Math 623: Problem set 6

1. Show that the Cauchy-Schwarz inequality $|(f,g)| \le \|f\|\|g\|$ is an equality if and only $f = cg$ for some $c \in \mathbf{C}$.

2. Exercise 4, p. 194

3. (a) Show that neither the inclusion $L^1(\mathbf{R}^d) \subset L^2(\mathbf{R}^d)$ nor the inclusion $L^2(\mathbf{R}^d) \subset L^1(\mathbf{R}^d)$ are valid.

   (b) Suppose E is a set of finite measure. Show then that $L^2(E) \subset L^1(E)$

4. Exercise 6, p. 194

5. Consider a vector space (real or complex) $\mathcal{B}$ with a norm $\|\cdot\|$. We may ask the question whether the norm $\|\cdot\|$ derive from a scalar product, i.e. is there a scalar product $(\cdot,\cdot)$ on $\mathcal{B}$ such that $(f,f) = \|f\|^2$.

   (a) Suppose that $\mathcal{B}$ is a **real** vector space with norm $\|\cdot\|$. Prove that the norm is induced by a scalar product if and only if the parallelogam law holds, i.e., we have
   $$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right] .$$
   and the scalar product is given by
   $$(x,y) := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \tag{1}$$

   Hint: To show that (x,y) given in eq. (1) is additive in the first variable show first that
   $$4(u+v,w) + 4(u-v,w) = 8(u,w)$$
   Deduce from this that $(x+y,z) = (x,z) + (y,z)$. Prove then that $(\alpha x, y) = \alpha(x,y)$ first for integers α then for rational α.

   Remark: The same result holds for **complex** scalar products but then the scalar product is given by
   $$(x,y) := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \tag{2}$$
   The proof is similar but more tedious....

   (b) Show that the norm on vector space $L^1(\mathbf{R}^d)$ does not derive from a scalar product

6. Exercise 9, p. 195

7. Exercise 24, p. 198
8. Exercise 25, p. 198
9. Exercise 28, p. 199
10. Exercise 32, p. 201