Math 645: Homework 7

1. Consider the system given, in polar coordinates, by

$$\begin{aligned} r' &= ar + r^3 - r^5 \,, \\ \theta' &= 1 \,. \end{aligned} \tag{1}$$

   Determine the phase plane for representative values of a and describe the bifurcations of the systems.

2. Consider the system

$$\begin{aligned} x' &= x - rx - ry + xy \,, \\ y' &= y + rx - ry - x^2 \,, \end{aligned} \tag{2}$$

   where $r = \sqrt{x^2 + y^2}$. Show that this system can be written in polar coordinates as

$$\begin{aligned} r' &= r(1 - r) \,, \\ \theta' &= r(1 - \cos\theta) \,, \end{aligned} \tag{3}$$

   Show that there are two critical points $(0, 0)$ (unstable source) and $(1, 0)$ (saddle node). Show that every solution $x(t)$ which does not pass through the origin satisfy $\lim_{t\to\infty} x(t) = (1, 0)$, but that $(1, 0)$ is not stable.

3. Consider the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \,, \\ y' &= x + y(1 - x^2 - y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \,. \end{aligned} \tag{4}$$

   Show that this system has infinitely many periodic orbits (limit cycles). Determine which ones are stable.

4. For the system $x' = f(y)$, $y' = g(x) + y^k$, give a sufficent condition for the system to have no periodic orbit.

5. Sketch the phase portrait of a system in the plane having

   (a) An orbit γ with $\alpha(\gamma) = \omega(\gamma) = \{x_0\}$ but $\gamma \neq \{x_0\}$.

   (b) An orbit γ where $\omega(\gamma)$ consists of one limit orbit.

   (c) An orbit γ where $\omega(\gamma)$ consists of one limit orbit and one critical point.

   (d) An orbit γ where $\omega(\gamma)$ consists of two limit orbits and one critical point.

   (e) An orbit γ where $\omega(\gamma)$ consists of two limit orbits and two critical points.

(f) An orbit γ where $\omega(\gamma)$ consists of four limit orbits and four critical points.

6. Consider the equation

$$x'' + (x^2 + x'^2 - 1)x' + x = 0\,. \tag{5}$$

Show that this system has a unique periodic orbit which is a stable limit cycle for every trajectory, except the one starting at the origin.

7. The system given in cylindrical coordinates by

$$\begin{aligned} r' &= r(1-r)\,, \\ \theta' &= 1\,, \\ z' &= -z\,. \end{aligned} \tag{6}$$

has exactly one periodic orbit. Determine this periodic orbit and compute the Poincaré map for the half-plane $y = 0, x > 0$ perpendicular to the periodic orbit. Show that this orbit is asymptotically stable.