Math 697: Homework 2

Exercise 1

The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3 someone takes all papers in the pile and put them in the recycling bin. Also if ever there at least five papers in the pile, Mr Smith, with probability one, take the papers to the bin in the afternoon.

1. Consider the number of papers in the pile in the evening and describe it with a Markov chain. What are the state space and transition probabilities?

2. Show that there is a unique limiting distribution. Compute it.

3. After a long time what would be the expected number of papers in the pile?

4. Assume that the piles starts with 0 papers. What is the expected time until the pile will have again 0 papers.

Exercise 2 Mrs X possesses r umbrellas which she uses going from her home to her office in the morning and vice versa in the evening. If it rains in the morning or in the evening she will take an umbrella with her provided there is one available. Assume that independent of the past it will rain in the morning or evening with probability p. Let X_n denote the number of umbrellas at her home before she gets to work.

1. Give the state space and the transition probabilities describing the Markov chain X_n.

2. Find the limiting probabilities π_j, $j = 0, 1, \ldots, r$.

3. In the long run, what fraction of the time does Ms Jenike gets wet?

Exercise 3 Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers. Define the sequence $\{b_n\}_{n \geq 1}$ by

$$b_n = \frac{a_0 + \cdots + a_{n-1}}{n} = \frac{1}{n} \sum_{j=0}^{n-1} a_j .$$

Show that if $\lim_{n \to \infty} a_n = a$ then $\lim_{n \to \infty} b_n = a$. The convergence of $\{b_n\}$ however does not imply the convergence of $\{a_n\}$.

Exercise 4

If the state space is large it can be difficult and tedious to compute the stationary distribution π for the Markov chain with transition probability P, i.e., to solve $\pi P = \pi$. The purpose of this exercise is to derive a simple algorithm to compute stationary distributions which reduces the problem to invert a matrix.

We define the following matrices: I is the $N \times N$ identity matrix and M is the $N \times N$ matrix whose entries are all 1, i.e.,

$$I = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

Algorithm for the stationary distribution π: Suppose π is the unique stationary distribution for the transition matrix P, then

$$\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}.$$

It is easy to invert a matrix on a computer and this makes the computation of π on a computer straightforward.

In order to justify this algorithm you need to prove the following statement

1. Assume that the matrix $(I-P+M)$ is invertible. Show then $\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}$.

2. Assume that π is the unique stationary distribution for P. Show then that the unique non-zero solution of $Px = x$ is the column vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

3. By definition to show that $(I - P + M)$ is invertible you need to show that the only solution of $(I - P + M)x = 0$ is $x = 0$. To do this multiply $(I - P + M)x = 0$ on the left by π and deduce from this that $(1, \cdots, 1)x = 0$. Show then that $Mx = 0$ and $Px = x$. Conclude.

Exercise 5

We consider the following inventory model for a certain item. Stock levels are inspected at fixed time intervals, say every evening. The following restocking policy is followed. Fix two number $0 \leq s \leq S$, if the inventory level is $\leq s$ then the stock is replenished up to level S. If the inventory level is $> s$ and $\leq S$, then no action is taken. Assume furthermore that the demand for the item during successive inspections and replenishments of the stocks is given by i.i.d random variables D_n. If the demand during a single day exceeds the number of items in stock, it is assumed that this demand remains unfulfilled. Let X_n to be the stock level at the inspection time just before restocking.

1. Describe this model using the state space $\{0, 1, \cdots S\}$ and determine the transition matrix P if $s = 1$, $S = 3$ and

$$P\{D_i = 0\} = .4, \quad P\{D_i = 1\} = .3,$$
$$P\{D_i = 2\} = .2, \quad P\{D_i = 3\} = .1$$

2. Compute the expected amount of items in stock at the moment of inspection in the long run.

3. If you want to determine if your restocking policy is efficient an important quantity to compute for this model is the average amount of command per day which goes unfulfilled. To do this consider the Markov chain Y_n where Y_n is defined similarly as in X_n but is also allowed to take negative value. For example $Y_n = -1$ means that 1 command was unfilled that day, this happens for example if $X_n = 2$ and $D_n = 3$. Determine the state space for Y_n and the transition matrix with the same distribution for D_n. Compute the average number of unfulfilled commands per day in the long run.

Exercise 6 Let P be a transition probability matrix and assume that there exists a stationary distribution $\pi = (\pi(1), \cdots, \pi(N))$ with $\pi(j) > 0$ for all j. We now define a new matrix, $\bar{P}$, called the *time reversed* transition probabilities, by

$$\bar{P}ij = \frac{\pi(j)}{\pi(i)} P_{ji}.$$

1. Show that $\bar{P}_{ij}$ is a stochastic matrix and that π is again a stationary distribution for $\bar{P}$.

2. Let X_n be the Markov chain with transition matrix P and initial distribution π. Let $\bar{X}_n$ be the Markov chain with transition matrix $\bar{P}$ and initial distribution π. Show that

$$P\left\{\bar{X}_0 = i_0, \bar{X}_1 = i_1, \cdots, \bar{X}_n = i_n\right\} = P\left\{X_n = i_0, X_{n-1} = i_1, \cdots X_0 = i_n\right\}.$$

   The Markov chain $\bar{X}_n$ is called the *time reversed* chain for the Markov chain X_n.

Exercise 7 The Markov property means that the future depends on the present but not on the past, i.e.,
$$P\{X_n = i_n \mid X_{n-1} = i_{n-1}, \cdots X_0 = i_0\} = P\{X_n = i_n \mid X_{n-1} = i_{n-1}\}.$$

1. Show that the Markov property implies that the past depends only on the present but not on the future, i.e.,

$$P\{X_0 = i_0 \mid X_1 = i_1, \cdots X_n = i_n\} = P\{X_0 = i_0 \mid X_1 = i_1\}.$$

2. Show that the Markov property also implies that, given the present, the past and the future are independent, i.e.,

$$P\{X_{n+1} = i_{n+1}, X_{n-1} = i_{n-1} \mid X_n = i_n\}$$
$$= P\{X_{n+1} = i_{n+1} \mid X_n = i_n\} P\{X_{n-1} = i_{n-1} \mid X_n = i_n\}.$$

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