STAT 315: Probability Basics (Section 2.2–2.5)

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September 2, 2025

What is probability?

- Probability describe an experiment whose outcome cannot be described with certainty. Keyword: random or stochastic
  - Roll of a dice
  - The price of GameStop stock or the price of Bitcoin tomorrow
  - The winner of Super Bowl LVIX: Chiefs of Eagles?
  - The amount of rain on Amherst due to Hurricane Ida
- Frequentist approach: deduce the probability by repeating the experiment N times (N very large) (Law of Large numbers)

$$P(\text{roll a } 5) \approx \frac{\text{number of 5 in } N \text{ rolls}}{N}$$

$\longrightarrow$ Can be simulated on a computer!

- Subjective (Bayesian) approach: Probability is a measures of one's belief in the occurrence of a future event
  $\longrightarrow$ You can't repeat the Super Bowl but you can bet on it!

Review of set notation

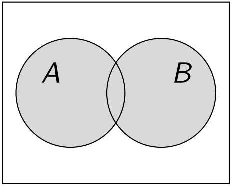
- Denote sets of points by capital letters, $A, B_1, B_2, S, \cdots$, and points by lower case, $a_1, a_2, b, c, x, \ \cdots$
- If the elements ins A are a_1, a_2, a_3 we write $A = \{a_1, a_2, a_3\}$
- Denote by S the sets of all elements (the sample space) and by $\emptyset$ the set with no element.

Sets operations

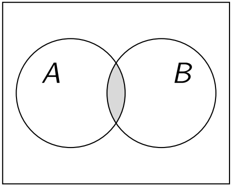
- $A \subset B$ (A is contained in B): every element in A is also in B.
- $A \cup B$ (the union of A and B): the set elements which belong either to A or to B.
- $A \cap B$ (the intersection of A and B): the sets of elements which belong both to A and B.
- $\overline{A}$ (the complement of A): the set of element in S which do not belong to A.
- $B \setminus A = B \cap \overline{A}$: the elements in B which are not in A.

Venn diagrams — Union, Intersection, Difference, Complement

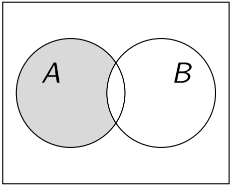
Union: $A \cup B$



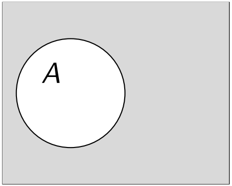
Intersection: $A \cap B$



Difference: $A \setminus B$



Complement: A^c



Laws of set algebra

Distributive Laws

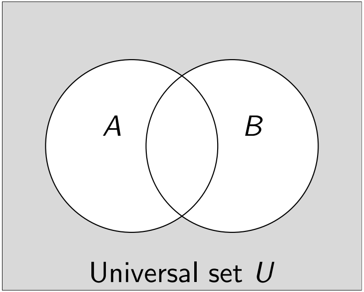
$$
\begin{aligned}
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)
\end{aligned}
$$

DeMorgan's Law

$$
\begin{aligned}
\overline{A \cap B} &= \overline{A} \cup \overline{B} \\
\overline{A \cup B} &= \overline{A} \cap \overline{B}
\end{aligned}
$$

Venn diagrams — visualizing De Morgan's laws

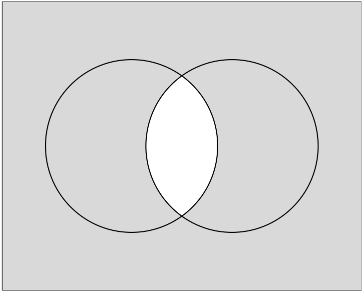
Complement of a union



$$(A \cup B)^c = A^c \cap B^c$$

- Shaded area: everything *outside both circles, including the intersection*.

Complement of an intersection



$$(A \cap B)^c = A^c \cup B^c$$

- Shaded area: everything *except* the overlap (the intersection).

Probabilistic experiment: discrete case

We use the language of set theory to describe an experiment with random outcomes:

Sample space and events

- S is called the sample space: the list of all possible outcomes of the experiment.

$$S = \{e_1, e_2, e_3, \cdots\} \quad \text{finite or countable set}$$

- A subset $A \subset S$ is called an event. Think of it as a question you ask about the experiment: does the event A occur?)

