Math 624: Problem set 5

1. Let μ be a signed (or complex measure) on $(X, \mathcal{M})$. Show that for any $E \in \mathcal{M}$

$$
\begin{aligned}
&|\nu|(E)| \\
&= \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| \,:\, n \geq 1, E_1, \cdots E_n \text{ disjoint}\,, E = \bigcup_{j=1}^{n} E_j \right\} \qquad (1) \\
&= \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| \,:\, E_1, E_2, \cdots \text{ disjoint}\,, E = \bigcup_{j=1}^{\infty} E_j \right\} \qquad (2) \\
&= \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\} . \qquad (3)
\end{aligned}
$$

   Hint: Prove that (1) $\leq$ (2) $\leq$ (3) $\leq$ (1).

2. Let F be of bounded variation on $[a, b]$ and let $G(x) = |\mu_F|([a, x])$. Show that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$. To do this prove

   (a) From the definition of T_F, we have $T_F \leq G$.

   (b) $|\mu_F(E)| \leq \mu_{T_F(E)}$ for any Borel set E. (Consider first an interval.)

   (c) Using the previous problem we have $|\mu_F| \leq \mu_{T_F}$ and hence $G \leq T_F$.

3. Consider a linear map $A : \mathbf{C}^d \to \mathbf{C}^d$, i.e., a $n \times n$ matrix $A = (a_{ij})_{i,j=1}^d$. Define

$$
\|A\|_p \;=\; \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \,,
$$

   i.e., $\|A\|_p$ is the operator norm of A as an operator on the Banach $\mathbf{C}^n$ equipped with $\| \cdot \|_p$. Show that

$$
\|A\|_1 = \max_j \sum_i |a_{ij}| \quad \|A\|_\infty = \max_i \sum_j |a_{ij}| \,.
$$

   and that $\|A\|_2$ is the square root of the largest eigenvalue of A^*A.

4. Let k be an integer and let $C^k([0, 1])$ denote the space of functions which have continuous derivatives up to order k, including one-sided derivatives at the endpoints.

   (a) Show that if $f \in C[0, 1])$, then $f \in C^k([0, 1])$ if and only if f is k-times continuously differentiable on $(0, 1)$ and $\lim_{x \to 0+} f^{(j)}(x)$ $\lim_{x \to 1-} f^{(j)}(x)$ exists for $j \leq k$. (Use the mean-value Theorem).

(b) Show that if $\{f_n\}$ is a sequence in $C^1([0,1])$ such that $f_n \to f$ uniformly and $f_n' \to g$ uniformly then $f \in C^1([0,1])$ and $f' = g$. (Show that $f(x) - f(y) = \int_y^x g(s)ds$.)

(c) Show that $\|f\|_{C^k} \equiv \sum_{j=0}^k \|f^{(j)}\|_\infty$ is a norm on $C^k([0,1])$ and that, with this norm $C^k([0,1])$ is a Banach space. (Use (b) and induction on k).

5. Let $0 < \alpha \leq 1$ and let $H_\alpha([0,1])$ denote the space of all functions satisfying a Lipschitz condition with exponent α, i.e., if $f \in H_\alpha([0,1])$ there exists a constant M such that $|f(x) - f(y)| \leq M|x-y|^\alpha$ for all $x, y \in [0,1]$. Define

$$\|f\|_{H_\alpha} \equiv \sup_{x\in[0,1]} |f(x)| + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Show that H_α with $\|\cdot\|_{H_\alpha}$ is a Banach space.

6. Let X be a Banach space and let $\mathcal{L}(X)$ be the set of all bounded operator from X to X.

(a) Let $T \in \mathcal{L}(X)$ and suppose that $\|I - T\| < 1$ (here I is the identity operator, i.e. $Ix = x$). Show that T is invertible with inverse $T^{-1} = \sum_{n=0}^\infty (I-T)^n$.

(b) Let $T \in \mathcal{L}(X)$ be invertible and suppose that $S \in \mathcal{L}(X)$ satisfies $\|S - T\| \leq \|T^{-1}\|^{-1}$, then S is invertible.

7. **Integral operators I**. Let $C([0,1])$ be the the Banach space of continuous functions on $[0,1]$ equipped with the uniform norm $\|f\|_u = \sup_{t\in[0,1]} |f(t)|$.

Let $k \in C([0,1]\times[0,1])$ be a continuous function. Define $A : C([0,1]) \to C([0,1])$ by

$$Af(t) = \int_0^1 k(t,s)f(s)\,ds.$$

Show that A is a bounded operator and $\|A\| = \max_{t\in[0,1]} \int_0^1 |k(t,s)|\,ds$. Compute the operator A^*.

8. **Integral Operators II** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $K(\cdot,\cdot)$ be a measurable function on $X \times X$ equipped with the product σ-algebra. Suppose that there exists a constant C such that

$$\int |K(x,y)|d\mu(x) \leq C \text{ for a.e } y, \quad \int |K(x,y)|d\mu(y) \leq C \text{ for a.e } x.$$

Let $1 \leq p \leq \infty$ and $f \in L^p(\mu)$. Show

(a) The integral $Tf(x) = \int K(x,y)f(y)d\mu(y)$ converges absolutely for a.e. x, i.e. $\int |K(x,y)f(y)|d\mu(y) < \infty$ for a.e. x.

(b) The function Tf defined in (a) is in $L^p(\mu)$ and $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p(\mu)$.

(c) Suppose $1 \leq p < \infty$. Compute the adjoint operator T^*.

9. Suppose X is a Banach space and X^* its dual space. Show that if X^* is separable then X is separable.
Hint: Let $\{l_n\}$ be a countable dense subset in X^*. Pick $x_n \in X$ such that $\|x_n\| = 1$ and $|l_n(x_n)| \geq \frac{1}{2}\|l_n\|$. Show that the linear combinations of x_n spans X.

10. **Weak convergence.** A sequence $\{x_n\}$ is said to *converge weakly* if

$$\lim_{n\to\infty} l(x_n) = l(x)$$

for any $l \in X^*$.

(a) Show that convergence implies weak convergence.

(b) Suppose $X = H$ is a Hilbert space and $\{x_n\}$ is an orthonormal basis of H. Show that x_n converges weakly to 0 but it does not converge strongly, in fact x_n has no convergent subsequence. *Hint: Bessel inequality*

(c) Suppose $X = l^1$, then if $\{x_n\}$ converges weakly in l^1 then it converges in l^1. *Hint:* What is the dual of l^1?

(d) Consider $X = L^2([0,2\pi])$ and $f_n = \cos(nx)$. Show that that f_n converges weakly to 0 but that f_n does not converge to 0 or in measure.

(e) Let $f_n = n\chi_{(0,\frac{1}{n})}$. Show that f_n converges to 0 in measure and a.e. but that f_n does not converge to 0 weakly in L^p for any p.