Math 645: Homework 2

1. Show the following: If the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, wher $f(t,x)$ is a continuous function, has a unique solution, then the Euler polygons converge to this solution.

2. Consider the Cauchy problem $x' = f(t,x)$, $x(0) = 0$, where
$$f(t,x) = \begin{cases} 4\text{sign}(x)\sqrt{|x|} & \text{if} \quad |x| \geq t^2 \\ 4\text{sign}(x)\sqrt{|x|} + 4(t - \frac{|x|}{t})\cos(\pi\frac{\log t}{\log 2}) & \text{if} \quad |x| < t^2 \end{cases} \tag{1}$$
The function f is continuous on $\mathbf{R}^2$. Consider the Euler polygons $x_h(t)$ with $h = 2^{-i}$, $i = 1, 2, 3, \cdots$. Show that $x_h(t)$ does not converge as $h \to 0$, compute its accumulation points, and show that they are solution of the Cauchy problem. *Hint:* $\pm 4t^2$.

3. Consider the the Cauchy problem $x' = f(t,x)$, $x(0) = 0$ where f is given by
$$f(t,x) = \begin{cases} 0 & \text{if} \quad t \leq 0, \quad x \in \mathbf{R} \\ 2t & \text{if} \quad t > 0, \quad x \leq 0 \\ 2t - \frac{4x}{t} & \text{if} \quad t > 0, \quad 0 \leq x < t^2 \\ -2t & \text{if} \quad t > 0, \quad t^2 \leq x \end{cases} \tag{2}$$
   (a) Show that f is continuous. What does that imply for the Cauchy problem?
   (b) Show that f does not satisfy a Lipschitz condition in any neighborhood of the origin.
   (c) Apply Picard-Lindelöf iteration with $x_0(t) \equiv 0$. Are the accumulation points solutions?
   (d) Show that the Cauchy problem has a unique solution. What is the solution?

   Note that this problem shows that existence and uniqueness of the solution does not imply that the Picard-Lindelöf iteration converges to the unique solution.

4. Consider the Cauchy problem $x' = \lambda x$, $x(0) = 1$, with $\lambda > 0$ on the interval $[0,1]$. Compute the Euler polygons $x_h(t)$. Show that
$$\frac{\lambda}{1 + \lambda h} x_h(t) \leq \frac{d}{dt} x_h(t) \leq \lambda x_h(t)\,. \tag{3}$$
and deduce from this the classical inequality
$$\left(1 + \frac{\lambda}{n}\right)^n \leq e^\lambda \leq \left(1 + \frac{\lambda}{n}\right)^{n+\lambda} \tag{4}$$

5. Consider the Cauchy problem for the Ricatti equation $x' = t^2 + x^2$, $x(0) = 1$. Show that the solution $x(t)$ satisfies the bounds
$$\frac{1}{1-t} \leq x(t) \leq \tan(t + \pi/4) \tag{5}$$
Hint: Consider the Cauchy problems $u' = u^2$ and $v' = 1 + v^2$ and compare $x(t)$ with $u(t)$ and $v(t)$.

6. Let $V(q)$ be a function of class $\mathcal{C}^2$ such that $\lim_{|q|\to\infty} V(q) = \infty$ and consider the Hamiltonian system $q' = p$, $p' = -dV(q)/dq$. Show that the solution exists for all positive as well as all negative times.

7. Let a, b, c, and d be positive constants. Consider the Predator-Prey equation $x' = x(a - by)$, $y' = y(cx - d)$ with positive initial conditions $x(t_0)$ and $y(t_0)$. Using the change of variables $p = \log(x)$ and $q = \log(y)$ express the equations as an Hamiltonian systems and deduce that the solution exist for all t.

8. Find the maximal interval of existence $I_{\max} = (\omega_-\,,\,\omega_+)$ for the following Cauchy problems.

   (a) $x' = x^3$.

   (b) $x' = e^{-x}$.

   (c) $x_1' = 1/x_1$, $x_2' = x_1$.

   (d) $x' = 1 - x^2$.

   (e) $x'' + x + x^3 = 0$.

   (f) $x'' + x' + x + x^3 = 0$.

   (g) $x' = \sin(2tx)x^3/(1 + t^2 + x^2)$.

9. Prove the generalized Gronwall Lemma. Let $h(t)$ and $g(t)$ be continuous nonnegative function and let $a > 0$ be a positive constant such that, for any $t \in [0, T]$

$$g(t) \,\leq\, a + \int_0^t h(s)g(s)\,ds\,. \tag{6}$$

   Then, for any $t \in [0, T]$

$$g(t) \,\leq\, ae^{\int_0^t h(s)\,ds}\,. \tag{7}$$

10. Consider the FitzHugh-Nagumo equation

$$\begin{aligned} x_1' &= f_1(x_1, x_2) = g(x_1) - x_2\,, \\ x_2' &= f_2(x_1, x_2) = \sigma x_1 - \gamma x_2\,, \end{aligned} \tag{8}$$

   where σ and γ are positive constants and the function g is given by $g(x) = -x(x - 1/2)(x - 1)$.

   (a) In the x_1-x_2 plane draw the graph of the curves $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$.

   (b) Consider the rectangles $ABCD$ whose sides are parallel to the X_1 and x_2 axis with two opposite corners located on the $f_2(x_1, x_2) = 0$. Show that if the rectangle is taken sufficiently large, a solution which start inside the rectangle stays inside the rectangle forever. Deduce from this that the equations for any initial conditions x_0 have a unique solutions for all time $t > 0$.

11. Show that the solutions of

$$\begin{aligned} x_1' &= x_1(3 - x_1 - 2x_2)\,, \\ x_2' &= x_2(4 - 2x_1 - 3x_2)\,, \end{aligned} \tag{9}$$

have a unique solution for all $t \geq 0$, for any initial conditions x_{10} , x_{20} wich are nonnegative. *Hint*: A possibility is to use a similar procedure as in the previous exercise.