Math 421 Solution of Fall 99 Final

1. Given that the first few terms of the Laurent series for the function $\cot(z)$ around $z = 0$ are:

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots$$

a) Find the principal part at $z = 0$ of the function $f(z) = \dfrac{(1+z)\cot z}{z^4}$.

Answer: $f(z) = z^{-5} + z^{-4} - \frac{1}{3}z^{-3} - \frac{1}{3}z^{-2} - \frac{1}{45}z^{-1} +$ terms of non-negative degree. The part shown is the principal part.

b) Find all the singularities of $f(z)$ in the disk $D = \{|z| < 5\}$. Determine the nature of each singularity (isolated, removable, pole of what order, essential).

Answer: Zero is a pole of order 5. The function $f(z)$ can be written in the form $\frac{\phi(z)}{\sin(z)}$, where $\phi(z) = \dfrac{(1+z)\cos(z)}{z^4}$. Integral multiples of π are simple zeroes (of order 1) of the function $\sin(z)$, of which $-\pi$, 0, and π are in the disk D. The function ϕ does not vanish at π and $-\pi$. Hence, π and $-\pi$ are simple poles of $f(z)$.

c) Find the residue at each isolated singularity in D.

Answer: Using the Laurent serries above, we see that $\mathrm{Res}_{z=0} f(z) = -\frac{1}{45}$. At the simple pole π, the residues is

$$\mathrm{Res}_{z\,=\,\pi} f(z) = \lim_{z\to 0}(z-\pi)f(z) = \lim_{z\to 0}\frac{z-\pi}{\sin(z)}\phi(z) = \frac{1}{\sin'(\pi)}\frac{(1+\pi)\cos(\pi)}{\pi^4} = \frac{1+\pi}{\pi^4},$$

$$\mathrm{Res}_{z=-\pi} f(z) = \frac{1-\pi}{\pi^4}.$$

2. Compute $\displaystyle\int_C \frac{\cos z}{e^{iz}-1}dz$, where C is the circle $\{|z| = 2\}$ (traversed counterclockwise).

Answer: $g(z) := e^{iz} - 1$ vanishes to order 1 at integer multiples of 2π (because $g'(2n\pi) = i \neq 0$). The numerator $\cos z$ has value 1 at $2n\pi$. Thus, $f(z) = \frac{\cos(z)}{e^{iz}-1}$ has a simple pole at $2n\pi$, $n \in \mathbb{Z}$. The only multiple of 2π enclosed by C is 0. Using the fact that 0 is a simple pole, we get

$$\mathrm{Res}_{z=0} f(z) = \frac{\cos(0)}{\frac{d}{dz}_{|z=0}(e^{iz}-1)} = \frac{1}{i} = -i.$$

Cauchy's Theorem yields,

$$\int_C f(z)dz = 2\pi i \mathrm{Res}_{z=0} f(z) = 2\pi.$$

3. Compute $I := \int_C (e^{\sin(z)} + \bar{z})dz$, where C is the circle $\{|z| = 2\}$ (traversed counterclockwise).

Answer: $e^{\sin(z)}$ is an entire function, so its integral, over any closed contour, is zero (by Cauchy-Goursat's Theorem). Using the parametrization $z = 2e^{i\theta}$, we get

$$I = \int_C \bar{z}dz = \int_0^{2\pi} 2e^{-i\theta}2ie^{i\theta}d\theta = 8\pi i.$$

4. Compute $I := \displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)}$.

Answer: Let $z = e^{i\theta}$. Then $\cos(\theta) = \frac{z+\frac{1}{z}}{2}$, $dz = ie^{i\theta}d\theta$ and $d\theta = \frac{dz}{iz}$. The integral gets converted to the contour integral over the unit circle C.

$$I = \int_C \frac{1}{2+\left[\frac{z+\frac{1}{z}}{2}\right]} \frac{dz}{iz} \;=\; (-2i)\int_C \frac{dz}{z^2+4z+1}.$$

The integrand has poles at $-2 \pm \sqrt{3}$. Only $-2+\sqrt{3}$ is enclosed by C. We get

$$I \;=\; (2\pi i)(-2i)\cdot \mathrm{Res}_{z=-2+\sqrt{3}} \frac{1}{z^2+4z+1} \;=\; 4\pi \cdot \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} \;=\; \frac{2\pi}{\sqrt{3}}.$$

5. Compute $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^6}dx$.

Answer: See the first Example in Section 60 of the text on page 205.

6. **a)** Find the Laurent series of the function $f(z) = \frac{\mathrm{Log} z}{z-i}$ around the point $z_0 = i$.

