Math 645: Homework 4

1. Compute the resolvent of

$$x' = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{pmatrix} x \,. \tag{1}$$

2. Transform the matrix

$$A = \frac{1}{9} \begin{pmatrix} 14 & 4 & 2 \\ -2 & 20 & 1 \\ -4 & 4 & 20 \end{pmatrix} \tag{2}$$

in Jordan normal form and compute the resolvent of $x' = Ax$. *Hint*: All eigenvalues are equal to 2.

3. The equation of motion of two coupled harmonic oscillators is

$$\begin{aligned} x_1'' &= -\alpha x_1 - \kappa(x_1 - x_2)\,, \\ x_2'' &= -\alpha x_2 - \kappa(x_2 - x_1)\,. \end{aligned} \tag{3}$$

Find a fundamental matrix for this system. You can either write it as a first order system and compute the chararcteristic polynomial or, better, stare at the equation long enough until you make a clever Ansatz. Discuss the solution in the case where $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = 0$.

4. Consider the scalar equation (i.e. $n = 1$) $x' = f(t)x$ where $f(t)$ is continuous and periodic of period p.

   (a) Determine $P(t)$ and R in Floquet Theorem.

   (b) Give necessary and sufficient conditions for the solutions to be bounded as $t \to \pm\infty$ or to be periodic

5. (a) Compute the resolvant $R(t, 0)$ (in real representation) for the ODE

$$\begin{aligned} x' &= \cos(t)x - \sin(t)y\,, \\ y' &= \sin(t)x + \cos(t)y\,. \end{aligned} \tag{4}$$

   Hint: Find an equation for the complex function $z = x + iy$.

   (b) Determine $P(t)$ and R in Floquet theorem.

6. Consider the differential equation $x'' + \epsilon f(t)x = 0$, where $f(t)$ is periodic of period 2π and

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < \pi \\ 0 & \text{if} \quad \pi < t \le 2\pi \end{cases} \,. \tag{5}$$

For both $\epsilon = 1/4$ and $\epsilon = 4$

   (a) Consider the fundamental solution $\Phi(t)$ which satisfies $\Phi(0) = \mathbf{I}$ and compute the corresponding transition matrix $C = e^{pR}$.

   (b) Compute the Floquet multipliers (the eigenvalues of C).

(c) Describe the behavior of solution.

7. Let $A(t)$ be periodic of period p and consider ODE $x' = A(t)x$.

   (a) Show that the transition matrix C depends, in general, on the fundamental solution, but that the eigenvalues of $C = e^{pR}$ are independent of this choice.

   (b) Show that for each Floquet multiplier λ (the eigenvalue of C), there exists a solution of $x' = A(t)x$ such that $x(t+p) = \lambda x(t)$, for all t.

8. Consider the equation $x' = A(t)x$ where $A(t)$ is periodic of period p.

   (a) Let $\Phi(t)$ be the fundamental solution with $\Phi(0) = 1$. Use Floquet Theorem and Liouville Theorem to show that

$$\det(e^{pR}) \;=\; e^{\int_0^p \mathrm{Trace}(A(s))\,ds}\,. \tag{6}$$

   (b) Deduce from (a) that the characteristic exponents μ_i satisfy

$$\mu_1 + \cdots + \mu_n = \frac{1}{p}\int_0^p \mathrm{Trace}(A(s))\,ds \tag{7}$$

9. Consider the linear differential equation

$$x' \;=\; A(t)x\,, \quad A(t) \;=\; S(t)^{-1}BS(t) \tag{8}$$

where

$$B \;=\; \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, \quad S(t) \;=\; \begin{pmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{pmatrix} \tag{9}$$

   (a) Show that, for any t, all eigenvalues of $A(t)$ have a negative real part.

   (b) Show, that for a suitable choice of a, the differential equation (8) has solutions $x(t)$ which satisfy $\lim_{t\to\infty} \|x(t)\| = \infty$.

Hint: Consider the transformation $y(t) = S(t)x(t)$.

10. Consider the equation for the mathematical pendulum

$$x'' + \sin(x) \;=\; 0\,, \quad x(0) = \epsilon\,, x'(0) = 0\,, \tag{10}$$

where ϵ is supposed to be small. Show that the solution can be written in the form

$$x(t) \;=\; \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + O(\epsilon^4)\,. \tag{11}$$

Compute $x_1(t)$, $x_2(t)$, and $x_3(t)$. *Hint:* Taylor expansion.

11. Consider the Mathieu equation

$$x'' + (a + \epsilon\cos(2t))x \;=\; 0\,, \quad a > 0\,. \tag{12}$$

Show that if $a \neq m^2$, (m an integer) and ϵ is small enough, then the Floquet multipliers of Mathieu equation have modulus 1 and the solutions are bounded uniformly in t.

To prove this denote by $\lambda_{1,2}(\epsilon)$ the Floquet multipliers for (12) and argue that $\lambda_{1,2}(\epsilon)$ are continuous function of ϵ.