Math 597/697: Homework 6

1. (a) Let X be normal random variable with mean μ and σ^2. Let $Y = aX + b$ and $a > 0$

$$\begin{aligned} P\{Y \le t\} &= P\{aX + b \le t\} = P\{X \le \frac{t-b}{a}\} \\ &= \int_{-\infty}^{\frac{t-b}{a}} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dx = \int_{-\infty}^{t} \frac{e^{-\frac{(y-b-\mu)^2}{2a^2\sigma^2}}}{\sqrt{2\pi a^2\sigma^2}}\, dy \quad (1) \end{aligned}$$

where we have used the change of variables $y = ax + b$ in the last equality. So the mean of Y is $\mu + b$ and the variance $a^2\sigma^2$. The case $a < 0$ is similar.

   (b) Let X_1 and X_2 be independent normal random variable with mean 0 and variance σ_i^2, $i = 1, 2$. Since X_2 has mean zero and the distribution of a normal RV is symmetric with respect to $x \mapsto -x$ we have $X_2 = -X_2$, meaning that they have the same density. If X_1 and X_2 are independent so are X_1 and $-X_2$ and so $X_1 + X_2 = X_1 - X_2$. This is again a normal R.V with mean 0 and variance $\sigma_1^2 + \sigma_2^2$ (this is seen most easily by considering the moment generating function).

2. Suppose X_t is a standard Brownian motion and let $Y_t = a^{-1/2}X_{at}$ with $a > 0$. We have

   (a) $Y_0 = X_0 = 0$

   (b) $Y_t - Y_s = a^{-1/2}(X_{at} - X_{as})$ so if X_t has independent increments so does Y_t.

   (c) $X_{at} - X_{as}$ is a normal RV with mean 0 and variance $a(t-s)$. By problem (1) $Y_t - Y_s = a^{-1/2}(X_{at} - X_{as})$ has variance $a^{-1}a(t-s) = (t-s)$.

   (d) If $t \mapsto X_t$ is continuous, so is $t \mapsto Y_t$.

   Therefore Y_t is a standard brownian motion.

3. Let X_t and Y_t be independent standard one-dimensional Brownian motion. Let $Z_t = X_t - Y_t$

   (a) We have

      i. $Z_0 = X_0 - Y_0 = 0$

ii. $Z_t - Z_s = (X_t - X_s) - (Y_t - Y_s)$ and so Z_t has independent increments since X_t and Y_t do.

iii. By (1) (b) $Z_t - Z_s$ is a normal random variable with parameter $2(t-s)$

iv. The map $t \mapsto Y_t$ is continuous.

So Z_t is a Brownian motion with variance parameter 2.

(b) Brownian motion is recurrent so that $P\{Z_t = 0 \text{ i.o.}\} = 1$. Therefore $P\{X_t = Y_t \text{ i.o.}\} = 1$.

4. Let X_t be a Brownian motion with variance parameter σ^2. Let $t \geq s$ then by the independent increments property

$$\begin{aligned} E[X_s X_t] &= E[X_s(X_t - X_s)] + E[(X_s)^2] \\ &= E[X_s]E[(X_t - X_s)] + E[(X_s)^2] \\ &= E[(X_s)^2] = s. \end{aligned} \tag{2}$$

Similarly if $s \geq t$ one finds $E[X_s X_t] = t$. Therefore

$$E[X_s X_t] = \min(s,t)\,. \tag{3}$$

5. Let X_t be a standard Brownian motion and let us define, for $\alpha > 0$,

$$V_t = e^{-\alpha t/2} X(e^{\alpha t})\,. \tag{4}$$

The process V_t is called an Ornstein-Uhlenbeck process. We have, using (5)

$$E[V_t] = e^{-\alpha t/2} E[X(e^{\alpha t})] = 0\,. \tag{5}$$

For $s \leq t$

$$\begin{aligned} E[V_s V_t] &= e^{-\alpha s/2} e^{-\alpha t/2} E[X(e^{\alpha s}) X(e^{\alpha t})] \\ &= e^{-\alpha s/2} e^{-\alpha t/2} e^{\alpha s} = e^{-\alpha(t-s)/2} \end{aligned} \tag{6}$$

