Math 697 Fall 2014: Week 1

Exercise 1 For positive numbers a and b, the Pareto distribution with parameter a, b, $P_{a,b}$ has the p.d.f $f(x) = ab^a x^{-a-1}$ for $x \geq b$ and $f(x) = 0$ for $x < b$. What is the inversion method to generate $P_{a,b}$.

Exercise 2 The standardized logistic distribution has the p.d.f $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$. What is the the inversion method to generate a random variable having this distribution.

Exercise 3 In class we have formulated the rejection method for continuous random variables but it can be extended to discrete random variables too.
(a) Suppose now X and Y are discrete random variables, both taking values in the same finite or countable set $S = \{s_1, s_2, \cdots\}$ with respective pdf $f(i) = P\{X = s_i\}$ and $g(i) = P\{Y = s_i\}$. Formulate and prove the rejection method in this case.
(b) Set-up an algorithm to simulate a Poisson random variable with parameter λ using a geometric random variable with parameter p. Discuss the choice of p for fixed λ.

Exercise 4 Consider the technique of generating a $\Gamma_{n,\lambda}$ random variable by using the rejection method with $g(x)$ being the p.d.f of an exponential with parameter λ/n.

1. Show that the average number of iterations to until acceptance is $n^n e^{1-n}/(n-1)!$.
2. Use Stirling formula to show that for large n the answer in part 1. is approximately $e\sqrt{(n-1)/2\pi}$.
3. Show that the rejection method is equivalent to the following
   - **Step 1:** Generate Y_1 and Y_2 independent exponentials with parameters 1.
   - **Step 2:** If $Y_1 < (n-1)[Y_2 - \log(Y_2) - 1]$ return to step 1. Otherwise go to step 3
   - **Step 3:** Set $X = nY_2/\lambda$.

   Hint: In Step 3, the transformation $X = nY/\lambda$ transform a $\Gamma_{n,n}$ into a $\Gamma_{n,\lambda}$. To show that Y_2 is $\Gamma_{n,n}$ use the rejection method with a T_1.

Exercise 5 ***(Computer exercise)*** In this problem you should implement some algorithms, in the language of your choice,

1. Write a program which implements the inversion method for, say, an exponential random variable T_λ, and use it to produce an approximate graph of the pdf T_λ.
2. Write a program to implement the rejection method for, say, generating a normal random variable $N_{0,1}$ using an exponential random variable T_1 (see class notes for details), again use it to approximately graph of the pdf $N_{0,1}$.

Exercise 6 (Generating a uniform distribution on the permutations) In this problem you will learn a way to shuffle a deck of cards perfectly. Consider a set of N objects numbered 1 to N and we wish to generate a random permutation of the N objects (there are $N!$ such permutations) objects such that they all have the same probability (that is $1/N!$). For a permutation we denote by $S(i)$ the element in position i. For example for the permutation $(2, 4, 3, 1, 5)$ of 5 elements we have $S(1) = 2$, $S(2) = 4$, and so on.

The algorithm is inductive and at the k-step you will have generated a random permutation of the set $\{1, 2, \cdots, k\}$. At step 1 you take the object 1 and set $S(1) = 1$. This is the permutation of 1 object. At step 2 you pick object 2 and with probability 1/2 put in position 2 that set $S(2) = 2$ and with probability 1/2 you exchange it with the objects in position 1, that is you set $S(1) = 2$ and $S(2) = 1$. At step k you add object k: pick an integer between 1 and k uniformly, say you pick j. If $j \leq k-1$ you set $S(j) = k$ and $S(k) = S(j)$ and if $j = k$ you set $S(k) = k$.

A pseudo-code for the algorithm is as follows: (We will use the following notation. If x is positive real number we denote by $\lfloor x \rfloor$ the integer part of x, i.e. $\lfloor x \rfloor$ is the greatest integer less than or equal x. For example $\lfloor 2.45 \rfloor = 2$.)

1. Set $k = 1$
2. Set $S(1) = 1$
3. If $k = n$ stop. Otherwise let $k = k + 1$.
4. Generate a random number U, and let

$$S(k) = S([kU] + 1)\,,$$

$$S([kU] + 1) = k\,.$$

   Go to step 3.

Show that at iteration k, – i.e. when the value of $S(k)$ is initially set– $S(1), S(2), S(k)$ is a random permutation of $1, 2, \cdots, k$, i.e., all permuation are equally likely and occur with probability $1/k!$.

Hint: Relate the probability P_k on the set of permutation of obtained at iteration k with the probability P_{k-1} obtained at iteration $k-1$.