Chapter 1

Existence and Uniqueness

1.1 Introduction

An ordinary differential equation (ODE) is given by a relation of the form

$$F(t, x, x', x'', \cdots, x^{(m)}) = 0\,, \tag{1.1}$$

where $t \in \mathbf{R}$, $x, x', \cdots, x^{(m)} \in \mathbf{R}^n$. The function F is defined in an open set of $\mathbf{R} \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$. We say that the ODE is of order m if the maximal order of the derivative occurring in (1.1) is m. A function $x : I \to \mathbf{R}^n$, where I is an interval in $\mathbf{R}$ is a solution of (1.1) if $x(t)$ is of class $\mathcal{C}^m$ (i.e. m-times continuously differentiable) and if

$$F(t, x(t), x'(t), x''(t), \cdots, x^{(m-1)}(t)) = 0 \quad \text{for all } t \in I\,. \tag{1.2}$$

Example 1.1.1 Clairaut equation (1734) Let us consider the first order equation

$$x - tx' + f(x') = 0\,, \tag{1.3}$$

where f is some given function. It is given in implicit form by a nonlinear equation in x'. It is easy to verify that the lines $x(t) = Ct - f(C)$ are solutions of (1.3) for any C. For example consider $f(z)z^2 + z$. One sees easily that given a point (t_0, x_0) there exists 0 or 2 solutions passing by the point (t_0, x_0).

As we see from this example, it is in general very difficult to obtain results on the uniqueness or existence of solutions for general equations of the form (1.1). We will restrict ourselves to situations where (1.1) can be solved as a function of $x^{(m)}$,

$$x^{(m)} = g(t, x, x', \cdots, x^{(m-1)})\,, \tag{1.4}$$

Such an equation is called *explicit*. If we introduce the new variables

$$x_1 = x\,, x_2 = x'\,, x_3 = x'' \cdots, x_m = x^{(m-1)}\,, \tag{1.5}$$

then we can rewrite (1.4) as the system

$$\begin{aligned} x_1' &= x_2\,, \\ x_2' &= x_3\,, \\ &\vdots \\ x_{m-1}' &= x_m\,, \\ x_m' &= g(t, x_1, x_2, \cdots, x_m)\,. \end{aligned} \tag{1.6}$$

This is an equation of order 1 for the supervector $x = (x_1, \cdots, x_m) \in \mathbf{R}^{nm}$ (each x_i is in $\mathbf{R}^n$) and it has the form $x' = f(t, x)$. Therefore, in general, it is sufficient to consider the case of first order equations (m=1).

Example 1.1.2 Predator-Prey equation Let us consider the equation

$$x' = x(\alpha - \beta y)\,, \quad y' = y(\gamma x - \delta)\,, \tag{1.7}$$

where $\alpha, \beta, \gamma, \delta$ are given positive constants. Here $x(t)$ is the population of the preys and $y(t)$ is the population of the predators. If the population of predators y is below the threshold α/β then x is increasing while if y is above α/β then x is decreasing. The opposite holds for the population z. In order to study the solutions, let us divide the first equation, by the second one and consider x as a function of y. We obtain

$$\frac{dx}{dy} = \frac{x}{y}\frac{(\alpha - \beta y)}{(\gamma x - \delta)} \quad \text{or} \quad \frac{(\gamma x - \delta)}{x} dx = \frac{(\alpha - \beta y)}{y} dy\,. \tag{1.8}$$

Integrating gives

$$\gamma x - \delta \log x = \alpha \log y - \beta y + \text{Const.} \tag{1.9}$$

One can verify that the level curves (1.9) are closed bounded curves and each solution $(x(t), y(t))$ stays on a level curve of (1.9) for any $t \in \mathbf{R}$. This suggests that the solutions are periodic.

Example 1.1.3 van der Pol equation. The van der Pol equation

$$x'' = \epsilon(1 - x^2)x' - x\,. \tag{1.10}$$

It can written as a first order system by setting $y = x'$

$$\begin{aligned} x' &= y\,, \\ y' &= \epsilon(1 - x^2)y - x\,. \end{aligned} \tag{1.11}$$

It is a perturbation of the harmonic oscillator ($\epsilon = 0$) $x'' + x = 0$ whose solutions are the periodic solution $x(t) = A\cos(t - \phi)$ and $y(t) = x'(t) = -A\sin(t - \phi)$ (circles). When $\epsilon > 0$ one observes that one periodic solution survives which is the deformation of a circle of radius 2 and all other solution are attracted to this periodic solution (limit cycle).

We will discuss these examples in more details later. For now we observe that, in both cases, the solutions curves in the (t, x)-plane never intersect. This means that there is never two solutions passing by the same point. Our first goal will be to find sufficient conditions for the problem

$$x' = f(t, x)\,, \quad x(t_0) = x_0\,, \tag{1.12}$$

to have a unique solution. We say that t_0 and x_0 are the *initial values* and the problem 1.12 is called a Cauchy Problem or an initial value problem (IVP).

1.2 Banach fixed point theorem

We will need some (simple) tools of functional analysis. Let E be a vector space with addition $+$ and multiplication by scalar λ in $\mathbf{R}$ of $\mathbf{C}$. A norm on E is a map $\|\cdot\| : E \to \mathbf{R}$ which satisfies the following three properties

- **N1** $\|x\| \geq 0 \quad \text{and} \|x\| = 0 \;$ if and only if $x = 0\,,$
- **N2** $\|\lambda x\| \;=\; |\lambda| \|x\|\,,$
- **N3** $\|x + y\| \;\leq\; \|x\| + \|y\| \quad$ (triangle inequality) .

If we have a norm we can define convergence of sequence in E.

We say that the sequence $\{x_n\}$ *converges* to $x \in E$, if for any $\epsilon > 0$, there exists $N \geq 1$ such that, for all $n \geq N$, we have $\|x_n - x\| \leq \epsilon$.

We say that $\{x_n\}$ is a *Cauchy sequence* if for any $\epsilon > 0$, there exists $N \geq 1$ such that, for all $n, m \geq N$, we have $\|x_n - x_m\| \leq \epsilon$.

Definition 1.2.1 A vector space equipped with a norm $\|\cdot\|$ is called a *normed vector space*. It is *complete* if every Cauchy sequence in E converges to an element of E. A complete normed vector space E is called a *Banach space*.

An example of Banach space is $\mathbf{R}^n$ with the euclidean norm $\|x\|_2 = (\sum_i x_i^2)^{1/2}$. One can also use the norm $\|x\|_1 = \sum_i |x_i|$ or $x_\infty = \sup_i |x_i|$. In both case $\mathbf{R}^n$ is a Banach space, since all norm are equivalent in a finite-dimensional space.

Proposition 1.2.2 *Let*

$$\mathcal{C}([0,1]) \;=\; \{f \;:\; [0,1] \to \mathbf{R}\,;\, f \text{ continuous}\}\;. \tag{1.13}$$

With the norm

$$\|f\|_\infty \;=\; \sup_{t\in[0,1]} |f(t)|\,. \tag{1.14}$$

$\mathcal{C}([0,1])$ *is a Banach space. With either of the norms*

$$\|f\|_1 \,=\, \int_0^1 |f(t)|\,dt\,, \qquad \text{or} \qquad \|f\|_2 \,=\, \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}\,, \tag{1.15}$$

$\mathcal{C}([0,1])$ *is not complete.*

Proof: We let the reader verify that $\|f\|_1$, $\|f\|_2$, and $\|f\|_\infty$ are norms.

Let $\{f_n\}$ be a Cauchy sequence for the norm $\|\cdot\|_\infty$. We have then

$$|f_n(t) - f_m(t)| \,\le\, \|f_n - f_m\|_\infty \,\le\, \epsilon \quad \text{for all } n, m \ge N\,. \tag{1.16}$$

This implies that, for any t, $\{f_n(t)\}$ is a Cauchy sequence in $\mathbf{R}$ which is complete. Therefore $\{f_n(t)\}$ converges to an element of $\mathbf{R}$ which we call $f(t)$. It remains to show that the function $f(t)$ is continuous. Taking the limit $m \to \infty$ in (1.16), we have

$$|f_n(t) - f(t)| \,\le\, \epsilon \quad \text{for all } n \ge N\,, \tag{1.17}$$

where N depends on ϵ but not on t. This means that $f_n(t)$ converges *uniformly* to $f(t)$ and therefore $f(t)$ is continuous.

Let us consider the sequence $\{f_n\}$ of piecewise linear function, where $f_n(t) = 0$ on $[0, 1/2 - 1/n]$ and $f_n(t) = 1$ on $[1/2 + 1/n, 1]$ and linearly interpolating in between. One verifies easily that for any $m \ge n$ we have $\|f_n - f_m\|_1 \le 1/n$ and $\|f_n - f_m\|_2 \le 1/\sqrt{n}$. Therefore $\{f_n\}$ is a Cauchy sequence. But the limit function is not continuous and therefore the sequence does not converge in $\mathcal{C}([0,1])$. ∎

We have also

Proposition 1.2.3 *Let X be an arbitrary set and let us consider the space*

$$\mathcal{B}(X) \,=\, \{f \,:\, X \to \mathbf{R}\,;\, f \text{ bounded}\}\,. \tag{1.18}$$

with the norm

$$\|f\|_\infty \,=\, \sup_{x\in X} |f(x)|\,. \tag{1.19}$$

Then $\mathcal{B}(X)$ is a Banach space.

Proof: The proof is almost identical to the first part the previous proposition and is left to the reader.

In a Banach space E we can define basic topological concepts as in $\mathbf{R}^n$.

- Two norms $\|\cdot\|$ and $|||\cdot|||$ are *equivalent* if there exists positive constants c and C such that $c\|x\| \le |||x||| \le C\|x\|$ for all $x \in E$.

- A *open ball* of radius r and center a is the set $B_\epsilon(a) \,=\, \{x \in E\,;\, \|x - a\| \,<\, r\}$.

- A neighborhood of a is a set V such that $B_\epsilon(a) \subset V$ for some $\epsilon > 0$.
- A set $U \subset E$ is *open* if U is a neighborhood of each of its element, i.e., for any $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.
- A set $V \subset E$ is *closed* if the limit of any convergent sequence $\{x_n\}$ is in V.
- A set K is compact if any sequence $\{x_n\}$ with $x_n \in K$ has a subsequence which converge in K.
- Let E and F be two Banach spaces and $U \subset E$. A function $f : U \to F$ is continuous at $x_0 \in U$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $x \in U$ and $\|x - x_0\| < \delta$ implies that $\|f(x) - f(x_0)\| < \epsilon$.
- The map $x \mapsto \|x\|$ is a continuous function of E to $\mathbf{R}$, since $|\|x\| - \|x_0\|| \leq \|x - x_0\|$ by the triangle inequality.

Certain properties which are true in finite dimensional Banach spaces cease to be true in infinite dimensional Banach space such as the function space we have considered in Propositions 1.2.2 and 1.2.3. For example we show that

- The closed ball $\{x \in E\,;\, \|x\| \leq 1\}$ is not necessarily compact.
- Two norms on a Banach space are not always equivalent.
- The theorem of Bolzano-Weierstrass which says each bounded sequence has a convergent subsequence is not necessarily true.
- The equivalence of K compact and K closed and bounded is not necessarily true.

The proposition 1.2.2 shows that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent. For, if they were equivalent, any Cauchy sequence for $\|\cdot\|_1$ would be a Cauchy sequence $\|\cdot\|_\infty$. But we have constructed explicitly a Cauchy sequence for $\|\cdot\|_1$ which is not a Cauchy sequence for $\|\cdot\|_\infty$. Let us consider the Banach space $\mathcal{B}([0,1])$ and let $f_n(t)$ to be equal to 1 if $1/(n+1) < t \leq 1/n$ and 0 otherwise. We have $\|f_n\|_\infty = 1$ for all n and $\|f_n - f_m\|_\infty = 1$ for any n, m. Therefore $\{f_n\}$ cannot have a convergent subsequence. This shows at the same time, that the unit ball is not compact, that Bolzano-WWeierstrass fails, and that closed bounded sets are not necessarily compact.

Let us suppose that we want to solve a nonlinear equation in a Banach space. Let f, g be functions from E to E then we might want to solve

$$
\begin{aligned}
g(x) &= 0 : \quad \text{find a zero of } g\,, && (1.20)\\
f(x) &= x : \quad \text{find a fixed point of } f\,. && (1.21)
\end{aligned}
$$

These two problems are equivalent, since we can set $f(x) = x + g(x)$. The next theorem will provide a sufficient condition for the existence of a fixed point.

Theorem 1.2.4 (Banach Fixed Point Theorem (1922)) *Let E be a Banach space, $D \subset E$ closed and $f : D \to E$ a map which satisfies*

1. *$f(D) \subset D$*
2. *f is a contraction on D, i.e., there exists $\alpha < 1$ such that,*

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|\,, \qquad \text{for all } x, y \in D\,. \tag{1.22}$$

Then f has a unique fixed point x in D, $f(x) = x$.

Proof: We first show uniqueness. Let us suppose that there are two fixed point x and y, i.e., $f(x) = x$ and $f(y) = y$. Since f is a contraction we have

$$\|x - y\| \;=\; \|f(x) - f(y)\| \leq \alpha \|x - y\| \tag{1.23}$$

with $\alpha < 1$, this is possible only if $x = y$.

To prove the existence we choose an arbitrary $x_0 \in D$ and we consider the iteration $x_1 = f(x_0), \cdots, x_{n+1} = f(x_n), \cdots$. Since $f(D) \subset D$ this implies that $x_n \in D$ for any n. Let us show that $\{x_n\}$ is a Cauchy sequence. We have $\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq \alpha \|x_n - x_{n-1}\|$. Iterating this inequality we obtain

$$\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|\,. \tag{1.24}$$

If $m > n$ this implies that

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots \|x_{n+1} - x_n\| \\
&\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots \alpha^n\right) \|x_1 - x_0\| \\
&\leq \frac{\alpha^n}{1 - \alpha} \|x_1 - x_0\|\,.
\end{aligned} \tag{1.25}$$

Therefore $\{x_n\}$ is a Cauchy sequence since $\alpha^n \to 0$. Since E is a Banach space, this sequence converges to $x \in E$. The limit x is in D since D is closed. Since f is a contraction, it is continuous and we have

$$x = \lim_{n \to \infty} x_{n+1} = f(\lim_{n \to \infty} x_n) = f(x)\,, \tag{1.26}$$

i.e., x is a fixed point of f. ∎

The proof of the theorem is constructive and provides the following algorithm to construct a fixed point.

Method of successive approximations: To solve $f(x) = x$,

- Choose an arbitrary x_0.

- Iterate: $x_{n+1} = f(x_n)$.

Even if the hypotheses of the theorem are difficult to check, one might apply this algorithm. If there is convergence this gives a fixed point, not necessarily unique.

Example 1.2.5 The function $f(x) = \cos(x)$ has a fixed point on $D = [0, 1]$. By the mean value theorem there is $\xi \in (x, y)$ such that $\cos(x) - \cos(y) = \sin(\xi)(y - x)$, thus $|\cos(x)-\cos(y)| \le \sup_{t\in[0,1]} |\sin(t)||x-y| \le \sin(1)|x-y|$, and $\sin(1) < 1$. One observes a quite rapid convergence to the solution $0.7390851\cdots$.

Example 1.2.6 Consider the Banach space $\mathcal{C}([0, 1])$ with the norm $\|\cdot\|_\infty$. Let $k(t, s)$ be a function of 2 variables continuous on $[0, 1]\times[0, 1]$. Consider the fixed point problem

$$x(t) \;=\; \lambda \int_0^1 k(t, s)x(s)\, ds\,. \tag{1.27}$$

and let us suppose that λ is such that $\alpha \equiv |\lambda| \sup_{0\le t\le 1} \int_0^1 |k(t, s)|\, ds < 1$. Consider the map $(Tx)(t) = \lambda \int_0^1 k(t, s)y(s)$. One obtains the bound

$$|(Tx)(t)| \;\le\; |\lambda| \int_0^1 |k(t, s)||x(s)|\, ds \;\le\; \|x\|_\infty|\lambda| \sup_{0\le t\le 1} \int_0^1 |k(t, s)|\, ds\,. \tag{1.28}$$

Taking the supremum over t on the left side gives

$$\|Tx\|_\infty \;\le\; \alpha\|x\|_\infty\,. \tag{1.29}$$

and a similar argument shows that

$$\|Tx - Ty\|_\infty \;\le\; \alpha\|x - y\|_\infty\,. \tag{1.30}$$

Let us now choose $D = \{x \in \mathcal{C}([0, 1])\,;\, \|x\| \le 1\}$, i.e the closed ball of radius 1 around 0. From (1.29) we see that $T(D) \subset D$, while from (1.30) we see that T is a contraction. Hence the Banach fixed point theorem implies the existence of a solution. Then the method of successive approximation applies and the iteration is, $y_0(t) = 0$ and

$$y_{n+1}(t) \;=\; \lambda \int_0^1 k(t, s)y_n(s)\, ds\,. \tag{1.31}$$

1.3 Existence and uniqueness for the Cauchy problem

Let us consider the Cauchy problem

$$x'(t) = f(t, x(t))\,, \quad x(t_0) = x_0\,, \tag{1.32}$$

where $f : U \to \mathbf{R}^n$ (U is open set of $\mathbf{R} \times \mathbf{R}^n$) is a continuous function. In order to find a solution we will rewrite (1.32) as a fixed point equation. We integrate the differential equation between t_0 an t, we obtain the *integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds\,. \tag{1.33}$$

Every solution of (1.32) is thus a solution of (1.33). The converse also holds. If $x(t)$ is a continuous function which verifies (1.33) on some interval I, then it is automatically of class $\mathcal{C}^1$ and it satisfies (1.32).

Let us define the function $T : \mathcal{C}(I) \to \mathcal{C}(I)$ given by

$$(Tx)(t) \;=\; x_0 + \int_{t_0}^{t} f(s, x(s))\, ds\,. \tag{1.34}$$

The integral equation (1.33) can then be written as

$$(Tx)(t) \;=\; x(t)\,, \tag{1.35}$$

i.e., we have transformed the differential equation (1.32) into a fixed point problem. The method of successive approximation for this problem is called

Picard-Lindelöf iteration:

$$\begin{aligned} x_0(t) &= x_0 \quad \text{(or any other function)}\,, \\ x_{n+1}(t) &= x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds\,. \end{aligned} \tag{1.36}$$

Example 1.3.1 Let us consider the Cauchy problem

$$x' = -x^2\,, \quad x(0) = 1\,. \tag{1.37}$$

The solution is $x(t) = \frac{1}{1+t}$. The Picard-Lindelöf iteration gives $x_0 = 1$, $x_1 = 1 - t$, $x_2 = 1 - t + t^2 - t^3/3$, and so on. If one studies this iteration in more details one see that it converges in a suitable interval $[0, a]$ and diverges for larger t.

The next results shows how to choose the interval I such that T maps a suitably chosen set D into itself. We have

Lemma 1.3.2 *Let $A = \{(t, x)\,; |t - t_0| \le a\,,\ \|x - x_0\| \le b\}$, $f : A \to \mathbf{R}^n$ a continuous function with $M = \sup_{(t,x)\in A} |f(t, x)|$. We set $\alpha = \min(a, b/M)$. The map T given by (1.34) is well-defined on the set*

$$B = \{x : [t_0 - \alpha, t_0 + \alpha] \to \mathbf{R}^n\,,\ x \text{ continuous and } \|x(t) - x_0\| \le b\}\ . \tag{1.38}$$

and it satisfies $T(B) \subset B$.

Proof: The lemma follows from the estimate

$$\|(Tx)(t) - x_0\| = \left\|\int_{t_0}^{t} f(s, x(s))\, ds\right\| \le M|t - t_0| \le M\alpha \le b\,. \tag{1.39}$$

∎

We say that a function $f : A \to \mathbf{R}^n$ (with A is in the previous lemma) satisfies a *Lipschitz condition* if

$$\|f(t, x) - f(t, y)\| \le L\|x - y\| \quad \text{for all } (t, x), (t, y) \in A\,. \tag{1.40}$$

The constant L is called the *Lipschitz constant.*

Remark 1.3.3 To elucidate the meaning of condition (1.40), let us suppose that $f(t, x) = f(x)$ does not depend on t and that we have $\|f(x) - f(y)\| \le L\|x - y\|$ whenever x and y are in the closed ball $\overline{B}_b(x_0)$. This clearly implies that f is continuous in $\overline{B}_b(x_0)$. The opposite does not hold, for example the function $f(x) = \sqrt{|x|}$ is continuous but not Lipschitz continuous at 0. On the other hand if f is of class $\mathcal{C}^1$, then f is Lipschitz continuous. To see this consider the line $z(s) = x + s(y - x)$ which interpolates between x and y. We have

$$\begin{aligned} \|f(y) - f(x)\| &= \left\|\int_0^1 \frac{d}{ds} f(z(s))\, ds\right\| = \left\|\int_0^1 f'(x(s))(y - x)\, ds\right\| \\ &\le \sup_v \|f'(v)\|\|y - x\|\,. \end{aligned} \tag{1.41}$$

and therefore f is Lipschitz continuous with $L = \sup_{\{x\,,\,\|x - x_0\| \le b\}} \|\nabla f(x)\|$. On the other hand Lipschitz continuity does not imply differentiability as the function $f(x) = |x|$ shows.

If $f(t, x)$ satisfy a Lipschitz condition we have, for any $t \in I = [t_0 - \alpha, t_0 + \alpha]$,

$$\begin{aligned} \|(Tx)(t) - (Tz)(t)\| &\le \int_{t_0}^{t} \|f(t, x(t)) - f(t, z(t))\|\, dt \\ &\le \int_{t_0}^{t} L\|x(t) - z(t)\|\, dt \\ &\le \alpha L \sup_{t \in I} \|x(t) - z(t)\| \le \alpha L\|x - z\|_\infty\,. \end{aligned} \tag{1.42}$$

Taking the supremum over t on the left side shows that $\|Tx - Tz\|_\infty \le \alpha L\|x - z\|$. If $\alpha L < 1$ we can apply the Banach fixed point theorem to prove existence of a fixed point and show the existence and uniqueness for the solution of the Cauchy problem for t in some interval around t_0.

In fact, working a little harder one can do without the condition $\alpha L < 1$. The following theorem is the basic result on existence of *local solutions* for the Cauchy problem (1.32). Here local means that we show the existence only of t is in some interval around t_0.

Theorem 1.3.4 (Existence and uniqueness for the Cauchy problem) *Let* $A = \{(t,x)\,;|t-t_0| \le a\,,\ \|x-x_0\| \le b\}$ *and let us suppose that* $f\,:\,A \to \mathbf{R}^n$

- *is continuous,*
- *satisfies a Lipschitz condition.*

Then the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$ *has a unique solution on* $I = [t_0-\alpha, t_0+\alpha]$, *where* $\alpha = \min(a, b/M)$ *with* $M = \sup_{(t,x)\in A} \|f(t,x)\|$.

Proof: Instead of using the Banach fixed point theorem we prove directly that the Picard-Lindelöf iteration converge uniformly on I to a solution of the Cauchy problem. In a first step we show, by induction, that

$$\|x_{k+1}(t) - x_k(t)\| \,\le\, ML^k \frac{|t-t_0|^{k+1}}{(k+1)!} \quad \text{for } |t-t_0| \le \alpha\,. \tag{1.43}$$

For $k=0$, we have $\|x_1(t) - x_0\| \,=\, \|\int_{t_0}^t f(s,x(s))\,ds\| \le M|t-t_0|$. Let us assume that 1.43 holds for $k-1$. Then we have

$$\begin{aligned}\|x_{k+1}(t) - x_k(t)\| \;&\le\; \int_{t_0}^t \|f(s,x_k(s)) - f(s,x_{k-1}(s)\|\,ds \,\le\, L\int_{t_0}^t \|x_k(s) - x_{k-1}(s)\|\,ds \\ &\le\; ML^k \int_{t_0}^t \frac{|s-t_0|^k}{k!}\,ds \,=\, ML^k \frac{|t-t_0|^{k+1}}{(k+1)!}\,. \end{aligned} \tag{1.44}$$

Using (1.43), we show that $\{x_k(t)\}$ is a Cauchy sequence for the norm $\|x\|_\infty = \sup_{t\in I}\|x(t)\|$: We have

$$\begin{aligned}\|x_{k+m}(t) - x_k(t)\| \;&\le\; \|x_{k+m}(t) - x_{k+m-1}(t)\| + \cdots + \|x_{k+1}(t) - x_k(t)\| \\ &\le\; \frac{M}{L}\left(\frac{L^{k+m}|t-t_0|^{k+m}}{(k+m)!} + \cdots + \frac{L^{k+1}|t-t_0|^{k+1}}{(k+1)!}\right) \\ &\le\; \frac{M}{L}\sum_{j=k+1}^{\infty}\frac{(L\alpha)^j}{j!}\,, \end{aligned} \tag{1.45}$$

and the right hand side is the reminder term of a convergent series and thus goes to 0 as k goes to ∞. The right hand side is independent of t so $\{x_k\}$ is a Cauchy sequence which converges to a continuous function $x : I \to \mathbf{R}^n$.

To show that $x(t)$ is a solution of the Cauchy problem we take the limit $n \to \infty$ in (1.36). The left side converges uniformly to $x(t)$. Since f is continuous and A is compact $f(t,x_k(t))$ converges uniformly to $f(t,x(t))$ on A. Thus one can exchange integral and the limit and $x(t)$ is a solution of the integral equation (1.33).

