Math 645: Homework 3

1. The spectral radius $\rho(A)$ of a $n \times n$ matrix A is defined as

$$\rho(A) \;=\; \max\{|\lambda|\,;\, \lambda \text{ eigenvalue of } A\}\,. \tag{1}$$

   Let $A \in \mathcal{L}(\mathbf{R}^n)$. Show that for any norm on $\mathbf{R}^n$ we have the inequality $\rho(A) \le \|A\|$, and that if A is symmetric we have the equality $\|A\|_2 = \rho(A)$.

2. Let

$$A \;=\; \begin{pmatrix} 0.999 & 1000 \\ 0 & 0.999 \end{pmatrix} \tag{2}$$

   (a) Compute the spectral radius of A as well as $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$.

   (b) Find a norm on $\mathbf{R}^n$ such that $\|A\| \le 1$.

3. Find 2 by 2 matrices A and B such that $e^{A+B} \ne e^A e^B$.

4. Show that if $A(t)$ is antisymmetric, i.e., $A^T = -A$, then the resolvant of $x' = A(t)x$ is orthogonal. *Hint:* Show that the scalar product of two solutions is constant.

5. Using the definition of the exponential matrix, compute the e^{tA} for the following matrices

   (a) $A = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix}$ with $\kappa > 0$.

   (b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

6. **(D'Alembert reduction method)**. Consider the ODE $x' = A(t)x$ where $A(t)$ is a $n \times n$ matrix and assume that we a non-trivial solution $x(t)$. Show that one can reduce the equation $x' = A(t)x$ to the problem $z' = B(t)z$ where $z \in \mathbf{R}^{n-1}$ and $B(t)$ is a $(n-1) \times (n-1)$ matrix. *Hint:* Without loss of generality you may assume that $x_n(t) \ne 0$. Look for solutions of the form $y(t) = \phi(t)x(t) + z(t)$, where $\phi(t)$ is a scalar function and z has the form $z = (z_1, \cdots, z_{n-1}, 0)^T$.

7. (a) Using the previous problem, compute the resolvent $R(t,1)$ of

$$x' \;=\; \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x\,, \tag{3}$$

   using the fact that $x(t) \;=\; (t^2, -t)^T$ is a solution. *Hint:* The solution is

$$\begin{pmatrix} t^2(1-\log t) & -t^2\log t \\ t\log t & t(1+\log t) \end{pmatrix} \tag{4}$$

(b) Compute the solution of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x + \begin{pmatrix} t \\ -t^2 \end{pmatrix}, \tag{5}$$

with initial condition $x(1) = (0,0)^T$.

8. Consider the system

$$x' = A(t)x\,, \qquad A(t) = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

(a) Show that $A(t)$ does not commute with $\int_{t_0}^t A(s)\,ds$.

(b) Compute the resolvent of (6).

(c) Show that $R(t,t_0) \neq \exp\left(\int_{t_0}^t A(s)\,ds\right)$.

9. (a) Consider the damped harmonic oscillator $x'' + bx' + \kappa x = 0$ where b, $\kappa > 0$. Write is a first order system $X = AX$, where $X = (x, x')$. Compute the eigenvalues of A. Show that every solution tends to 0 as $t \to +\infty$.

(b) Consider now the forced damped harmonic oscillator $x" + bx' + \kappa x = \cos(t)$, i.e. with a periodic forcing of period 2π. Show the system has a unique periodic solution of period 2π. *Hint:* Use the use the variation of constants formula to show that the existence of a periodic solution $X(t)$ of period 2π with $X(0) = X_0$ reduce to the existence of a solution to the equation

$$X_0 = e^{2\pi A} X_0 + W \tag{7}$$

for asuitable W. Show that this equation has a unique solution.

(c) Deduce from (a) and (b) that any solution tends the periodic solution as $t \to \infty$. *Hint:* Write an arbitrary solution with initial condition Y_0 using the variation of constants formula and compare it to the periodic solution found in (b).

Hint: There is no need to actually solve the linear equation here!