Math 697 Fall 2014: Week 8

Exercise 1 Consider the Markov chain on $S = \{0, 1, 2, 3, \cdots\}$ with transition probabilities

$$P(0,0) = 1 - p_0\,, \quad P(0,1) = p_0$$
$$P(j,j-1) \;=\; 1 - p_j\,, \quad P(j,j+1) = p_j\,.$$

(a) Under which conditions on p_j is the Markov chain positive recurrent. *Hint:* Use detailed balance.

(b) Suppose that you are given a probability distribution $\pi(j)$ on S. Under which conditions can you choose $p(0), p(1), \cdots$ such that the stationary distribution of the Markov chain in **(a)** is π?

(c) Find p_i such that π is a Poisson distribution with parameter λ.

(d) Choose $p_0 = 1/2$ and $p_i = 1/2$ for $i \ge 1$ and use this as the proposal matrix (i.e Q) in Metropolis-Hastings to generate a Poisson distribution with parameter λ. Compare with **(c)**.

Exercise 2 Discrete queueing model with state space $S = \{0, 1, 2, 3, \cdots\}$ and transition probabilities

$$P(0,0) = 1 - p\,, \quad P(0,1) = p$$
$$P(j,j-1) \;=\; q(1-p)\,, \quad P(j,j) = pq + (1-p)(1-q)\,, \quad P(j,j+1) = p(1-q)\,.$$

Determine when the Markov chain is transient, recurrent, or positive recurrent. In the positive recurrent case compute the stationary distribution and the length of the queue in equilibrium. In the transient case compute the probability $\alpha(j)$ of ever reaching 0 starting from i.

Exercise 3 For each of the following Markov chain with state space $\{0, 1, 2, \cdots\}$ determine if it transient, recurrent, positive recurrent. (If it is positive recurrent compute the stationary distribution π).

- $p(j,0) = \frac{1}{j+2}, \quad p(j,j+1) \;=\; \frac{j+1}{j+2}$
- $p(j,0) = \frac{j+1}{j+2}, \quad p(j,j+1) \;=\; \frac{1}{j+2}$
- $p(j,0) = \frac{1}{j^2+2}, \quad p(j,j+1) \;=\; \frac{j^2+1}{j^2+2}$

Exercise 4 Let $p(k)$, $k = 0, 1, 2, 3, \cdots$ be such that $\sum_{k=0}^{\infty} p(k) = 1$. Consider the Markov chain on $S = \{0, 1, 2, 3, \cdots\}$ with transition probabilities

$$P(0,k) = p(k)\,, \quad k = 0, 1, 2, 3, \cdots$$
$$P(k,k-1) = 1\,, \quad k \ge 1\,.$$

Under which conditions on $p(k)$ is the Markov chain positive recurrent? In that case compute the stationary distribution π.

Exercise 5 Consider the Markov chain with state space $\{0, 1, 2, \cdots\}$ with transition probabilities

$$P(0,0) = 1-p, P(0,2) = 1-p, P(j, j+2) = p, P(j, j-1) = 1-p\,.$$

For which values of p is the chain transient?

Exercise 6 A mouse is performing a symmetric random walk on the positive integer $\{0, 1, 2, 3, \cdots\}$: if it is in state i it is equally likely to move to state $i-1$ or $i+1$. The state 0 is the mouse's home filled with lots of tasty cheese. If the mouse ever reaches its home it will stay there forever. On the other hand there is a bad cat who tries to catch the mouse and each time the mouse moves there is a probability $1/5$ that the cat will kill the mouse.

To describe this process as a Markov chain consider an extra state $*$ which corresponds to the mouse being dead. The state space is $S = \{*, 0, 1, 2, \cdots\}$ and X_n denotes the position of the mouse at time n. Compute the corresponding transition matrix.

Compute the probability that the mouse reaches safety if it starts in state i, i.e.,

$$p_i \;\equiv\; P\{X_n = 0 \text{ for some n} \mid X_0 = i\}\,, i = 0, 1, 2, \cdots$$