ second with probability q (with $0 < q < 1$).

1. Let $X_n = 1$ if the light is functioning at time n seconds, and $X_n = 0$ otherwise. Let T be the time of failure of the light (in seconds), i.e.,

$$T = \inf\{n\,;\, X_n = 0\}\,. \tag{1}$$

   Determine $E[T]$.

2. A building contains m lights of the type described above, which behave independently of each other. At time 0 they are all functioning. Let Y_n denote the number of lights functioning at time n. Specify the transition matrix of Y_n.

3. Find the moment generating function

$$\phi_n(s) = E[s^{Y_n}] \tag{2}$$

   of Y_n. *Hint*: Express ϕ_n in terms of ϕ_{n-1} and solve the recursion relation.

4. Use the moment generating function to find $P\{Y_n = 0\}$ and $E[Y_n]$.

Exercise 11 Jamie is working in a bookstore, ordering books that are not in store and that the customers request. Each order takes 5 minutes to complete. While each order is being filled there is a probability p_j that j more customers arrive with $p_0 = .2$, $p_1 = .2$, $p_2 = .6$. Jamie cannot take a coffee break until a service is completed and no one is waiting in line to order a book. When Jamie starts her shift there is one customer waiting. What is the probability that she ever will take a coffee break.