Evolutionary game theory, Problem set 3

1. Consider the transformation

$$q_i = \frac{p_i c_i}{\sum_j c_j p_j}$$

with $c_j > 0$. Show that if p_i satisfies the replicator equation with payoff matrix $\boldsymbol{\pi}$ then q_i satisfies a replicator equation with a new payoff matrix $\tilde{\pi}(i,j) = \pi(i,j)c_j^{-1}$. Show that if there is an interior NE in intΔ then one can always move it to $(1/n, \cdots, 1/n)$ by such a transformation.

2. Consider the generalized Rock-Paper-Scissor with payoff matrix

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{pmatrix}$$

and $a_i > 0$ and $b_i > 0$.

(a) Show that the only stationary solutions of the replicator equation are the extremal points of the simplex and the interior point

$$\mathbf{p}^* = \alpha(b_2b_3 + b_2a_3 + a_2a_3, b_3b_1 + b_3a_1 + a_3a_1, b_1b_2 + b_1a_2 + a_1a_2)$$

where $\alpha > 0$ is such that $\mathbf{p}^*$ is normalized. Moreover show that

$$\langle \mathbf{p}^*, \boldsymbol{\pi}\mathbf{p}^* \rangle = \alpha \det(\boldsymbol{\pi})$$

(b) Consider the special case where

$$b_1 - a_2 = b_2 - a_3 = b_3 - a_1 = C\,.$$

Show that $\mathbf{p}^*$ is asymptotically stable if $C > 0$, stable if $C = 0$ and unstable if $C < 0$. *Hint: Consider* $V(p) = H(\mathbf{p}^*|\mathbf{p})$.

(c) Using Exercise 1 show that you can find c_1, c_2, and c_3 such that you can transform a general RPS replicator equation into RPS replicator equation for the special case discussed in (b). *Hint: The equation that* $m_i = \frac{1}{c_i}$ *is the equation that an interior NE must statisfy for the payoff matrix*

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & b_2 & -a_3 \\ -a_1 & 0 & b_3 \\ b_1 & -a_2 & 0 \end{pmatrix}$$

and so you can use (a) or a slight variation thereof.

(d) Conclude that the Nash equilibrium $\mathbf{p}^*$ for the generalized RPS is asymptotically stable if and only if $det(A) > 0$.

3. Consider the replicator equation for the symmetric game with payoff matrix

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

Show that there exists a NE $\mathbf{p}^*$ in intδ and that it is stable. Show that $\mathbf{p}^*$ is a not an ESS. *Hint: (a) Study the stability of the interior NE by linearization. (b) Look for all NE and ESS.* We have seen in class in every ESS state is asymptotically stable and that if the ESS is in the interior of the simplex then every orbit starting in the interior converge to the ESS. This example shows that the converse statement is wrong.

4. Show that generically a assymetric game with 2 strategies for each player is Nash equivalent to either a potential game or a zero-sum game. This is equivalent to show that there exists $b(j)$, $c(i)$, a (either postive of negative), and $\pi(i,j)$ such that

$$\pi_\alpha(i,j) = \pi(i,j) + b(j)\,, \quad \pi_\beta(i,j) = a\pi(i,j) + d(i)\,.$$

$a > 0$ corresponds to a potential game while $a < 0$ corresponds to a zero-sum game after rescaling.

5. (Convergence of time-averages under the replicator dynamics). Suppose that the solution of the replicator equation

$$\frac{dp_i}{dt} = p_i\left[\boldsymbol{\pi}(e_i,p) - \boldsymbol{\pi}(p,p)\right]\,.$$

stays bounded uniformly away from the boundary of the simplex for all time, i.e., there exists $\delta > 0$ such that $p_i(t) \geq \delta$ for all $t \geq 0$ and all i.

Let us consider the time average

$$\overline{p}_i(t) = \frac{1}{t}\int_0^t p_i(s)ds\,.$$

We want to show that there exists then NE in intΔ and that $\overline{\mathbf{p}}(t)$ converges to the set of interior Nash equilibria as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \min_{\mathbf{p}^*,NE} \|\mathbf{p}(t) - \mathbf{p}^*\| = 0\,. \tag{1}$$

In order to do this prove

(a) $y_i(t) = \log p_i$ satisfies

$$\frac{1}{t}(y_i(t) - y(0)) = \int_0^t \left[\pi(e_i, p(s)) - \pi(\mathbf{p}(s),\mathbf{p}(s))\right] ds\,.$$

(b) We can assume that $\overline{p}_i(t)$ has a convergent subsequence $\overline{p}_i(t_l)$ (why?) and thus a limit point $\overline{p}^*$. Use (a) and the hypothesis to show that $\boldsymbol{\pi}(e_i,\mathbf{p}^*)$ is independent of i and thus $\mathbf{p}^*$ is a NE.

(c) Prove equation (1).

6. Consider a symmetric zero-sum game and suppose that p^* is a Nash equilibrium in the interior of the simplex intΔ. Show that if n is even then p^* is contained in line segment consisting entirely of Nash equilibria. (This shows that isolated interior Nash equilibria for zero-sum game can occur only for odd number of strategies.) *Hint: Suppose λ is an eigenvalue for the payoff matrix $\boldsymbol{\pi}$. Use the fact that the payoff matrix is anti-sefadjoint $\boldsymbol{\pi}^T = -\boldsymbol{\pi}$ and that $\boldsymbol{\pi}$ has real elements to show that that λ is purely imaginary, that the complex conjugates $\bar{\lambda}$ is also an eigenvalue, and that the number of nonzero eigenvalues is even.*

7. Consider a 2 player potential game with payoff $\boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\beta = \boldsymbol{\pi}$ and potential function $V(\mathbf{p}_\alpha, \mathbf{p}_\beta) = \boldsymbol{\pi}(\mathbf{p}_\alpha, \mathbf{p}_\beta) = \langle \mathbf{p}_\alpha , \boldsymbol{\pi}\mathbf{p}_\beta \rangle$. Show that a strict maximum of V corresponds to a pure strategy and deduce that every potential game has a pure Nash equilibrium. Use this result to show that the replicator for a potential two-population game has an asymptotically stable equilibrium it must be pure.