It remains to prove uniqueness of the solution. Let $x(t)$ and $z(t)$ be two solutions of (1.33). By recurrence we show that

$$\|x(t) - y(t)\| \,\le\, 2ML^k \frac{|t-t_0|^{k+1}}{(k+1)!}\,. \tag{1.46}$$

We have $x(t) - y(t) = \int_{t_0}^{t}(f(s, x(s)) - f(s, y(s)))\, ds$ and therefore $\|x(t) - y(t)\| \leq 2M|t - t_0|$ which (1.46) for $k = 0$. If (1.46) for $k - 1$ holds we have

$$\begin{aligned} \|x(t) - y(t)\| \leq \int_{t_0}^{t} L\|x(s) - y(s)\|\, ds &\leq 2ML^k \int_{t_0}^{t} \frac{|s - t_0|^k}{k!}\, dt \\ &\leq 2ML^k \frac{|t - t_0|^{k+1}}{(k+1)!}\,, \end{aligned} \tag{1.47}$$

and this proves (1.46). Since this holds for all k, this shows that $x(t) = y(t)$. ∎

1.4 Peano Theorem

In the previous section we established a local existence result by assuming a Lipschitz condition. Simple examples show that this condition is also necessary.

Example 1.4.1 Consider the ODE

$$x' = 2\sqrt{|x|}\,. \tag{1.48}$$

We find that $x(t) = (t - c)^2$ for $t > c$ and $x(t) = -(c - t)^2$ for $t < c$ is a solution for any constant c. But $x(t) \equiv 0$ is also a solution. The Cauchy problem with, say, $x(0) = 0$ has infinitely many solutions. For $t > 0$, $x(t) \equiv 0$ is one solution, $x = t^2$ is another solution, and more generally $x(t) = 0$ for $0 \leq t \leq c$ and then $x(t) = (t - c)^2$ for $t \geq c$ is also a solution for any c. This phenomenon occur because $\sqrt{x}$ is not Lipschitz at $x = 0$.

We are going to show that, without Lipschitz condition, we can still obtain existence of solutions, but not uniqueness. Instead of using the Picard-Lindelöf iteration we are using another approximation scheme. It turns out to be the simplest algorithm used for numerical approximations of ODE's (in theory but not in practice).

Euler polygon (1736) Fix some $h \neq 0$, the idea is to approximate the solution locally by $x(t+h) \simeq x(t) + hf(t, x(t))$. Let us consider now the sequence $\{t_n, x_n\}$ given recursively by

$$t_{n+1} = t_n + h\,, \quad x_{n+1} = x_n + hf(t_n, x_n)\,. \tag{1.49}$$

We then denote by $x_h(t)$ the piecewise linear function which passes through the points (t_n, x_n). It is called the Euler polygon and is an approximation to the solution of the Cauchy problem.

Lemma 1.4.2 *Let $A = \{(t, x)\,; |t - t_0| \leq a\,, \|x - x_0\| \leq b\}$, $f : A \to \mathbf{R}^n$ a continuous function with $M = \sup_{(t,x)\in A} \|f(t, x)\|$. We set $\alpha = \min(a, b/M)$. If $|h| = \alpha/N$, N an* *integer, the Euler Polygon satisfies* $(t, x_h(t)) \in A$ *for* $t \in [t_0 - \alpha, t_0 + \alpha]$ *and we have the bound*

$$\|x_h(t) - x_h(t')\| \leq M|t - t'| \quad \text{for } t, t' \in [t_0 - \alpha, t_0 + \alpha]\,. \tag{1.50}$$

Proof: Let us consider first the interval $[t_0, t_0 + \alpha]$ and choose $h > 0$. We show first, by induction that $(t_n, x_n) \in A$ for $n = 0, 1, \cdots, N$. We have $\|x_{n+1} - x_n\| \leq hM$ and so $\|x_{n+1} - x_0\| \leq (n + 1)hM \leq \alpha M \leq b$ if $n + 1 \leq N$. Since $x_h(t)$ is piecewise linear $(t, x_h(t)) \in A$ for any $t \in [t_0, t_0 + \alpha]$. The estimate (1.50) follows from the fact that the slope of $x_h(t)$ is nowhere bigger than M. On $[t_0 - \alpha, t_0]$ the argument is similar. ∎

Definition 1.4.3 A family of functions $f_j : [a, b] \to \mathbf{R}^n$, $j = 1, 2, \cdots$, is *equicontinuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for, for all j, $|t - t'| < \delta$ implies that $\|f_j(t) - f_j(t')\| \leq \epsilon$.

Equicontinuity means that all the functions f_j are uniformly continuous and that, moreover, δ can be chosen to depend only on ϵ, but *not* on j. The estimate (1.50) shows that the family $x_h(t)$, with $h = \alpha/N$, $N = 1, 2, \cdots$, is equicontinuous.

Theorem 1.4.4 (Arzelà-Ascoli 1895) *Let* $f_j : [a, b] \to \mathbf{R}^n$ *be a family of functions such that*

- $\{f_j\}$ *is equicontinuous.*
- *For any* $t \in [a, b]$*, there exists* $M(t) \in \mathbf{R}$ *such that* $\sup_j \|f_j(t)\| \leq M(t)$*.*

Then the family $\{f_j\}$ *has a convergent subsequence* $\{g_n\}$ *which converges uniformly to a continuous function* g *on* $[a, b]$*.*

Remark 1.4.5 As we have seen a bounded closed set in $\mathcal{C}([a, b])$ is not always compact. The Arzelà-Ascoli theorem shows that a bounded set of equicontinuous function is a compact set in $\mathcal{C}([a, b])$, it is a generalization of Bolzano-Weierstrass to $\mathcal{C}([a, b])$.

Proof: The subsequence is constructed proof via a trick which is referred to as "diagonal subsequence". The set of rational numbers in $[a, b]$ is countable and we write it as $\{t_1, t_2, t_3, \cdots\}$. Consider the sequence $\{f_j(t_1)\}$, by assumption it is bounded in $\mathbf{R}^n$ and, by Bolzano-Weierstrass, it has a convergent subsequence which we denote by $\{f_{1i}(t_1)\}_{i\geq 1}$ and therefore

$$f_{11}(t), f_{12}(t), f_{13}(t) \cdots \quad \text{converges for } t = t_1\,. \tag{1.51}$$

Consider next the sequence $\{f_{1i}(t_2)\}_{i\geq 1}$. Again, by Bolzano-Weierstrass, this sequence has a convergent subsequence denoted by $\{f_{2i}(t_2)\}_{i\geq 1}$. We have

$$f_{21}(t), f_{22}(t), f_{23}(t) \cdots \quad \text{converges for } t = t_1, t_2\,. \tag{1.52}$$

After n steps we find a sequence $\{f_{ni}(t)\}_{i\geq 1}$ of $\{f_j(t)\}$ such that

$$f_{n1}(t), f_{n2}(t), f_{n3}(t)\cdots \quad \text{converges for } t = t_1, t_2, \cdots, t_n\,. \tag{1.53}$$

Next we consider the diagonal sequence $g_n(t) = f_{nn}(t)$. This sequence converges for any t_l, since $\{g_n(t_l) = f_{nn}(t_l)\}_{n\geq l}$ is a subsequence of $\{f_{ln}(t_l)\}_{n\geq l}$ which converges.

By equicontinuity, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $n \geq 1$, $|t-t'| < \delta$ implies that $\|g_n(t) - g_n(t')\| < \epsilon$. Let us choose rational points $t_1, t_2, \cdots t_{q-1}$ such that $a = t_0 < t_1 < \cdots < t_{q-1} < t_q = b$ and $t_{i+1} - t_i < \delta$. For $t \in [t_l, t_{l+1}]$ we have

$$\|g_n(t) - g_m(t)\| \;\leq\; \|g_n(t) - g_n(t_l)\| + \|g_n(t_l) - g_m(t_l)\| + \|g_n(t) - g_n(t_l)\|\,. \tag{1.54}$$

By equicontinuity $\|g_n(t) - g_n(t_l)\|$ and $\|g_n(t) - g_n(t_l)\|$ are smaller than ϵ. By the convergence of $\{g_n(t_l)\}$ there exists $N(l)$ such that $\|g_n(t_l) - g_m(t_l)\| \leq \epsilon$ if $n, m \geq N_l$. If we choose $N = \max_l N_l$ we have that $\|g_n(t) - g_m(t)\| \leq 3\epsilon$ for all $t \in [a, b]$ and $n, m \geq N$. This shows that $g_n(t)$ converges uniformly to some $g(t)$ which is then continuous. ∎

From this we obtain

Theorem 1.4.6 (Peano 1890) *Let $A = \{(t, x)\,; |t - t_0| \leq a\,,\ \|x - x_0\| \leq b\}$, $f : A \to \mathbf{R}^n$ a continuous function with $M = \sup_{(t,x)\in A} \|f(t, x)\|$. Set $\alpha = \min(a, b/M)$. The the Cauchy problem (1.32) has a solution on $[t_0 - \alpha, t_0 + \alpha]$.*

Proof: Let us consider the Euler polygons with $h = \alpha/N$, $N = 1, 2, \cdots$. The sequence is bounded since $\|x_h(t)-x_0\| \leq M|t-t_0| \leq M\alpha$ and equicontinuous by Lemma 1.4.2. By Arzelà-Ascoli Theorem, the family $x_h(t)$ has a subsequence which converges uniformly to a continuous function $x(t)$ on $[t_0 - \alpha, t_0 + \alpha]$. It remains to show that $x(t)$ is a solution.

Let $t \in [t_0, t_0 + \alpha]$ and let (t_n, x_n) the approximation obtained by Euler method for $x_h(t)$. If $t \in [t_l, t_{l+1}]$ we have

$$x_h(t) - x_0 \;=\; hf(t_0, x_0) + hf(t_1, x_1) + \cdots + hf(t_{l-1}, x_{l-1}) + (t - t_l)f(t_l, x_l)\,. \tag{1.55}$$

Since $f(t, x(t))$ is a continuous function of t it is Riemann integrable and, using a Riemann sum with left-end points have

$$\begin{aligned}\int_{t_0}^{t} f(s, x(s))\,ds \;&=\; hf(t_0, x(t_0)) + hf(t_1, x(t_1)) + \cdots \\ &\cdots + hf(t_{l-1}, x(t_{l-1})) + (t - t_l)f(t_l, x(t_l)) + r(h)\,,\end{aligned} \tag{1.56}$$

where $\lim_{h\to 0} \|r(h)\| = 0$. By the uniform continuity of f on A and the uniform convergence of the subsequence of $\{x_h(t)\}$ to $x(t)$ we have that $\|f(t, x_h(t) - f(t, x(t))\| \leq$

ϵ if h is sufficiently small (and h is such that x_h belongs to the convergent subsequence). Using that $x_h(t_j) = x_j$ and subtracting (1.56) from (1.55) we find that

$$\|x_h(t) - x_0 - \int_{t_0}^{t} f(s, x(s))\, ds\| \leq (l+1)h\epsilon + \|r(h)\| \leq \alpha\epsilon + \|r(h)\| \tag{1.57}$$

which converges to 0 as $h \to 0$. Therefore $x(t)$ is a solution of the Cauchy problem in integral form (1.33). ∎

1.5 Continuation of solutions

So far we only considered local solutions, i.e. solutions which are defined in a neighborhood of (t_0, x_0). Simple examples shows that the solution $x(t)$ may not exist for all t, for example the equation $x' = 1 + x^2$ has solution $x(t) = \tan(t - c)$ and this solution does not exist beyond the interval $(c - \pi/2, c + \pi/2)$, and we have $x(t) \to \pm\infty$ as $t \to c \pm \pi/2$. This is typical.

To extend the solution we solve the Cauchy problem locally, say from t_0 to $t_0 + \alpha$ and then we can try to continue the solution by solving the Cauchy problem $x' = f(t, x)$ with new initial condition $x(t_0 + \alpha)$. In order do this we should be able to solve it locally everywhere and we will assume that f satisfy a *local Lipschitz condition.*

Definition 1.5.1 A function $f : U \to \mathbf{R}^n$ (where U is an open set of $\mathbf{R} \times \mathbf{R}^n$) satisfies a *local Lipschitz condition* if for any $(t_0, x_0) \in U$ there exist a neighborhood $V \subset U$ such that f satisfies a Lipschitz condition on V, see (1.40).

If the function f is of class $\mathcal{C}^1$ in U, then it satisfies a local Lipschitz condition.

Lemma 1.5.2 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and let us assume that $f : U \to \mathbf{R}^n$ is continuous and satisfies a local Lipschitz condition. Then for any $(t_0, x_0) \in U$ there exists an open interval $I_{\max} = (\omega_-, \omega_+)$ with $-\infty \leq \omega_- < t_0 < \omega_+ \leq \infty$ such that*

- *The Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ has a unique solution on $I_{\max}$.*
- *If $y : I \to \mathbf{R}^n$ is a solution of $x' = f(t, x)$, $y(t_0) = x_0$, then $I \subset I_{\max}$ and $y = x|_I$.*

Proof: a) Let $x : I \to \mathbf{R}^n$ and $z : J \to \mathbf{R}^n$ be two solutions of the Cauchy problem with $t_0 \in I, J$. Then $x(t) = z(t)$ on $I \cap J$. Suppose it is not true, there is point $\bar{t}$ such that $x(\bar{t}) \neq z(\bar{t})$. Consider the first point where the solutions separate. The local existence theorem 1.3.4 shows that it is impossible.

b) Let us define the interval

$$I_{\max} = \{I\,;\, I \text{ open interval}\ ,\ t_0 \in I\ ,\ \text{ there exists a solution on } I\}\ . \tag{1.58}$$

This interval is open and we can define the solution on $I_{\max}$ as follows. If $t \in I_{\max}$, then there exists I where the Cauchy problem has a solution and we can define $x(t)$. The part (a) shows that $x(t)$ is uniquely defined on $I_{\max}$. ∎

Theorem 1.5.3 *Let $U \subset \mathbf{R}\times\mathbf{R}^n$ be an open set and let us assume that $f : U \to \mathbf{R}^n$ is continuous and satisfies a local Lipschitz condition. Then every solution of $x' = f(t,x)$ has a continuation up to the boundary of U. More precisely, if $x : I_{\max} \to \mathbf{R}^n$ is the solution passing through $(t_0, x_0) \in U$, then for any compact $K \subset U$ there exists $t_1, t_2 \in I_{\max}$ with $t_1 < t_0 < t_2$ such that $(t_1, x(t_1)) \notin K$, $(t_2, x(t_2)) \notin K$.*

Remark 1.5.4 If $U = \mathbf{R} \times \mathbf{R}^n$, Theorem 1.5.3 means that either

- $x(t)$ exists for all t,
- There exists t^* such that $\lim_{t\to t^*} \|x(t)\| = \infty$,

that is the solution exists globally or the solution "blows up" at a certain point.

Proof: Let $I_{\max} = (\omega_-\, , \omega_+)$. If $\omega_+ = \infty$, clearly there exists a point t_2 such that $t_2 > t_0$ and $(t_2, x(t_2)) \notin K$ because K is bounded. If $\omega_+ < \infty$, let us assume that there exist a compact K such that $(t, x(t)) \in K$ for any $t \in (t_0, \omega_+)$. Since $f(t,x)$ is bounded on the compact set K, we have, for t, t' sufficiently close to ω_+

$$\|x(t) - x(t')\| \;=\; \left\|\int_{t'}^{t} F(t, x(t))\, dt\right\| \le M|t - t'| < \epsilon\,. \tag{1.59}$$

This shows that $\lim_{t\to\omega_+} x(t) = x_+$ exists and $(\omega_+, x_+) \in K$, since K is closed. Theorem 1.3.4 for the Cauchy Problem with $x(\omega_+) = x_+$ implies that there exists a solution in a neighborhood of ω_+. This contradicts the maximality of the interval $I_{\max}$. For t_1 the argument is similar. ∎

1.6 Global existence

In this section we derive sufficient conditions for *global existence* of solutions, i.e absence of blow-up for $t > t_0$ or for all t.

Lemma 1.6.1 (Gronwall Lemma) *Suppose that $g(t)$ is a continuous function with $g(t) \ge 0$ and that there exits constants $a, b > 0$ such that*

$$g(t) \;\le\; a + b\int_{t_0}^{t} g(s)\, ds\,, \quad t \in [t_0, T]\,. \tag{1.60}$$

Then we have

$$g(t) \le a e^{b(t-t_0)} \quad t \in [t_0, T]\,. \tag{1.61}$$

Proof: Set $G(t) = a + b\int_0^t g(s)\,ds$. Then $G(t) \geq g(t)$, $G(t) > 0$, for $t \in [t_0, T]$, and $G'(t) = bg(t)$. Therefore

$$\frac{G'(t)}{G(t)} = \frac{bg(t)}{G(t)} \leq \frac{bG(t)}{G(t)} = b\,, \quad t \in [t_0, T]\,, \tag{1.62}$$

or, equivalently,

$$\frac{d}{dt}\log G(t) \leq b\,, \quad t \in [t_0, T]\,, \tag{1.63}$$

or

$$\log G(t) - logG(0) \leq b(t - t_0)\,, \quad t \in [t_0, T]\,, \tag{1.64}$$

or

$$G(t) \leq G(0)e^{b(t-t_0)} = ae^{b(t-t_0)}\,, \quad t \in [t_0, T]\,, \tag{1.65}$$

which implies that $g(t) \leq ae^{b(t-t_0)}$, for $t \in [t_0, T]$. ∎

The first condition is rather restrictive, but has the advantage of being expressed in terms of f and so easy to check.

Definition 1.6.2 We say that the function $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is *linearly bounded* if there exists a constant C such that

$$\|f(t,x)\| \leq C(1 + \|x\|)\,, \qquad \text{for all } (t,x) \in \mathbf{R} \times \mathbf{R}^n\,. \tag{1.66}$$

Obviously if $f(t,x)$ is bounded on $\mathbf{R}\times\mathbf{R}^n$, then it is linearly bounded. The functions $x\cos(x^2)$, or $x/\log(2 + |x|)$ are examples of linearly bounded function. The function $f(x,y) = (x + xy, y^2)^T$ is not linearly bounded.

Theorem 1.6.3 *Let $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous, locally Lipschitz (see definition 1.5.1) and linearly bounded (see (1.66)). Then the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, has a unique solution for all t.*

Proof: . Since f is locally Lipschitz, there is a unique local solution $x(t)$. We have the a-priori bound on solutions

$$\|x(t)\| = \|x_0\| + \int_{t_0}^t \|f(x(s))\|\,ds \leq \|x_0\| + C\int_{t_0}^t (1 + \|x(s)\|)\,ds\,, \tag{1.67}$$

Using Gronwall Lemma for $g(t) = 1 + \|x(t)\|$, we find that

$$1 + \|x(t)\| \leq (1 + \|x_0\|)e^{C(t-t_0)}\,, \quad \text{or} \quad \|x(t)\| \leq \|x(0)\|e^{C(t-t_0)} + (e^{C(t-t_0)} - 1)\,. \tag{1.68}$$

This shows that the norm of the solution grows at most exponentially fast in time. From Remark 1.5.4 it follows that the solution does not blow up in finite time. ∎

We will formulate more sufficient conditions but, for simplicity we restrict ourselves to *autonomous* equations: we consider Cauchy problems of the form

$$x' = f(x)\,, \quad x_{t_0} = x_0\,, \tag{1.69}$$

where $f(t,x)$ does not depend explicitly on t.

Theorem 1.6.4 (Liapunov functions) *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be locally Lipschitz. Suppose that there exists a function $V(x) : \mathbf{R}^n \to \mathbf{R}$ of class $\mathcal{C}^1$ such that*

- $V(x) \geq 0$ *and* $\lim_{\|x\|\to\infty} V(x) = \infty$.
- $\langle \nabla V(x)\,,\, f(x) \rangle \leq a + bV(x)$

Then the Cauchy problem $x' = f(x)$, $x(t_0) = x_0$, has a unique solution for $t_0 < t < +\infty$.

Proof: Since f is locally Lipschitz, there is a unique local solution $x(t)$ for the Cauchy problem. We have

$$\frac{d}{dt}V(x(t)) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}\frac{dx_j}{dt} = \langle \nabla V(x(t))\,,\, f(x(t)) \rangle \leq a + bV(x(t) \tag{1.70}$$

or, by integrating,

$$V(x(t)) \leq V(x(t_0)) + \int_{t_0}^{t} (a + bV(x(s))\, ds\,. \tag{1.71}$$

Applying Gronwall lemma to $g(t) = a + bV(x(t))$ gives the bound

$$a + bV(x(t)) \leq (a + bV(x(0)))\, e^{b(t-t_0)}\,. \tag{1.72}$$

Therefore $V(x(t)$ remains bounded for all t. Since $\lim_{\|x\|\to\infty} V(x) = \infty$, $\|x(t)\|$ stays finite for all $t > t_0$ too. ∎

Remark 1.6.5 The function V in Theorem 1.6.4 is usually referred to as a *Liapunov function*. We will use such function to study the stability of solutions. Note that there is no general method to construct such Liapunov function, but it involves some trial and error and some a-priori knowledge on the equation.

Example 1.6.6 (Hamiltonian systems.) Let $H(p,q) = p^2/2 + V(q)$ where $p, q \in \mathbf{R}$ and assume that $V(q)$ is of class $\mathcal{C}^2$. The function $H(p,q)$ is called a Hamiltonian and the ODE

$$p' = -\frac{\partial H}{\partial q}(p,q) = -\frac{dV}{dq}\,, \qquad q' = \frac{\partial H}{\partial q}(p,q) = p\,. \tag{1.73}$$

is called the Hamiltonian equation for the Hamiltonian $H(p,q)$. Since V is of class $\mathcal{C}^2$, $f(p,q) = (-\frac{dV}{dq}, p)^T$ is locally Lipschitz so that we have local solutions.

Let us assume further that $\lim_{|q|\to\infty} V(q) = \infty$, in particular $V(q)$ is bounded below, i.e. $V(q) \geq -c$ from some $c \in \mathbf{R}$. Then the system 1.73 has a unique solution for all $t > 0$: we have

$$\frac{d}{dt}H(p(t),q(t)) \;=\; p(t)p'(t) + \frac{dV}{dq}(q(t))q'(t) \;=\; -p(t)\frac{dV}{dq}(q(t)) + \frac{dV}{dq}(q(t))p(t) \;=\; 0\,.$$

and $H(p,q) + c$ is a Liapunov function.

Example 1.6.7 (van der Pol equations) The second order equation $x'' = \epsilon(1 - x^2)x' - x$ is written as the first order system

$$\begin{aligned} x' &= y \\ y' &= \epsilon(1-x^2)y - x \end{aligned} \tag{1.74}$$

and is a perturbation of the Hamiltonian system $x''+x = 0$ with Hamiltonian $H(y,x) = y^2 + x^2$ (harmonic oscillator). Taking the Hamiltonian as the Liapunov function we have

$$\langle \nabla H(y,x)\,,\, f(y,x)\rangle \;=\; \epsilon(1-x^2)y^2 \;=\; \begin{cases} \leq 0 & \text{if} \quad x^2 \geq 1 \\ \leq \epsilon p^2 & \text{if} \quad x^2 \leq 1 \end{cases}\,. \tag{1.75}$$

Therefore $\nabla H \cdot f \leq \epsilon H$ and H is a Liapunov function and we have global existence.

Theorem 1.6.8 (Dissipative systems) *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be locally Lipschitz. Suppose that there exists $v \in \mathbf{R}^n$ and positive constants a and b such that*

$$\langle f(x)\,,\, x - v\rangle \;\leq\; a - b\|x\|^2\,. \tag{1.76}$$

Then the Cauchy problem $x' = f(x)$, $x(t_0) = x_0$, has a unique solution for $t_0 < t < +\infty$.

Proof: Consider the balls $B_0 = \{x \in \mathbf{R}^n\,;\, \|x\|^2 \leq a/b\}$ and $B = \{x \in \mathbf{R}^n\,;\, \|x - v\|^2 \leq R\}$ with $R \geq \|v\| + \sqrt{a/b}$ is chosen so large that B_0 is contained in the interior of B. If $y \in \partial B$, we have $\langle f(x)\,,\, x - v\rangle < 0$ which means that the vector $f(y)$ points toward the interior of B. As a consequence the solutions stay inside B forever. If we choose R so large that $y_0 \in B$, we obtain global existence. ∎

Example 1.6.9 The Lorentz equations are given by

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= -x_1 x_3 + r x_1 - x_2 \\ x_3' &= x_1 x_2 - b x_3 \end{aligned} \tag{1.77}$$

Despite its apparent simplicity, the Lorentz equations exhibits, for suitable values of the parameters, a very complex behavior. All solutions are attracted to a compact invariant set on which the motion is chaotic. Such an invariant set is called a strange attractor.

We show that the system is dissipative, we take $v = (0, 0, \gamma)$. Choosing $\gamma = b + r$ and using the inequality $2\gamma x_3 \le \gamma^2 + x_3^2$, we find

$$\begin{aligned} \langle f(x)\,,\, x - v\rangle &= -\sigma x_1^2 - x_2^2 - b y_3^2 + (\sigma + r - \gamma) x_1 x_2 + b\gamma x_3 \\ &= -\sigma x_1^2 - x_2^2 - b y_3^2 + b\gamma x_3 \\ &\le -\sigma x_1^2 - x_2^2 - \frac{b}{2} y_3^2 + b\frac{\gamma^2}{2}\,. \end{aligned} \tag{1.78}$$

If $\gamma = b + r$, then (1.76) is satisfied with $\alpha = b\frac{\gamma^2}{2}$ and $\beta = \min(\sigma, 1, b/2)$ and the solution of Lorentz systems exists for all $t > 0$.

There are many variants to Theorem 1.6.8 (see the homework). The basic idea is to find a family of sets (large balls in Theorem 1.6.8 but the set could have other shapes) such that, on the boundary of the sets the vector f points innards. This implies that solutions starting on the boundary will move inward the set. If one proves this for all sufficiently large sets, then one obtains global existence for all initial data. Note also that if the system has a Liapunov function V which satisfy $\nabla V \cdot f \le 0$, then the system is dissipative in the sense that the vector f points inward the level set of V.

Chapter 2

Linear systems

In this chapter we consider *linear differential equations* which have the general form

$$x' = A(t)x + g(t)\,. \tag{2.1}$$

Here $x \in \mathbf{R}^n$, $g : \mathbf{R} \to \mathbf{R}^n$, and $A(t) : \mathbf{R}^n \to \mathbf{R}^n$ is a $n \times n$ matrix

$$A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}. \tag{2.2}$$

and $A(t)x$ denote the vector with i-th component $(A(t)x)_i = \sum_{j=1}^n a_{ij}(t)x_j$.

