STAT 315: Normal Random Variables

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Normal Random Variables

PDF of normal random variables

A continuous random variable Y is normal random variable with parameters $-\infty < \mu < \infty$ and $\sigma > 0$ if it has the density

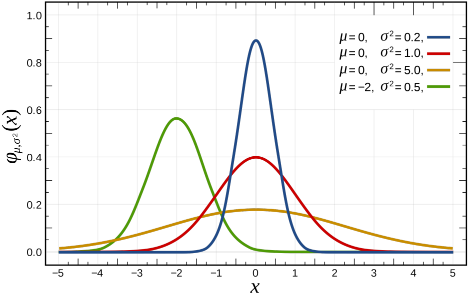
$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \qquad -\infty < y < +\infty$$

We write $Y \sim N(\mu, \sigma^2)$

The CDF

$$F(Y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

has no closed form. Compute it using technology.
See e.g `https://www.webassign.net/tparise/beta/stats/distributionIndex.html`



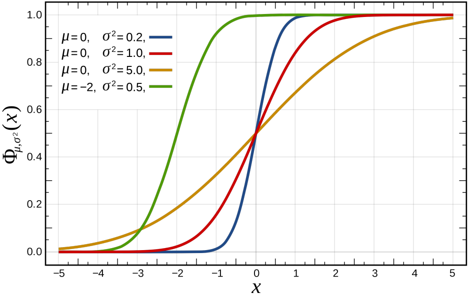
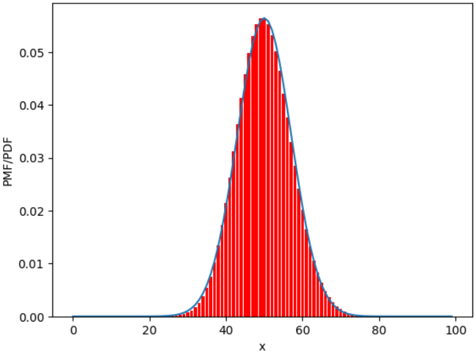


Figure: Top: Pdf, Bottom:Cdf

Why "normal"?

- Normal random variables are everywhere, at least in good approximation.
- Many random variables look very close to a normal (under suitable rescaling), e.g

Poisson PMF ($\lambda = 50$) versus a Normal PDF with $\mu = 50$ and $\sigma^2 = 50$



Y Poisson with large λ

- This come from the Central Limit Theorem (Chapter 7)

Standard normal random variable

Standard normal random variable

A normal random variables is called standard normal random variable Z if $\mu = 0$ and $\sigma^2 = 1$. The density is

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \qquad -\infty < y < +\infty$$

Usually we reserve the letter Z is $Z \sim N(0,1)$

Theorem

If Z is standard normal, $Z \sim N(0,1)$, then

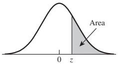
$$Y = \sigma Z + \mu$$

is normal and $Y \sim N(\mu, \sigma^2)$

See proof later

Normal table $P(Z \geq z)$

Table 4 Normal Curve Areas
Standard normal probability in right-hand tail
(for negative values of z, areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).

- Standardize your RV. If $Y \sim N(\mu, \sigma^2)$ then $Z = \frac{Y-\mu}{\sigma}$ is standard normal.

$$P(a \leq Y \leq b)$$
$$= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$$

and compute from the table.

- Also use that by symmetry $P(z \geq a) = P(Z \leq -a)$.

Examples

- Write each of the probabilities in parts (1)–(3) in terms of the standard normal variable $Z = \frac{X-50}{10}$.
  1. Compute the probability that X lies between 45 and 65:
  2. Compute the probability that $X > 70$.
  3. Find the value x_0 such that $P(X \leq x_0) = 0.9$.
- A company that manufactures and bottles apple juice uses a machine that automatically fills 16 ounce bottles. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation of .5 ounce.
  1. What is the probability that bottles contains less than 14.5 ounces?
  2. Suppose bottle the big enough to contain 17 ounces. What is the probability that the bottles overflow.
  3. How big should the bottles be so less than one percent of bottles overflow?

Mean, variance

Mean and Variance of normal RV

If $Y \sim N(\mu, \sigma^2)$ then

$$E[Y] = \mu \qquad V[Y] = \sigma^2$$

Proof: By the previous theorem $Y = \sigma Z + \mu$ so we can assume $\mu = 0$ and $\sigma^2 = 1$ and Z standard normal

$$E[Z] = \int_{-\infty}^{\infty} y \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = 0 \quad \text{(the integrand is odd)}$$

$$E[Z^2] = \int_{-\infty}^{\infty} y \times y \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = -\int_{-\infty}^{\infty} y \frac{d}{dy} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$\underbrace{=}_{IBP} -y \frac{e^{-y^2/2}}{\sqrt{2\pi}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} = 1$$