Math 624: Problem set 1

1. Show that if f and g belong to $L^1(\mathbf{R}^d)$ then we have

$$\widehat{f \star g}(\xi) \;=\; \hat{f}(\xi)\,\hat{g}(\xi)$$

   where $f \star g$ denote the convolution product.

2. Problem 22, p. 94

3. Problem 23, p. 94,

4. Problem 25, p. 95 (This provides examples of L^1 function whose Fourier transform is not in L^1.

5. Show that $c_n \left[\cos(\frac{\pi x}{2})\right]^n$, $-1 \le x \le 1$, for suitable constants n is an approximation of the identity. Use this and suitable trigonometric formulas to prove a version of the Weierstrass approximation theorem for periodic continuous function on $[-\pi, \pi]$ by trigonometric polynomials.

6. Let $\{a_n\}$ be a sequence. Show that if $\lim_n a_n = a$ then the limit $\lim_n \frac{1}{n}\sum_{k=1}^n a_n$ exists and is equal to a (convergence in the sense of Cesaro). Show that if a_n converges in the sense of Cesaro then the sequence $\{a_n\}$ is in general not a convergent.

7. Let $f \in L^1(\mathbf{R})$. Use the Fourier transform to solve the equation

$$u(x) - \frac{d^2}{dx^2}u(x) \;=\; f(x)$$

   Hint: It is useful to know what the Fourier transform of $e^{-a|x|}$ is, so compute it!

8. Use Fourier series solve the wave equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) \;=\; 0\,, \quad u(0,x) = f(x)\,, \quad \frac{\partial}{\partial t}u(0,x) = g(x)\,,$$

   where $x \in \mathbf{R}$, $t \in [0,\infty)$ and f, g and $u(t,\cdot)$ are periodic functions of x of period 2π.

9. Using Parseval equality it is easy to to see that if $f \in L^2([-\pi,\pi])$ then its Fourier coefficients satisfy $\lim_{|n|\to\infty} c_n = 0$. Extend this to all $f \in L^1([-\pi,\pi])$ by proving the more general results know as Riemann-Lebesgue Lemma: If $f \in L^1(\mathbf{R})$ then

$$\lim_{n\to\infty}\int f(x)\cos(nx)dx \;=\; \lim_{n\to\infty}\int f(x)\sin(nx)dx \;=\; 0$$

   Hint: Prove it first for an appropriate dense set of functions.

10. Consider the function $f(x) = \frac{\pi - x}{2}$ on the interval $0 < x < 2\pi$ and extend f to be periodic on $\mathbf{R}$. Show that $\frac{\pi-x}{2} = \sum_{n=1}^{\infty} \frac{\sin(nx)}{x}$ for $0 < x < 2\pi$ and deduce from this the formulas

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and

$$\frac{\pi^2}{6} = 1 + + \frac{1}{4} + \frac{1}{9} + \cdots$$

11. Suppose that f is periodic of period 2π and Hölder continuous with exponent α, i.e. we have $|f(x) - f(y)| \leq C|x-y|^{\alpha}$. Show that the Fourier coefficients of f satisfy the bound

$$|c_n| \leq \frac{C}{|n|^{\alpha}}$$

Hint: In the integral expression for c_n do the substitution $x \to x + \frac{\pi}{n}$.