The equation is called *homogeneous* if $g(t) \equiv 0$, and *inhomogeneous* otherwise. If $A(t) = A$ is independent of t and $g \equiv 0$, the linear ODE $x' = Ax$ is called a *system with constant coefficients*.

2.1 Linear maps from $\mathbf{R}^n$ to $\mathbf{R}^n$

A $n \times n$ matrix A defines a linear map $A : \mathbf{R}^n \to \mathbf{R}^n$ and we usually write Ax instead of $A(x)$. If we equip $\mathbf{R}^n$ with a norm for example,

$$\|x\|_1 = \sum_{i=1}^n |x_i|\,, \quad \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\,, \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|\,, \tag{2.3}$$

then $\mathbf{R}^n$ is a Banach space. All norms are equivalent on $\mathbf{R}^n$ so that the choice is a matter of convenience. In $\mathbf{R}^n$ all linear maps are continuous, uniformly continuous and differentiable (this not true anymore if the Banach space has infinite dimension).

Definition 2.1.1 We denote by $\mathcal{L}(\mathbf{R}^n)$ the set of all linear maps from $\mathbf{R}^n$ into $\mathbf{R}^n$. For $A \in \mathcal{L}(\mathbf{R}^n)$ we define

$$\|A\| = \sup_{\|x\|\le 1} \|Ax\| = \sup_{x\ne 0} \frac{\|Ax\|}{\|x\|}\,. \tag{2.4}$$

The number $\|A\|$ is called the *operator norm* of A.

This definition means that $\|A\|$ is the smallest real number such that

$$\|Ax\| \;\le\; \|A\|\,\|x\|\,, \qquad \text{for all } x \in \mathbf{R}^n\,. \tag{2.5}$$

We verify that $\|A\|$ is indeed a norm and it makes $\mathcal{L}(\mathbf{R}^n)$ a Banach space.

Lemma 2.1.2 *The space $\mathcal{L}(\mathbf{R}^n)$ equipped with (2.4) is a Banach space.*

Proof: The properties **N1** and **N2** are easily verified. For the triangle inequality, we have for $A, B \in \mathcal{L}(\mathbf{R}^n)$

$$\|(A+B)x\| \le \|Ax\| + \|Bx\| \;\le\; (\|A\| + \|B\|)\,\|x\|\,. \tag{2.6}$$

Dividing by $\|x\|$ and taking the supremum over all $x \neq 0$ one obtains the triangle inequality $\|A+B\| \le \|A\| + \|B\|$.

Let $\{A_n\}$ is a Cauchy sequence in $\mathcal{L}(\mathbf{R}^n)$. If $\|x\| \le 1$, we have

$$\|A_n x - A_m x\| \le \|A_n - A_m\|\,\|x\| \le \epsilon\|x\| \le \epsilon\,, \qquad \text{for } n, m \ge N\,. \tag{2.7}$$

Therefore $\{A_n x\}$ is a Cauchy sequence in $\mathbf{R}^n$ which has a limit denoted by Ax. The linearity of A is obvious and so we obtain $A \in \mathcal{L}(\mathbf{R}^n)$. Taking the limit $m \to \infty$ in (2.7) and dividing by $\|x\|$ we see that A is the limit of $\{A_n\}$. ∎

Note that the vector space $\mathcal{L}(\mathbf{R}^n)$ is isomorphic to $\mathbf{R}^{n^2}$ and that $\|A\|$ can be considered as a norm on $\mathbf{R}^{n^2}$, although of particular type. Important properties of $\|A\|$ are summarized in

Lemma 2.1.3 *Let $I \in \mathcal{L}(\mathbf{R}^n)$ be the identity map ($Ix = x$) and let $A, B \in \mathcal{L}(\mathbf{R}^n)$. Then we have*

1. $\|I\| = 1$.
2. $\|AB\| \le \|A\|\,\|B\|$. .
3. $\|A^n\| \le \|A\|^n$.

Proof: 1. is immediate, 3. is a consequence of 2. To estimate $\|AB\|$, we apply twice (2.5)

$$\|(AB)x\| \;\le\; \|A\|\,\|Bx\| \;\le\; \|A\|\|B\|\|x\|\,. \tag{2.8}$$

To conclude we divide by $\|x\|$ and take the supremum over $x \neq 0$. ∎

Example 2.1.4 Let us denote by $\|A\|_p$, $p = 0, 1, \infty$ the operator norm of A acting on $\mathbf{R}^n$ with the norm $\|x\|_p$, $p = 0, 1, \infty$ (see (2.3)). Then we have the formulas

$$\begin{aligned} \|A\|_1 &= \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| \,, \\ \|A\|_\infty &= \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \,, \\ \|A\|_2 &= \sqrt{\text{biggest eigenvalue of } A^T A} \,. \end{aligned} \tag{2.9}$$

Proof: For $\|x\|_1$ we have

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} x_j \right| \le \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}| \, |x_j| = \sum_{j=1}^{n} |x_j| (\left(\sum_{i=1}^{m} |a_{ij}| \right) \le \max_{1 \le j \le n} \left(\sum_{i=1}^{m} |a_{ij}| \right) \|x\|_1 \,, \tag{2.10}$$

and therefore $\|A\|_1 \le \max_j (\sum_{i=1}^m |a_{ij}|)$. To prove the equality, choose j_0 such that $\sum_{i=1}^m |a_{ij_0}| = \max_j (\sum_{i=1}^m |a_{ij}|)$ and then set $x = (0, \cdots, 1, \cdots, 0)^T$ where the 1 is in position j_0. Then for such x we have equality in (2.10). This shows that $\|A\|_1$ cannot be smaller than $\max_j (\sum_{i=1}^m |a_{ij}|)$. The formula for $\|A\|_\infty$ is proved similarly.

For the norm $\|\cdot\|_2$ note that the matrix $A^T A$ is symmetric and positive semi-definite ($\langle x, A^T A x \rangle = \|Ax\|_2 \ge 0$). We can diagonalize it: there exists an orthogonal matrix U ($U^T U = 1$) such that $U^T A^T A U = \text{diag}(\lambda_1, \cdots, \lambda_n)$, where $\lambda_i \ge 0$. With $x = Uy$ ($\|x\| = \|y\|$) we obtain

$$\|Ax\|_2^2 \;=\; \langle x, A^T A x \rangle \;=\; \langle y, U^T A^T A U y \rangle \;=\; \sum_{i=1}^{n} \lambda_i |y_i|^2 \;\le\; \lambda_{\max} \|y\|^2 \;=\; \|x\|^2 \,. \tag{2.11}$$

This implies that $\|A\|_2 \le \sqrt{\lambda_{\max}}$. To show equality choose x to be an eigenvector for $\lambda_{\max}$. ∎

In order to solve linear ODE's we will need to construct the exponential of a $n \times n$ matrix A which we will denote by e^A. We will define it using the series representation of the exponential function. If $\{C_k\}$ is a sequence with $C_k \in \mathcal{L}(\mathbf{R}^n)$ we define infinite series as usual: $C = \sum_{k=0}^\infty C_k$ if and only if the partial sums converge. The convergence of $C = \sum_{k=0}^\infty C_k$ is equivalent to the convergence of the n^2 series of the coefficients $\sum_k c_{ij}^{(k)}$. We say that the series converges absolutely if the real series $\sum \|C_k\|$ converges. For any norm, there exist positive constants a and A such that $a|c_{ij}^{(k)}| \le \|C_k\| \le A \sum_{ij} |c_{ij}^{(k)}|$ and therefore absolute convergence of the series is equivalent to the absolute convergence of the n^2 series $\sum_k c_{ij}^{(k)}$.

Proposition 2.1.5 *Let $A \in \mathcal{L}(\mathbf{R}^n)$. Then*

1. *For any $T > 0$, the series*
$$e^{tA} := \sum_{j=0}^{\infty} \frac{t^j A^j}{j!}\,, \tag{2.12}$$
converges absolutely and uniformly on $[-T, T]$, e^{tA} is a continuous function of t and we have
$$\left\| e^t A \right\| \leq e^{t\|A\|}\,. \tag{2.13}$$

2. *The map $t \to e^{tA}$ is everywhere differentiable and*
$$\frac{d}{dt} e^t A = A e^{tA} = e^{tA} A\,. \tag{2.14}$$

Proof: Item 1. For $t \in [-T, T]$ we have
$$\left\| \frac{t^j A^j}{j!} \right\| \leq \frac{|t|^j \|A\|^j}{j!} \leq \frac{T^j \|A\|^j}{j!}\,. \tag{2.15}$$

Let us denote by $S_n(t)$ the partial sum $\sum_{j=0}^{n} \frac{t^j A^j}{j!}$. Then, for $m > n$, we have
$$\|S_n(t) - S_m(t)\| \leq \sum_{j=n+1}^{m} \frac{T^j \|A\|^j}{j!} \leq \sum_{j=n+1}^{\infty} \frac{T^j \|A\|^j}{j!}\,. \tag{2.16}$$

This implies that $S_n(t)$ is a Cauchy sequence in $\mathcal{L}(\mathbf{R}^n)$, uniformly in $t \in [-T, T]$, since the right side (2.16) is the remainder term for the series $e^{T\|A\|}$. The function $S_n(t)$ are continuous function, they converge uniformly on $[-T, T]$ and $\mathcal{L}(\mathbf{R}^n)$ is a Banach space so that the limit e^{tA}, exists and is continuous. The bound (2.13) follows immediately from (2.15).

Item 2. The partial sum $S_n(t)$ are differentiable function of t with
$$S_n'(t) = A S_{n-1}(t) = S_{n-1}(t) A\,. \tag{2.17}$$

The same argument as in 1. shows that $S_n'(t)$ converges uniformly on $[-T, T]$. Since both $S_n(t)$ and $S_n'(t)$ converge uniformly we can exchange limit and differentiation. If we take the limit $n \to \infty$ in (2.17) we obtain (2.14). ∎

We summarize some properties of the exponential in

Proposition 2.1.6 *Let $A, B, C \in \mathcal{L}(\mathbf{R}^n)$. Then*

1. *If $AB = BA$ then $e^{A+B} = e^A e^B$.*

2. *If C is invertible then $e^{C^{-1}AC} = C^{-1} e^A C$.*

3. *If $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ then $e^A = \mathrm{diag}(e_1^{\lambda}, \cdots, e_n^{\lambda})$.*

Proof: If $AB = BA$ then, using the binomial theorem, we obtain

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \tag{2.18}$$

and therefore

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k B^{n-k}}{k!(n-k)!} = \sum_{p=0}^{\infty} \frac{A^p}{p!} \sum_{q=0}^{\infty} \frac{B^q}{q!} = e^A e^B, \tag{2.19}$$

and this proves 1.

It is easy to see that $C^{-1}A^kC = (C^{-1}AC)^k$. Dividing by $k!$, summing and taking the limit proves 2. If $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, then $A^k = \mathrm{diag}(\lambda_1^k, \cdots, \lambda_n^k)$ and this proves 3. ∎

As a consequence we obtain

Corollary 2.1.7 *Let $A \in \mathcal{L}(\mathbf{R}^n)$. Then*

1. $(e^{tA})^{-1} = e^{-tA}$.
2. $e^{(t+s)A} = e^{tA}e^{sA}$.
3. $e^{\lambda I + A} = e^{\lambda}e^{A}$.

2.2 General theory

We discuss first general properties of the differential equations $x' = A(t)x + g(t)$.

Theorem 2.2.1 (Global existence and uniqueness) *Let $I = [a, b]$ be an (arbitrary) interval and suppose that $A(t)$ and $g(t)$ are continuous function on I. Then the Cauchy problem $x' = A(t)x + g(t)$, $x(t_0) = x_0$ (with $t_0 \in I$, $x_0 \in \mathbf{R}^n$) has a unique solution on I.*

Proof: The function $f(t, x) = A(t)x + g(t)$ is continuous and satisfy a Lipschitz condition on $I \times \mathbf{R}^n$. Therefore the solution is unique wherever it exists. Moreover on $I \times \mathbf{R}^n$ we have the bound $\|f(t, x)\| \leq a\|x\| + b$ where $a = \sup_{t \in I} \|A(t)\|$ and $b = \sup_{t \in I} \|g(t)\|$. Therefore we have the bound $\|x(t)\| \leq \|x_0\| + \int_{t_0}^{t} (a\|x(s)\| + b)\, ds$ for $t_0, t \in I$. Gronwall lemma implies that $\|x(t)\|$ remains bounded if $t \in I$. ∎

Example 2.2.2 Let us apply Picard-Lindelöf iteration in the special case where $A(t)$ commutes with $B(t) := \int_{t_0}^t A(s)ds$ and $g(t) \equiv 0$. This is the case if A does not depends on t, since $B = (t - t_0)A$ commutes with A, or if $x \in \mathbf{R}$ is one dimensional, in which case $A(t)$ reduces to a number. From the commutativity of A with B it follows that

$$\frac{d}{dt} B^m(t) \;=\; mA(t)B^{m-1}(t)\,. \tag{2.20}$$

We start the Picard iteration with $x_0(t) \equiv x_0$. The first iteration is

$$x_1(t) \;=\; x_0 + \int_{t_0}^t A(s)x_0(s)\,ds \;=\; (I + B(t))x_0\,, \tag{2.21}$$

or, equivalently,

$$\frac{d}{dt} x_1(t) \;=\; A(t)x_0\,, \quad x_1(0) = x_0\,. \tag{2.22}$$

The m-th iteration $x_m(t) \;=\; x_0 + \int_{t_0}^t A(s)x_{m-1}(s)\,ds$ satisfy the differential equation

$$\frac{d}{dt} x_m(t) \;=\; A(t)x_{m-1}(t)\,, \quad x_m(0) = x_0\,. \tag{2.23}$$

By induction, using (2.20), one sees that the solution is given by

$$x_m(t) \;=\; I + B(t) + \frac{B^2(t)}{2!} + \cdots \frac{B^m(t)}{m!}\,. \tag{2.24}$$

The solution converges to the solution

$$x(t) \;=\; e^{B(t)}x_0 \;=\; e^{\int_{t_0}^t A(s)\,ds}x_0\,. \tag{2.25}$$

Be aware that, in general however, it is **not true** that A commutes with $\int_{t_0}^t A(s)\,ds$ and the solution does not have the simple form (2.25).

In the special case where $A(t) \equiv A$ does not depend on t then we have from (2.25)

$$x(t) \;=\; e^{(t-t_0)A}x_0\,, \tag{2.26}$$

and this is the solution of a system with constant coefficients. In one dimension we can also use (2.25), for example the ode $x'(t) \;=\; 2\cos(t)x$, $x(\pi) = 5$ has the solution $x(t) = (5/e^2)e^{2\sin(t)}$.

Theorem 2.2.3 (Superposition principle) *Let I be an interval and let $A(t)$, $g_1(t)$, $g_2(t)$ be continuous function on I. If*

$$\begin{aligned} x_1 \,:\, I \to \mathbf{R}^n \quad & \text{is a solution of } x' = A(t)x + g_1(t)\,, \\ x_2 \,:\, I \to \mathbf{R}^n \quad & \text{is a solution of } x' = A(t)x + g_2(t)\,, \end{aligned}$$

then

$$x(t) := c_1x_1(t) + c_2x_2(t) \,:\, I \to \mathbf{R}^n \quad \text{is a solution of } x' = A(t)x + (c_1g_1 + c_2g_2(t))\,.$$

Proof: : This a simple exercise. ∎

This theorem has very important consequences.

Homogeneous equations Let us consider homogeneous Cauchy problem $x' = A(t)x$, $x(t_0) = x_0$ and let denote its solutions $x(t, t_0, x_0)$ to indicate explicitly the dependence on the initial data.

- (a) The solution $x(t, t_0, x_0)$ depends linearly on the initial condition x_0, i.e.,

$$x(t, t_0, c_1 x_0 + c_2 y_0) \;=\; c_1 x(t, t_0, x_0) + c_2 x(t, t_0, y_0)\,. \tag{2.27}$$

  This follows by noting that, by linearity, both sides are solutions of the ODE and have the same initial conditions. The uniqueness of the solutions implies then the equality. As a consequence there exists a linear map $R(t, t_0) : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$R(t, t_0) x_0 \;=\; x(t, t_0, x_0)\,. \tag{2.28}$$

  It maps the initial condition x_0 at time t_0 to the position at time t. The linear map $R(t, t_0)$ is called the *resolvent* of the differential equation $x' = A(t)x$. The i-th column of $R(t, t_0)$ is a solution $x' = A(t)x$ with initial condition $x_0 = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ where 1 is in i-th position.

- (b) If $x_0 = 0$, then $x(t) \equiv 0$ for all $t \in I$ (The point 0 is called a *critical point*). As a consequence if $x(t)$ is a solution and it vanishes at some point t, then it is in fact identically 0.

- (c) The set of solutions of $x' = A(t)x$ form a vector space. We call a set of solutions $x_1(t), \cdots, x_k(t)$ *linearly dependent* if there exists constants $c_1, \cdots, c_k$, with at least one $c_i \neq 0$, such that

$$c_1 x_1(t) + \cdots + c_k x_k(t) \;=\; 0\,. \tag{2.29}$$

  Note that by (b), if (2.29) holds at one point t, it holds at any point t. Therefore if the initial condition $x_1(t_0), \cdots x_k(t_0)$ are linearly dependent, then the corresponding solutions are linearly dependent. The k solutions are called *linearly independent* if they are not linearly dependent, i.e., $c_1 x_1(t) + \cdots + c_k x_k(t) = 0$ implies that that $c_1 = \cdots = c_k = 0$.

- (d) From (c) it follows that there exist exactly n linearly independent solutions, $x_1, \cdots, x_n$. Every such set of n linearly independent solutions is called a *fundamental system* of solutions. Any solution x of $x' = A(t)x$ can be written, in a unique way, as a linear combination

$$x(t) = a_1 x_1(t) + \cdots + a_n x_n(t)\,. \tag{2.30}$$

- (e) A system of n linearly independent solutions can be arranged in a matrix $\Phi(t) = (x_1(t), \cdots, x_n(t))$. In this notation the i-th column of $\Phi(t)$ is the vector $x_i(t)$. The matrix $\Phi(t)$ is called a *fundamental matrix* or a *Wronskian* for $x' = A(t)x$. It satisfies the differential equation

$$\frac{d}{dt}\Phi(t) \;=\; A(t)\Phi(t)\,. \tag{2.31}$$

- (f) If $\Phi(t)$ is a fundamental matrix then the resolvent is given by

$$R(t,t_0) \;=\; \Phi(t)\Phi(t_0)^{-1}\,. \tag{2.32}$$

  Indeed $x(t) = \Phi(t)\Phi(t_0)^{-1}x_0$ satisfies $x' = A(t)x$ (because of (2.31)) and $x(t_0) = x_0$.

Theorem 2.2.4 (Properties of the resolvent) *Let $A(t)$ be continuous on the interval I. Then the resolvent of $x' = A(t)x$ satisfies*

1. $\frac{\partial}{\partial t}R(t,t_0) \;=\; A(t)R(t,t_0)$.
2. $R(t_0,t_0) = I$ *(the identity matrix).*
3. $R(t,t_0) \;=\; R(t,t_1)R(t_1,t_0)$.
4. $R(t,t_0)$ *is invertible and* $R(t,t_0)^{-1} = R(t_0,t)$.

Proof: We have $\frac{\partial}{\partial t}R(t,t_0)x_0 \;=\; \frac{\partial}{\partial t}x(t,t_0,x_0) \;=\; A(t)R(t,t_0)x_0$ and $R(t_0,t_0)x_0 = x_0$ for any $x_0 \in \mathbf{R}^n$. This proves 1. and 2. Item 3 simply says that $x(t,t_0,x_0) = x(t,t_1,x(t_1,t_0,x_0))$. Item 4 follows from 2. and 3. by setting $t = t_0$.

Example 2.2.5 The harmonic oscillator $x'' + \kappa x = 0$ can be written with $x_1 = x$ and $x_2 = x'$ as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2.33}$$

One can find easily two linearly independent solutions, namely

$$\begin{pmatrix} \cos(\sqrt{\kappa}t+\phi) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t+\phi) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sin(\sqrt{\kappa}t+\phi) \\ \sqrt{\kappa}\cos(\sqrt{\kappa}t \end{pmatrix} \tag{2.34}$$

By defintion, the resolvent is the fundamental solution $(x_1(t), x_2(t))$ with $x_1(t) = (1,0)^T$ and $x_2(t_0) = (0,1)^T$ so that we have

$$R(t,t_0) \;=\; \begin{pmatrix} \cos(\sqrt{\kappa}(t-t_0)) & \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}(t-t_0)) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}(t-t_0)) & \cos(\sqrt{\kappa}(t-t_0)) \end{pmatrix}. \tag{2.35}$$

Note that the relation $R(t,t_0) = R(t,s)R(s,t_0)$ is simply the addition formula for sine and cosine.

Theorem 2.2.6 (Liouville) *Let $A(t)$ be continuous on the interval I and let $\Phi(t)$ be a fundamental matrix of $x' = A(t)x$. Then*

$$\det \Phi(t) \;=\; \det \Phi(t_0) \exp\left(\int_{t_0}^{t} \operatorname{trace} A(s)\, ds\right) , \tag{2.36}$$

where $\operatorname{trace} A(t) := a_{11}(t) + \cdots + a_{nn}(t)$.

Proof: Let $\Phi(t) = (\phi_{ij}(t))_{i,j=1}^{n}$. From linear algebra we know that $\det(A)$ is a multilinear function of the rows of A. It follows that

$$\frac{d}{dt}\Phi(t) \;=\; \sum_{i=1}^{n} \det D_i(t) \quad \text{where} \quad D_i(t) \;=\; \begin{pmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \cdots & & \cdots \\ \phi'_{i1}(t) & \cdots & \phi'_{in}(t) \\ \cdots & & \cdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{pmatrix} . \tag{2.37}$$

The matrix $D_i(t)$ is obtained from $\Phi(t)$ by replacing the i-th line by its derivative. We have $\Phi'(t) = A(t)\Phi(t)$, i.e., $\phi'_{ij}(t) \;=\; \sum_{k=1}^{n} a_{ik}(t)\phi_{kj}(t)$. Using the multilinearity of the determinant we find

$$\begin{aligned} \frac{d}{dt}\Phi(t) \;&=\; \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(t) \det \begin{pmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \cdots & & \cdots \\ \phi_{k1}(t) & \cdots & \phi_{kn}(t) \\ \cdots & & \cdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{pmatrix} \longleftarrow i - \text{th line} \\ &=\; \left(\sum_{i=1}^{n} a_{ii}(t)\right) \det \Phi(t) \,. \end{aligned} \tag{2.38}$$

Solving this differential equation for $\det \Phi(t)$, see Example 2.2.2, proves the theorem. ∎

Remark 2.2.7 The Liouville theorem has the following useful interpretation. If $V = (v_1, \cdots, v_n)$ is a matrix whose columns are the vectors $v_1, \cdots, v_n$, then $|\det V|$ is the volume of the parallelepiped spanned by $v_1, \cdots, v_n$. Using $\det A^{-1} = 1/\det A$, Liouville Theorem is equivalent to

$$\det R(t, t_0) \;=\; \exp\left(\int_{t_0}^{t} \operatorname{trace} A(s)\, ds\right) . \tag{2.39}$$

If, at time t_0 we start with a set of initial conditions B of volume, say, 1 (e.g. a unit cube), at time t the set B is mapped to a set a parallelepiped $R(t, t_0)B$ of volume $\exp\left(\int_{t_0}^{t} \operatorname{trace} A(s)\, ds\right)$.

In particular, if $\operatorname{tr} A(t) \equiv 0$, then the flow defined by the equation $y' = A(t)y$ preserves volume. We have such a situation in Example 2.2.5, see (2.33).

Inhomogeneous equations We consider the equation

$$x' = A(t)x + g(t)\,. \tag{2.40}$$

Theorem 2.2.8 *Let $\bar{x}(t)$ be a fixed solution of the inhomogeneous equation (2.40). If $x(t)$ is a solution of the homogeneous equation, then $x(t) + \bar{x}(t)$ is a solution of the homogeneous equation and all solutions of the inhomogeneous equation are obtained in this way.*

Proof: : This is an easy exercise. ∎

If we know how to solve the homogeneous problem, i.e. if we know the resolvent $R(t, t_0)$, our task is then to find just *one* solution of the inhomogeneous equation.

Theorem 2.2.9 (Variation of constants or Duhamel formula) *Let $A(t)$ and $g(t)$ be continuous on the interval I and let $R(t, t_0)$ be the resolvent of the homogeneous equation $x' = A(t)x$. Then the solution of the Cauchy problem $x' = A(t)x + g(t)$ is given by*

$$x(t) \;=\; R(t, t_0)x_0 + \int_{t_0}^{t} R(t, s)g(s)\,ds\,. \tag{2.41}$$

Proof: The general solution of the homogeneous equation has the form $R(t, t_0)c$ with $c \in \mathbf{R}^n$. The idea is to "vary the constants" and to look for a solution of the inhomogeneous problem of the form

$$x(t) = R(t, t_0)c(t)\,. \tag{2.42}$$

We must then have, using 1. of Theorem 2.2.4,

$$\begin{aligned} x'(t) &= R'(t, t_0)c(t) + R(t, t_0)c'(t) \;=\; A(t)R(t, t_0)c(t) + R(t, t_0)c'(t) \\ &= A(t)R(t, t_0)c(t) + g(t)\,. \end{aligned} \tag{2.43}$$

Thus me must have

$$c'(t) \;=\; R(t, t_0)^{-1}g(t) \;=\; R(t_0, t)g(t)\,, \tag{2.44}$$

and, integrating, this gives $c(t) = c(t_0) + \int_{t_0}^{t} R(t_0, s)g(s)\,ds$. Inserting this formula in (2.42) gives the result. ∎

In the special case, see example 2.2.2, where $A(t) \equiv A$ is independent of t, the variation of constants gives

$$x(t) \;=\; e^{(t-t_0)A} + \int_{t_0}^{t} e^{(t-s)A}g(s)\,ds\,. \tag{2.45}$$

and this formula is referred to as Duhamel formula.

