Homework #3

Prob 1

Define $X_n = S_n \pmod 8$. We have $X_0 = 0$, the state space is $S = \{0,1,2,\dots,7\}$, the transition matrix is

$$P = \begin{pmatrix} 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 \\ 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 \\ \frac16 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 \\ \frac16 & \frac16 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 \\ \frac16 & \frac16 & \frac16 & 0 & 0 & \frac16 & \frac16 & \frac16 \\ \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & \frac16 & \frac16 \\ \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & \frac16 \\ \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 \end{pmatrix}$$

In terms of X_n, T_1 is the same as τ_0 the first return time to 0 and T_2 is τ_1 the first return time to 1

- $E[T_1] = E[\tau_0 \mid X_0 = 0] = \dfrac{1}{\pi(0)}$

  Since the sum of the columns is 1, $\pi(i) = \frac18 \quad i = 0,\dots,7$

  and $E[T_1] = 8$

- $E[T_2] = E[\tau_1 \mid X_0 = 0]$. This can be used by making 1 an absorbing state and computing then the mean time spent in transient states.

Reordering the states as $\{1, 0, 2, 3, 4, \dots, 7\}$ we need to compute

$$M = (I - Q)^{-1}$$

where

$$Q = \begin{array}{c} \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

Do it with your computer.

$$E[T_1 \mid X_0 = 0] = \sum_K M_{0K} \quad \text{(Sum of the first line).}$$

Prob 2

$X_0 = 0, \; Y_0 = 2$

$$T = \inf\{n \,;\, X_n = Y_n\} = \inf\{n,\ Z_n = (0,0),\ (1,1) \text{ or } (2,2)\}$$

The transition matrix for Z_n is easily computed from P

$$\begin{array}{c} (0,0) \\ (1,1) \\ (2,2) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,2) \\ (2,0) \\ (2,1) \end{array} \begin{pmatrix} \frac{1}{4} & \frac{1}{16} & \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{4} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{4} & 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

(a) We think of (0,0) (1,1) (2,2) as absorbing states. To find $E[T]$ we must then compute the mean time spent in transient states starting from (0,2).

$E[T]$ = Sum of the second row of

$$M = (I - Q)^{-1} \text{ where}$$

$$Q = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & \frac{1}{4} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

(b) $P\{X_T = 2\} = P\{ Z_n$ reaches (2,2) before reaching (0,0) and (1,1)$\}$ is computed using the matrix

$$A = MS$$

where M is taken from (a) and

$$S = \begin{pmatrix} \frac{1}{8} & \frac{1}{16} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

$P\{X_T = 2\}$ is the

$$\begin{array}{cc} \cdots \; (2,2) & \\ \begin{pmatrix} & \\ & * \\ & \\ & \\ & \end{pmatrix} & \begin{array}{c} (0,1) \\ (0,2) \\ \vdots \\ \end{array} \end{array}$$

entry of A.

(c) In the long run the percentage of time both chains spends in the same state is

$$\pi((0,0)) + \pi((1,1)) + \pi((2,2))$$

where π is the stationary distribution for Z_n.

Note that if $\pi(i)$ is stationary for X_n then $\pi((i,j)) = \pi(i)\pi(j)$ is stationary for Z_n.
We $\pi = \left(\frac{2}{11}, \frac{4}{11}, \frac{5}{11}\right)$ and so the answer is

$$\frac{4+16+25}{121} = \frac{45}{121}$$

Prob 3

This is the gambler's ruin problem with $p = \frac{3}{5}$ $j = 5$, $N = 25$.

So $P(\text{Wipe out your friend}) = \dfrac{1-\left(\frac{2}{3}\right)^5}{1-\left(\frac{2}{3}\right)^{25}} \cong 1-\left(\frac{2}{3}\right)^5 = .86$

If you start with \$10 then $j = 10$, $N = 30$ and

so $P(\text{Wipe out your friend}) = \dfrac{1-(2/3)^{10}}{1-(2/3)^{30}} \cong 1-\left(\frac{2}{3}\right)^{10} = .98$

