Math 597/697: Homework 6

1. At a popular BBQ place, customers arrive according to a Poisson process with a rate λ. There is a single service station at which they order and wait for their meal. The service time is exponential with parameter μ. The cook, however, is quite lousy: there is a probability α, $0 < \alpha < 1$ that the customer is dissatisfied and sends his meal back. If he is dissatisfied, he is asking for his meal to be prepared again. Subsequent service is again exponential with parameter μ and independently of what happened before the customer will be again dissatisfied with probability α and so on....

   (a) Show that the length of times it takes for a customer to be satisfied with his meal is exponential. What is the parameter?

   (b) Assume that dissatisfied customers are served again immediately until they are satisfied. Compute the birth and death rates and determine when the corresponding Markov chain is recurrent or transient.

   (c) Assume that dissatisfied customers are not served again immediately, but they are rather sent to the back of the line. How does that affect the the birth and death rates? How are answers in (b) affected?

2. A population of organisms consists of both males and females. Any particular male is likely to mate with any particular female during the time interval h with probability $\lambda h + o(h)$. Each mating produces immediately an offspring which is male with probability p and female with probability $1 - p$. Let M_t and F_t denote the numbers of males and females in the population. Derive the parameters of the Markov chain $\{F_t, M_t\}$.

3. For a general birth and death process, find a differential equation for the expected population at time t, $E[X_t]$. Solve this equation for the $M/M/1$ and $M/M/\infty$ queues.

4. Consider a continuous time branching process defined as follows. A organism lifetime is exponential with parameter λ and upon death, it leaves k offspring with probability p_k, $k \geq 0$. The organisms act independently of each other. We assume assume that $p_1 = 0$. Let X_t be the population at time t, find the generator of X_t. Specialize your

results to the binary splitting case where the particle either splits with probability p or vanishes with probability $(1-p)$. Find the stationary distribution π of $\{X_t\}$. Is it a limiting distribution?

5. Consider a queuing system with one single server, arrival rate λ and serving rate μ.

   (a) When N customers are in the system, the arriving customer give up and do no enter the system. What is the state space and the generator of the system? What is the limiting distribution?

   (b) When n customers are in the system, an arriving customer will join the system with probability $1/(n+1)$. What is the state space and the generator of the system? What is the limiting distribution? Does it look familiar?

6. Consider the $M/M/s$ queue where customers arrive according to a Poisson process with rate λ and are served at one of s service station with an exponential service time with parameter μ. However when a customer arrives and finds all s service stations busy, he leaves immediately. Model this as a birth and death process and find the corresponding parameters. Let Q_t be the number of busy servers at time t. Find

$$p_s = \lim_{t\to\infty} P\{Q(t) = s\}\,.$$

   This represents the probability that an arriving customer finds the system blocked and is lost. Why is this also the long run percentage of customers who are lost?

7. Consider a birth and death process with birth rate $\lambda_n = 1 + \frac{1}{n+1}$ and death rate $\nu_n = 1$. Is this process positive recurrent, null recurrent, or transient. What if $\lambda_n = 1 - \frac{1}{n+2}$?

8. Consider the population model with immigration, i.e., $\lambda_n = n\lambda + \nu$ and $\mu_n = n\mu$. Find for which values of λ, μ, ν the process is positive recurrent, null recurrent, transient.

9. Consider a Yule process (pure birth with linear growth) and let T_i be the time it takes for a population of size i to reach $i+1$. In this problem we indicate a method to actually compute the transition probabilities $P_{mn}(t)$ which is not based on solving directly the backward of forward equation.

   (a) Argue that T_i is exponential with rate $i\lambda$.

   (b) Let $X_1, X_2, \ldots, X_n$ be n exponentials random variables with parameter λ. Show

$$\max(X_1, \ldots, X_n) = T_n + T_{n-1} + \ldots T_1 \tag{1}$$

Hint: Interpret X_i as time the failure of a component, T_n as the time of the first failure, T_{n-1} as the time between the first and second failure, etc....

(c) Deduce from (b) that $P\{T_1 + \ldots + T_n < t\} = (1 - e^{-\lambda t})^n$.

(d) Use (c) and (a) to obtain that

$$P^t_{1n} = (1 - e^{-\lambda t})^{n-1} - (1 - e^{-\lambda t})^n = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \quad (2)$$

and hence that $P\{X_t = n \,|\, X_0 = 1\}$ has a geometric distribution.

(e) Deduce from (d) that for $n > m$

$$P^t_{mn} = \binom{n-1}{m-1}(e^{-\lambda t})^m (1 - e^{-\lambda t})^{n-m} \quad (3)$$

10. FEDEX in town Y has 35 truck. Times between breakdowns for each truck are exponentially distributed with mean 30 days. If a vehicle breaks down it is sent (equally likely) to one of 2 mechanical facility at which it is repaired at an exponential rate of 1 per day. After repair it is sent to a cleaning facility at which it is cleaned at a rate of 3 per day. Model this with a closed network of queues and give an expression for the stationary distribution. Can you compute the expected number of vehicles in operation?(If you want to compute it you will have to use Mathematica or do some programming.)

11. Consider the following network of queues. There are k nodes and you move through the nodes from 1 to 2, 2 to 3, and so on up to node k. From node k you move to node j with probability p_j, $j = 1, \cdots, k$. Assume that the rate of service at node j is $\beta(j)$.

(a) Consider a closed network of queues with m customers. What are the traffic equations and the stationary distributions? Specialize your results to the case where $\beta(j) = \beta$.

(b) Consider an open network of queues where customers enter the system in node 1 with a rate λ. Moreover the customers quit the system after being serviced at node k with probability $q > 0$. Find the traffic equations and the condition for the stationary distribution to exist. Give the stationary distribution when it exists. Specialize your results to the case where $\beta(j) = \beta$.