It should be noted that, in general, the computation of the resolvent for $x' = A(t)x$ is not easy and can rarely be done explicitly if A depends on t.

Example 2.2.10 Forced harmonic oscillator We consider the differential equation $x'' + x = f(t)$, or equivalently the first order system $x' = y$, $y' = -x - f(t)$. The resolvent is given by (2.35). The solution of the above system with initial conditions $(x(0), y(0))^T = (x_0, y_0)^T$ is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = R(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t \begin{pmatrix} f(s)\sin(t-s) \\ f(s)\cos(t-s) \end{pmatrix} ds\,, \tag{2.46}$$

so that

$$x(t) = \cos(t)x_0 + \sin(t)y_0 + \int_0^t f(s)\sin(t-s)\,ds\,. \tag{2.47}$$

For example if $f(t) = \cos(\sqrt{\kappa}t)$ we find

$$x(t) = \cos(t)x_0 + \sin(t)y_0 + \begin{cases} \frac{\sqrt{\kappa}}{1-\kappa}(\cos(t) - \cos(\sqrt{\kappa}t)) & \kappa \neq 1 \\ \frac{1}{2}t\sin(t) & k = 1 \end{cases}\,. \tag{2.48}$$

The motion is quasi-periodic if $\sqrt{\kappa}$ is irrational, periodic if $\sqrt{\kappa}$ is rational (and $\neq 1$), and the solution grows as $t \to \infty$ if $\kappa = 1$ (resonance).

2.3 Linear systems with constant coefficients

We compute the resolvent of

$$x' = Ax\,, \tag{2.49}$$

where A is a constant matrix of order n. In general the coefficients a_{ij} and the solutions can be either real or complex. As we have seen in example 2.2.2 using Picard-Lindelöf iteration we have

Theorem 2.3.1 *The resolvent of the linear equation with constant coefficients $x' = Ax$ is given by*

$$R(t, t_0) = e^{(t-t_0)A}\,. \tag{2.50}$$

Proof: We give another proof. From proposition 2.1.5 we obtain

$$\frac{d}{dt}e^{(t-t_0)A} = Ae^{(t-t_0)A}\,. \tag{2.51}$$

Thus the j-th column of $e^{(t-t_0)A}$ is the solution of the Cauchy problem $x' = Ax$, $x(t_0) = (0, \cdots, 0, 1, 0, \cdots)^T$ where the 1 is in j-th position. ■

Solving $x' = Ax$ is thus reduced to the problem of computing the exponential of a matrix A.

The basic idea to find solutions is try to find solutions of the form $x(t) = e^{\lambda t}v$ where v is a nonzero vector. Inserting we find

$$x'(t) \;=\; \lambda e^{\lambda t} v \;=\; e^{\lambda t} A v \,, \tag{2.52}$$

from which we deduce that $e^{\lambda t}$ is a solution if and only if

$$Av \;=\; \lambda v \,, \tag{2.53}$$

i.e., λ is an eigenvalue of A with a corresponding eigenvector v. From (2.53) one sees that λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial $\det(A - \lambda I)$.

As is well known from linear algebra there are two cases. Let λ be an eigenvalue, if λ is a k-fold zero of the characteristic polynomial, we say that its *algebraic multiplicity* is equal to k. The dimension of the eigenspace of λ, i.e. the maximal number of its linearly independent eigenvector is called the *geometric multiplicity* of λ. In general the geometric multiplicity is smaller than the algebraic multiplicity.

We will distinguish two cases

(a) (A is diagonalizable) For all eigenvalues of A the algebraic multiplicity is equal to the geometric multiplicity. In this situation there exist n linearly independent vectors $v_1, \cdots, v_n$ with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$. To find the resolvent we consider the matrix $V = (v_1, \cdots, v_n)$. We have

$$AV \;=\; (\lambda_1 v_1, \cdots, \lambda_n v_n) \;=\; V \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \,. \tag{2.54}$$

Therefore we have

$$V^{-1} A V \;=\; \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \,. \tag{2.55}$$

Since $\exp\left(t \mathrm{diag}(\lambda_1, \cdots, \lambda_n)\right) \;=\; \mathrm{diag}(e^{t\lambda_1}, \cdots, e^{t\lambda_n})$ we find that the resolvent is

$$e^{(t-t_0)A} \;=\; V \mathrm{diag}(e^{(t-t_0)\lambda_1}, \cdots, e^{(t-t_0)\lambda_n}) V^{-1} \,. \tag{2.56}$$

We discuss real and complex solutions. If the matrix is complex, then there are n linearly independent (complex) solutions corresponding to the the n eigenvectors of A. If the matrix A is real, i.e., the coefficients a_{ij} are real, then complex eigenvalues can occur, but they must come in pairs of complex conjugate $\lambda = \mu + i\nu$ and $\overline{\lambda} = \mu - i\nu$. This follows from the fact that if $Av = \lambda v$, then $\overline{Av} = A\overline{v} = \overline{\lambda}\overline{v}$. Corresponding to a pair of complex conjugate eigenvalue, λ and $\overline{\lambda}$, there are therefore two complex conjugate solutions $z(t) = e^{\lambda t}v$ and $\overline{z}(t) = e^{\overline{\lambda} t}\overline{v}$. By linearity we can construct real solutions by taking the real and imaginary part of $z(t)$. If $v = a + ib$, then

$$\begin{aligned} x(t) &= \mathrm{Re} z(t) \;=\; e^{\mu t}\left(a\cos(vt) - b\sin(\nu t)\right) \,, \\ y(t) &= \mathrm{Im} z(t) \;=\; e^{\mu t}\left(a\sin(vt) + b\cos(\nu t)\right) \,, \end{aligned} \tag{2.57}$$

are two linearly independent real solutions. In this way we obtain n linearly independent real solutions. The resolvent is constructed as before using the vectors a and b, instead of v and $\overline{v}$. We have

$$A(a+ib) \;=\; Aa+iAb \;=\; (\mu+i\nu)(a+ib) \;=\; (\mu a-\nu b)+i(\mu b+\nu a)\,. \tag{2.58}$$

Let us consider first a 2 by 2 matrix with complex conjugate eigenvalues. Thus, with $U=(a,b)$, we have

$$AU \;=\; (\mu a-\nu b, \mu b+\nu a) \;=\; \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}(a,b)\,, \tag{2.59}$$

so that

$$A \;=\; U\begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}U^{-1}\,. \tag{2.60}$$

We also find that

$$\begin{aligned} e^{tA}U &= e^{\mu t}\left(\cos(\nu t)a-\sin(\nu t)b\,,\,\cos(\nu t)a+\sin(\nu t)b\right) \\ &= U\begin{pmatrix} e^{\mu t}\cos(\nu t) & e^{\mu t}\sin(\nu t) \\ -e^{\mu t}\sin(\nu t) & e^{\mu t}\cos(\nu t) \end{pmatrix}\,. \end{aligned} \tag{2.61}$$

so that

$$e^{tA} \;=\; U\begin{pmatrix} e^{\mu t}\cos(\nu t) & e^{\mu t}\sin(\nu t) \\ -e^{\mu t}\sin(\nu t) & e^{\mu t}\cos(\nu t) \end{pmatrix}U^{-1} \tag{2.62}$$

The general case is treated by considering the eigenspace for each pair of eigenvectors separately: suppose that A has $2k$ complex eigenvalues $\lambda_1=\mu_1+i\nu_1$, $\lambda_1-i\nu_1$, $\cdots$ with eigenvectors $v_1=a_1+ib_1$, $v_2=a_2-ib_2$, $\cdots$ and $n-2k$ real eigenvalues with eigenvectors $v_{n-2k+1},\cdots v_n$. Then we set $V=(a_1,b_1,\cdots v_{n-2k+1},\cdots v_n)$ and the resolvent is

$$R(t,t_0) \;=\; V\begin{pmatrix} e^{\mu_1 t}\cos(\nu_1 t) & e^{\mu_1 t}\sin(\nu_1 t) & 0 & \cdots & 0 \\ -e^{\mu_1 t}\sin(\nu_1 t) & e^{\mu_1 t}\cos(\nu_1 t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_{n-1}t} & 0 \\ 0 & 0 & \cdots & 0 & e^{\lambda n t} \end{pmatrix}V^{-1} \tag{2.63}$$

Example 2.3.2

$$\begin{array}{rcrcrcr} x_1' &=& x_1 &-& 2x_2 && \\ x_2' &=& 2x_1 &&&-& x_3\;, \\ x_3' &=& 4x_1 &-& 2x_2 &-& x_3 \end{array} \qquad A \;=\; \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{pmatrix}\,. \tag{2.64}$$

The eigenvalues are the root of $\det(A - \lambda I) = 2 - \lambda - \lambda^3 = (1-\lambda)(\lambda^2 + \lambda + 2)$. The eigenvalues are $\lambda_{1,2} = -1/2 \pm i\sqrt{7}/2$, and $\lambda_3 = 1$. The eigenvectors are computed to be $v_{1,2} = (3/2 \pm i\sqrt{7}/2, 2, 4)^T$ and $v_3 = (1, 0, 2)^T$ We have then

$$V = \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{7}}{2} & 1 \\ 2 & 0 & 0 \\ 4 & 0 & 2 \end{pmatrix}. \tag{2.65}$$

(b) (*A* is not diagonalizable) This is the case if, for at least one eigenvalue, the geometric multiplicity is smaller than the algebraic multiplicity. The idea is then to transform A into a simpler form, for example in a triangular form or in Jordan normal form, i.e., one finds an invertible T such that $T^{-1}AT$ has such form. With the transformation $x = Ty$ and $x' = Ty'$ the ODE $x' = Ax$ becomes $y' = Sy$. For example if S has a triangular form we have the system

$$\begin{array}{lcccccccc} x_1' & = & s_{11}x_1 & + & s_{12}x_2 & + & \cdots & + & s_{1n}x_n \\ x_2' & = & & & s_{22}x_2 & + & \cdots & + & s_{2n}x_n \\ \vdots & & & & & & & & \vdots \\ x_n' & = & & & & & & & s_{nn}x_n \end{array} \tag{2.66}$$

One can then solve the system iteratively: one solves first the equation for y_n, then the one for y_{n-1}, and so on up to the equation for y_1 (see the example below). Finally one obtains $x = Ty$.

Example 2.3.3 Consider the system of equations

$$\begin{array}{lcccccc} x_1' & = & -3x_1 & + & 2x_2 & + & 5x_3 \\ x_2' & = & & + & x_2 & - & x_3 \\ x_3' & = & & & & & 2x_3 \end{array}, \qquad A = \begin{pmatrix} -3 & 2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.67}$$

with initial conditions $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30})$. The third equations has solution $x_3(t) = e^{2t}x_{30}$. Inserting into the second equations gives the inhomogeneous equation $x_2' = 2x_2 - e^{2t}x_{30}$ and is solved using Duhamel formula

$$x_2(t) = e^t x_{20} - \int_0^t e^{t-s}e^{2s}x_{30} = e^t x_{20} + (e^t - e^{2t})x_{30}. \tag{2.68}$$

Inserting the the solutions $x_2(t)$ and $x_3(t)$ into the first equations gives the equation $x_1' = -3x_1 + 2e^t x_{20} + (2e^t + 3e^{2t})x_{30}$. Again with Duhamel formula one finds

$$x_1(t) = e^{-3t}x_{10} + 2(e^t - e^{-3t})x_{20} + (2e^t + 3e^{2t} - 5e^{-3t})x_{30}. \tag{2.69}$$

The resolvent is then

$$R(t, t_0) = \begin{pmatrix} e^{-3(t-t_0)} & 2e^{(t-t_0)} - 2e^{-3(t-t_0)} & 2e^{(t-t_0)} + 3e^{2(t-t_0)} - 5e^{-3(t-t_0)} \\ 0 & e^{(t-t_0)} & e^{(t-t_0)} - e^{2(t-t_0)} \\ 0 & 0 & e^{2(t-t_0)} \end{pmatrix}. \tag{2.70}$$

The resolvent can be computed easily if S is in Jordan normal form:

$$S = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} \tag{2.71}$$

where each Jordan block J_i has the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix} \tag{2.72}$$

It is easy to see that

$$e^{tS} = \begin{pmatrix} e^{tJ_1} & & \\ & \ddots & \\ & & e^{tJ_k} \end{pmatrix}, \tag{2.73}$$

so that it is enough to compute e^{tJ} where J is a Jordan block. A Jordan block has the form $J = \lambda I + K$ where K has the form

$$K = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \tag{2.74}$$

The matrix K is a *nilpotent* matrix. If J and K are $l \times l$ matrices then $K^m = 0$ for all $m \geq l$, we have

$$K^2 = \begin{pmatrix} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} \cdots K^{l-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 1 \\ & 0 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}. \tag{2.75}$$

and $K^l = 0$. Using these properties one can compute the resolvent easily

$$e^{tJ} = e^{t(\lambda I + K)} = e^{\lambda t} e^{tK} = e^{t\lambda}\left(1 + tK + \frac{t^2K^2}{2!} + \cdots + \frac{t^{l-1}K^{l-1}}{(l-1)!}\right)$$

$$
= \begin{pmatrix}
e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \cdots & & \frac{t^{l-1}}{(l-1)!}e^{\lambda t} \\
 & e^{\lambda t} & te^{\lambda t} & \cdots & & \frac{t^{l-2}}{(l-2)!}e^{\lambda t} \\
 & & \ddots & \ddots & & \vdots \\
 & & & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\
 & & & & e^{\lambda t} & te^{\lambda t} \\
 & & & & & e^{\lambda t}
\end{pmatrix} \tag{2.76}
$$

Example 2.3.4 The system of equations

$$
\begin{array}{rcrcrcr}
x_1' & = & -2x_1 & + & x_2 & & \\
x_2' & = & & + & -2x_2 & & \\
x_3' & = & & & & & 2x_3
\end{array} , \tag{2.77}
$$

is already in Jordan normal form and its resolvent is

$$
e^{tS} = \begin{pmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} . \tag{2.78}
$$

The calculation of solution is easily accomplished once the Jordan normal form $S = T^{-1}AT$ and the transformation matrix T has been determined. The computations of the k solutions associated with the eigenvalue λ as well as the matrix T can be obtained, one step at a time.

(a) Compute all the eigenvalues $\lambda_1 \cdots, \lambda_k$ of A and determine their algebraic mutli-plicities $m_1, m_2, \cdots, m_k$. For each eigenvalue compute the all the linearly independent eigenvectors and determine the lagebraic multiplicity of λ_i. If n eigenvectors $v_1, \cdots, v_n$ are found in this way, then one finds n linearly indpendent solutions for $x' = Ax$ of the form $e^{\lambda_i t} v_j$, with $1 \leq i \leq k$, $1 \leq j \leq n$.

(b) Suppose that for some eigenvalue $\lambda \equiv \lambda_i$ the algebraic multiplicity, m, is bigger than the geometric multiplicity l. Let $v_1, \cdots, v_l$ be the corresponding eigenvectors which leads to solutions of the form $e^{\lambda t} v_1, \cdots e^{\lambda t} v_l$.

In order to find the remaining independent solutions one tries the Ansatz

$$
(u_1 + v_1 t)\, e^{\lambda t} \tag{2.79}
$$

Inserting the first Ansatz in the equation gives

$$
e^{\lambda t} \left(\lambda u_1 + v_1 + t\lambda v_1\right) \;=\; e^{\lambda t} \left(Au_1 + tAv_1\right) . \tag{2.80}
$$

Comparing the coefficients of t for the first Ansatz in (2.79) we obtain

$$
(A - \lambda)v_1 = 0\,, \quad (A - \lambda)u_1 = v_1\,. \tag{2.81}
$$

If this equation has no solution stop here. If this equation has a solution, then one has a solution of the form (2.79). If one needs more solutions try the Ansatz

$$\left(s_1 + u_1 t + v_1 \frac{t^2}{2}\right) e^{\lambda t} . \tag{2.82}$$

Inserting into the equation gives

$$(A - \lambda)v_1 = 0 , \quad (A - \lambda)u_1 = v_1 , \quad (A - \lambda)s_1 = u_1 \tag{2.83}$$

If this equation has no solution, or you have found all the solutions needed stop here. Otherwise try the ansatz

$$\left(r_1 + s_1 t + u_1 \frac{t^2}{2!} + v_1 \frac{t^3}{3!}\right) e^{\lambda t} . \tag{2.84}$$

This gives the method to compute the solutions and/or compute the Jordan normal form. For example for a 3×3 with one eigenvalue of algebraic multiplicity 3 and geometric multiplicity 1, one has with $T = (v_1, u_1, s_1)$

$$AT = (\lambda v_1, \lambda u_1 + v_1, \lambda s_1 + u_1) = (v_1, u_1, s_1) \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} , \tag{2.85}$$

so that $T^{-1}AT$ has Jordan normal form.

Example 2.3.5

$$\begin{array}{rcl} x' & = & x - y \\ y' & = & 4x - 3y \end{array} , \quad A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \tag{2.86}$$

The eigenvalue $\lambda = -1$ has algebraic multiplicity 2, $v = (1, 2)^T$ is the only (up to a constant) eigenvector of A so that the geometric multiplicity is 1. We obtain one solution $(x(t), y(t))^T = (e^{-t}, 2e^{-t})^T$. To obtain a second solution we make the Ansatz $(x(t), y(t)) = e^{-t}(a + t, b + 2t)^T$ and find the equation $A(a, b)^T = (-1, -2)$ which has $u = (0, -1)$ as a solution. The corresponding solution is then $(x(t), y(t))^T = e^{-t}(t, -1 + 2t)$. The matrix T is given by $T = (v, u)$.

2.4 Floquet theory

The case where $A(t)$ is periodic can be reduced, at least in principle, to the case of constants coefficients. We consider the system $x' = A(t)x$ where $A(t)$ is a continuous and periodic of period p, i.e., there exists $p > 0$ such that

$$A(t + p) = A(t) . \tag{2.87}$$

Before we state the results we will need to prove some fact on logarithm of matrices.

Proposition 2.4.1 (Logarithm of a matrix) *Let C be an invertible matrix, then there exists a (complex) matrix R such that*

$$C = e^R . \tag{2.88}$$

Proof: Since (2.88) is equivalent to $QCQ^{-1} = e^{QRQ^{-1}}$, it is sufficient to suppose that C is in Jordan normal form. One can treat each Jordan block separately so that we may assume that C has the form $J = \lambda I + K$, see (2.72) and (2.74). Recall that

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}, \quad \text{for } |t| < 1 \tag{2.89}$$

and that the formal rearrangement of power series

$$\sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \frac{(-t)^{j+1}}{j} \right)^n \frac{1}{n!} = 1 + t \tag{2.90}$$

is valid for $|t| \le 1$ (this is simply the identity $e^{\log(1+t)} = 1 + t$).

Let us write $J = \lambda(I + K/\lambda)$ and set $S = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} K^j}{j\lambda^j}$. Note that S is defined in fact by a finite sum, since $K^n = 0$ if n is sufficiently large. Since all powers of K commute, we can use the formal rearrangement (2.90) with t replace by K/λ to conclude that

$$e^S = I + K/\lambda . \tag{2.91}$$

There is no convergence issue at all, since S is defined by a finite sum. From this we obtain that

$$J = \lambda(I + K/\lambda) = \lambda e^S, = e^{\log(\lambda) I + S} . \tag{2.92}$$

This concludes the proof of Proposition 2.4.1 with $R = \log(\lambda)I + S$. ∎

Theorem 2.4.2 (Floquet) *Let $A(t)$ be a continuous periodic function of period p. Then any fundamental matrix $\Phi(t)$ for $x' = A(t)x$ has a representation of the form*

$$\Phi(t) = P(t)e^{Rt} , \quad P(t+p) = p(t) , \tag{2.93}$$

where R is a constant matrix.

Remark 2.4.3 The theorem 2.4.2 gives us the form of the solutions. If x_0 is an eigenvector of R for the eigenvalue λ, then the solution $x(t)$ has the form $z(t)e^{\lambda t}$, where $z(t) = P(t)x_0$ is periodic with period p. More generally, by the discussion of Jordan blocks in Section 2.3, a general solution will have components which is a linear combination of terms of the form $\alpha(t)t^k e^{\lambda t}$, where $\alpha(t)$ is a vector periodic in t.

Proof of Theorem 2.4.2: (a) We note first that if $\Phi_1(t)$ and $\Phi_2(t)$ are two fundamental matrices, then there exists an invertible matrix C such that

$$\Phi_1(t) \;=\; \Phi_2(t)C\,. \tag{2.94}$$

This follows from the fact that

$$R(t,t_0) \;=\; \Phi_1(t)\Phi_1^{-1}(t_0) \;=\; \Phi_2(t)\Phi_2^{-1}(t_0)\,, \tag{2.95}$$

i.e.,

$$\Phi_1(t) \;=\; \Phi_2(t)\Phi_2^{-1}(t_0)\Phi_1(t_0)\,. \tag{2.96}$$

(b) If $x(t)$ is a solution of $x' = A(t)x$, then one verifies easily that $y(t) = x(t+p)$ is also a solution. Therefore if $\Phi(t)$ is a fundamental matrix, then $\Psi(t) \;=\; \Phi(t+p)$ is also a fundamental matrix. By (a) and Proposition 2.4.1, there exist a matrix R such that

$$\Phi(t+p) \;=\; \Phi(t)e^{pR}\,. \tag{2.97}$$

We now set

$$Q(t) \;=\; \Phi(t)e^{-tR} \tag{2.98}$$

and $Q(t)$ is periodic of period p since

$$Q(t+p) \;=\; \Phi(t+p)e^{-(t+p)R} \;=\; \Phi(t)e^{pR}e^{-(t+p)R} \;=\; Q(t)\,. \tag{2.99}$$

This concludes the proof. ∎

The matrix C is called the *transition* matrix. The eigenvalues, λ_i, of $C = e^{pR}$ are called the Floquet multipliers. The matrix C depends on the fundamental matrix $\Phi(t)$ but the eigenvalues do not (see homework). The eigenvalues of R, μ_i are given by $\lambda_i = e^{p\mu_i}$ are called the *characteristic exponents.* They are unique up to a multiple of $2\pi i/p$.

Remark 2.4.4 For the equation $x' = A(t)x$, let us consider the transformation

$$x(t) \;=\; P(t)y(t) \tag{2.100}$$

where $P(t)$ is the periodic matrix given by Floquet theorem. We obtain

$$x'(t) \;=\; P'(t)y(t) + P(t)y'(t) \;=\; A(t)P(t)y(t)\,. \tag{2.101}$$

On the other hand $P(t) = \Phi(t)e^{-Rt}$, so that

$$P'(t) \;=\; \Phi'(t)e^{-Rt} - \Phi(t)e^{-Rt}R \;=\; A(t)P(t) - P(t)R\,. \tag{2.102}$$

Thus we find

$$y'(t) = Ry(t)\,. \tag{2.103}$$

The transformation $x = P(t)y$ reduces the linear equation with periodic coefficients $x' = A(t)x$ to the system with constant coefficients $y' = By$. Nevertheless there are, in general, no methods available to compute $P(t)$ or the Floquet multipliers. Each equation has to be studied for itself and entire books are devoted to quite simple looking equations. The Floquet theory is very useful to study the stability of periodic solutions.

Chapter 3

Dependence on initial conditions and parameters

3.1 The flow defined by ODE's : dynamical systems

For the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, let us denote the solution by $x(t,t_0,x_0)$. Let us consider the map $\phi^{t,t_0} : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$\phi^{t,t_0}(x_0) \,=\, x(t,t_0,x_0)\,. \tag{3.1}$$

The map ϕ^{t,t_0} maps the initial position x_0 at time t_0 to the position at time t, $x(t,t_0,x_0)$. By definition the maps ϕ^{t,t_0} satisfy the relation

$$\phi^{t+s,t_0}(x_0) \,=\, \phi^{t+s,t}(\phi^{t,t_0}(x_0))\,. \tag{3.2}$$

Let us assume that the ODE is autonomous, i.e., $f(x)$ does not depend explicitly on t. We have then

Lemma 3.1.1 (Translation property) *Suppose that $x(t)$ is a solution of $x' = f(x)$, then $x(t-t_0)$ is also a solution.*

Proof: If $x'(t) = f(x(t))$, then $\frac{d}{dt}x(t-t_0) = x'(t-t_0) = f(x(t-t_0))$. ∎

This implies that, if $x(t) = x(t,0,x_0)$ is the solution of the Cauchy problem $x' = f(x)$, $x(0) = x_0$, then $x(t-t_0)$ is the solution of the Cauchy problem $x' = f(x)$, $x(t_0) = x_0$. In other words $x(t-t_0) = x(t,t_0,x_0)$ and so the solution depends only on $t-t_0$. For autonomous equations we can always assume that $t_0 = 0$. In this case we will denote then the map $\phi^{t,t_0} = \phi^{t-t_0,0}$ simply by ϕ^{t-t_0}. The map ϕ^t has the following group properties

- **(a)** $\phi^0(x) = x$.

- **(b)** $\phi^t(\phi^s(x)) = \phi^{t+s}(x)$.
- **(c)** $\phi^t(\phi^{-t}(x)) = \phi^{-t}(\phi^t(x)) = x$.

If the solutions exists for all $t \in \mathbf{R}$, the collection of maps ϕ^t is called the *flow* of the differential equations $x' = f(x)$. Note that Property (c) implies that the map $\phi^t : \mathbf{R}^n \to \mathbf{R}^n$ is invertible. More generally, a map $\phi^{\cdot} : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ which satisfies Properties (a)-(b)-(c) is called a *dynamical system*. If the maps ϕ^t are continuous or of class $\mathcal{C}^k$ we will say that the dynamical system is continuous or of class $\mathcal{C}^k$ (see next section) and we call ϕ^t a flow of homeomorphisms or a flow of $\mathcal{C}^k$-diffeomorphisms.

3.2 Dependence on initial conditions

For real systems the initial conditions are only determined with a finite precision. It is thus important to establish that the solution changes continuously or smoothly if we make a small error in the initial condition. A problem for which the solution depends continuously of the initial position and time is called *well-posed*. If f is smooth, we show that the solutions depend then smoothly on (t_0, x_0). If $f(t, x, c)$ depends smoothly on parameters, then we also show that the solutions depend smoothly on the parameters.

