Math 597/697: Midterm

You can use your classnotes and Ross textbook and that's it. Don't talk to each other, talk to me. The exam is due on monday 11:15, no extension. You must **EXPLAIN** your answers to obtain full credit. All problems are worth the same number of points. And yes, you must **EXPLAIN** your answers to obtain full credit.

1. Ms Jenike possesses r umbrellas which she uses going from her home to her office in the morning and vice versa in the evening. If it rains in the morning or in the evening she will take an umbrella with her provided there is one available. Assume that independent of the past it will rain in the morning or evening with probability p. Let X_n denote the number of umbrellas at her home before she gets to work.

   (a) Give the state space and the transition probabilities describing the Markov chain X_n.

   (b) Find the limiting probabilities π_j, $j = 0, 1, \ldots, r$.

   (c) In the long run, what fraction of the time does Ms Jenike gets wet?

2. Find all communication classes for the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 1/3 & 0 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2/3 & 1/6 & 0 & 1/6 \end{pmatrix} . \qquad (1)$$

   Determine which classes are recurrent and which ones are transient. Determine also the period of state 2.

3. For this problem you must explain your results in terms of recurrence, transience, positive recurrence, periodicity, etc... *Hint*: Write P in standard form and analyze each communication class separately.

   (a) Consider the Markov chain with state space {0,1,2,3,4} and with transition matrix

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 1/5 & 1/2 & 3/10 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 2/3 \end{pmatrix}. \tag{2}$$

Compute $\lim_{n\to\infty} P_{i0}^n$ for $i = 0, 1, 2, 3, 4$.

(b) Consider the Markov chain with state space $\{0, 1, 2, 3, 4\}$ with transition probabilities

$$P = \begin{pmatrix} 1/10 & 0 & 0 & 9/10 & 0 \\ 0 & 0 & 3/10 & 0 & 7/10 \\ 0 & 1/5 & 1/2 & 0 & 3/10 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Compute $\lim_{n\to\infty} P_{ij}^n$ for all i and j.

4. Jamie is working in a bookstore, ordering books that are not in store and that the customers request. Each order takes 5 minutes to complete. While each order is being filled there is a probability p_j that j more customers arrive with $p_0 = .2$, $p_1 = .2$, $p_2 = .6$. Jamie cannot take a coffee break until a service is completed and no one is waiting in line to order a book. When Jamie starts her shift there is one customer waiting. What is the probability that she ever will take a coffee break. *Hint:* Think of some process we have seen in class.

5. The weather on a single day in a grape growing region is modeled by a four state Markov chain with state 0 (sunny and clear), 1 (cool and muggy), 2 (gray and dreary), 3 (raining). The transition matrix for this chain is

$$P = \begin{pmatrix} .4 & .2 & .1 & .3 \\ .4 & .3 & .2 & .1 \\ .6 & .1 & .1 & .2 \\ .2 & .4 & .3 & .1 \end{pmatrix}. \tag{4}$$

Today it is gray and dreary and the grapes are not quite ready for picking. Another sunny day would bring them to perfection, but two or more rainy days would ruin them. What would you compute to make a decision? Compute it.

6. A boy and a girl move into a town with two bars on the same day (this is day 0). Each night the boy visits one of the two bars (denoted by 1 and 2), starting in bar 1 according to a Markov chain with transition matrix

$$P = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix} . \tag{5}$$

Likewise and independently of the boy the girl visits one or the other bar according to a Markov chain with transition matrix

$$P = \begin{pmatrix} .4 & .6 \\ .6 & .4 \end{pmatrix} . \tag{6}$$

but starting in bar 2. Of course if the boy and the girl are in the same bar they meet.

(a) Compute the probability that they meet on the day 2.

(b) Let N denote the first time they meet. Compute the expected value of N.

(c) Compute the probability that they meet the first time in bar 2.

(d) In bar 1 they each have meal for an amount of \$10 each and in bar 2 they each have a drink for an amount of \$5 each. If they meet in bar 1 the boy pays for both, if they meet in bar 2 the girl pays for both, and if they are not in the same bar they each pay their bills. Who do you think is paying more in the long run? Estimate the average difference of money they spend in the long run.

7. Consider the Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probabilities

$$P_{i,i+1} = \frac{\lambda}{i+1}, \quad P_{i0} = 1 - P_{i,i+1}, \tag{7}$$

where $\lambda > 0$ is a constant. Show that the chain is irreducible, aperiodic and positive recurrent and compute the stationary distribution.

8. An electric light that has survived n seconds fails during the $(n+1)st$ second with probability q (with $0 < q < 1$).

(a) Let $X_n = 1$ if the light is functioning at time n seconds, and $X_n = 0$ otherwise. Let T be the time of failure of the light (in seconds), i.e.,

$$T = \inf\{n \,;\, X_n = 0\} . \tag{8}$$

Determine $E[T]$.

(b) A building contains m lights of the type described above, which behave independently. At time 0 they are all functioning. Let Y_n denote the number of lights functioning at time n. Specify the transition matrix of Y_n.

(c) Find the moment generating function

$$\phi_n(s) = E[s^{Y_n}] \tag{9}$$

of Y_n. *Hint*: Express ϕ_n in terms of ϕ_{n-1} and solve the recursion relation.

(d) Use the moment generating function to find $P\{Y_n = 0\}$ and $E[Y_n]$.