Lecture 12: Combinatorics

In many problems in probability one needs to count the number of outcomes compatible with a certain event. In order to do this we shall need a few basic facts of combinatorics

Permutations: Suppose you have n objects and you make a list of these objects. There are

$$n! = n(n-1)(n-2)\cdots 1$$

different way to write down this list, since there are n choices for the first on the list, $n-1$ choice for the second, and so on.

The number $n!$ grows very fast with n. Often it is useful to have a good estimate of $n!$ for large n and such an estimate is given by Stirling's formula

$$\text{Stirling's formula} \quad n! \sim n^n e^{-n}\sqrt{2\pi n}$$

where the symbols $a_n \sim b_n$ means here that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Combinations: Suppose you have a group of n objects and you wish to select a j of the n objects. The number of ways you can do this defines the *binomial coefficients*

$$\binom{n}{j} = \#\text{ of ways to pick } j \text{ objects out of } n \text{ objects}$$

and this pronounced "n choose j".

Example: The set $U = \{a, b, c\}$ has 3 elements. The subsets of U are

$$\emptyset\,, \quad \{a\}\,, \quad \{b\}\,, \quad \{c\}\,, \quad \{a,b\}\,, \quad \{a,c\}\,, \quad \{b,c\}\,, \quad \{a,b,c\}$$

and there are $\binom{3}{0} = 1$ subset with 0 elements, $\binom{3}{1} = 3$ subset with 0 elements, $\binom{3}{2} = 3$ subset with 2 elements, and $\binom{3}{3} = 1$ subset with 3 elements. ■

Recursion relation for the binomial coefficients: You can compute the binomial coefficients $\binom{n}{j}$ if you know the binomial coefficients $\binom{n-1}{k}$ for all $k = j$ and $k = j-1$. We have the relation

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1} \tag{1}$$

for $0 < j < n$. For $j = 0$ and n we have $\binom{n}{0} = \binom{n}{n} = 1$.

To see why the formula (1) holds think of n objects, let us say n people. The left hand side of (1) is the number of ways to form groups of j people. Now choose one individual among the n, let us say we pick Bob. Then $\binom{n-1}{j}$ is the number of way to choose a group of j people which do not include Bob (pick j out of the remaining $n-1$) while $\binom{n-1}{j-1}$ is the number of ways to pick a group of j people which does include Bob (pick Bob and then pick $j-1$ out of the remaining $n-1$.

To find an explicit formula for $\binom{n}{j}$ we note first that

$$n(n-1)\cdots n-(j-1)$$

is the number of ways to write an *ordered* list of j objects out of n objects since there are n choices for the first one on the list, $n-1$ choices for the second one and so on. Many of these lists contain the same objects but arranged in a different order and there are $j!$ ways to write a list of the same j objects in different orders. So we have

$$\binom{n}{j} = \frac{n(n-1)\cdots n-(j-1)}{j!}$$

which we can rewrite as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

Poker hands. We will compute the probability of certain poker hands. A poker hands consists of a 5 randomly chosen cards out of a deck of 52. So we have

$$\text{Total number of poker hands} = \binom{52}{5} = 2598960$$

Four of a kind: This hands consists of 4 cards of the same values (say 4 seven). To compute the probability of a four of a kinf note that there are 13 choices for the choice of values of the four of a kind. Then there are 48 cards left and so 48 choice for the remaining cards. So

$$\text{Probability of a four of a kind} = \frac{13 \times 48}{\binom{52}{5}} = \frac{624}{2598960} = 0.00024$$

Full house: This hands consists three cards of the same value and two cards of an another value (e.g. 3 kings and 2 eights). There are 13 ways to choose the value of three of a kind and once this value is chose there is $\binom{4}{3}$ to select the three cards out of the four of same value. There are then 12 values left to choose from for the pair and there $\binom{4}{2}$ to select the the pair. So we have

$$\text{Probability of a four of a full house} = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960} = 0.0014$$

So the full house is 6 times as likely as the four of a kind.

Three of a kind: There are $13\binom{4}{3}$ ways to pick a three of kind. There are then 48 cards left from which to choose the remaining last 2 cards and there are $\binom{48}{2}$ ways to do this. But we are then also allowing to pick a pair for the remaining two cards which would give a full house. Therefore we have

$$\text{Probability of a three of a kind} = \frac{13 \times \binom{4}{3} \times \binom{48}{2} - 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}}$$

Another way to compute this probability is to note that among the 48 remaining cards we should choose two different values (so as not to have a pair) and then pick a card of that value. This gives

$$\text{Probability of a three of a kind} = \frac{13 \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1} \times \binom{4}{1}}{\binom{52}{5}}$$

Either way this gives a probability $\frac{54912}{2598960} = 0.0211$

Exercise 1: A six card hand is dealt from an ordinary deck of 52 cards. Find the probability that

1. All six cards are hearts
2. There are three aces, two kings and one queen.
3. There three cards of one suit and three of another suit.

Exercise 2: Compute the probabilities to obtain the following poker hands

1. Two pairs
2. A straight flush: fives cards of the same suit in order (e.g. 6, 7, 8, 9, 10 of hearts).
3. A flush : five cards of the same suit but not in order (e.g. 3, 5,6, queen,and king of spades).

Exercise 3: Explain why the identity

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$$

holds.
Hint: Think of a group of consisting of n boys and n girls.