3.2.1 Continuous dependence

We first investigate the continuous dependence of the solution $x(t, t_0, x_0)$ on (t_0, x_0).

Lemma 3.2.1 *Let $f : U \to \mathbf{R} \times \mathbf{R}^n$ (U an open set of $\mathbf{R} \times \mathbf{R}^n$) be continuous and satisfy a local Lipschitz condition. Then for any compact $K \subset U$ there exists $L \geq 0$ such that*

$$\|f(t, y) - f(t, x)\| \leq L\|x - y\|\,, \quad \text{for all } (t, x), (t, y) \in K \tag{3.3}$$

Proof: Let us assume the contrary. There there exists sequences (t_n, x_n) and (t_n, y_n) such that

$$\|f(t_n, x_n) - f(t_n, y_n)\| > n\|x_n - y_n\|\,. \tag{3.4}$$

By Bolzano-Weierstrass, the sequence (t_n, x_n) has an accumulation point (t, x). By assumption the function $f(t, x)$ satisfies a Lipschitz condition in a neighborhood V of (t, x). Since f is bounded on K with $M = \max_{(t,x)\in K} \|f(t, x)\|$, it follows from (3.4) that $\|y_n - z_n\| \leq 2M/n$. Therefore there exists infinitely many indices n such that $(t_n, x_n), (t_n, y_n) \in V$. Then (3.4) contradicts the Lipschitz condition on V. ∎

Theorem 3.2.2 *Let $f : U \to \mathbf{R} \times \mathbf{R}^n$ (U an open set of $\mathbf{R} \times \mathbf{R}^n$) be continuous and satisfy a local Lipschitz condition. Then the solution $x(t, t_0, x_0)$ of the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ is a continuous function of (t_0, x_0).*

Proof: We choose a closed interval $[a,b] \subset I_{\max}(t_0,x_0)$ such that $t, t_0 \in [a,b]$. We choose ϵ small enough such that the tubular neighborhood K of $x(t,t_0,x_0)$

$$K = \{(t,x)\,;\, t \in [a,b]\,,\, \|x - x(t,t_0,x_0)\| \le \epsilon\}\,, \tag{3.5}$$

is contained in the open set U. By Lemma 3.2.1 $f(t,x)$ satisfies a Lipschitz condition on K with a Lipschitz constant L. The set V

$$V = \left\{(t_1,x_1)\,;\, t_1 \in [a,b] \;\; \|x_1 - x(t_1,t_0,x_0)\| \le \epsilon e^{-L(b-a)}\right\}\,, \tag{3.6}$$

is a neighborhood of (t_0,x_0) which satisfies $V \subset K \subset U$. If $(t_1,x_1) \in V$ we have

$$\begin{aligned} \|x(t,t_1,x_1) - x(t,t_0,x_0)\| &= \|x(t,t_1,x_1) - x(t,t_1,x(t_1,t_0,x_0))\| \\ \le \|x_1 - x(t_1,t_0,x_0)\| &+ L\int_{t_1}^{t} \|x(s,t_1,x_1) - x(s,t_1,x(t_1,t_0,x_0))\|\,. \end{aligned} \tag{3.7}$$

From Gronwall lemma we conclude that

$$\|x(t,t_1,x_1) - x(t,t_0,x_0)\| \le e^{L|t-t_1|}\|x_1 - x(t_1,t_0,x_0)\| \le \epsilon \tag{3.8}$$

and this concludes the proof. ∎

Corollary 3.2.3 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfy a Lipschitz condition and let us assume that the solutions of $x' = f(x)$ exist for all time. Then, for any t, the map $\phi^t : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and ϕ^t is a continuous dynamical system.*

Remark 3.2.4 Theorem 3.2.2 shows the following: For fixed t, $x(t,t_0,x_0)$ can be made arbitrarily close to $x(t,t_0,x_0+\xi)$ provided ξ is small enough (depending on t!). This does not mean however that the solutions which start close to each other will remain close to each other, What we proved is a bound $\|x(t,t_0,x_0+\xi) - x(t,t_0,x_0)\| \le K\|\xi\|e^{L|t-t_0|}$ which show that two solutions can separate, although not faster than at an exponential rate.

Example 3.2.5 For the Cauchy problem

$$x' = \begin{pmatrix} -1 & 0 \\ 0 & \kappa \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.9}$$

the solution is $(e^{-t},0)^T$. The solution with initial condition $(1,\xi)^T$ is $(e^{-t},\xi e^{\kappa t})^T$. If $\kappa \le 0$ both solutions stay a distance less than $|\xi|$ for all time $t \ge 0$, if $\kappa > 0$ the solutions diverge from each other exponentially with time. For given t, we can however make them arbitrarily close up to tome t by choosing ξ small enough, hence the continuity.

3.2.2 Smooth dependence with respect to x_0

.

We turn next to the question whether supplementary conditions on $f(t,x)$ such as differentiability do imply that the solution $x(t,t_0,x_0)$ is differentiable. We consider first the differentiability with respect to x_0.

In order to obtain an idea of the form of the derivative $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ we write the Cauchy problem as

$$\frac{\partial x}{\partial t}(t,t_0,x_0) \;=\; f(t,x(t,t_0,x_0))\,, \quad x(t_0,t_0,x_0)=x_0\,, \tag{3.10}$$

and differentiate *formally* with respect to x_0. Exchanging the derivatives with respect to t and x_0 we find

$$\frac{\partial}{\partial t}\frac{\partial x}{\partial x_0}(t,t_0,x_0) \;=\; \frac{\partial f}{\partial x}(t,x(t,t_0,x_0))\frac{\partial x}{\partial x_0}(t,t_0,x_0)\,, \quad \frac{\partial x}{\partial x_0}(t_0,t_0,x_0)=I\,. \tag{3.11}$$

This formal calculation shows that the n by n matrix $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ is a solution of the linear equation

$$\Psi' \;=\; \frac{\partial f}{\partial x}(t,x(t,t_0,x_0))\Psi\,, \quad \Psi(t_0)=I\,. \tag{3.12}$$

This equation is called the *variational equation* for the Cauchy problem $x'=f(t,x)$, $x(t_0)=x_0$. It is a linear equation of the form $y'=A(t,t_0,x_0)y$, where the matrix A depends on the parameters (t_0,x_0). The resolvent also depends on this parameters and let us denote it by $R(t,s,t_0,x_0)$. The formal calculation shows that

$$\frac{\partial x}{\partial x_0}(t,t_0,x_0) \;=\; R(t,t_0,t_0,x_0)\,. \tag{3.13}$$

This formal calculation which we will justify below means that for small ξ the solution of $x'=f(t,x)$, $x(t_0)=x_0+\xi$ is

$$x(t,t_0,x_0+\xi) \;=\; x(t,t_0,x_0)+R(t,t_0,t_0,x_0)\xi+o(\|\xi\|)\,, \tag{3.14}$$

where $R(t,t_0,t_0,x_0)$ is the resolvent of the *linear equation* (3.12). The right hand side of (3.14) is called the *linearization* around the solution $x(t,t_0,x_0)$.

An important special case is the linearization around equilibrium solution. Let us suppose $f(t,x)=f(x)$ is smooth and a is such that $f(a)=0$ (a is a critical point). Then $x(t,0,a)=a$ is a solution. The variational equation is then

$$\Psi' \;=\; \frac{df}{dx}(a)\Psi\,, \quad \Psi(0)=I\,, \tag{3.15}$$

whose solution is e^{tA} where $A=\frac{df}{dx}(a)$. Therefore, for small ξ we have

$$x(t,0,a+\xi) \;=\; a+e^{tA}\xi+o(\|\xi\|)\,. \tag{3.16}$$

Example 3.2.6 Consider the mathematical pendulum $x'' + \sin(x) = 0$ or

$$\begin{aligned} x' &= y \\ y' &= -\sin(x) \end{aligned} \tag{3.17}$$

There are two equilibrium solution $a = (\pi, 0)^T$ and $b = (0,0)^T$ (modulo 2π). The linearization around a and b gives, for small ξ,

$$x(t, 0, (\pi, 0)^T + \xi) \approx \begin{pmatrix} \pi \\ 0 \end{pmatrix} + e^{At}\xi\,, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.18}$$

where A has eigenvalues ± 1 and

$$x(t, 0, \xi) \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix} + e^{Bt}\xi\,, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.19}$$

where B has eigenvalues $\pm i$.

Theorem 3.2.7 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ be an open set, $f : U \to \mathbf{R}^n$ be continuous. Assume that $\frac{\partial f}{\partial x}(t,x)$ exists and is continuous on U. Then the solution $x(t, t_0, x_0)$ of $x' = f(x)$, $x(t_0) = x_0$ is continuously differentiable with respect to x_0 and its derivative $\frac{\partial x}{\partial x_0}(t, t_0, x_0)$ is a solution of the variational equation*

$$\Psi' = \frac{\partial f}{\partial x}(t, x(t, t_0, x_0))\Psi\,, \quad \Psi(t_0) = I\,. \tag{3.20}$$

Proof: For given (t, t_0, x_0) let us choose $[a, b] \subset I_{\max}$ such that $t, t_0 \in (a, b)$. Let $\xi \in \mathbf{R}^n$, we need to show that for fixed (t, t_0, x_0),

$$x(t, t_0, x_0 + \xi) - x(t, t_0, x_0) - R(t, t_0, t_0, x_0)\xi = r(\|\xi\|)\|\xi\|\,, \tag{3.21}$$

where $r(\|\xi\|) \to 0$ as $\xi \to 0$. The integral equations for $x(t, t_0, x_0 + \xi)$, $x(t, t_0, x_0)$ and $R(t, t_0, t_0, x_0)\xi$ are

$$\begin{aligned} x(t, t_0, x_0 + \xi) &= x_0 + \xi + \int_{t_0}^t f(s, x(s, t_0, x_0 + \xi))\, ds \\ x(t, t_0, x_0) &= x_0 + \int_{t_0}^t f(s, x(s, t_0, x_0))\, ds \\ R(t, t_0, t_0, x_0)\xi &= \xi + \int_{t_0}^t \frac{\partial f}{\partial x}(s, x(s, t_0, x_0))R(s, t_0, t_0, x_0)\xi\, ds \end{aligned} \tag{3.22}$$

By Theorem 3.2.2 (see in particular (3.8)), there exists $\delta > 0$ such that $\|\xi\| \le \delta$ implies that there exists a constant D such that $\|x(s, t_0, x_0 + \xi) - x(s, t_0, x_0)\| \le D\|\xi\|$ for $t_0 \le s \le t$. We use the Taylor approximation

$$f(s, z) - f(s, y) - \frac{\partial f}{\partial x}(s, y)(z - y) = r(s, y, z - y)\|z - y\|\,, \tag{3.23}$$

where $\lim_{z\to y} r(s,y,z-y) = 0$ uniformly for (s,y) in any compact set K. Applying this to $z = x(s,t_0,x_0+\xi)$ and $y = x(s,t_0,x_0)$ we see that there exists a exists a function $r_1(d)$ with $\lim_{d\to 0} r_1(d) = 0$ such that

$$\|r(s,x(s,t_0,x_0),x(s,t_0,x_0+\xi)-x(s,t_0,x_0))\| \le r_1(\|\xi\|) \tag{3.24}$$

for all $t_0 \le s \le t$, provided $\|\xi\| \le \delta$. Using this estimate together with the integral equations we obtain that

$$\begin{aligned}
&\|x(t,t_0,x_0+\xi)-x(t,t_0,x_0)-R(t,t_0,t_0,x_0)\xi\| \\
&\le \int_{t_0}^{t} \frac{\partial f}{\partial x}(s,x(s,t_0,x_0))(x(s,t_0,x_0+\xi)-x(s,t_0,x_0)-R(s,t_0,t_0,x_0)\xi)\,ds \\
&+ \int_{t_0}^{t} r(s,x(t,t_0,x_0),x(s,t_0,x_0+\xi)-x(s,t_0,x_0))\|x(s,t_0,x_0+\xi)-x(s,t_0,x_0)\|\,ds\,. \\
&\le C\int_{t_0}^{t} \|x(s,t_0,x_0+\xi)-x(s,t_0,x_0)-R(s,t_0,t_0,x_0)\xi\|\,ds + (b-a)r_1(\|\xi\|)\|\xi\|\,.
\end{aligned}$$

where $C = \sup_{s\in[a,b]}\left\{\left\|\frac{\partial f}{\partial x}(s,x(s,t_0,x_0))\right\|\right\}$. By Gronwall Lemma we conclude that

$$\|x(t,t_0,x_0+\xi)-x(t,t_0,x_0)-R(t,t_0,t_0,x_0)\xi\| \le \epsilon(b-a)e^{(b-a)}r_1(\|\xi\|)\|\xi\|\,, \tag{3.25}$$

and this shows that the derivative exists and satisfies the variational equation. It remains to show that the derivative $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ is a continuous function. We cannot apply Theorem 3.2.2 directly, since (t_0,x_0) are not the initial conditions for the variational equation, but are parameters of the equation. We will show this in Lemma 3.2.8. ∎

Lemma 3.2.8 *Let I is an open interval and V an open set in $\mathbf{R}^q$. Assume that $A(t,c)$ is continuous on $I\times V$. Then the resolvent $R(t,t_0,c)$ for the differential equation $x' = A(t,c)x$ is a continuous function of c.*

Proof: For given (t,t_0), let us choose $[a,b]\subset I$ such that $t,t_0\in(a,b)$. Given $\epsilon > 0$, let δ be such that $\|A(t,c_1)-A(t,c)\| \le \epsilon$ whenever $c_1\in B_\delta(c)$. Furthermore, see Theorem 2.2.1, for $c_1\in B_\delta(c)$ we have $\sup_{t,t_0\in[a,b]}\|R(t,t_0,c_1)\| \le e^{(b-a)L}$ where $L = \sup_{c_1\in B_\delta(c)}\sup_{t\in[a,b]}\|A(t,c_1)\|$.

$$\begin{aligned}
R(t,t_0,c_1)-R(t,t_0c) &= \int_{t_0}^{t} A(s,c_1)R(s,t_0,c_1)-A(s,c)R(s,t_0,c) \\
&= \int_{t_0}^{t} (A(s,c_1)-A(s,c))R(s,t_0,c)+A(s,c_1)(R(s,t_0,c_1)-R(s,t_0,c))\,ds
\end{aligned} \tag{3.26}$$

and thus

$$\|R(t,t_0,c_1)-R(t,t_0c)\| \le \epsilon(b-a)e^{L(b-a)} + L\int_{t_0}^{t}\|R(s,t_0,c_1)-R(s,t_0c)\|\,. \tag{3.27}$$

By Gronwall lemma we have

$$\|R(t,t_0,c_1) - R(t,t_0c)\| \leq \epsilon(b-a)e^{2L(b-a)}\,, \tag{3.28}$$

for all $t, t_0 \in [a,b]$ which proves the Lemma. ∎

If the ODE is autonomous we obtain

Corollary 3.2.9 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be of class $\mathcal{C}^1$ and let us assume that the solutions of $x' = f(x)$ exist for all time. Then, for any t, the map $\phi^t : \mathbf{R}^n \to \mathbf{R}^n$ is of class $\mathcal{C}^1$ and ϕ^t is a $\mathcal{C}^1$ dynamical system.*

3.2.3 Smooth dependence with respect to parameters and t_0

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Let us consider a Cauchy problem $x' = f(t,x,c)$ where $f : U \to \mathbf{R}^n$ is differentiable with respect to x and c (U is an open set of $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^q$). The solution is denoted $x(t,t_0,x_0,c)$

In order to study the differentiability with respect to c we consider the extended system

$$\left(\begin{array}{c} x \\ c \end{array}\right)' = \left(\begin{array}{c} f(t,x,c) \\ 0 \end{array}\right)\,, \quad \left(\begin{array}{c} x \\ c \end{array}\right)(t_0) = \left(\begin{array}{c} x_0 \\ c \end{array}\right) \tag{3.29}$$

If we set $z = (x,c)^T$ and $F(t,z) = (f(t,x,c),0)^T$, then this system becomes $z' = F(t,z)$, $z(0) = (x_0,c)^T$ and c appears only in the initial condition. Therefore we can apply Theorem 3.2.7. The function $z(t,t_0,z_0)$ and then also $y(t,t_0,x_0,c)$ is continuously differentiable with respect to $(x_0,c)^T$. By deriving the equation

$$\frac{\partial x}{\partial t}(t,t_0,x_0,c) = f(t,x(t,t_0,x_0,c),c)\,, \quad x(t_0,t_0,x_0,c) = x_0 \tag{3.30}$$

with respect to c we find a linear inhomogeneous equation for $\Psi(t) = \frac{\partial x}{\partial c}(t,t_0,x_0,c)$

$$\Psi' = \frac{\partial f}{\partial x}(t,x(t,t_0,x_0,c),c)\Psi + \frac{\partial f}{\partial c}(t,x(t,t_0,x_0,c),c)\,, \quad \Psi(0) = 0\,. \tag{3.31}$$

Example 3.2.10 The solution of the problem

$$x' = f(t,x) + \epsilon g(t,x)\,, \quad x(t_0) = x_0\,. \tag{3.32}$$

is given, for small $|\epsilon|$ by $x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + o(\epsilon)$ where

$$\begin{aligned} x_0'(t) &= f(t,x_0(t))\,, \quad x(t) = x_0\,, \\ x_1'(t) &= \frac{\partial f}{\partial x}(t,x_0(t))x_1(t) + g(t,x_0(t))\,, \quad x_1(t) = 0\,. \end{aligned} \tag{3.33}$$

If we solve the first equation and find $x_0(t)$, the second equation is a linear inhomogeneous equation for $x_1(t)$

Example 3.2.11 Consider the equation $x' + x - \epsilon x^3$ with $x(0) = 1$. For $\epsilon = 0$ the solution is $x_0(t) = e^{-t}$. Expanding around this solution we find that $x(t) = x_0(t) + \epsilon x_1(t) + o(\epsilon)$ where $x_1(t)$ is a solution of

$$x_1' = -x_1 + e^{-3t}\,, \quad x_1(0) = 0 \tag{3.34}$$

so that $x(t) = e^{-t} + \epsilon(e^{-t} - \frac{1}{2}e^{-3t}) + o(\epsilon)$.

Let us assume that $f(t,x)$ is continuously differentiable with respect to t and x. Let us consider the extended system

$$\begin{pmatrix} x \\ t \end{pmatrix}' = \begin{pmatrix} f(t,x) \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x \\ t \end{pmatrix}(t_0) = \begin{pmatrix} x_0 \\ t_0 \end{pmatrix} \tag{3.35}$$

If we set $z = (x,t)^T$ and $F(z) = (f(t,x),1)^T$, then this system becomes $z' = F(z)$, $z(t_0) = (x_0, t_0)^T$ and the dependence on t_0 can be studied as the dependence on x_0. By Theorem 3.2.7, the solution $x(t,t_0,x_0)$ is continuously differentiable with respect to t_0. Differentiating the equation $\frac{\partial x}{\partial t}(t,t_0,x_0) = f(t, x(t,t_0,x_0))$ with $x(t_0,t_0,x_0,) = x_0$ with respect to t_0, one finds a linear equation for $\Psi(t) = \frac{\partial x_0}{\partial t}(t,t_0,x_0)$

$$\Psi' = \frac{\partial f}{\partial x}(t, x(t,t_0,x_0))\Psi\,, \quad \Psi(t_0) = -f(t_0,x_0)\,. \tag{3.36}$$

This is the same differential equation as in the variational equation for $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ but with a different initial condition. Since $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ is the resolvent of (3.36) we obtain the relation

$$\frac{\partial x}{\partial t_0}(t,t_0,x_0) = -\frac{\partial x}{\partial x_0}(t,t_0,x_0) f(t_0,x_0)\,. \tag{3.37}$$

We can summarize the result of this section by

Theorem 3.2.12 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and let $f : U \to \mathbf{R}^n$ be of class $\mathcal{C}^k$, then the solution $x(t,t_0,x_0)$ is of class $\mathcal{C}^k$ in the variables (t,t_0,x_0).*

Proof: We have proved that $x(t,t_0,x_0)$ is differentiable with respect to x_0 and t_0. The differentiability with respect to t is automatic. We can apply this argument iteratively to the variational equation and see by recurrence that x is of class $\mathcal{C}^k$.

Chapter 4

Stability analysis

4.1 Orbits, critical points and stability

In this chapter we will consider (mostly) autonomous equations. Let U be an open set in $\mathbf{R}^n$ and let $f : U \to \mathbf{R}^n$ satisfy a local Lipschitz condition. The the equation $x' = f(x)$, $x(t_0) = x_0$ or more explicitly

$$\begin{aligned} x_1' &= f_1(x_1, \cdots, x_n), \quad x_1(t_0) = x_{10}, \\ &\vdots \\ x_n' &= f_n(x_1, \cdots, x_n), \quad x_n(t_0) = x_{n0}, \end{aligned} \tag{4.1}$$

has a unique solution in some time interval $I_{\max}$. The set U is called the *phase space* of (4.1), and $x(t)$ is called a *phase point*.

Since the equation is autonomous, one can eliminate t and use one component of x, say x_1, as a new independent variable. In the domain where $f_1(x)$ does not vanish, we obtain, using the chain rule,

$$\begin{aligned} \frac{dx_2}{dx_1} &= \frac{f_2(x)}{f_1(x)}, \\ &\vdots \\ \frac{dx_n}{dx_1} &= \frac{f_n(x)}{f_1(x)}. \end{aligned} \tag{4.2}$$

The solutions of the system of equations (4.2) are called *orbits*. We have excluded the singularities in (4.2) by assuming that $f_1(x)$ has no zeros. If f_1 does have zeros, we can take x_2 as an independent variables provided f_2 does not vanish where f_1 does, and so on. Therefore we can construct orbits except at points $a = (a_1, a_2, \cdots, a_n)$ such that

$$f_1(a) = f_2(a) = \cdots = f_n(a) = 0, \tag{4.3}$$

that is the vector $f(a) = 0$.

The points where $f(a) = 0$ are called are called *critical points* of $f(x)$. They are also called *equilibrium points*, since for any critical point, the constant $x(t, t_0, a) = a$ is a solution of (4.1). They represent a state where the system is at rest. No orbit can pass through a or rather the orbit degenerates to a single point.

Example 4.1.1 The harmonic oscillator $x'' + x = 0$ is equivalent to the equations $x_1' = x_2$ and $x_2' = -x_1$. The orbits in the two dimensional phase-space are given by the equation

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}\,. \tag{4.4}$$

which by integration gives the orbits $x_1^2 + x_2^2 = c$, i.e the orbits are circles. The point $(0, 0)$ is the only critical point.

The equation $x'' + x = 0$ is treated similarly and the orbits in this case are the hyperbolas $x_1^2 - x_2^2 = c$. The point $(0, 0)$ is the only critical point.

The solutions of (4.2) are called *orbits* or *characteristic curves*. Note that orbits and solutions are not synonymous: an orbit in U can be represented parametrically by several solutions. For example if the $x(t)$ represents an orbit C, then $x(t - t_0)$ also represents the same orbit C. All solutions representing C can be written in that way by a suitable choice of t_0. Therefore it follows that for every solution of (4.1) the direction of increase of t along the orbit is the same. An orbit is a *directed curve* and usually we represent the direction of increasing t by an arrow.

If the conditions of existence an uniqueness apply to (4.1), then they also apply to (4.2). The uniqueness of solutions imply that the orbits in phase space will not intersect.

If there exist two values t_0 and t_1 such that $x(t_0) = x(t_1)$, it follows, bu uniqueness of solutions, that for all t, $x(t_1 + t) = x(t_0 + t)$. This is a *periodic solution* and this means that the orbit is a *closed curve*. The closed orbit C will have period p if p is the least number such that $x(t + p) = x(t)$. Note that the converse statement is also true

Lemma 4.1.2 *A periodic solution of (4.1) corresponds to a closed orbit in phase space and a closed orbit corresponds to a periodic orbit.*

Proof: Let C be a closed orbit and a point $x_0 \in C$ and denote by $x(t)$ which starts at x_0 at time 0. By uniqueness of solutions, C is a directed curve which does not contain any critical point of f. Since C is compact we have $\|x'(t)\| \geq b > 0$ on the curve so that at the speed at which the solution moves along the curve is strictly bounded away from zero. This implies that at a certain time p, $x(t)$ returns to x_0: $x(T) = x_0$. It follows that $x(t)$ is periodic solution of period p, if p is the first tome of return to x_0. ∎

Example 4.1.3 The Predator-Prey equations are given by

$$x' = x(\alpha - \beta y)\,, \quad y' = y(\gamma x - \delta)\,, \tag{4.5}$$

where $\alpha, \beta, \gamma, \delta$ are given positive constants and we assume that $x \geq 0$ and $y \geq 0$.

$$\frac{dx}{dy} = \frac{x}{y}\frac{(\alpha - \beta y)}{(\gamma x - \delta)} \quad \text{or} \quad \frac{(\gamma x - \delta)}{x}dx = \frac{(\alpha - \beta y)}{y}dy\,. \tag{4.6}$$

Integrating gives

$$\gamma x - \delta \log x + \beta y - \alpha \log y = c \tag{4.7}$$

which are closed curves corresponding to periodic solutions. There are two critical points $a = (0,0)$ and $b = (\delta/\gamma, \alpha/\beta)$. The linearization around $(0,0)$ yields

$$A = \frac{df}{dx}(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}, \tag{4.8}$$

with eigenvalues α and $-\delta$, and

$$B = \frac{df}{dx}(\delta/\gamma, \alpha/\beta) = \begin{pmatrix} 0 & -\beta\delta/\gamma \\ \gamma\alpha/\beta & 0 \end{pmatrix}, \tag{4.9}$$

with eigenvalues $\lambda = \pm i\alpha\delta$.