Example (Roll a pair of dice)

$$\begin{aligned} S &= \{(1,1),(1,2),(1,3),\cdots,(6,6)\} \quad S \text{ has 36 elements} \\ A &= \{\text{The sum of the dice is } 4\} = \{(1,3),(2,2)(3,1)\} \\ B &= \{\text{Exactly one } 6\} = \{(1,6),(6,1),(2,6),(6,2),\cdots(5,6)(6,5)\} \end{aligned}$$

Set operations in probabilistic language

Intuitive meaning of set operations

- $A \cap B =$ "A and B" $\longrightarrow$ both A and B occur.
- $A \cup B =$ "A or B" $\longrightarrow$ either A or B occur.
- $A \subset B =$ "A implies B" $\longrightarrow$ if A occurs then B occurs.
- $\overline{A} =$ "not A" $\longrightarrow$ A does not occur.
- $A \cap B = \emptyset \longrightarrow$ A and B are mutually exclusive, they cannot occur simultaneously.
- $B \setminus A = B \cap \overline{A} =$ "B but not A", B occurs but not A

Laws of probability I

Laws of Probability

S is the sample space. To every event A in S (i.e., $A \subset S$) we assign a number $P(A)$, called the probability of A, with the following properties

- Axiom 1: $0 \leq P(A) \leq 1$.
- Axiom 2: $P(S) = 1$.
- Axiom 3: If $A_1, A_2, A_3, \cdots$ are pairwise mutually exclusive $A_i \cap A_j = \emptyset$ if $i \neq j$ then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

For the discrete case $S = \{e_1, e_2, \cdots\}$ simply assign numbers $P(\{e_i\}) = p_i$ with $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. We have then

$$P(A) = \sum_{i : e_i \in A} p_i .$$

Laws of probability II

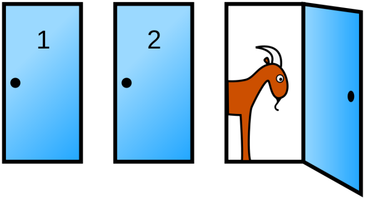
Simple consequence of the laws of probability

1. $P(\emptyset) = 0$
2. If $A \cap B = \emptyset$ (mutually exclusive) then $P(A \cup B) = P(A) + P(B)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Inclusion-Exclusion)
4. If $A \subset B$ then $P(A) \leq P(B)$ (Monotonicity)
5. $P(\overline{A}) = 1 - P(A)$ (Complement)

Examples/Exercises

- Flip 2 coins. Let $A = \{\text{1st coin is heads}\}$ and $B = \{\text{2nd coin is tails}\}$. Compute
  1. $P(A \cap B)$
  2. $P(A \cup B)$
- Draw one card from a standard deck of 52 cards. Let $A = \{\text{red card}\}$ and $B = \{\text{face card}\}$. Compute
  1. $P(A \cap B)$
  2. $P(A \cup B)$
- There are 5 computers, two of which are defective. You select 2 computers at random out of the 5. What is the probability you have no defective computers.
- You like book 1 with probability .5, you like book 2 with probability .4 and you like both books 1 and 2 with probability .3. Determine the probability you like none of the books.

The Monty Hall Problem



- There are 2 goats G_1 and G_2 and \$1 million hidden behind 3 doors.
- You pick a door (without opening it), say door 1.
- The game hosts open of the other two door (on the picture, door 3) and reveals a goat behind it.
- You are given the following xchoice
  - Keep your door?
  - Switch?
- What should you do to maximize your probability of winning?

The Monty Hall Problem Solution

- Idea: write down the sample spaces carefully!
- Before you pick a door the sample space S describe the distribution of goats G1 and G2 and \$ behind the 3 closed doors

$$S = \{(G_1, G_2, \$), (G_2, G_1, \$), (G_1, \$, G_2), (G_2, \$, G_1), (\$, G_1, G_2)), (\$, G_2, G_1))\}$$

- Suppose you pick door 1. Then the host opens a door to reveal a goat. There are two closed doors left (door 1 and one of the doors 2 or 3). This is the new sample space S' with

$$S' = \{(G_1, \$), (G_2, \$), (G_1, \$), (G_2, \$), (\$, G_{12}), (\$, G_{21})\}$$

  For example if we have $(G_1, G_2, \$)$ then the host opens door 2 and we have $(G_1, \$)$. Here G_{12} means one of the 2 goats.

- Out of the 6 states in S', 2 states have the \$ hidden behind door 1 while 4 states have the \$ hidden under the other door. So switching door will make you win with probability $\frac{2}{3}$

Any questions?