Prob 4 (a)

$$M_{ij} = E[\sigma_j \mid X_0 = i] = \sum_{n=1}^{\infty} P\{\sigma_j \geq n \mid X_0 = i\}$$

$$= P\{\sigma_j \geq 1 \mid X_0 = i\} + \sum_{n=2}^{\infty} \sum_{\substack{k \\ k \neq j}} P\{\sigma_j \geq n, X_1 = k \mid X_0 = i\}$$

$$= 1 + \sum_{k, k \neq j} \sum_{n=2}^{\infty} P\{\sigma_j \geq n \mid X_1 = k\} P_{ik}$$

$$= 1 + \sum_{k, k \neq j} \sum_{n=1}^{\infty} P\{\sigma_j \geq n \mid X_0 = k\} P_{ik}$$

$$= 1 + \sum_{k, k \neq j} P_{ik} M_{kj}$$

(b)

$$\sum_i \pi(i) M_{ij} = \sum_i \pi(i) + \sum_{k, k \neq j} \sum_i \pi(i) P_{ik} M_{kj}$$

$$= 1 + \sum_{k, k \neq j} \pi(k) M_{kj}$$

$$\Rightarrow \pi(j) M_{jj} = 1 \quad \text{or} \quad M_{jj} = E[\sigma_j \mid X_0 = j] = \frac{1}{\pi(j)}$$

Prob 5

(a) $\{1, 3\}$ $\{2, 5\}$ $\{4, 6\}$

Recurrent Recurrent transient

(b)

$$\begin{array}{c} 1 \\ 3 \\ 2 \\ 5 \\ 4 \\ 6 \end{array} \begin{pmatrix} 1/3 & 2/3 & & & & \\ 1/4 & 3/4 & & 0 & & 0 \\ & & 1/10 & 9/10 & & \\ & 0 & 3/10 & 7/10 & & 0 \\ \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{1}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

(c)

$$\pi^{(1)} = \left(\frac{\frac{1}{4}}{\frac{1}{4} + \frac{2}{3}}, 0, \frac{\frac{2}{3}}{\frac{1}{4} + \frac{2}{3}}, 0, 0, 0 \right)$$

$$\pi^{(2)} = \left(0, \frac{\frac{3}{10}}{\frac{3}{10} + \frac{9}{10}}, 0, 0, \frac{\frac{9}{10}}{\frac{3}{10} + \frac{9}{10}}, 0 \right)$$

(d) $M = (I - Q)^{-1}$, $Q = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$ $I - Q = \begin{pmatrix} 4/5 & -\frac{1}{10} \\ -\frac{1}{8} & \frac{7}{8} \end{pmatrix}$

$$M = \frac{16}{11} \begin{pmatrix} \frac{7}{8} & \frac{1}{10} \\ \frac{1}{8} & \frac{4}{5} \end{pmatrix}$$

Mean time spent in transient states starting from $4 = \frac{16}{11}\left(\frac{7}{8} + \frac{1}{10}\right)$

" " " " " " " $5 = \frac{16}{11}\left(\frac{1}{8} + \frac{4}{5}\right)$

(e) <u>Absorption probabilities</u>

Change recurrent classes into absorbing states

$$\tilde{P} = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ \frac{3}{10} & \frac{4}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

$$\qquad\qquad S \qquad\quad Q$$

$$A = MS = \begin{pmatrix} \frac{24}{55} & \frac{31}{55} \\ \frac{27}{55} & \frac{28}{55} \end{pmatrix}$$

so $P\{$ absorbed in $\{1,3\}$ starting from $4 \}$ is $\frac{24}{55}$

and so on ....

(f) The non-zero limits are

- $\lim_{n\to\infty} P_{11}^n = \lim_{n\to\infty} P_{31}^n = \dfrac{\frac{1}{4}}{\frac{1}{4}+\frac{2}{3}} = \dfrac{3}{11}$ (use (c))

  $\lim_{n\to\infty} P_{13}^n = \lim_{n\to\infty} P_{33}^n = \dfrac{\frac{2}{3}}{\frac{1}{4}+\frac{2}{3}} = \dfrac{8}{11}$ (use (c))

- $\lim_{n\to\infty} P_{25}^n = \lim_{n\to\infty} P_{55}^n = \dfrac{\frac{9}{10}}{\frac{3}{10}+\frac{9}{10}} = \dfrac{3}{4}$ (use (c))

  $\lim_{n\to\infty} P_{22}^n = \lim_{n\to\infty} P_{52}^n = \dfrac{1}{4}$

- $\lim_{n\to\infty} P_{41}^n = P\{\text{enter the class } \{1,3\}\} \times$ average time spent in 1

  $= \dfrac{24}{55} \times \dfrac{3}{11}$

  $\lim_{n\to\infty} P_{43}^n = \dfrac{24}{55} \times \dfrac{8}{11}$

- Similarly for P_{42}^n, P_{45}^n, P_{51}^n, ...