Example 4.1.4 Let us assume that f is of class $\mathcal{C}^1$ and consider the equation $x'' + f(x) = 0$, or equivalently $x' = y$ and $y' = -f(x)$. The critical points are all the points $(a, 0)$ where a is a critical point of f, $f(a) = 0$. The equations for the orbits are

$$\frac{dy}{dx} = -\frac{f(x)}{y}\,. \tag{4.10}$$

which is easily integrated to give the orbits

$$\frac{y^2}{2} + \int_0^x f(z)dz = c\,. \tag{4.11}$$

For example for the mathematical pendulum $x'' + \sin(x) = 0$ the orbits are the level curves of $F(x,y) = y^2/2 - \cos(x)$.

We define next the concept of *stability* of a solution.

Definition 4.1.5 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ and let $f : U \to \mathbf{R}^n$ be continuous and locally Lipschitz. Let $x(t, t_0, x))$ be the solution of the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ which we assume to exist for times $t > t_0$.*

1. *The solution* $x(t, t_0, x_0)$ *is* stable *(in the sense of Liapunov) if for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* ξ *with* $\|\xi\| \leq \delta$ *we have*

$$\|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)\| \leq \epsilon \quad \text{for all } t \geq t_0 \,. \tag{4.12}$$

2. *The solution* $x(t, t_0, x_0)$ *is* asymptotically stable *if there exists* $\delta > 0$ *such that hat for all* ξ *with* $\|\xi\| \leq \delta$ *we have*

$$\lim_{t \to \infty} \|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)\| \, = \, 0 \,. \tag{4.13}$$

3. *The solution* $x(t, t_0, x_0)$ *is* unstable *if it is not stable.*

If a is a critical point, we will say the critical point a is stable or unstable if the solution $x(t) \equiv a$ is stable or unstable.

Example 4.1.6 The solution $x(t) = 0$ of $x' = \lambda x$ is asymptotically stable of $\lambda < 0$, stable if $\lambda = 0$, unstable if $\lambda > 0$.

Example 4.1.7 The solutions of the equation $x'' + x = 0$ are stable (but not asymptotically stable). The general solution is $(x_0 \cos(t) + y_0 \sin(t), -x_0 \sin(t) + y_0 \cos(t))$ is a periodic solution, the orbits are on the circle of radius $x_0^2 + y_0^2$. A solution starting at a point (x_1, y_1) near (x_0, y_0) will remain close forever.

Example 4.1.8 The solution $x(t, 0, x_0) = \frac{x_0}{1 - x_0 t}$ of $x' = x^2$ is asymptotically stable for $x_0 < 0$ but unstable for $x_0 \geq 0$.

Example 4.1.9 For the ODE $x' = x^2 - 1$, there are two critical points 0 and 1. The solution $x(t) = 0$ is unstable and the solution $x(t) = 1$ is stable.

4.2 Stability and phase portraits of linear systems

Let us study the stability of a solution $x_0(t)$ for the linear homogeneous equation $x' = A(t)x$, i.e, we want to study the difference $x(t) - x_0(t)$ for $t \to \infty$. By linearity the difference $y(t) = x(t) - x_0(t)$ satisfy the equation $y' = A(t)y$. Therefore it suffices to study the stability of the critical point $\equiv 0$.

If we consider the stability of a solution $x_0(t)$ for a linear inhomogeneous equation $x' = A(t)x + f(t)$, then the transformation $y(t) = x(t) - x_0(t)$ yields again the equation $y' = A(t)x$, so that the stability properties of a solution $x_0(t)$ of the inhomogeneous problem are the same as the stability of the trivial solution of the homogeneous problem. Therefore, in the case of linear differential equations, all the solutions have the same stability properties and one can talk about the stability of the differential equation.

Theorem 4.2.1 *Let $x' = Ax$ be a linear system with constant coefficients $x' = Ax$, and let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of A.*
(a) *The critical point 0 is asymptotically stable if and only if all the eigenvalues of A have a negative real part: $\mathrm{Re}\,\lambda_i < 0$ for $i = 1, \cdots, k$.*
(b) *The critical point 0 is stable if and only*

1. *All the eigenvalues have a nonpositive real part $\mathrm{Re}\,\lambda_i \le 0$ for $i = 1, \cdots, k$.*

2. *If $\mathrm{Re}\lambda_i = 0$ the Jordan blocks have dimension 1.*

Proof: Let us transform A in Jordan normal form $S = VAV^{-1}$. Then

$$\|e^{tA}\| \le \|V^{-1}\|\,\|e^{tS}\|\,\|V\| \;\le\; C\|e^{tS}\|\,. \tag{4.14}$$

If $\mathrm{Re}\lambda < 0$, then for any $k \ge 0$, we have

$$\lim_{t\to\infty} |t^k e^{\lambda t}| \;=\; \lim_{t\to\infty} |t^k| e^{\mathrm{Re}\lambda t} \;=\; 0\,. \tag{4.15}$$

Since all matrix elements of e^{tS} have this form this implies that $\lim_{t\to\infty} \|e^{tA}\| = 0$ and therefore 0 is stable.

If $\mathrm{Re}\lambda = 0$ then $|e^{\lambda t}| = 1$. If the Jordan blocks corresponding to such a λ have dimension 1, then all matrix elements in e^{tS} are bounded uniformly in t and therefore $\|e^{tA}\| \le K$. This implies stability of 0.

If some eigenvalue λ has a positive real part, or if $\mathrm{Re}\lambda = 0$ and λ has a Jordan block of dimension at least 2 then there exists solutions $x(t)$ with $\|x(t)\| \to \infty$ as $t \to \infty$. In this case 0 is unstable. ∎

If $A(t)$ depends on t, in general it is not enough to look at the eigenvalues of A. One can construct examples of matrices $A(t)$ whose eigenvalues are negative but for which 0 is unstable (see homework). One needs stronger condition. As an example we prove

Theorem 4.2.2 *Let $A(t)$ be symmetric, i.e., $A^T(t) = A(t)$ and continuous on $[t_0, \infty)$. If the eigenvalues $\lambda_i(t)$ of $A(t)$ satisfy $\lambda_i(t) \le \alpha$ for $t \in [t_0, \infty)$, then the solution $x(t)$ of $x' = A(t)x$ satisfy*

$$\|x(t)\|_2 \;\le\; e^{\alpha(t-t_0)}\|x(t_0)\|_2\,, \quad t > t_0\,. \tag{4.16}$$

In particular, if $\alpha \le 0$, then 0 is stable and if $\alpha < 0$ then 0 is asymptotically stable.

Proof: Since $A(t)$ is symmetric, its eigenvalues are real and it is diagonalizable with an orthogonal matrix: there exists a matrix $Q(t)$ with $Q^T(t) = Q^{-1}(t)$) such that $Q^T(t)A(t)Q(t) = \mathrm{diag}(\lambda_1(t), \cdots, \lambda_n(t))$. We show that, for all v and all $t > t_0$ we have

$$\langle v\,,\,Av\rangle \;\le\; \alpha\langle v\,,\,v\rangle\,. \tag{4.17}$$

We set $v = Qw$ and then we have $\langle v \,,\, v\rangle = \langle w \,,\, w\rangle$ and

$$\langle v \,,\, Av\rangle \;=\; \langle w \,,\, Q^T AQw\rangle \;\leq\; \alpha\langle w \,,\, w\rangle \;=\; \alpha\langle v \,,\, v\rangle\,. \tag{4.18}$$

For a solution $x(t)$ of $x' = A(t)x$ we obtain

$$\frac{d}{dt}\|x(t)\|_2^2 \;=\; 2\langle x(t) \,,\, A(t)x(t)\rangle \;\leq\; 2\alpha\|x(t)\|^2\,. \tag{4.19}$$

Integrating gives

$$\|x(t)\|_2^2 \;\leq\; \|x(t_0)\|_2^2 + 2\alpha\int_{t_0}^{t}\|x(s)\|_2^2\,ds\,, \tag{4.20}$$

and therefore, by Gronwall Lemma

$$\|x(t)\|_2^2 \;\leq\; \|x(t_0)\|_2^2\,e^{2\alpha(t-t_0)}\,. \tag{4.21}$$

∎

4.3 Stability of critical points of nonlinear systems

Consider the autonomous equation $x' = f(x)$ and let us assume that a is an *isolated singularity*, i.e. $f(a) = 0$ and there exists a neighborhood $B_r(a)$ of a such that $B_r(a)$ contains no other singularities of a. We will study the behavior of solutions in a neighborhood of a.

It is convenient to change variable and set $y = x - a$ and define $g(y) \equiv f(y + a)$ so that $g(0) = 0$. Then we have $y' = x' = f(x) = f(y + a) = g(y)$. Therefore we can and will always assume that the critical point is $a = 0$.

If f is of class $\mathcal{C}^1$, we can linearize around 0. We write $f(x)$ as

$$f(x) = \frac{df}{dx}(0)x + g(x) \quad \text{or} \quad g(x) = f(x) - \frac{df}{dx}(0)x\,. \tag{4.22}$$

We have $g(0) = 0$ and

$$g(x) \;=\; g(x) - g(0) \;=\; \int_0^1 \frac{d}{ds}g(sx)\,ds \;=\; \int_0^1 \left(\frac{df}{dx}(sx) - \frac{df}{dx}(0)\right)x\,ds \tag{4.23}$$

and thus

$$\|g(x)\| \;=\; \sup_{\{y\,;\,\|y\|\leq\|x\|\}}\left\|\frac{df}{dx}(y) - \frac{df}{dx}(0)\right\|\|x\|\,, \tag{4.24}$$

and so

$$\lim_{\|x\|\to 0}\frac{\|g(x)\|}{\|x\|} \;=\; 0\,. \tag{4.25}$$

So the differential equation has the form

$$x' = Ax + g(x) \tag{4.26}$$

where $A = \frac{df}{dx}(0)$ and g satisfies (4.25).

Theorem 4.3.1 (Stability Theorem) *Let $g : (t_0, \infty) \times U$ be continuous and locally Lipschitz in U where U is a neighborhood of 0. Let us assume that*

$$\lim_{\|x\| \to 0} \sup_{t > t_0} \frac{\|g(t,x)\|}{\|x\|} = 0\,. \tag{4.27}$$

Let A be a $n \times n$ matrix whose all eigenvalues have negative real part, $\mathrm{Re}\lambda_i < 0$. *Then the zero solution of*

$$x' = Ax + g(t,x) \tag{4.28}$$

is asymptotically stable.

Proof: Since g continuous and locally Lipschitz, we have existence of solutions $x(t) = x(t, t_0, x_0)$ if x_0 is in a neighborhood of 0. We use the following generalization of Duhamel formula: $x(t)$ is solution of the integral equation

$$x(t) = e^{At} x_0 + \int_{t_0}^{t} e^{A(t-s)} g(x(s))\, ds\,. \tag{4.29}$$

One can verify this formula by differentiation. Since the real parts of the eigenvalues of A are negative we conclude that there exists constants $K > 0$ and $\mu > 0$ such that

$$\|e^{A(t-t_0)}\| \leq K e^{-\mu(t-t_0)}\,. \tag{4.30}$$

From this we deduce the estimate

$$\|x(t)\| \leq K e^{-\mu(t-t_0)} \|x_0\| + K \int_{t_0}^{t} e^{-\mu(t-s)} \|g(x(s))\|\, ds\,. \tag{4.31}$$

Since $\|g(t,x)\|/\|x\| \to 0$ uniformly in t, for any $b > 0$ there exists $\epsilon > 0$ such that $\|g(t,x)\| \leq b\|x\|$ provided $\|x\| \leq \epsilon$. In the sequel we choose $b = \frac{\mu}{2K}$.

As long as the solution $x(t)$ stays in $\{x\,;\, \|x\| \leq \epsilon\}$ we have the bound

$$e^{\mu(t-t_0)} \|x(t)\| \leq K\|x_0\| + Kb \int_{t_0}^{t} e^{\mu(s-t_0)} \|x(s)\|\, ds\,. \tag{4.32}$$

By Gronwall Lemma we obtain

$$e^{\mu(t-t_0)} \|x(t)\| \leq K\|x_0\| e^{bK(t-t_0)}\,. \tag{4.33}$$

or, with $b = \frac{\mu}{2K}$,

$$\|x(t)\| \le K\|x_0\|e^{-\frac{\mu}{2}t}. \tag{4.34}$$

We set $\delta := \frac{\epsilon}{K}$. If $\|x_0\| \le \delta$, then the estimate shows that $x(t)$ stays in $\{x\,;\,\|x\| \le \epsilon\}$ for all $t > 0$. This shows that the zero solution is asymptotically stable. ∎

We prove next an unstability result. We will consider the equation $x' = Ax + g(t,x)$ and assume that A has at least one eigenvalue has a positive real part.

Theorem 4.3.2 (Instability Theorem) *Let $g : (t_0, \infty)\times U$ be continuous and locally Lipschitz in U where U is a neighborhood of 0. Let us assume that*

$$\lim_{\|x\|\to 0} \sup_{t>t_0} \frac{\|g(t,x)\|}{\|x\|} = 0. \tag{4.35}$$

Let A be a $n \times n$ matrix and let us suppose that at least one eigenvalue of A has a positive real part. Then the zero solution of

$$x' = Ax + g(t,x) \tag{4.36}$$

is unstable.

Proof: We first transform the differential equation into a form which is better suited to our purposes. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A (counting multiplicities). There exists an invertible matrix S such that $B = S^{-1}AS$ is in Jordan normal form, i.e., $b_{ii} = \lambda_i$ and $b_{i,i+1} = 1$ or 0 and all other $b_{ij} = 0$. Let H be the diagonal matrix $H = \mathrm{diag}(\eta, \eta^2, \cdots, \eta^n)$ and so $H^{-1} = \mathrm{diag}(\eta^{-1}, \eta^{-2}, \cdots, \eta^{-n})$. It is easy to check that for the matrix $C = H^{-1}BH$ we have $c_{ii} = \lambda_i$ and $c_{i,i+1} = \eta$ or 0 and and all other $c_{ij} = 0$.

We now set $x(t) = SHy(t)$, then the equation (4.36) transforms into

$$y' = Cy + h(t,y), \tag{4.37}$$

where

$$h(t,y) \equiv H^{-1}S^{-1}g(t, SHy). \tag{4.38}$$

Since g satisfies condition (4.35), so does h. Indeed from $\|g(t,x)\| \le b\|x\|$ for $\|x\| \le \delta$ it follows that

$$\|h(t,y)\| \le \|H^{-1}S^{-1}\|\,\|SH\|\,b\|y\|, \quad \text{for } \|y\| \le \frac{\delta}{\|SH\|}. \tag{4.39}$$

The i^{th} component of (4.37) has either the form

$$y_i' = \lambda_i y_i + h_i(t,y), \tag{4.40}$$

or

$$y_i' = \lambda_i y_i + \eta y_{i+1} + h_i(t,y)\,. \tag{4.41}$$

Let us denote by j the indices for which $\mathrm{Re}\lambda_j > 0$ and by k the indices for which $\mathrm{Re}\lambda_k \leq 0$. We set

$$R(t) = \sum_j |y_j(t)|^2\,, \quad r(t) = \sum_k |y_k(t)|^2\,. \tag{4.42}$$

Let us choose η such that

$$0 < 6\eta < \mathrm{Re}\lambda_j\,, \qquad \text{for all } j \tag{4.43}$$

and δ so small that

$$\|h(t,y)\| \leq \eta \|y\| \qquad \text{for } \|y\| \leq \delta\,. \tag{4.44}$$

If $y(t)$ is a solution of (4.40) or (4.41) with

$$\|y_0\| \leq \delta\,, \quad r(0) \leq R(0)\,, \tag{4.45}$$

then as long as $\|y(t)\| \leq \delta$ and $r(t) \leq R(t)$ we have

$$\begin{aligned} R'(t) &= 2\sum_j \mathrm{Re}\, y_j' \bar{y}_j \\ &= \sum_j 2\mathrm{Re}\, \lambda_j y_j \bar{y}_j + \{\eta \mathrm{Re}\, y_{j+1}\bar{y}_j\} + \mathrm{Re}\, \bar{y}_j h_j(t,y) \end{aligned} \tag{4.46}$$

where the term in brackets appears or not depending on j. By Cauchy-Schwartz inequality we have

$$\left|\sum_j \mathrm{Re}\, y_{j+1}\bar{y}_j\right| \leq \sum_j |y_{j+1} y_j| \leq \sqrt{\sum_j |y_j|^2 \sum_j |y_{j+1}|^2} \leq R\,. \tag{4.47}$$

and

$$\left|\sum_j \mathrm{Re}\, \bar{y}_j h_j(t,y)\right| \leq \sqrt{\sum_j |y_j|^2 \sum_j |h_j|^2} \leq R^{1/2}\|h\|.\,, \tag{4.48}$$

Since we assumed that $r(t) \leq R(t)$ and $\|y(t)\| \leq \delta$ we have

$$R^{1/2}\|h\| \leq R^{1/2}\eta\|y\| \leq \eta R^{1/2}\sqrt{R+r} \leq 2\eta R \tag{4.49}$$

and

$$\sum_j \mathrm{Re}\, \lambda_j y_j \bar{y}_j > 6\eta R\,. \tag{4.50}$$

Therefore we have the equation

$$\frac{1}{2}R'(t) > 6\eta R - \eta R - 2\eta R = 3\eta R\,. \tag{4.51}$$

A similar equation holds for r. Using $\mathrm{Re}\lambda_k \le 0$ one obtains

$$\frac{1}{2}r'(t) \; < \; \eta r + 2\eta R\,. \tag{4.52}$$

. As long as $r(t) < R(t)$ we have

$$\frac{1}{2}\left(R' - r'\right) \; > \; \eta(3R - r - 2R) \; = \; \eta(R - r) \; > \; 0\,, \tag{4.53}$$

i.e., the difference $R - r$ is increasing as long as it is positive. Therefore

$$\|y(t)\|^2 \; \ge \; R^2 - r^2 \; \ge \; (R^2(t_0) - r^2(t_0))e^{2\eta(t-t_0)}\,. \tag{4.54}$$

So this solution leaves the domain given by $\|y\| \le \delta$, this means that the trivial solution is unstable. ∎

From Theorems 4.3.1 and 4.3.2 we obtain immediately

Corollary 4.3.3 *Let $f(x)$ be a function of class $\mathcal{C}^2$ and let a be a critical point of f, i.e. $f(a) = 0$.*

1. *If the eigenvalues of $A = \frac{df}{dx}(a)$ have all a negative real part, then the critical point a is asymptotically stable.*
2. *If at least one of the eigenvalues of $A = \frac{df}{dx}(a)$ has all a positive real part, then the critical point a is asymptotically stable.*

Example 4.3.4 (Competing species) Consider the set of equations

$$\begin{aligned} x' \; &= \; x - ax^2 - cxy\,, \quad x \ge 0\,, \\ y' \; &= \; y - by^2 + dxy \quad y \ge 0\,, \end{aligned} \tag{4.55}$$

where $a, b, c, d > 0$. This models the competition of two species living in a certain territory. If, say $y = 0$, then $y' = 0$ and $x' = x - ax^2$ (logistic equation) then the population x has linear growth rate with a natural limit ($x = 1/a$ is an asymptotically stable equilibrium). A similar situation holds if $x = 0$. This also implies that if x_0 and y_0 are nonnegative they remain so forever. The third term on the right side of (4.55) favors species y over species x if they are interacting.

The critical points with their linearization are given by

$$(0,0)\,,\; (0, 1/b)\,,\; (1/a, 0)\,,\; \left(\frac{b-c}{ab+cd}\,,\, \frac{a+d}{ab+cd}\right) \tag{4.56}$$

If $b \geq c$ (weak interaction) the fourth critical point is found in the domain of interest $(x, y > 0)$ while if $b < c$ (strong interaction) we have only three relevant critical points. For the first three critical points linearization gives gives

crit.point	linearization	eigenvalues	$b > c$	$b < c$
$(0,0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\lambda_1 = 1, \lambda_2 = 1$	source	source
$(0, \frac{1}{b})$	$\begin{pmatrix} 1 - \frac{c}{b} & 0 \\ \frac{d}{b} & -1 \end{pmatrix}$	$\lambda_1 = -1, \lambda_2 = 1 - \frac{c}{b}$	saddle	sink
$(\frac{1}{a}, 0)$	$\begin{pmatrix} -1 & -\frac{c}{a} \\ 0 & 1 + \frac{d}{a} \end{pmatrix}$	$\lambda_1 = -1, \lambda_2 = 1 + \frac{d}{a}$	saddle	saddle

(4.57)

For the last critical point linearization around $\left(\frac{b-c}{ab+cd}, \frac{a+d}{ab+cd}\right)$ gives

$$A = \frac{1}{ab+cd} \begin{pmatrix} -a(b-c) & -c(b-c) \\ d(a+d) & -b(a+d) \end{pmatrix} . \tag{4.58}$$

For $b > c$ we have $\lambda_1 \lambda_2 = \det(A) > 0$ and $\lambda_1 + \lambda_2 = \mathrm{Trace}(A) < 0$ from which we conclude that A has 2 eigenvalues with negative real part and so we have a stable critical point (both a sink or a spiral are possible).

In the case of strong interaction ($b < c$) the species y will die out while for weak interaction ($b > c$) there exists a postive stable equilibrium where the two species coexist.

4.4 Stability by Liapunov functions

The linear stability analysis of the previous section allow to determine asymptotic stability or unstability of a critical point by inspection of the linear part of f. There are several questions which cannot be answered by this analysis. If a is an asymptotically stable critical point of linear system, one might want to determine which portion of phase space actually converges to the critical point (i.e., determine its basin of attraction). Another question is to determine the stability of critical point for which some of the eigenvalues of the linearization have a zero real part. In that case the linear stability analysis of the previous section is inconclusive.

Let us consider the autonomous equation

$$x' = f(x) , \tag{4.59}$$

where f is locally Lipschitz in an open set U. We assume that $0 \in U$ and $f(0) = 0$. The zero solution $x(t) = 0$ is an equilibrium state. The extension to an equilibrium state $x(t) = a$ is elementary.

We call a function $V : D \to \mathbf{R}$, where D is an open neighborhood of 0, a *Liapunov function* if $V(0) = 0$ and $V(x) > 0$ for $x \in D$, $x \neq 0$.

We recall that the derivative of V along a solution $x(t)$ is given by

$$\frac{d}{dt}V(x(t)) = \equiv \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x(t))x_i'(t) = \langle \nabla V(x(t)) \, , \, f(x(t)) \rangle \, . \tag{4.60}$$

We set

$$LV(x) = \langle \nabla V(x) \, , \, f(x) \rangle \, . \tag{4.61}$$

We also introduce the concept of *exponential stability* of a solution.

Definition 4.4.1 *A solution $x(t, t_0, x_0)$ is exponentially stable if there exists constants c, $\gamma > 0$ and $\delta > 0$ such that $\|\xi\| \leq \delta$ implies that*

$$\|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)\| \leq ce^{-\gamma(t-t_0)} \, . \tag{4.62}$$

Clearly exponential stability implies asymptotic stability. For linear equation asymptotic stability and exponential stability are equivalent.

Theorem 4.4.2 (Stability Theorem of Liapunov) *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be locally Lipschitz with $f(0) = 0$. Let V be a Liapunov function defined in an open neighborhood D of 0*

1. *If $LV \leq 0$ in D then 0 is a stable critical point.*

2. *If $LV < 0$ in $D \setminus \{0\}$ then 0 is asymptotically stable critical point.*

3. *if $LV \leq -\alpha V$ and $V(x) \geq a\|x\|^\beta$ in D (α, β, a are postive constants) then 0 an exponentially stable critical point.*

Proof: 1. Choose η so small that the set $V_\eta \equiv \{x \, ; \, V(x) < \eta\}$ is contained in D. Since $V(x) > 0$ for $x \neq 0$ V_η is an open neighborhood of 0. Since $LV \leq 0$ in $V_\eta \subset D$, if $x_0 \in V_\eta$ then the solution $x(t, 0, x_0)$ exists for all $t \geq t_0$ and it does not leave V_η ever. This implies stability since for η small enough we can choose δ and ϵ such that $\{\|x\| < \delta\} \subset V_\eta \subset \{\|x\| < \epsilon\}$.

2. By 1. we know that for η sufficiently small $x(t, t_0, x_0)$ stay in V_η for all times. Moreover $V(x(t, t_0, x_0))$ is a positive decreasing function so that $\lim_{t\to\infty} V(x(t)) = V^*$ exists. Let us assume that $V^* \neq 0$. The set $M = \{V^* \leq V(x) \leq \eta\}$ is a compact set which does not contain 0. We have $x(t) \in M$, for all $t > t_0$ and $\max_{x\in M} LV(x) \leq -\alpha < 0$. So $\frac{dV}{dt}(x(t)) \leq -\alpha$ for all $t > t_0$ which is a contradiction. So we must have $V^* = 0$ and $\lim_{t\to\infty} \|x(t)\| = 0$.

3. We have $\frac{dV}{dt}(x(t)) \leq -\alpha V(x(t))$and thus, by Gronwall Lemma, $V(x(t)) \leq V(x(t_0))e^{-\alpha(t-t_0)}$. Since $b\|x(t)\|^\beta \leq V(x(t))$ we obtain $\|x(t)\| \leq \frac{V(x(t_0))}{b}e^{-\frac{\alpha}{\beta}(t-t_0)}$ and this proves exponential stability. ∎

Theorem 4.4.3 (Instability theorem of Liapunov) *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be locally Lipschitz with $f(0) = 0$. Let V be a function defined in an open neighborhood D of 0 (V is not necessarily positive) which satisfies*

1. $\lim_{\|x\| \to 0} V(x) = 0$.
2. $LV(x) > 0$ *if* $x \in D \setminus \{0\}$.
3. $V(x)$ *takes positive values in each sufficiciently small neighborhood of* 0.

Then 0 *is unstable.*

Remark 4.4.4 The conditions 1 and 3 of Theorem 4.4.3 are satisfied, in particular, if V is a Liapunov function, i.e., $V(0) = 0$ and $V(x) > 0$, $x \in D \setminus \{0\}$.

Proof: Stability means that for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x_0\| \le \delta$ implies that $\|x(t, 0, x_0)\| \le \epsilon$ for all $t > 0$.

