Math 645: Homework 1

1. Derive the following *error estimate* for the method of successive approximations. Let x be a fixed point given by this method. Show that
$$\|x - x_k\| \leq \frac{\alpha}{1-\alpha}\|x_k - x_{k-1}\| . \tag{1}$$

2. Consider the function $f(x) = e^x/4$ on the interval $[0,1]$. Show that f has a fixed point on $[0,1]$. Do some iterations and estimate the error rigorously.

3. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by
$$f(x) \;=\; \begin{cases} x + e^{-x/2} & \text{if } x \geq 0 \\ e^{x/2} & \text{if } x \leq 0 \end{cases} . \tag{2}$$
   (a) Show that $|f(x) - f(y)| < |x - y|$ for $x \neq y$.
   (b) Show that f does not have a fixed point.

   Explain why this does not contradict the Banach fixed point theorem.

4. Show that the assumption that "D is closed" cannot be omitted in general in the fixed point theorem. Find a set D which is not closed and a map $f : D \to E$ such that $f(D) \subset D$, f is a contraction, but f does not have a fixed point in D.

5. Show that $\|f\|_2$ is a norm on $\mathcal{C}([0,1]$.

6. (a) Consider the norm of $\mathcal{C}([0,a])$ given by
$$\|f\|_{\mathrm{e}} \;=\; \max_{0\leq t\leq a} |f(t)|e^{-t^2} . \tag{3}$$
   (Why is it a norm?) Let
$$Tf(t) \;=\; \int_0^t sf(s)\,ds . \tag{4}$$
   Show that $\|Tf\|_\infty \leq \frac{a^2}{2}\|f\|_\infty$ and $\|Tf\|_{\mathrm{e}} \leq \frac{1}{2}\|f\|_{\mathrm{e}}$.

   (b) Show that the integral equation
$$x(t) \;=\; \frac{1}{2}t^2 + \int_0^t sx(s)\,ds\,, \quad t \in [0,a]\,, \tag{5}$$
   has exactly one solution. Determine the solution (i) by rewriting the equation as an initial value problem and solving it, (ii) by using the methods of successive approximations starting with $x_0 \equiv 0$.

7. Let us consider $\mathbf{R}^2$ with the norm $\|x\| = \max\{|x_1|, |x_2|\}$. Let $f\,;\,\mathbf{R}^2 \to \mathbf{R}^2$ be given by
$$f(x_1, x_2) \;=\; \begin{pmatrix} x_1^2 + 2x_2^2 + 5 \\ 4x_1x_2 + 3 \end{pmatrix} \tag{6}$$
   Let $K = \{(x_1, x_2)\,,\, |x_1| < 1, |x_2| \leq 2\}$. Find a Lipschitz constant for f.

8. Apply the Picard-Lindelöf iteration to

$$x' = x^2 , \quad x(0) = 1 . \tag{7}$$

Compute the first three iterations $x_1(t)$, $x_2(t)$, $x_3(t)$ and show, by induction, that $x_n(t) = 1 + t + \cdots t^n + O(t^{n+1})$.

9. Apply the Picard-Lindelöf iteration to the Cauchy problem

$$x_1' = x_1 + 2x_2 , \qquad x_1(0) = 0 \tag{8}$$

$$x_2' = t^2 + x_1 , \qquad x_2(0) = 0 \tag{9}$$

Compute the first five terms in the taylor series of the solution.

10. Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfy a *global Lipschitz condition*, i.e., there exists a positive $L > 0$ such that

$$\|f(x) - f(y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbf{R}^n . \tag{10}$$

Consider the Banach space $E = \{g : [t_0, \infty) \to \mathbf{R}^n , \, g(t) \text{ continuous } \}$ with the norm

$$\|g\|_\kappa = \sup_{t_0 \le t < \infty} \|g(t)\| e^{-\kappa t} . \tag{11}$$

Using the Banach fixed point theorem, show that the Cauchy problem $x' = f(x)$, $x_{t_0} = x_0$ has a unqiue solution. *Hint*: Choose κ such as to obtain a contraction.

11. (a) Let $f : U \to \mathbf{R}^n$ where $U \subset \mathbf{R}^n$ is an open set and suppose that f satisfies a Lipschitz condition on U. Show that f is uniformly continuous on U.

    (b) Show that $f(x) = 1/x$ does not satisfy a Lipschitz condition on $(0, 1)$. *Hint:* Is f uniformly continuous on $(0, 1)$?.

    (c) Does the Cauchy problem $x' = 1/x$, $x(0) = x_0 > 0$ have a unique solution? Solve it and determine the maximal interval of existence. What is the behavior of the solution at the boundary of this interval.

12. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be of class $\mathcal{C}^1$ and satisfy $f(0, 0) = 0$. Consider the ODE

$$x'' = f(x, x') . \tag{12}$$

Show that every non-zero solution of this equation has simple zeros. Examples: the harmonic oscillator $x'' + x =$ or the mathematical pendulum $x'' + \sin(x) = 0$.