Math 523H–Homework 2

1. Suppose $\{s_n\}$ is a sequence such that for all n we have
$$|s_n - s_{n+1}| \leq \frac{1}{\alpha^n}$$
for some $\alpha > 1$. Then prove that $\{s_n\}$ is a Cauchy sequence.
Hint: Use geometric series to bound $|s_n - s_{n+k}|$.

2. Suppose we are given a decimal expansion $k.d_1 d_2 d_3 \cdots$ of a real number where k is an integer and $d_i \in \{0, 1, 2, \cdots, 9\}$. Show that
$$s_n = k + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n}$$
is a Cauchy sequence. *Hint: Use Problem 1.*

3. Consider the sequence
$$s_n = \frac{1}{1 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{5 \cdot 9} + \frac{1}{7 \cdot 11} + \cdots + \frac{1}{(2n-1)(2n+3)}$$
Show that $\{s_n\}$ is a Cauchy sequence and compute its limit.
Hint: Compute the partial fraction expansion of the $\dfrac{1}{(2j-1)(2j+3)}$.

4. Consider two rational Cauchy sequences $\{s_n\}$ and $\{t_n\}$ and denote by $\{s_n \cdot t_n\}$ the sequence obtained by the products term by term. Show
   (a) The sequence $\{s_n \cdot t_n\}$ is again a Cauchy sequence.
   (b) If $\{s_n\} \sim \{s'_n\}$ and $\{t_n\} \sim \{t'_n\}$ show that $\{s_n \cdot t_n\} \sim \{s'_n \cdot t'_n\}$. This shows that the product of two real numbers is independent of the choice of the representatives in the equivalence classes.

5. Consider a sequence s_n and assume that $\displaystyle\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ exists.
   (a) Show that if $L < 1$ then $\lim_{n\to\infty} s_n = 0$.
   Hint: Pick b such that $L < b < 1$ and choose N so large that $\left|\frac{s_{n+1}}{s_n}\right| < a < 1$ for all $n \geq N$. Then show that for $n \geq N$ we have $|s_n| \leq b^{n-N}|s_N|$.
   (b) Show that if $L > 1$ then $\lim_{n\to\infty} |s_n| = \infty$.
   Hint: You can either use a similar trick as in part (a) or you can deduce it directly from part (a) by considering the sequence $t_n = \dfrac{1}{|s_n|}$.

6. Let $p > 0$. Show that

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \le 1 \\ +\infty & \text{if } a > 1 \\ \text{ does not exists} & \text{if } a < -1 \end{cases}$$

   Hint: Use Problem 5

7. Show that for any number a we have $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.
   Hint: Use Problem 5.

8. Suppose $\{a_n\}$ is a convergent sequence with $\lim_{n\to\infty} a_n = a$ and define a new sequence b_n by

$$b_n = \frac{1}{n}\sum_{k=1}^{n} a_k \; = \; \frac{1}{n}\left(a_1 + \cdots a_n\right)$$

   that is b_n is the average of the first n terms of the sequence a_n. Show that $\{b_n\}$ is a convergent sequence and $\lim_{n\to\infty} b_n = a$.