Let us choose $\epsilon > 0$ such that the set $\{x\,;\, \|x\| \le \epsilon\} \subset D$. Now for arbitrary $\delta > 0$, by assumptions 1 and 3 we can find x_0 such that $\|x_0\| \le \delta$ and $V(x_0) = \alpha > 0$. By assumption 2, $LV > 0$, so that $V(x(t)) \ge \alpha$ for all $t \ge 0$. The set $\{x\,;\, \|x\| \le \epsilon$ and $V(x) \ge \alpha\}$ does not contain the origin and therefore there exists $\beta > 0$ such that, in this set, $LV(x) \ge \beta$. We thus obtain that

$$V(x(t)) - V(x(0)) \;=\; \int_0^t LV(x(s))\, ds \ge \beta t\,, \tag{4.63}$$

or

$$V(x(t)) \ge \alpha + \beta t \tag{4.64}$$

as long as $\|x(t)\| \le \epsilon$. This implies that $x(t)$ actually exits the balls $\{x\,;\, \|x\| \le \epsilon\}$. Since δ is arbitrary we conclude that 0 is unstable. ∎

Example 4.4.5 Consider the system of equations

$$\begin{aligned} x' &= ax - y + kx(x^2 + y^2)\,, \\ y' &= x - ay + ky(x^2 + y^2) \end{aligned} \tag{4.65}$$

where $a > 0$ and k are constants. Clearly $(0, 0)$ is a critical point and the linearization gives

$$A \;=\; \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix} \text{ with eigenvalues } \lambda \;=\; \pm(a^2 - 1)\,. \tag{4.66}$$

and so we have

1. If $a^2 > 1$ then 0 is a saddle point and it is unstable by Theorem 4.3.2.

2. If $a^2 = 1$ the system is degenerate.

3. If $a^2 < 1$ then the eigenvalues are purely imaginary and we have a center (vortex).

In order to study the case $a^2 < 1$ we construct a Liapunov function. The linearized system $x' = Ax$ has orbits which are the ellipses

$$V(x, y) = x^2 - 2axy + y^2 = c\,. \tag{4.67}$$

It is quite natural to take $V(x, y)$ a Liapunov function to study the effect of the nonlinear terms. We have

$$\begin{aligned} LV &= (2x - 2ay)kx(x^2 + y^2) + (2y - 2ax)ky(x^2 + y^2) \\ &= 2k(x^2 + y^2)(x^2 + y^2 - 2axy)\,. \end{aligned} \tag{4.68}$$

We conclude from Theorem 4.4.3 that 0 is unstable if $k > 0$ and from Theorem 4.4.2 that 0 is asymptotically stable for $k < 0$.

If a is a asymptotically stable critical point, then all solutions starting in a neighborhood of a converge to a as t goes to infinity. We call a an *attracting point* or an *attractor*. The *basin of attraction* of a is the set of point y such that $x(t, 0, y) \to a$ as $t \to \infty$. Liapunov functions are useful to determine, or at least estimate, the basin of attraction of a critical point.

For example, if V is a Liapunov function in a neighborhood D of a and $LV < 0$ in $D \setminus a$, then D is the basin of attraction of a.

Example 4.4.6 Consider the system

$$\begin{aligned} x' &= -x^3\,, \\ y' &= -y(x^2 + z^2 + 1) \\ x' &= -\sin(z) \end{aligned} \tag{4.69}$$

The critical points are $(0, 0, n\pi)$ with $n = 0, \pm 1, \pm 2, \cdots$. Note that if $z = n\pi$ then $z' = 0$ and thus the planes $z = n\pi$ are invariant. Any solution which starts in such plane stays in this plane for all time $t \in \mathbf{R}$. This implies that any solution wich starts in the region $|z| < \pi$ stays in this region for ever. Let us study the stability of $0 = (0, 0, 0)$. The linearization around 0 gives the linear system with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.70}$$

and this tells us nothing about the stability of this equilibrium point. However let us consider the function $V(x, y, z) = x^2 + y^2 + z^2$. We have

$$LV = -2x^4 - 2y^2(x^2 + y^2 + z^2) - 2z\sin(z) \tag{4.71}$$

For $|z| < \pi$, $LV < 0$ except at the origin. It follows from Theorem 4.4.2 that the basin of attraction of 0 is the entire region $\{(x, y, z), |z| < \pi\}$.

In many interesting examples, however, $LV = 0$ in some subset of D but nevertheless a is asymptotically stable and to study this we are going to prove a stronger version of the Liapunov stability theorem.

Theorem 4.4.7 **(Lasalle Stability Theorem)** *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be locally Lipschitz with $f(0) = 0$. Let V be a Liapunov function defined in an open neighborhood D of 0 and let us assume that the set $G = \{x\,;\, V(x) \le \alpha, x \in D\}$ is compact for some $\alpha > 0$. Let us assume that $LV(x) \le 0$ for $x \in G$ and that there is no solution $x(t)$, $t \in \mathbf{R}$ with $x_0 \in G$ on which V is constant. Then 0 is asymptotically stable and V_α is contained in the basin of attraction of 0.*

Proof: Let $x_0 \in G$, then $x(t) = x(t, 0, x_0) \in G$ for all $t > 0$. This follows from the fact that $LV \le 0$ in G and from the compactness of G (the distance from G to the boundary of D is positive).

The proof is by contradiction. Let us assume that $x(t)$ does not tend to 0. Since $x(t)$ stays in the compact set G there exists a sequence $t_n \to \infty$ so that $\lim_{n\to\infty} x(t_n) = x^*$ with $x^* \in G$.

We claim that the solution $y(t) = x(t, 0, x^*)$ starting at x^* exists for all $t \in \mathbf{R}$ and stays in the set G. Clearly $y(t)$ exists for all positive t. On the other hand $x(t, 0, x(t_n))$ is defined for all $t \in [-t_n, 0]$. Since t_n is an increasing sequence then, for any $k \ge 1$, $x(t, 0, x(t_{n+k}))$ is also defined for all $t \in [-t_n, 0]$. By the continuous dependence on initial conditions we have that $y(t) = x(t, 0, x^*)$ is defined for $t \in [-t_n, 0]$. Since n is arbitrary and $t_n \to \infty$ this proves the claim.

We show next that V is constant on the solution $y(t) = x(t, 0, x^*)$. If $V(x^*) = b$ then $V(x(t_n)) \ge b$ and $\lim_{n\to\infty} V(x_{t_n}) = b$. More generally for any sequence s_n with $\lim_{n\to\infty} s_n = \infty$ we have $\lim_{n\to\infty} V(x_{s_n}) = b$. This follows from the fact that V is nonincreasing along a solution. With $s_n = s + t_n$ we have $\lim_{n\to\infty} x(s_n) = y(s) = x(s, 0, x^*)$ and therefore $V(x(s, 0, x^*)) = b$. Since s is arbitrary this proves the claim.

This contradicts our assumption that V is not constant on any solution. ∎

Example 4.4.8 Let us consider the stability of the solution $(0, 0)$ for the mathematical pendulum with friction

$$q'' + q' + \sin(q) \;=\; 0\,. \tag{4.72}$$

The pendulum without friction $q'' + \sin(q) = 0$ has equilibrium position at $(p, q) = (0, 0)$ (stable) and $(0, -\pi)$, $(0, \pi)$ (unstable). We could us a linear stability analysis to show that $(0, 0)$ is asymptotically stable if $\epsilon > 0$. Instead we use a Liapunov function and estimate, at the same time, the size of its basin of attraction.

We consider the Liapunov function $V(p,q) = H(p,q) + 1 = p^2/2 + 1 - \cos(q)$. We have $V(0,0) = 0$ and $LV = -p^2 \leq 0$. Since $LV = 0$ for $y = 0$ the Liapunov function is not strict and we will use Theorem 4.4.7.

Fix a number $c < 2$ and let us consider the set

$$P_c = \{(p,q) \,|\, V(p,q) \leq c \text{ and } |q| < \pi\} . \tag{4.73}$$

For $c < 2$, the set $\{V(p,q) \leq c\}$ consist of infinitely many disjoints closed regions given by the conditions $\{|q - 2n\pi| < \pi\}$, $n \in \mathbf{Z}$. Thus P_c is compact.

We next show that there is no solution on which V is constant, except the 0 solution. Let us assume that there is such a solution, then we have $\frac{d}{dt}V(p(t), q(t)) = -p^2(t) = 0$ and thus $p(t) \equiv 0$. Thus $q'(t) = p(t) = 0$ so $q(t)$ is constant. We also have $p' = -\sin(q) = 0$ and therefore $q(t) \equiv 0$. This is a contradiction. By Theorem 4.4.7 we conclude that $(0,0)$ is asymptotically stable and that P_c is contained in its basin of attraction.

4.5 Gradient and Hamiltonian systems

There are several classes of systems where the use of Liapunov functions is very natural.

4.5.1 Gradient systems

Let $V : U \to \mathbf{R}$ (U is an open set of $\mathbf{R}^n$) be a function of class $\mathcal{C}^2$. A gradient system on $U \subset \mathbf{R}^n$ is a differential equation of the form

$$x' = -\nabla V(x) . \tag{4.74}$$

(The negative sign is a traditional convention). The equilibrium points for (4.74) are the critical points of V, i.e., the points a for which $\nabla V(a) = 0$.

Consider the level sets of the function V, $V^{-1}(c) = \{x \,; V(x) = c\}$. If $x \in V^{-1}(c)$ is a *regular point*, i.e., if $\nabla V(x) \neq 0$, then, by the implicit function Theorem, locally near x, $V^{-1}(c)$ is a smooth hypersurface surface of dimension $n-1$. For example, if $n = 2$, the level sets are smooth curves.

We summarize the properties of the gradient systems in

Proposition 4.5.1 *Let $V : U \to \mathbf{R}^n$ (U an open set in $\mathbf{R}^n$) be of class $\mathcal{C}^2$ and let $x' = -\nabla V(x)$ be a gradient system.*

1. *If x is a regular point of the level curve $V^{-1}(c)$, then the solution curve $x(t)$ is perpendicular to the level surface $V^{-1}(c)$.*

2. *If a is an isolated minimum of V, then it a is an asymptotically stable critical point.*

3. *If a is an isolated minimum of V or a saddle point of V, then a is an unstable critical point.*

Proof: Let y be a vector wich is tangent to the level surface $V^{-1}(c)$ at the point x. For any curve $\gamma(t)$ in the level set $V^{-1}(c)$ with $\gamma(0) = x$ and $\gamma'(0) = y$ we have

$$0 = \frac{d}{dt} V(\gamma(t))|_{t=0} = \langle \nabla V(x) \, , \, y \rangle \, , \tag{4.75}$$

and so $\nabla V(x)$ is perpendicular to any tangent vector to the level set $V^{-1}(c)$ at all regular points of V. This proves 1.

If $x(t)$ is a solution of (4.74), then we have

$$\frac{d}{dt} V(x(t)) = -\langle \nabla V(x(t)) \, , \, \nabla V(x(t)) \rangle \leq 0 \, . \tag{4.76}$$

If a is an isolated minimum of V, then consider the Liapunov function $W(x) = V(x) - V(a)$. We have $LW(x) < 0$ in a neighborhood of a and so, by Theorem 4.4.2, a is an asymptotically stable equilibrium point. This proves 2. If If a is an isolated minimum of V or a saddle point of V, we consider the function $W(x) = V(a) - V(x)$. If a is a isolated minimum $W(x)$ is a Liapunov function and if a is a saddle point then $W(x)$ satisfy the conditions 1 and 3 of Theorem 4.4.3. In both case we have $LW > 0$ in a neighborhood of a and thus, by Theorem 4.4.3, a is unstable. This proves 3. ∎

Example 4.5.2 Let $V : \mathbf{R}^2 \to \mathbf{R}$ be the function $V(x, y) = x^2(x-1)^2 + y^2$. The the gradient system is given by

$$\begin{aligned} x' &= -2x(x-1)(2x-1) \, , \\ x' &= -2y \, . \end{aligned} \tag{4.77}$$

There are 3 critical points $(0, 0)$, $(1/2, 0)$, and $(1, 0)$. From the form of V one concludes that $(0, 0)$ and $(1, 0)$ are asymptotically stable with basins of attraction $\{-\infty < x < 1/2 \, , \, -\infty < y < \infty\}$ and $\{1/2 < x < \infty \, , \, -\infty < y < \infty\}$ respectively. The critical point $(1, 0)$ is unstable (saddle point). The solution with initial conditions $x(0) = 1$ and $y(0) = y_0$ satisfy $(x(t), y(t)) \to (1, 0)$.

We also have

Proposition 4.5.3 *Let $V : U \to \mathbf{R}^n$ (U an open set in $\mathbf{R}^n$) be of class $\mathcal{C}^2$ and let $x' = -\nabla V(x)$ be a gradient system. If a is a critical point, then the linearization around A has only real eigenvalues.*

Proof: Let a be a critical point. Then the linearized system is given by the matrix $A = (a_i j)$ where

$$a_{ij} = -\frac{\partial^2}{\partial x_i \partial x_j}(a). \tag{4.78}$$

Since V is of class $\mathcal{C}^2$ we have $a_{ij} = a_{ji}$. Therefore A is a symmetric matrix and its eigenvalues are real. ∎

4.5.2 Hamiltonian systems

Let $p = (p_1, \cdots, p_n) \in \mathbf{R}^n$ and $q = (q_1, \cdots, q_n) \in \mathbf{R}^n$ and let $H : U \to \mathbf{R}$ (U an an open set of $\mathbf{R}^{2n}$) be a function of class $\mathcal{C}^2$. For mechanical systems q are the coordinates of the particles and p are the momenta of the particles. The function $H(p, q)$ is called the energy of the system.

A Hamiltonian system for the Hamiltonian H is a system of differential equations of the form

$$\begin{aligned} q_i' &= \frac{\partial H}{\partial p_i} \qquad i = 1, \cdots, n\,, \\ p_i' &= -\frac{\partial H}{\partial p_i} \qquad i = 1, \cdots, n\,. \end{aligned} \tag{4.79}$$

A simple but very important property of Hamiltonian is conservation of energy

Proposition 4.5.4 *Let $H(p, q)$ be a function of class $\mathcal{C}^2$. Let $(p(t), q(t))$ be a solution of the Hamitlon's equations (4.79), then $H(p(t), q(t))$ is constant.*

Proof: We have

$$\begin{aligned} \frac{d}{dt} H(p(t), q(t)) &= \sum_{i=1}^n \frac{\partial H}{\partial p_i} p_i' + \frac{\partial H}{\partial q_i} q_i' \\ &= \sum_{i=1}^n -\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} = 0\,. \end{aligned} \tag{4.80}$$

Hence H is constant along any solution. ∎

Let $x = (p, q)$ and define $J : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ as the linear map given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4.81}$$

where I is the $n \times n$ identity matrix. Note that we have $J^T = J^{-1} = -J$. With $x = (p, q)$ we can rewrite (4.80) as

$$x' = J^{-1} \nabla H\,. \tag{4.82}$$

Theorem 4.5.5 *Let $U \subset \mathbf{R}^{2n}$ be an open set and $H : U \to \mathbf{R}$ be of class $\mathcal{C}^2$. Then the flow $\phi^t(x) = x(t,0,x)$ for (4.82) satisfy*

$$\left(\frac{\partial \phi^t}{\partial x}\right)^T J \left(\frac{\partial \phi^t}{\partial x}\right) = J\,. \tag{4.83}$$

Remark 4.5.6 A map $T : U \to \mathbf{R}^{2n}$ ($U \subset \mathbf{R}^{2n}$ open) which satisfies $\frac{\partial T}{\partial x}^T J \frac{\partial T}{\partial x} = J$ is called a *symplectic transformation.*

Proof: The derivative $\frac{\partial \phi^t}{\partial x}$ is the solution for the variational equation, which is

$$\Psi'(t) = J^{-1}\frac{d^2H}{dx^2}(x(t,0,x))\Psi(t)\,, \quad \Psi(0) = I\,, \tag{4.84}$$

where

$$\frac{d^2H}{dx^2} = \left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right)_{1\le i,j\le n}\,. \tag{4.85}$$

is a symmetric matrix. We have then

$$\begin{aligned}\frac{d}{dt}\left(\left(\frac{\partial \phi^t}{\partial x}\right)^T J \left(\frac{\partial \phi^t}{\partial x}\right)\right) &= \left(\left(\frac{\partial \phi^t}{\partial x}\right)^T\right)' J \left(\frac{\partial \phi^t}{\partial x}\right) + \left(\frac{\partial \phi^t}{\partial x}\right)^{T'} J \left(\frac{\partial \phi^t}{\partial x}\right)' \\ &= \left(\frac{\partial \phi^t}{\partial x}\right)^T \frac{d^2H}{dx^2}(x(t,0,x))J^2\left(\frac{\partial \phi^t}{\partial x}\right) + \left(\frac{\partial \phi^t}{\partial x}\right)^T \frac{d^2H}{dx^2}(x(t,0,x))\left(\frac{\partial \phi^t}{\partial x}\right) \\ &= 0\,. \end{aligned} \tag{4.86}$$

Since (4.83) holds for $t = 0$ it holds thus for all t. ∎

Theorem 4.5.7 *Let $U \subset \mathbf{R}^{2n}$ be an open set and $H : U \to \mathbf{R}$ be of class $\mathcal{C}^2$. The the flow ϕ^t for (4.82) is volume preserving, i.e., for any $A \subset \mathbf{R}^n$ compact with ∂A negligeable we have*

$$\mathrm{vol}\left(\phi^t(A)\right) = \mathrm{vol}(A)\,. \tag{4.87}$$

Proof: The map ϕ^t is of class $\mathcal{C}^1$ by Theorem (3.2.7) and injective by the uniqueness of solutions. Since $\det J = 1$, from Theorem 4.5.5 we have

$$\det\left(\frac{\partial \phi^t}{\partial x}\right) = 1\,. \tag{4.88}$$

By the change of variables formula we have

$$\mathrm{vol}\left(\phi^t(A)\right) = \int\int_{\phi^t(A)} dp\,dq = \int\int_{\phi^t(A)} dp\,dq = \int\int_A \det\frac{\partial \phi^t}{\partial x}\,dp\,dq = \mathrm{vol}(A)\,. \tag{4.89}$$

and so ϕ^t preserves volume. ∎

Remark 4.5.8 This can be derived directly from Liouville Theorem 2.2.6 (see homework). The property of symplecticity is a much stronger than volume preserving though, at least for $n \geq 1$.

Remark 4.5.9 Because of the volume preserivng property critical points of Hamiltonian systems cannot be asymptotically stable.

Theorem 4.5.10 *Consider Hamilton's equation (4.79) where* H *is of class* $\mathcal{C}^2$*. Let* a *be a critical point for (4.79). If* $H(p,q) - H(a)$ *is positive (or negative) in a neighborhood of* a *then* a *is a stable critical point.*

Proof: Without loss of generality we may assume that $a = 0$. By Proposition (4.5.4) $V(p,q) = H(p,q) - H(0)$ (or $V(p,q) = H(0) - H(p,q)$) is a Liapunov function with $LV = 0$. The theorem follows immediately from item 1. of Theorem 4.4.2. ∎

In mechanical systems, the Hamiltonian has (usually) the form $H = T + W$ where T is the kinetic energy and U is the potential energy. We have

$$T = \sum_{i=1}^{n} \frac{p_i^2}{2}, \quad W = W(q), \tag{4.90}$$

in particular T is positive. The Hamiltonian equations have then the form (Newton's 2^{nd} law)

$$q_i'' = -\nabla W(q). \tag{4.91}$$

Equilibrium solutions then correspond to

$$p_i = 0, \quad \frac{\partial W}{\partial q_i} = 0, \quad i = 1, \cdots, n, \tag{4.92}$$

We have

Theorem 4.5.11 *Let* $W : U \to \mathbf{R}$ *(*$U \subset \mathbf{R}^n$ *an open set) be a function of class* $\mathcal{C}^2$*. Assume that* 0 *be a critical point of of* U*, i.e.,* $\nabla W(0) = 0$*. If* 0 *is a local (strict) minimum of* W *then* a *is a stable critical point for (4.91). If* 0 *is a local maximum of* W *and*

$$W(q) = -\sum a_i q_i^{2m} + O(\|(q-a)\|^{2m+1}) \tag{4.93}$$

then a *is an unstable critical point for (4.91).*

Proof: The stability of strict minima follows from Theorem 4.5.10. For local maxima we consider the function

$$V(p,q) = \sum_{i=1}^{n} p_i q_i \tag{4.94}$$

and apply Theorem 4.4.3. It satisfies Condition 1 and 3 of this theorem and we have

$$\begin{aligned} LV(p,q) &= \sum_{i=1}^{n} (p_i' q_i + q_i' p_i) = \sum_{i=1}^{n} -\frac{\partial W}{\partial q_i} q_i + p_i^2 \\ &= \sum_{i=1}^{n} \left(2ma_i q_i^{2m} + p_i^2\right) + O(\|q\|^{2m+1}) \end{aligned} \tag{4.95}$$

and so $LV > 0$ is positive in a neighborhood of 0. Therefore 0 is unstable.

Example 4.5.12 Let $H : \mathbf{R}^2 \to \mathbf{R}$ be given by $H(p,q) = \frac{p^2}{2} + q^2(q-1)^2$. The Hamilton's equation of motion are

$$q'' = -2q(q-1)(2q-1) \tag{4.96}$$

There are 3 critical points $(0,0)$, $(0,1/2)$, and $(0,1)$. Both $(0,0)$ and $(1,0)$ are local minima of $W(q) = q^2(q-1)^2$ and therefore they are stable. The point $(0,1/2)$ is a local maximum of $W(q)$ and thus is unstable.

Example 4.5.13 Consider the equation $q'' + q' + q^3 = 0$, or $q' = p$ and $p' = -p - q^3$ The equation $q'' + q^3$ is Hamiltonian with Hamiltonian $p^2/2 + q^4/4$. The term $q' = p$ is a friction term and one expects that 0 is an attracting fixed point with basin of attraction $\mathbf{R}^2$. This can be shown using Lasalle Theorem 4.4.7, as in Example 4.4.8, since we have $LH = -p^2$. We use instaed a very smartly chosen Liapunov function (due to Nevelson). Let

$$V(p,q) = \frac{p^2}{2} + (q - \alpha \arctan(q))p + \frac{q^4}{4} + \int_0^q (x - \alpha \arctan(x))\, dx\,. \tag{4.97}$$

We obtain

$$\begin{aligned} LV(p,q) &= (p + (q - \alpha \arctan(q)))\left(-p - q^3\right) \\ &\quad + \left(\left(1 - \alpha\frac{1}{1+q^2}\right)p + q^3 + (q - \alpha\arctan(q))\right)p \\ &= -q^3(q - \alpha\arctan(q)) - \frac{1}{1+q^2}p^2\,, \end{aligned} \tag{4.98}$$

so that $LV < 0$ if $(p,q) > 0$. You can also verify that $V(0,0) = 0$ and that V is positive. This implies that the basin of attraction of $(0,0)$ is $\mathbf{R}^2$.

Chapter 5

Poincaré-Bendixson Theorem

We discuss in this chapter the long time limit of two dimensional autonomous systems. In particular, we discuss in the problem of the existence of periodic solutions for two dimensional systems. Closed orbits correspond to periodic solutions and according Jordan closed curve theorem they separate $\mathbf{R}^2$ into two connected components, the interior of and the exterior of the orbits. This makes the 2-dimensional case special and much more tractable then the general case.

5.1 Limit sets and attractors

Let us consider the autonomous equation $x' = f(x)$ where $x \in \mathbf{R}^n$ and $f(x)$ is locally Lipschitz. We denote by $\gamma(x_0)$ the *orbit* corresponding to the solution with $x(0) = x_0$. In particular if $x(t_1) = x_1$ then we have $\gamma(x_0) = \gamma(x_1)$. We denote by $\gamma^+(x_0)$ the *positive orbit* defined by $x(t)$, $t \geq 0$ and by $\gamma^-(x_0)$ the *negative orbit* defined by $x(t)$, $t \geq 0$. We have $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$ and for a periodic solution we have $\gamma^+(x_0) = \gamma^-(x_0)$.

We call a set M invariant, if $x(0) \in M$ implies that $x(t) \in M$ for all $t \in \mathbf{R}$ and we call a set *positively invariant* , if $x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

Definition 5.1.1 A point x^* is called a *positive limit point* of the orbit $\gamma(x_0)$ if there exists an increasing sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that $x^* = \lim_{n\to\infty} x(t_n)$. A *negative limit point* of $\gamma(x_0)$ is defined similarly.

Example 5.1.2 For the equation $x' = -2x$ and $y' = -y$, for all orbits $(0,0)$ is the unique limit point.

Example 5.1.3 If $x' = f(x)$ has a periodic orbit C, every point on the periodic orbit is a positive and negative limit point of the orbit.

Definition 5.1.4 We denote by $\omega(\gamma)$ the set of all positive limit points for the orbit γ, it is called the *ω-limit set* of γ. Similarly we denote $\alpha(\gamma)$ the set of all negative limit points for the orbit γ (the *α-limit set* of γ).

The basic properties of limit points and limit sets are summarized in

Theorem 5.1.5 *The sets $\omega(\gamma)$ and $\alpha(\gamma)$ are closed and invariant. If the positive orbit γ^+ is bounded, then $\omega(\gamma)$ is a compact, connected and non-empty set. If $\gamma^+ = \gamma^+(x_0)$ then we have* $\lim_{t\to\infty} \mathrm{dist}\,(x(t,0,x_0)\,,\,\omega(\gamma)) = 0$. *Analogous properties hold for γ^- and $\alpha(\omega)$.*

Proof: We first prove that $\omega(\gamma)$ is closed. Let $\{y_m\}$ be a convergent sequence in $\omega(\gamma)$ with limit y. We show that y is a limit point. For any $\epsilon > 0$, there exists M such that, for any $m \ge M$, $\|y_m - y\| \le \epsilon/2$ and there exist sequences t_n^m with $\lim_n t_n^{(m)} = \infty$ such that $\|x(t_n^{(m)}) - y_m\| \le \epsilon/2$. This implies that y is a limit point.

