Math 623: Problem set 2

1. Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of measurable subsets of $\mathbf{R}^n$. We define

$$\begin{aligned}
\limsup_{n\to\infty} E_n &= \left\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for infinitely many } n\right\} \\
\liminf_{n\to\infty} E_n &= \left\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for all but finitely many } n\right\}.
\end{aligned}$$

   (a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\,, \quad \text{and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k\,.$$

   (b) Show that

$$\begin{aligned}
m\left(\liminf_{n\to\infty} E_n\right) &\leq \liminf_{n\to\infty} m\left(E_n\right) \\
\limsup_{n\to\infty} m\left(E_n\right) &\leq m\left(\limsup_{n\to\infty} E_n\right) \quad \text{provided } m\Big(\bigcup_{n=1}^{\infty} E_j\Big) < \infty\,.
\end{aligned}$$

   (c) Exercise 16, p.42. (Borel-Cantelli Lemma).

2. Suppose that A is a measurable set in $\mathbf{R}^d$ with $m(A) > 0$. Show that for any $q < m(A)$ there exist a measurable set $B \subset A$ with $m(B) = q$. *Hint:* Prove it first for the case that $m(A) = p < \infty$. Use then the intermediate value theorem for $A \cap B_R(0)$.

3. Exercise 28, p.43

4. Exercise 32, p.43

5. Exercise 33, p.43

6. Show that if $f : \mathbf{R}^d \to \mathbf{R}$ is measurable, then $|f|$ is measurable. Show that the converse is not always true.

7. Suppose $f : \mathbf{R}^d \to \mathbf{R}$ is finite-valued. Show that f is measurable if and only if $f^{-1}(A)$ is measurable for every Borel set A.

8. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable. Show that f and f' are measurable functions.

9. (a) Suppose $f : \mathbf{R} \to \mathbf{R}$ is a monotone function. Show that $f^{-1}(A)$ is a Borel set for every Borel set A. In particular f is measurable.

(b) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a one to one continuous function. Show that f maps Borel sets onto Borel sets.

10. (a) Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ and a measurable set A such that $f(A)$ is not measurable.

    (b) Give an example of a function $g : \mathbf{R} \to \mathbf{R}$ and a measurable set A such that $g^{-1}(A)$ is not measurable.

    (c) Give an example of a measurable set such which is not a Borel set.

    (d) Give an example of a continuous function g and a measurable function h such that $h \circ g$ is not measurable.

Hint: Let $F : [0,1] \to [0,1]$ be the Cantor Lebesgue function constructed in Exercise 2, chapter 1, and extend it to $\mathbf{R}$ by setting $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$. Finally consider the monotone function

$$f(x) = x + F(x) .$$

Use problem 9(b) to show that if C is the middle third cantor set then $m(f(C)) = 1$ and so f maps a set of measure 0 onto a set of positive measure.

Using this fact, problem 4 (Exercise 32 (b)), and problem 9 again, you can deduce (a), (b), (c), (d).