Math 697 Fall 2014: Week 6

Exercise 1 Consider a Markov chain with state space $\{0, \cdots, 5\}$ and transition matrix

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix}.$$

1. What are the communication classes. Which ones are recurrent and which ones are transient?
2. Suppose $X_0 = 5$. What is the probability that X_n visits the state 1 before the state 3?
3. Suppose $X_0 = 5$. Compute $\lim_{n\to\infty} P^n(5, j)$ for all j?

Exercise 2 Suppose we flip a fair coin repeatedly until we have flipped four consecutive heads. What is the expected number of flips that are needed?
Hint: Consider a Markov chain with state space $\{0, 1, 2, 3, 4\}$

Exercise 3 Consider the Markov chain with state space $\{0, \cdots, 6\}$ and transition matrix

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

1. Is this chain irreducible? Aperiodic?
2. Suppose that this chain has been running for a very long time. What is the probability that the next three states are 4, 5, 0 in that order?
3. Suppose the chain starts in state 1. What is the probability it reaches state 6 before reaching state 0?
4. Suppose the chain starts in state 3. What is the expected number of steps until it reaches 3 again?
5. Suppose the chain starts in state 0. What is the expected number of steps until it reaches state 6?

Exercise 4 ***(Random walk on the complete graph)*** The complete graph on $\{1, \cdots, N\}$ is the simple graph with these vertices such that any two vertex are neighbors. Let X_n the the simple random walk on this graph and let $\tau^{(1)}$ be the first time the chain returns to state 1.

1. Suppose $X_0 = 1$, compute the p.d.f of $\tau^{(1)}$. Use this to compute $E[\tau^{(1)}|X_0 = 1]$.
2. Compute $E[\tau^{(1)}|X_0 = 2]$
3. Find the expected number of steps until every state has been visited at least once.

Exercise 5 ***(Partially observed Markov chains.)*** Let X_n be an irrreducible Markov chain with a finite state space $S = \{1, \cdots, N\}$ and transition matrix P. Let T be a subset of states, $T \subset S$, $T \neq S$. Let ν_j, $j \geq 0$, denote the successive times at which the Markov chain visits one of the states in T, i.e.

$$\begin{aligned} \nu_0 &= \inf\{n \geq 0 \,:\, X_n \in T\}\,, \\ \nu_1 &= \inf\{n > \nu_0 \,:\, X_n \in T\}\,, \\ &\vdots \end{aligned}$$

We define a new stochastic process Y_j with state space T which is given by

$$Y_j \,=\, X_{\nu_j}\,.$$

Think of this process as follows: you can only observe X_n if X_n is in one of the states in T. Moreover you don't have a watch and thus have no way to keep track of the time elapsed between successive visits to T.

1. Argue that Y_j is a Markov process.
2. Reordering the state if necessary we can assume that the transition matrix has the form as
$$P \,=\, \begin{array}{c} T \\ T^c \end{array} \begin{pmatrix} R & U \\ S & Q \end{pmatrix}$$
Let $D = (d_{ij})$ be the transition matrix for the Markov chain Y_j, i.e. $d_{ij} = P\{Y_1 = j \,|\, Y_0 = i\}$ for $i, j \in T$. Compute the matrix D in terms of the matrix R, U, S, Q.
3. Suppose that the Markov chain X_n has a stationary distribution $\pi = (\pi(1), \cdots, \pi(N))$. What is the stationary distribution for the Markov chain Y_n.
4. Compute
$$\lim_{k\to\infty} \frac{\nu_k}{k}\,.$$