We next show that $\omega(\gamma)$ is an invariant invariant. Let $x^* \in \omega(\gamma)$, then there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\lim_n x(t_n) = x^*$. For arbitrary t, we have $x(t+t_n,0,x_0) = x(t,0,x(t_n,0,x_0))$ so that, using the continuous dependence on initial conditions we have $\lim_n x(t+t_n,x_0) = x(t,0,x^*)$. Therefore for all t, the orbit which contains x^* lies in $\omega(\gamma)$ and this proves invariance.

Let us assume that γ^+ is bounded, then it has at least one accumulation point and so $\omega(\gamma)$ is non empty. Since $\omega(\gamma)$ is closed it is also compact.

Since, for any t, $\lim_n x(t+t_n,0,x_0) = x(t,x^*)$ we have $\gamma(x^*) \subset \omega(\gamma(x_0))$. It follows that $\lim_{t\to\infty} \mathrm{dist}\,(x(t,0,x_0)\,,\omega(\gamma)) = 0$. This also implies that $\omega(\gamma)$ is connected. ∎

Definition 5.1.6 We say that a closed invariant set A is an *attracting set* if there is a open neighborhood U of A such that for all points $x_0 \in U$, $x(t,0,x_0) \in U$ for all $t > 0$ and $\lim_{t\to\infty} \mathrm{dist}\,(x(t)\,,\,A) = 0$. An *attractor* is an attracting set which contains a dense orbit. A *stable limiting cycle* is an attractor which consists of a single periodic orbit.

Example 5.1.7 For a critical point a, we always have $\gamma(a) = a$ and so $\{a\}$ contains a dense orbit. If a is is asymptotically stable then it is an attractor.

Example 5.1.8 Consider the system $x' = x - x^3$ and $y' = -y$. The system has three critical points $(-1,0)$ and $(1,0)$ which are asymptotically stable and $(0,0)$ which is a saddle point. The set $A = \{y = 0\,,\,-1 \le x \le 1\}$ is an attracting set, every orbit is attracted to A. But A does not contain a dense orbit and it is not an attractor.

Example 5.1.9 Consider the system

$$\begin{aligned} x' &= -y + x(1-x^2-y^2)\,, \\ y' &= x + y(1-x^2-y^2)\,. \end{aligned} \tag{5.1}$$

It is convenient to write the equation in polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$. A simple computation shows that (5.1) is equivalent to

$$\begin{aligned} r' &= r(1-r^2)\,, \\ \theta' &= 1\,. \end{aligned} \tag{5.2}$$

We see that the origin is an equilibrium point of the system. The set $r = 1$ is invariant and consists of a periodic orbit of period 2π. The origin is unstable and the flow spirals outward (counterclockwise) for $0 < r < 1$ and inward for $r > 1$. The circle $\{r = 1\}$ is a stable limiting cycle. Each point in the open neighborhood $U = \{a < r < A\}$ where $a > 0$ is attracted to the limiting cycle.

Example 5.1.10 Consider the system (in polar coordinates)

$$\begin{aligned} r' &= r(1-r)\,, \\ \theta' &= \sin^2(\theta) + (1-r)^3\,. \end{aligned} \tag{5.3}$$

Both the origin and the circle $\{r = 1\}$ are invariant set. The circle $\{r = 1\}$ is the ω-limit set for all orbits starting outside the origin and outside the circle. The invariant set $\{r = 1\}$ consists of four orbits given by $\theta = 0$, $\theta = \pi$, and the arcs $0 < \theta < \pi$, and $\pi < \theta < 2\pi$.

5.2 Poincaré maps

Let C be a periodic orbit of the system

$$x' = f(x)\,, \tag{5.4}$$

which passes through the point x_0. Let Ξ be an hyperplane perpendicular to the orbit C at x_0. If $x \in \Xi$ is sufficiently close to x_0, then the solution starting at x at $t = 0$ will cross the hyperplane Ξ at a point $P(x)$ near x_0. The mapping $x \mapsto P(x)$ is called a *Poincaré map*.

There is nothing special about hyperplanes and so the Poincaré map can be defined in a similar way if Ξ is a smooth hypersurface through x_0 which is not tangent to the periodic orbit C.

Theorem 5.2.1 *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be of class $\mathcal{C}^1$ and let $\phi^t(x)$ denote the flow defined by the differential equation $x' = f(x)$. Assume that $\phi^t(x_0)$ is a periodic solution of period p and the orbit $\{\phi^t(x_0)\}_{0\le t\le p}$ is contained in U. let Ξ be the hyperplane orthogonal to C at x_0, i.e.*

$$\Xi = \{x \in \mathbf{R}^n\,,\, \langle (x - x_0)\,,\, f(x_0)\rangle = 0\}\,. \tag{5.5}$$

Then there exists $\delta > 0$ and a unique function $\tau(x)$ which is defined and continuously differentiable in $N_\delta(x_0) = \{x \in \Xi\,;\, \|x - x_0\| < \delta\}$ so that $tau(x_0) = p$ and

$$\phi^{\tau(x)}(x) \in \Xi\,, \quad \text{for all} \quad x \in N_\delta(x_0)\,. \tag{5.6}$$

Proof: This is an consequence of the smooth dependence of $\phi^t(x)$ with respect to x and t (see Theorem 3.2.7) and from the implicit function theorem. We define the function

$$F(t,x) \;=\; \Big\langle (\phi^t(x) - x_0)\,,\, f(x_0) \Big\rangle\,. \tag{5.7}$$

The function $F(t,x)$ is of class $\mathcal{C}^1$ for $(t,x) \in \mathbf{R} \times U$. Since $\phi^p(x_0) = x_0$ we have

$$F(p, x_0) \;=\; 0\,. \tag{5.8}$$

Since $\frac{\partial \phi^t(x_0)}{\partial t}|_{t=p} = f(x_0)$ we have

$$\frac{\partial F}{\partial t}(p, x_0) \;=\; \left\langle \frac{\partial \phi^t(x)}{\partial t}|_{t=p}\,,\, f(x_0) \right\rangle \;=\; \langle f(x_0)\,,\, f(x_0) \rangle \;\neq\; 0\,, \tag{5.9}$$

as x_0 is not an equilibrium point. From the implicit function theorem there exists a function $\tau(x)$ of class $\mathcal{C}^1$ defined in a neighborhood $B_\delta(x_0)$ such that $\tau(x_0) = x_0$ and

$$F(\tau(x), x) \;=\; \Big\langle (\phi^{\tau(x)}(x) - x_0)\,,\, f(x_0) \Big\rangle \;=\; 0\,, \tag{5.10}$$

i.e.,

$$\phi^{\tau(x)}(x) \in \Xi\,. \tag{5.11}$$

This concludes the proof by taking $N_\delta(x_0) = B_\delta(x_0) \cap \Xi$. ∎

Theorem 5.2.1 implies that the *Poincaré map* $P \,:\, N_\delta(x_0) \to \Xi$ given by

$$P(x) \;=\; \phi^{\tau(x)}(x)\,, \tag{5.12}$$

is well defined and of class $\mathcal{C}^1$. Fixed points of the Poincaré map $P(x) = x$ correspond to periodic orbits $\phi^t(x)$ for (5.4). One can also show easily that the map $P(x)$ is invertible with an inverse of class $\mathcal{C}^1$ given $P^{-1}(x) \;=\; \phi^{-\tau(x)}(x)$.

Example 5.2.2 For the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2)\,, \\ y' &= x + y(1 - x^2 - y^2)\,. \end{aligned} \tag{5.13}$$

or equivalently, in polar coordinates

$$\begin{aligned} r' &= r(1 - r^2)\,, \\ \theta' &= 1\,. \end{aligned} \tag{5.14}$$

we can compute the Poincaré map explicitly. The solution to equations (5.14) with $r(0) = r_0$ and $\theta(0) = \theta_0$ are

$$\begin{aligned} r(t, r_0) &= \left[1 + \left(\frac{1}{r_0^2} - 1 \right) e^{-2t} \right]^{-\frac{1}{2}}\,, \\ \theta(t, \theta_0) &= t + \theta_0\,. \end{aligned} \tag{5.15}$$

Let Ξ be the ray $\theta = \theta_0$ through the origin, then Ξ is perpendicular to the closed orbit $r = 1$. Any trajectory starting at (r_0, θ_0) intersects Ξ at time 2π. Therefore the Poincaré map is given by

$$P(r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-4\pi}\right]^{-\frac{1}{2}} . \tag{5.16}$$

with $P(1) = 1$ and

$$P'(r_0) = e^{-4\pi} \frac{1}{r_0^3} \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-4\pi}\right]^{-\frac{3}{2}} \tag{5.17}$$

so that $P'(1) = e^{-4\pi} < 1$.

For two-dimensional system, Ξ is a line segment intersecting C orthogonally. We can parametrize the line Ξ by the distance s to the intersection point x_0. We can then write the Poincaré map as $P = P(s)$ as a a function of s which is defined in a neighborhood of the origin and satisfy $P(0) = 0$.

The stability of the periodic orbit C is determined by $P'(0)$. To see this let us introduce the *displacement function*

$$d(s) = P(s) - s . \tag{5.18}$$

Then $d(0) = 0$ and $d'(s) = P'(s) - 1$. If $d'(0) \neq 0$ then, by continuity the sign of $d(s)$ will be the same in a neighborhood of 0. Thus if $d'(0) < 0$ (i.e $P'(0) < 1$) therefore the periodic orbit C is a stable limit cycle since the successive intersection of the orbit with Ξ approach 0. Similarly if $d'(0) < 0$ (i.e., $P'(0) < 1$) C is an unstable limit cycle.

Example 5.2.3 For the example 5.2.2 we have seen that $P'(1) < 1$ which means that the circle $r = 1$ is a stable limit cycle.

In general it is of course very difficult to compute the Poincaré map explicitly. It is useful however to investigate stability properties (as briefly discussed above in the 2-dimensional case) and also the concept of a Poincaré map will central in establishing the existence of periodic orbits in the Poincaré-Bendixson theorem.

5.3 Bendixson criterion

We consider the autonomous system

$$x' = f(x) \tag{5.19}$$

where f is of class $\mathcal{C}^1$ in an open set $U \subset \mathbf{R}^2$. We derive a simple sufficient condition criterion for a two dimensional not to have a periodic orbit.

Theorem 5.3.1 (Bendixson criterion) *Suppose that the open $D \subset \mathbf{R}^2$ is simply connected and f is of class $\mathcal{C}^1$. The equation (5.19) can have a periodic solution only if* $\operatorname{div} f$ *changes sign in D or* $\operatorname{div} f = 0$ *in D.*

Proof: Suppose that we have a closed orbit C in D and let G be the interior of the orbit C. By the divergence theorem (Gauss Theorem) we have,

$$\int\int_G \operatorname{div} f \, dx \;=\; \int_C f \cdot ds \;=\; \int_C (f_1 dx_2 - f_2 dx_1) \;=\; \int_C \left(f_1 \frac{dx_2}{dt} - f_2 \frac{dx_1}{dt} \right) dt \;=\; 0 \quad (5.20)$$

where the last equality follows from the fact that the closed curve C can be parametrized by a solution of (5.19). So the integral on the left side vanishes which implies that $\operatorname{div} f$ either vanishes or changes sign in D. ∎

Example 5.3.2 The Lienard equation is given $x'' + f(x)x' + g(x) \;=\; 0$ where $f(x)$ and $g(x)$ are Lipschitz continuous. The vector fields $h(x, y) = (y, -f(x)y - g(x))^T$ satisfies $\operatorname{div} f(x, y) = -f(x)$. If $f(x)$ is either positive or negative then Theorem (5.3.1) implies that there is no periodic solution.

Example 5.3.3 For the the van der Pol equation $x'' + \epsilon(x^2 - 1)x' + x \;=\; 0$ we have $\operatorname{div} f(x, y) \;=\; -\epsilon(x^2 - 1)$. So a periodic solution, if it exists, must intersect with the lines $x = 1$ or $x = -1$.

5.4 Poincaré-Bendixson Theorem

In this section we prove

Theorem 5.4.1 *Let $f : U \to \mathbf{R}^2$ (U an open set of $\mathbf{R}^2$) be of class $\mathcal{C}^1$. Assume that the positive orbit γ^+ for the system*

$$x' \;=\; f(x)\,, \qquad (5.21)$$

is bounded and that the limit set $\omega(\gamma^+)$ does not contain critical points. Then $\omega(\gamma^+)$ is a periodic orbit.

We will split the proof of the theorem in a sequence of lemmas. We first introduce the concept of a transversal line and regular points for (5.21).

Definition 5.4.2 A finite closed segment of a straight line l contained in U is *transversal* for (5.21) if l does not contain any critical points of (5.21) and if the vector field f is not tangent to l at any point of l.

Definition 5.4.3 A point $x \in U$ is *regular* for (5.21) if it is not a critical point for (5.21).

Lemma 5.4.4 *Let x^* be an interior point of a transversal l, then for any $\epsilon > 0$, there exists $\delta > 0$ such that every trajectory passing through a point x with $\|x - x^*\| \leq \delta$ crosses l at some time t with $|t| \leq \epsilon$.*

Proof: This follows from the continuity of the vector field f. Since l contains only regular points, the vector field on l always points on the same side of l. A sufficiently small neighborhood of l contains also only regular points. This implies that an orbit starting close enough from l will actually crosses it for some positive or negative time. The lemma follows then by the continuous dependence of the solution from initial conditions. ∎

Lemma 5.4.5

1. *If a finite closed arc $\{x(t) : a \leq t \leq b\}$ of a trajectory γ intersects a transversal l, it does so at a finite number of points.*
2. *If γ is a periodic orbit which intersects l, it does intersect l only once.*
3. *The successive intersections with l form a monotonic sequence (with respect to the order on the line l).*

Proof: Let us assume that for $t \in [a, b]$, $x(t)$ intersects the transversal at infinitely many points $x_n = x(t_n)$. Then the sequence t_n will have an accumulation point $t^* \in [a, b]$. Passing to a subsequence, also denoted by $\{t_n\}$ we have that $x(t_n)$ converges to $x(t^*)$ in l. But on the other hand we have

$$\lim_{n \to \infty} \frac{x(t_n) - x(t^*)}{t_n - t^*} = x'(t^*) = f(x(t^*)) . \tag{5.22}$$

which is a vector tangent to l at $x(t^*)$. Since $x(t_n), x(t^*) \in l$ this is a contradiction to the fact that l is transversal. This proves 1 and thus a fine segment of an orbit meets l finitely many times.

Let now $x_1 = x(t_1)$ and $x_2 = x(t_2)$ with $t_1 < t_2$ be two successive point of intersection of the orbit with l. Suppose that x_1 is distinct from x_2. Then the arc $\{x(t) : t_1 \leq t \leq t_2\}$ together with the closed segment $\overline{x_1 x_2}$ on l forms a closed Jordan curve J. By Jordan closed curve Theorem J separates the plane into two regions, the interior of J and the exterior of J. The points $x(t)$ with $t < t_1$ (and t close to t_1) and the points $x(t)$ with $t > t_2$ (and t close to t_2) will be on the opposite sides of the curve J. It, in fact, remains so for all times $t < t_1$ and $t > t_2$. Suppose for example $x(t)$, $t > t_2$ is inside J. The orbit cannot cross J on the transversal because the flow points inward on the transversal, it cannot cross either on the orbit part of J by uniqueness of solutions. Therefore $x(t)$ must remains inside J for all $t > t_2$. A similar argument holds if $x(t)$, $t < t_1$ is inside J.

From this 2 and 3 follows immediately. ∎

This lemma gives some insight in the structure of possible ω-limit sets in $\mathbf{R}^2$.

Lemma 5.4.6 *If an orbit γ and its ω-limit set $\omega(\gamma)$ have a point in common, then γ is either a critical point or a periodic orbit.*

Proof: Let $x_1 = x(t_1) \in \gamma \cap \omega(\gamma)$. If x_1 is a critical point then $x(t) = x_1$ for all $t \in \mathbf{R}$. If x_1 is a regular point, then we can find a transversal l such that x_1 is an interior point of l. Since $x_1 \in \omega(\gamma)$, there exists a point $x^* = x(t^*)$ with, say, $t^* > t_1 + 2$ which is at distance no more than δ from x_1. By lemma (5.4.4) the orbits passing through x^* will intersect l at a time $t_2 > t_1 + 1$. If $x_2 = x_1$ then $\omega(\gamma)$ is periodic orbit. If $x_2 \neq x_1$, then the piece of orbit $\{x(t)\,;\, t_1 \leq t \leq t_2\}$ intersect l only at a finite number of points, by lemma 5.4.5, item 1. By Lemma 1, item 3, the successive intersections of γ with l are a monotone sequence which tend away from x_1. This contradicts our assumption that $x_1 \in \omega(\gamma)$. ∎

Lemma 5.4.7 *If the ω-limit set $\omega(\gamma)$ of an orbit γ intersect a transversal l, it does so in one point only. If x^* is such an intersection point, we have $\omega(\gamma) = \gamma$ with γ a periodic orbit or there exists a sequence t_n with $\lim_n t_n = \infty$ and $x(t_n)$ tends to x^* monotonically on l.*

Proof: Suppose that $\omega(\gamma)$ intersects l in x^*. If the orbits γ passes through x^* then γ and $\gamma(\omega)$ have x^* in common which is regular point. Thus, by lemma 5.4.6 $\omega(\gamma)$ is a periodic orbit. If not, x^* being in the ω-limit set, there exists a sequence $\{t'_n\}$ with $\lim_n t'_n = \infty$ such that $\lim_n x(t'_n) = x^*$. Then, by Lemma 5.4.4, there exists a sequence t_n with $\lim_n t_n = \infty$ such that $x(t_n) \in l$ and $\lim_n x(t_n) = x^*$ monotonically in l by Lemma 5.4.5.

Now suppose that there exists another intersection point y^* of $\omega(\gamma)$ with l. By the same argument we can construct a sequence $x(s_n)$ which tends monotonically in l to y^*. But in that case we can construct a sequence of intersection of the orbit with l which is not monotone. This contradicts Lemma 5.4.5. ∎

Lemma 5.4.8 *If $\omega(\gamma)$ contains no critical point and $\omega(\gamma)$ contains a periodic orbit C, then $\omega(\gamma) = C$.*

Proof: Let us suppose $\omega(\gamma) \setminus C$ is not empty. Since $\omega(\gamma)$ is connected, C must contain a limit point x^* of the set $\omega(\gamma) \setminus C$. Let l be a transversal which contains x^*, from lemma 5.4.7 it follows that that $\omega(\gamma)$ intersects l only at x^*. Since x^* is a limit point of $\omega(\gamma) \setminus C$, then there exist a point y of $\omega(\gamma) \setminus C$ in arbitrarily small neighborhoods of x^*. By Lemma 5.4.4 an orbit through y will intersect l, since $y \in \omega(\gamma)$ and $\omega(\gamma)$ is an invariant set, an orbit through y belongs also to $\omega(\gamma)$ and this contradicts lemma 5.4.7. ∎

With these preparations we can now conclude the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1: Since γ^+ is bounded, $\omega(\gamma^+)$ is compact, connected and not empty. If γ^+ is a periodic orbit then, by lemma 5.4.8, $\gamma^+ = \omega(\gamma^+)$. Suppose $\gamma^+ \neq \omega(\gamma^+)$. There exists an orbit $C \subset \omega(\gamma^+)$ and the orbit C is contained in compact bounded set F. Therefore the orbit C has itself a limit point x^* which is in $\omega(\gamma^+)$ since $\omega(\gamma^+)$ is closed. Let l be a transversal through x^*, then by Lemma 5.4.7, l intersects $\omega(\gamma^+)$ only at x^*. Since x^* is a limit point of C, by Lemma (5.4.4), l must also intersects C at some point which must be then x^*. This implies that C and $\omega(C)$ have a point in common and so, by lemma 5.4.6, C must be a periodic orbit. By lemma 5.4.8 this implies that $\omega(\gamma^+) = C$. ∎

We can derive several consequences from Theorem (5.4.1). If $\omega(\gamma)$ is a periodic orbit C, we call it a *ω-limit cycle*. We have seen that $\text{dist}(x(t), \omega(gamma)) \to 0$ as $t \to \infty$, which means that the orbit spirals toward C.

Limit cycles possess a kind of (at least one-sided) stability.

Corollary 5.4.9 *Let C be an ω-limit cycle. If $C = \omega(\gamma(x))$, then there exists a neighborhood O of x such that $C = \omega(\gamma(y))$ for all $y \in O$. The set*

$$\{y \,|\, \omega(\gamma(y)) = C\} \setminus \gamma \tag{5.23}$$

is an open set.

Proof: Let l be a transversal to C. Then there exists a line segment f in l, which is disjoint from C and is bounded by $x(t_1)$ and $x(t_2)$ with $t_1 < t_2$ and such that $x(t)$ does not meet this segment for $t_1 < t < t_2$. Consider now the region A bounded one side by C and on the other side by the closed curve consisting of the orbit segment $\{x(t) t_1 \leq t \leq t_2\}$ and the line segment f. This region is positively invariant and if $y \in A \setminus \gamma$ the orbit passing through y spirals toward C. ∎

If there exists orbits attracted to C starting on both sides of C, then C is an attractor.

Corollary 5.4.10 *Let K be a compact set which is positively invariant, then K contains either a limit cycle or a critical point.*

Proof: If K is positively invariant, then any orbit starting in K is bounded. The corollary follows then from Theorem 5.4.1. ∎

Corollary 5.4.11 *Let C be a periodic orbit and let O be the open region in the interior of C. Then C contains either an equilibrium point or a limit cycle.*

Proof: The $D = O \cup C$ is positively and negatively invariant. If O contains no limit cycle or critical point, then for all $x \in U$ we must have by Poincaré-Bendixson Theorem

$$\omega(\gamma(x)) = \alpha(\gamma(x)) = C \tag{5.24}$$

so the orbit spirals toward C both for positive and negative times. This is a contradiction. ∎

5.5 Examples: van der Pol equation

The theorem of Poincaré-Bendixson implies that the existence of a positively invariant set which does not contain any critical point, must contain some periodic orbit (limit cycles). Analyzing particular systems can be quite complicated. We will give a class of examples, the so-called Lienard equations for which we establish the existence of periodic orbits. More can be said on such systems with a finer analysis, for example one can determine the number of periodic orbits.

The Lienard equation has the general form

$$x'' + f(x)y + g(x)\,. \tag{5.25}$$

where $f(x)$ and $g(x)$ are Lipschitz continuous.

We will restrict ourselves to the (generalizied) van der Pol equation

$$x'' + (x^2 - 1)x' + x^{2n-1}\,, \tag{5.26}$$

where $n \geq 1$, i.e., $f(x) = x^2 - 1$ and $g(x) = x^{2n-1}$. It will be useful to introduce the functions

$$F(x) = \int_0^x f(z)\,dz = \frac{x^3}{3} - x\,, \quad G(x) = \int_0^x g(x) = \frac{x^{2n}}{2n}\,. \tag{5.27}$$

We will show that a suitable "annular region" around 0 is positively invariant. We will do this using two suitably chosen Liapunov function.

We first consider the function

$$V(x,y) = \frac{(y + F(x))^2}{2} + G(x)\,, \tag{5.28}$$

We have

$$\begin{aligned} LV(x,y) &= (y + F(x))\,(-f(x)y - g(x)) + ((y + F(x))f(x) + g(x))\,y \\ &= -g(x)F(x) = -x^{2n-1}(x^3 - x)\,. \end{aligned} \tag{5.29}$$

The function $F(x) = x^3 - x$ is positive for $0 < x < 1$ and negative for $-1 < x < 0$ and thus $LV > 0$ if $|x| < 1$ and $x \neq 0$. This implies that orbits starting on the boundary of a circular domain $\{\|(x, y)\|_2 < a\}$ with $a < 1$ cannot enter this domain.

Next we consider the function

$$W(x, y) \;=\; \frac{y^2}{2} + (F(x) - \arctan(x))\, y + G(x) + \int_0^x f(z)(F(z) - \arctan(z))\, dz\,. \quad (5.30)$$

It is not difficult to see that $\lim_{\|(x,y)\|\to\infty} W(x, y) = \infty$ (look for example along the lines $y = \alpha x$). We have

$$\begin{aligned} LW(x, y) \;&=\; (y + F(x) - \arctan(x))\,(-f(x)y - g(x)) \\ &\quad + \left(\left(f(x) - \frac{1}{1+x^2}\right) y + g(x) + f(x)(F(x) - \arctan(x))\right) y \\ &=\; -(F(x) - \arctan(x))g(x) - \frac{1}{1+x^2} y^2\,. \end{aligned} \quad (5.31)$$

We note that

$$\lim_{\|(x,y)\|_2\to\infty} LW(x, y) \;=\; -\infty\,. \quad (5.32)$$

This implies that there exists b_0 such that on the level sets of $\{W(x, y) = b\}$ for $b \geq b_0$, the vector field points inward the level set. This implies that the system is dissipative, all orbits starting outside the level sets $\{W(x, y) = b_0\}$ eventually enter it and never leave it again.

This implies that the annular shaped domain

$$A \;=\; \{x\,;\; \|(x, y\|_2 \geq a\,,\; W(x, y) \leq b_0\} \quad (5.33)$$

is positively invariant. The only critical point is $(0, 0)$ so that the Poincaré-Bendixson theorem implies the existence of (at least) one periodic orbit in A. We have also proved that every orbit starting at any point except $(0, 0)$ will eventually enter the set A.

A somewhat finer analysis shows that, for (5.26) there exist only one periodic orbit which is a stable limiting cycle.

To conclude note that the same argument works for more general function f and g. For example if we assume that there exists $a < b < c$ such that

1. All the zeros of g are contained in $(-a, a)$.
2. In $(-b, -a)$ and (a, b) we have $g(x)F(x) < 0$
3. For $x < -c$ and $x > c$ we have $g(x)F(x) > 0$.
4. $\lim_{x\to\infty} G(x) = \infty$.
5. For large x, $F(x) = x^{2n+1} +$ lower order terms.

then we have the existence of a periodic orbit. The number of periodic orbits depends on the detailed behavior of F, in particular how many times it changes signs between b and c.