Math 697 Fall 2014: Week 4

Exercise 1 ***(More on the Markov property)*** The Markov property means that the future depends on the present but not on the past, i.e.,

$$P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}, \cdots X_0 = i_0\} = P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}\}\,.$$

1. Show that the Markov property implies that the past depends only on the present but not on the future, i.e.,

$$P\{X_0 = i_0 \,|\, X_1 = i_1, \cdots X_n = i_n\} = P\{X_0 = i_0 \,|\, X_1 = i_1\}\,.$$

2. Show that the Markov property also implies that, given the present, the past and the future are independent, i.e.,

$$\begin{aligned} &P\{X_{n+1} = i_{n+1}\,,\, X_{n-1} = i_{n-1} \,|\, X_n = i_n\} \\ &= P\{X_{n+1} = i_{n+1} \,|\, X_n = i_n\} P\{X_{n-1} = i_{n-1} \,|\, X_n = i_n\}\,. \end{aligned}$$

Exercise 2 ***(2 steps Markov chains Part 1)*** Suppose that X_n is a Markov chain with state space S, transition probabilities $P(i,j)$ and stationary distribution $\pi(i)$. Show that

$$Z_n = (X_{n+1}, X_n)$$

is a Markov chain. What are (a) the state space, (b) the transition probabilities, and (c) the stationary distribution?

Exercise 3 ***(2 steps Markov chains Part 2)***

1. Instead of the Markov property let us assume X_n depends on the state of the previous two steps: i.e

$$P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}, \cdots X_0 = i_0\} = P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}\}\,.$$

   This is called a 2-Markov chain and if it is time-homogeneous it is specified by the numbers $Q_{i,j,k} = P\{X_n = k \,|\, X_{n-1} = j, X_{n-2} = i\}$. Show that

$$Z_n = (X_{n+1}, X_n)$$

   is a Markov chain. Describe its state space and transition probabilities.

2. The weather is in two states: rainy or not rainy and it depends on the weather in the previous two days. If it was rainy yesterday and and the day before yesterday it will be rainy today with probability 0.8. If it was not rainy yesterday and the day before yesterday it will be rainy today with probability 0.2. If it was rainy yesterday but not rainy the day before yesterday, then the weather will be rainy today with probability 0.5. If it was not rainy yesterday but rainy the day before yesterday, then the weather will be rainy today with probability 0.4.

- Construct the corresponding Markov chain as in 1.
- If it rained yesterday and the day before yesterday, what is the probability it will rain tomorow?

Exercise 4

The Smiths receive the paper every morning, they read it during breakfast, and place it on a pile after reading it. Each afternoon, with probability 1/3 someone takes all papers in the pile and put them in the recycling bin. Also if ever there at least five papers in the pile, Mr Smith, with probability one, take the papers to the bin in the afternoon.

1. Consider the number of papers in the pile in the evening and describe it with a Markov chain. What are the state space and transition probabilities?

2. Find the stationary distribution.

3. After a long time what would be the expected number of papers in the pile?

Exercise 5 Jane possesses r umbrellas which she uses going from her home to her office in the morning and vice versa in the evening. If it rains in the morning or in the evening she will take an umbrella with her provided there is one available. Assume that independent of the past it will rain in the morning or evening with probability p. Let X_n denote the number of umbrellas at her home before she gets to work.

1. Give the state space and the transition probabilities describing the Markov chain X_n.

2. Find the stationary distribution $\pi(j)$, $j = 0, 1, \ldots, r$.

3. Estimate the number of times in a year where Jane gets wet.

4. Estimate the number of times in a year there are 0 umbrellas at home for two consecutive days.

Exercise 6 ***(Computer exercise)*** Given a initial distribution μ and a transition matrix P, write a code which produces the distribution of X_n. Using your code to investigate numerically the distribution of X_n for the following transition probabilities

$$P_1 = \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} & \frac{2}{5} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad P_3 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{5} & 0 & \frac{2}{5} & \frac{2}{5} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

1. Does the distribution of X_n converges to a limit as $n \to \infty$?

2. Does the limit (if it exists) depends on the initial condition?