Math 597/697: Homework 3

You are encouraged to work in groups. But please write your solutions yourself. If you have questions about the homework, please ask them in class, the other students will profit from them too.

1. The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3 someone takes all papers in the pile and put them in the recycling bin. Also if ever there at least five papers in the pile, Mr Smith, with probability one, take the papers to the bin in the afternoon. Consider the number of papers in the pile in the evening and describe it with a Markov chain.

   (a) What are the state space and the transition matrix?

   (b) After a long time what would be the expected number of papers in the pile?

   (c) Assume that the piles starts with 0 papers. What is the expected time until the pile will have again 0 papers.

   (d) Same question as (c) but assume now that the pile starts with 2 papers.

2. Consider an irreducible Markov chain with a finite state space. Let

$$M_{ij} = E[\tau_j | x_0 = i] \tag{1}$$

   denote the expected return time to the state j, starting from i.

   (a) Analyzing the first step show that

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj} \tag{2}$$

   (b) Let π_i be the stationary distribution. Multiplying both sides by π_i and summing over i show that

$$\pi_j = \frac{1}{M_{jj}} . \tag{3}$$

3. *The Ehrenfest urn model*, continued from Problem 3 of HWK #2. This is a Markov chain on $\{-a, -a+1, \ldots, a-1, a\}$ with nonzero transition probabilities given by $P_{ii+1} = (a-i)/2a$ and $P_{ii-1} = (a+i)/2a$.

(a) Let $\mu_n = E[X_n]$. Show that $\mu_n = (\frac{a-1}{a})^n \mu_0$ and so $\lim_{n\to\infty} \mu_n = 0$. Hint: Use conditional expectation to express μ_n in terms of μ_{n-1}.

(b) Show that

$$\pi_j = \binom{2a}{a+j} 2^{-2a} \tag{4}$$

is the stationary distribution for X_j.

4. Consider the simple queuing model introduced in class. During each time period exactly a new customer arrives with probability p and no customer arrives with probability $1-p$. During each time period exactly one customer is served with probability q and zero is served with probability $(1-q)$. The transition probabilites are

$$\begin{aligned} &P_{00} = (1-p)\,, \quad P_{01} = p \\ &P_{ii-1} = (1-p)q \quad P_{ii} = pq + (1-p)(1-q) \quad P_{ii+1} = p(1-q) \end{aligned} \tag{5}$$

Determine for which p and q the Markov chain is positive recurrent.

5. Consider independent trials which result in successs S with probability p and failure F with probability $q = 1-p$. We say that a success run of length r happened at trial n if the outcomes in the preceeding $r+1$ trials, including the the present trial as the last, were F, S, S, ... S. Now let us denote by X_n the length of the trial run at trial n. This is markov chain with state space $\{0, 1, 2, 3, \ldots\}$.

(a) Verify that the transition probabilities are $p_{i,0} = (1-p)$ and $p_{ii+1} = p$

(b) Show that 0 is positive recurrent by computing $E[\tau_0 | x_0 = 0]$.

(c) To compute the stationary distribution, first use (b) to determine π_0 and then use the equation $\pi P = \pi$ to determine $\pi_1, \pi_2, \ldots$.

6. Consider the Markov chain with state space $\{0, 1, 2, 3\}$ and transition probabilities

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .3 & 0 & .7 & 0 \\ 0 & .3 & 0 & .7 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

Compute

(a) The expected number of visits to the state 1, starting from state 2.

(b) The expected number of visits to states 1 and 2 prior to absorption.

(c) The probability of absorption into state 0 starting from state 1.

7. Let X_n and Y_n be two independent Markov chains with state space $\{0,1\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} . \tag{7}$$

Suppose that $X_0 = 1$ and $Y_0 = 0$ and let

$$T = \inf\{n\,;\, X_n = Y_n\} \tag{8}$$

(a) Find $E[T]$.

(b) What is $P\{X_T = 1\}$?

(c) In the long run, what percentage of the time are both chains in the same state?

Hint: Consider the Markov chain $\xi_n = (X_n, Y_n)$.

8. You start to play backgammon with a friend. You have \$5 and your friend \$20. If you win a game your friend gives you \$1 and if you loose you give him \$1. Being a better player, the probability that you win any single game is .6. What is the probability that you wipe out your friend. What if you start with \$10?

9. For a branching process, calculate the probability a that the population eventually dies out starting with one individual when

(a) $p_0 = \frac{1}{4}$, $p_2 = \frac{3}{4}$

(b) $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$

(c) $p_0 = \frac{1}{6}$, $p_1 = \frac{1}{2}$, $p_3 = \frac{1}{3}$

10. Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers. Prove that if $\lim_{n\to\infty} a_n = a$ then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_k = a\,. \tag{9}$$