For $t \leq s$ one finds

$$E[V_s V_t] = \, = e^{+\alpha(t-s)/2} \tag{7}$$

and so

$$E[V_s V_t] = \, = e^{-\alpha|t-s|/2}\,. \tag{8}$$

6. The motion of an oil spill (assume it is pointwise) on the surface of the ocean is a 2-dimensional Brownian motion with variance parameter $\sigma^2 = 1/2$. Let C be a square with side lengths 2 and assume that the oil spill, at time 0, is at the center of the square C. We assume that 0 is the center of the square and so the motion of the oil spill is a two dimensional Brownian $X_t = (X_t^{(1)}, X_t^{(2)})$ motion with variance parameter 1/2. The desired probability P_S is

$$P_S = P\{|X_t^{(1)}| \geq 1 \ \text{ or } |X_t^{(2)}| \geq 1 \ \text{ for some } 0 \leq t \leq 1\} \tag{9}$$

Using the Reflection principle this can be estimated by

$$\begin{aligned} P_S &\leq P\{|X_t^{(1)}| \geq 1 \ \text{ for some } 0 \leq t \leq 1\} \\ &\quad + P\{|X_t^{(2)}| \geq 1 \ \text{ for some } 0 \leq t \leq 1\} \\ &\leq 4P\{X_1^{(1)} \geq 1\} = 8\int_1^\infty \frac{e^{-x^2}}{\sqrt{\pi}} \end{aligned} \tag{10}$$

7. Let X_t be a standard Brownian motion. Let M be the random variable given by

$$M = \sup_{0 \leq t \leq 1} X_t \tag{11}$$

It is the maximum of X_t on the time interval $[0, 1]$. We have, by the reflection principle,

$$\begin{aligned} P\{M \geq a\} &= P\left\{\sup_{0 \leq t \leq 1} X_t \geq a\right\} = P\{X_t \geq a \text{ for some } 0 \leq t \leq 1\} \\ &= 2P\{X_1 \geq a\} = 2\int_a^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \end{aligned} \tag{12}$$

Hence the cumulative distribution function of M is

$$P\{M < a\} = 2\int_0^a \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \tag{13}$$

and its density is

$$f(a) = 2\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad a \geq 0\,. \tag{14}$$

8. Let X_t be a Brownian motion with drift parameter μ and variance parameter σ^2.

(a) Since $X_t = Y_t + \mu_t$ where Y_t is a Brownian motion with parameter σ^2

$$E[X_s X_t] \;=\; \mu^2 ts + \sigma^2 \min(s,t) \tag{15}$$

(b) We expand $g(x)$ in a taylor series around X_t

$$= \lim_{h\to 0} \frac{1}{h} E\Big[g'(X_t)(X_{t+h} - X_t) + \frac{1}{2} g\text{''}(X_t)(X_{t+h} - X_t)^2 + \\ + o((X_{t+h} - X_t)^2) \,|\, X_0 = x\Big] \tag{16}$$

By the independent increments property, for any function $h(x)$, $h(X_t)$ is independent of $X_{t+h} - X_t$. We have

$$E[X_{t+h} - X_t \,|\, X_0 = x] \;=\; h\mu\,. \tag{17}$$

and, using that, $X_t \;=\; \mu t + Y_t$

$$\begin{aligned} & E\Big[(X_{t+h} - X_t)^2 \,|\, X_0 = x\Big] \\ &= E\Big[(\mu h + (Y_{t+h} - Y_t))^2 \,|\, Y_0 = x\Big] \\ &= \mu^2 h^2 + \sigma^2 h \end{aligned} \tag{18}$$

and we have

$$E\Big[o((X_{t+h} - X_t)^2)\Big] \;=\; o(h)\,. \tag{19}$$

Therefore we have

$$\begin{aligned} \frac{d}{dt} g(t,x) &= \lim_{h\to 0} \frac{1}{h} \Big(E\left[g'(X_t \,|\, X_0 = x\right] \mu h \\ &\qquad + \frac{1}{2} E\left[g''(X_t \,|\, X_0 = x\right] (\mu^2 h^2 + h) + o(h)\Big) \\ &= \mu \frac{\partial g}{\partial x}(t,x) + \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x^2}(t,x) \end{aligned} \tag{20}$$