Answer: Let $\sum_{n=0}^{\infty} a_n(z-i)^n$ be the Taylor series of $\mathrm{Log}(z)$ centered at i. Taylor's Theorem states, in particular, that $a_n = \dfrac{\mathrm{Log}^{(n)}(i)}{n!}$. Now, $\mathrm{Log}'(z) = \frac{1}{z}$ and $\mathrm{Log}^{(n)}(z) = \dfrac{(-1)^{n+1}(n-1)!}{z^n}$ for $n \geq 1$. We get, for $n \geq 1$,

$$a_n \;=\; \frac{\mathrm{Log}^{(n)}(i)}{n!} \;=\; \frac{(-1)^{n+1}}{ni^n} \;=\; \frac{(-1)^{n+1}(-i)^n}{n} \;=\; -\frac{i^n}{n}.$$

$a_0 = \mathrm{Log}(i) = \frac{\pi i}{2}$. Summerizing, for $|z-i| < 1$ we get

$$\mathrm{Log}(z) \;=\; \frac{\pi i}{2} + \sum_{n=1}^{\infty} -\left(\frac{i^n}{n}\right)(z-i)^n, \quad \text{and setting} \quad k = n-1, \quad \text{we get}$$

$$\frac{\mathrm{Log}(z)}{z-i} \;=\; \frac{\pi i}{2}\frac{1}{z-i} + \sum_{k=0}^{\infty} -\left(\frac{i^{(k+1)}}{k+1}\right)(z-i)^k.$$

b) Find the Taylor series of the function $f(z) = \frac{1}{z^2-3z+2}$ around the point $z_0 = 0$.

Answer: Use the partial fraction decomposition

$$\frac{1}{z^2-3z+2} \;=\; \frac{-1}{z-1} + \frac{1}{z-2}.$$

Now, use the Taylor series of $\frac{1}{1-w}$ to obtain

$$\frac{-1}{z-1} \;=\; \frac{1}{1-z} \;=\; \sum_{n=0}^{\infty} z^n,$$

$$\frac{1}{z-2} \;=\; -\left(\frac{1}{2}\right)\frac{1}{1-\frac{z}{2}} \;=\; -\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^n,$$

$$f(z) \;=\; \sum_{n=0}^{\infty}\left[1-\left(\frac{1}{2}\right)^{n+1}\right] z^n.$$

7. Determine whether the following statements are true or false. Justify your answers.

a) The limit $\lim_{z\to 0} \frac{e^{\bar{z}}-1}{z}$ exists and is equal to 1.

Answer: False, $e^{\bar{z}}$ is not analytic at 0. The above limit is the derivative limit, which does not exists. A direct argument, that the limit doesn't exists, consists of letting z approach 0 along the x and y axis:

$$\begin{aligned} \lim_{x\to 0} \frac{e^{x}-1}{x} &= 1, \\ \lim_{iy\to 0} \frac{e^{-iy}-1}{iy} &= -1. \end{aligned}$$

b) There is a function $f(z)$, analytic in the disk $D = \{|z| < 1\}$, such that

$$|f(z)|^2 = 4 - |z|^2, \quad \text{for all } z \text{ in } D.$$

Answer: False. Such a function f would be a non-constant analytic function, whose absolute value achieves its maximum at the interior point 0 of D. This contradicts the *Maximum Modulus Principle.*

c) If $f(z)$ has an isolated singularity at z_0 and $\mathrm{Res}_{z=z_0}(f) = 0$, then z_0 is a removeable singularity.

Answer: False. Take $f(z) = \frac{1}{z^2}$ and $z_0 = 0$ as a counter example.

8. Compute $\cos(\frac{\pi}{2} - i\ln 2)$. Simplify your answer as much as possible.

Answer:

$$\cos(\frac{\pi}{2} - i\ln 2) = \frac{e^{i\frac{\pi}{2}+\ln 2} + e^{-i\frac{\pi}{2}-\ln 2}}{2} = \frac{2i - \frac{i}{2}}{2} = \frac{3}{4}i.$$

9. Prove that $\left|\int_C e^{iz^2}dz\right| < 5$, where C is the piece of the circle $|z| = 2$ going from 2 to $2i$ counter-clockwise.

Answer: The curve C is parametrized by $z = 2e^{i\theta}$, $0 \le \theta \le \frac{\pi}{2}$. Thus, $z^2 = 4e^{i2\theta}$ has a positive imaginary part. Consequently, iz^2 has a negative real part equal to $-\sin(2\theta)$, where $0 \le \theta \le \frac{\pi}{2}$, and

$$\left|e^{iz^2}\right| = e^{-\sin(2\theta)} < 1.$$

We conclude, that

$$\left|\int_C e^{iz^2}dz\right| \le \int_C \left|e^{iz^2}\right||dz| \le 1 \cdot \mathrm{length}(C) = \frac{4\pi}{4} = \pi < 5.$$

10. Find an entire function $f(z)$ such that $\mathrm{Re}(f) = 4x^3y - 4xy^3 - y$.

Answer: Note the equality

$$(x+iy)^4 = x^4 + 4ix^3y - 6x^2y^2 - i4xy^3 + y^4.$$

Take $f(z) = -iz^4 + iz$.