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Prob 6

Find the probability to reach 0 before reaching 3 starting from 2

Transform 0 and 3 into absorbing states

$$\tilde{P} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .4 & .1 & .3 & .2 \\ .6 & .2 & .1 & .1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 3 \\ 1 \\ 2 \end{array}$$

Compute

$$A = MS = (I-Q)^{-1} S = \begin{pmatrix} .7 & -.2 \\ -.1 & .9 \end{pmatrix}^{-1} \begin{pmatrix} .4 & .1 \\ .6 & .2 \end{pmatrix}$$

$$= \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 3 \end{array} \\ \begin{pmatrix} \frac{48}{61} & \frac{13}{61} \\ \frac{46}{61} & \frac{15}{61} \end{pmatrix} \end{array}$$

The desired probability is $\frac{46}{61}$

Prob 7

$$\begin{pmatrix} 0 & p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \end{pmatrix} = P$$

$P\{\tau_0 = 2 \mid X_0 = 0\} = p_1$

$P\{\tau_0 = 3 \mid X_0 = 0\} = p_2$

$\vdots$

So $E[\tau_0 \mid X_0 = 0] = \sum_{K=2}^{\infty} K p_{K-1} = \sum_{K=1}^{\infty} (K+1) p_K$

$= 1 + \sum_{K=1}^{\infty} K p_K$

$\boxed{\text{Positive recurrent if } \sum_{K=1}^{\infty} K p_K < \infty.}$

Prob 8

$$P = \begin{pmatrix} \frac{1}{3} & 2/3 & & 0 \\ \frac{1}{3} & 0 & 2/3 & \\ \frac{1}{3} & 0 & 0 & 2/3 \\ \vdots & & & \ddots \end{pmatrix}$$

$\pi P = \pi$ gives

$\frac{1}{3}(\pi_0 + \pi_1 + \cdots) = \pi_0$

$\frac{2}{3}\pi_0 = \pi_1$

$\frac{2}{3}\pi_1 = \pi_2$

$\vdots$

Since $(\pi_0 + \pi_1 + \cdots) = 1 \Rightarrow \pi_0 = \frac{1}{3}, \; \pi_1 = \frac{1}{3}\frac{2}{3}, \cdots$

$\pi_n = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1}$

Positive recurrent since $\pi P = \pi$ has a solution

Prob 9 Set $A^{(n)} = \frac{I + P + \cdots + P^{n-1}}{n}$

Then $A^{(n)}P = \frac{P + P^2 + \cdots P^n}{n} = A^{(n)} + \frac{P^n - I}{n}$

Since P^n is a stochastic matrix, all its eigenvalues are less than 1 in absolute values and so

$$\frac{P^n}{n} \to 0 .$$

If $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{k} = \pi(j)$ then

$$\lim_{n\to\infty} A^{(n)} = \Pi = \begin{pmatrix} \pi(1), \pi(2) \cdots \\ \pi(1), \pi(2), \cdots \\ \vdots \quad \vdots \end{pmatrix}$$

So $\Pi P = \lim_{n\to\infty} A^{(n)} P \underset{\text{see above}}{=} \lim_{n\to\infty} A^{(n)} = \Pi$

This means $\pi P = \pi$

Prob 10 If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ then for any $\varepsilon > 0$ there exists N s.t.

$b_n(1-\varepsilon) \le a_n \le b_n(1+\varepsilon)$ if $n \ge N$.

This implies that $(1-\varepsilon)\sum_{n\ge N} b_n \overset{①}{\le} \sum_{n\ge N} a_n \overset{②}{\le} (1+\varepsilon)\sum_{n\ge N} b_n$

① implies that $\sum a_n < \infty \Rightarrow \sum b_n < \infty$

② " " $\sum b_n < \infty \Rightarrow \sum a_n < \infty$.