Math 597/697: Spring 2006 Midterm Exam

This is a take-home exam. You can use your classnotes and the text book. Don't talk to each other, talk to me. Write your answers clearly and in such a way that I can understand your reasoning.

1. Ms Smith goes for a run every morning. When she leaves her house for her run she is equally likely to go out either the front or the back door; and similarly when she comes back she is equally likely to enter either through the front or the back door. Ms Smith owns k pairs of running shoes which she takes off after the run at whichever door she happens to be. We want to determine the proportion of time she has to run barefooted.

   (a) Set this up as a Markov chain and determine the transition probabilities.

   (b) Determine the proportion of days she runs barefooted.

2. Let X_n be a positive recurrent Markov chain on the state space S with stationary distribution π. Consider the stochastic process $Y_n = (X_n, X_{n+1})$ with state space $S \times S$.

   (a) Show that Y_n is a Markov chain.

   (b) Determine the stationary distribution of Y_n.

   (c) Consider the Markov chain with state space $1, 2, 3$ and transition probability

   $$P = \begin{pmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

   Compute the long run proportion of steps for which $X_{n+1} \geq X_n$.

3. **Partially observed Markov chains.** Let X_n be a recurrent Markov chain with a finite state space $S = \{1, \cdots, N\}$ and transition matrix P. Let T be a subset of states, $T \subset S$, $T \neq S$. Let ν_j, $j \geq 0$, denote the successive times at which the Markov chain visits one of the states in T, i.e.

$$\begin{aligned} \nu_0 &= \inf \{n \geq 0 \,:\, X_n \in T\} \,, \\ \nu_1 &= \inf \{n > \nu_0 \,:\, X_n \in T\} \,, \\ &\vdots \end{aligned}$$

We define a new stochastic process Y_j with state space T which is given by

$$Y_j \,=\, X_{\nu_j} \,.$$

You can think of this process as follows: you can only observe X_n if X_n is in one of the states in T. Moreover you don't have a watch and thus have no way to keep track of the time elapsed between successive visits to T.

(a) Argue that Y_j is a Markov process.

(b) Reordering the state if necessary we can assume that the transition matrix has the form as

$$P \,=\, \begin{array}{c} T \\ T^c \end{array} \begin{pmatrix} R & U \\ S & Q \end{pmatrix}$$

Let $D = (d_{ij})$ be the transition matrix for the Markov chain Y_j, i.e. $d_{ij} = P\{Y_1 = j \,|\, Y_0 = i\}$ for $i, j \in T$. Compute the matrix D in terms of the matrix R, U, S, Q.

(c) Suppose that the Markov chain X_n has a stationary distribution $\pi = (\pi(1), \cdots, \pi(N))$. What is the stationary distribution for the Markov chain Y_n.

(d) Compute

$$\lim_{k \to \infty} \frac{\nu_k}{k} \,.$$

4. A mouse is performing a symmetric random walk on the positive integer $\{0, 1, 2, 3, \cdots\}$: if it is in state i it is equally likely to move to state $i-1$ or $i+1$. The state 0 is the mouse's home filled with lots of tasty cheese. If the mouse ever reaches its home it will stay there forever. On the other hand there is a bad cat who tries to catch the mouse and each time the mouse moves there is a probability $1/5$ that the cat will kill the mouse.

   (a) To describe this process as a Markov chain consider an extra state $*$ which corresponds to the mouse being dead. The state space is $S = \{*, 0, 1, 2, \cdots\}$ and X_n denotes the position of the mouse at time n. Compute the corresponding transition matrix.

   (b) Compute the probability that the mouse reaches safety if it starts in state i, i.e.,

   $$P_i \;=\; P\{X_n = 0 \text{ for some n} \mid X_0 = i\}\,, i = 0, 1, 2, \cdots$$

   Hint: At some point it might be useful to consider the quantity $Q_i \equiv P_{i+1} - P_i$