Math 523H–Homework 1

1. Prove, by induction, that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

2. Show that $\sqrt{3}$ is not a rational number.

3. (a) Show that $|b| \leq a$ if and only if $-a \leq b \leq a$.

   (b) Show that $||a| - |b|| \leq |a - b|$ (this sometimes called the *reverse triangle inequality*).

   (c) Show that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

4. Compute the following limits. In this problem you should justify your claims using the definitions of limits.

   (a) $\lim_{n\to\infty} \frac{1}{n^{1/3}}$.

   (b) $\lim_{n\to\infty} \frac{7n-19}{3n+7}$.

   (c) $\lim_{n\to\infty} \frac{n^2+3}{n+1}$.

5. Let $\{t_n\}$ be a bounded sequence and $\{s_n\}$ be a convergent sequence with $\lim_{n\to\infty} s_n = 0$. Show that $\lim_{n\to\infty}(s_n t_n) = 0$.

6. (a) Show that if $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n = t > 0$ then $\lim_{n\to\infty}(s_n t_n) = +\infty$.

   (b) Construct examples of sequences $\{s_n\}$ and $\{t_n\}$ such that $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} t_n = 0$ and

      i. $\lim_{n\to\infty}(s_n t_n) = +\infty$

      ii. $\lim_{n\to\infty}(s_n t_n) = c$ for any arbitrary constant c.

      iii. The sequence $s_n t_n$ is bounded but not convergent.

7. Show using the appropriate limit laws that

   (a) $\lim_{n\to\infty}(\sqrt{n^2+1} - n) = 0$

   (b) $\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \frac{1}{2}$

8. Show that if $\lim_{n\to\infty} s_n = s$ then $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$.