Math 697: MIDTERM

Problem 1 (**General random walk on** $\{0, \cdots, N\}$) Let X_n be a Markov chain on the state space $\{0, \cdots, N\}$ with a transition probabilities

$$
\begin{aligned}
p(0,0) &= q_0\,, \;\; p(0,1) = p_0 \\
p(j,j-1) &= q_j\,, \;\; p(j,j) = r_j\,, \;\; p(j,j+1) = p_j\,, \quad j = 1, \cdots, N-1 \\
p(N,N-1) &= q_N\,, \;\; p(N,N) = p_N\,,
\end{aligned}
\tag{1}
$$

with $p_0 + q_0 = p_N + q_N = 1$ and $p_j + r_j + q_j = 1$ for $j = 1, \cdots, N-1$ and we assume that $p_j > 0$ and $q_j > 0$ for all j.

1. Show that the Markov chain X_n satisfies detailed balance, i.e., show that there exists positive number $\nu(0), \cdots, \nu(N)$ such that

   $$\nu(i)p(i,j) = \nu(j)p(j,i)\,.$$

   Use this to give a formula for the stationary distribution for x_n in terms of the p_j's, q_j's and r_j's.

2. Consider the following Markov chain. An urn contains N balls which are either white or black. At each step one picks a ball in the urn at random and it is replaced with probability p by a white ball and with probability $1-p$ by a black ball. Let X_n denotes the number of white balls after n steps. Compute the transition probabilities and the stationary distribution.

Problem 2 Let X_n be a positive recurrent Markov chain on the state space S with stationary distribution π. Consider the stochastic process $Y_n = (X_n, X_{n+1})$ with state space $S \times S$.

1. Show that Y_n is a Markov chain.

2. Compute the transition probabilities and the stationary distribution of Y_n.

3. Consider the Markov chain with state space $1, 2, 3$ and transition probability

   $$P = \begin{pmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

   Compute the long run proportion of steps for which $X_{n+1} \geq X_n$.

Problem 3 (Partially observed Markov chains) Let X_n be an irreducible Markov chain with a finite state space S and transition matrix $P = (p(i,j))$. Let T be a subset of states, $T \subset S$, $T \neq S$. Let ν_j, $j \geq 0$, denote the successive times at which the Markov chain visits one of the states in T, i.e.

$$
\begin{aligned}
\nu_0 &= \inf\{n \geq 0 \,:\, X_n \in T\}\,, \\
\nu_1 &= \inf\{n > \nu_0 \,:\, X_n \in T\}\,, \\
&\vdots
\end{aligned}
$$

Define a new stochastic process Y_j with state space T which is given by

$$Y_j = X_{\nu_j} .$$

You can think of this process as follows: you can only observe X_n only if X_n is in one of the states in T. Moreover you don't have a watch and thus have no way to keep track of the time elapsed between successive visits to T.

1. Show Y_j is a Markov process.

2. Reordering the state if necessary we can assume that the transition matrix has the form as

$$P = \begin{array}{c} T \\ T^c \end{array} \begin{pmatrix} R & U \\ S & Q \end{pmatrix}$$

   Let $D = (d_{ij})$ be the transition matrix for the Markov chain Y_j, i.e. $d_{ij} = P\{Y_1 = j \,|\, Y_0 = i\}$ for $i, j \in T$. Compute the matrix D in terms of the matrix R, U, S, Q.

3. Suppose that the Markov chain X_n has a stationary distribution $\pi = (\pi(1), \cdots, \pi(N))$. What is the stationary distribution for the Markov chain Y_n.

4. Compute

$$\lim_{k \to \infty} \frac{\nu_k}{k} .$$