Lecture 6: Mixed strategies Nash equilibria and reaction curves

Nash equilibrium: The concept of Nash equilibrium can be extended in a natural manner to the mixed strategies introduced in Lecture 5. First we generalize the idea of a *best response* to a mixed strategy

Definition 1. *A mixed strategy $\widehat{\sigma}_R$ is a* best response *for R to some mixed strategy σ_C of C if we have*

$$\langle \widehat{\sigma}_R, P_R \sigma_C \rangle \geq \langle \sigma_R, P_R \sigma_C \rangle \quad \text{for all } \sigma_R \,.$$

A mixed strategy $\widehat{\sigma}_C$ is a best response *for C to some strategy σ_R of R if we have*

$$\langle \sigma_R, P_C \widehat{\sigma}_C \rangle \geq \langle \sigma_R, P_C \sigma_C \rangle \quad \text{for all } \sigma_C$$

We can then extend the definition to Nash equilbrium

Definition 2. *The mixed strategies $\widehat{\sigma}_R, \widehat{\sigma}_R$ are a Nash equilibrium for a two-player game with payoff matrices P_R and P_C if*

$$\widehat{\sigma}_R \text{ is a best response to } \widehat{\sigma}_C \quad \text{and} \quad \widehat{\sigma}_C \text{ is a best response to } \widehat{\sigma}_R$$

or in other words

$$\begin{aligned} \langle \widehat{\sigma}_R, P_R \widehat{\sigma}_C \rangle &\geq \langle \sigma_R, P_R \widehat{\sigma}_C \rangle \quad \text{for all } \sigma_R \,. \\ \langle \widehat{\sigma}_R, P_C \widehat{\sigma}_C \rangle &\geq \langle \widehat{\sigma}_R, P_C \sigma_C \rangle \quad \text{for all } \sigma_R \end{aligned}$$

Reaction curves: For games with *two strategies* one can compute the best responses and the Nash equilbria in terms of the *reactions curves*. To explain the idea let us start with a example

Example: Matching Pennies The payoff matrices are given by

$$P_R = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad P_R = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and let us write the mixed strategies as

$$\sigma_R = (p, 1-p) \quad \sigma_C = (q, 1-q)$$

To find the best response to σ_C we compute

$$\begin{aligned} \langle \sigma_R, P_R \sigma_C \rangle &= \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 2q-1 \\ 1-2q \end{pmatrix} \right\rangle \\ &= p(2q-1) + (1-p)(1-2q) = (2q-1) + p(4q-2) \end{aligned}$$

Since we are computing the best response for R to the strategy of C we consider the payoff

$$(2q-1) + p(4q-2)$$

for **fixed** q and **variable p** with $0 \le p \le 1$. This is a linear function of p and the maximum will depend on the slope of this function (here $4q-2$), whether it is positive, negative, or 0.

Best response for R:

- $4q-2 > 0$ (or $q < 1/2$) The slope is positive so the maximum is at $p = 1$.
- $4q-2 < 0$ (or $q > 1/2$) The slope is negative so the maximum is at $p = 0$
- $4q-2 = 0$ (or $q = 1/2$) The slope is 0 so the maximum is at any p between 0 and 1.

To find the best response for C we compute

$$\begin{aligned}
\langle \sigma_R, P_R \sigma_C \rangle &= \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 1-2q \\ 2q-1 \end{pmatrix} \right\rangle \\
&= p(1-2q) + (1-p)(2q-1) \\
&= (2p-1) + q(2-4p)
\end{aligned}$$

which we now consider has a function of the **variable** q and for **fixed** p. By maximizing over $0 \le q \le 1$ we find

Best response for C:

- $2-4p > 0$ (or $p > 1/2$) The slope is positive so the maximum is at $q = 1$.
- $2-4p < 0$ (or $p < 1/2$) The slope is negative so the maximum is at $q = 0$
- $2-4p = 0$ (or $p = 1/2$) The slope is 0 so the maximum is at any q between 0 and 1.

To find the Nash equilibria we argue as follows.

- If $q > 1/2$ then the best response is $p = 0$ but the best response to $p = 0$ is $q = 0$ which contradicts $q > 1/2$ and this does not lead to a Nash equilbrium.
- If $q < 1/2$ then the best response is $p = 1$ but the best response to $p = 1$ is $q = 1$ and again this does not lead to a Nash equilbrium.
- If $q = 1/2$ then the best response is any p and so if we choose $p = 1/2$ then the best response to $p = 1/2$ is any q, in particular $q = 1/2$.

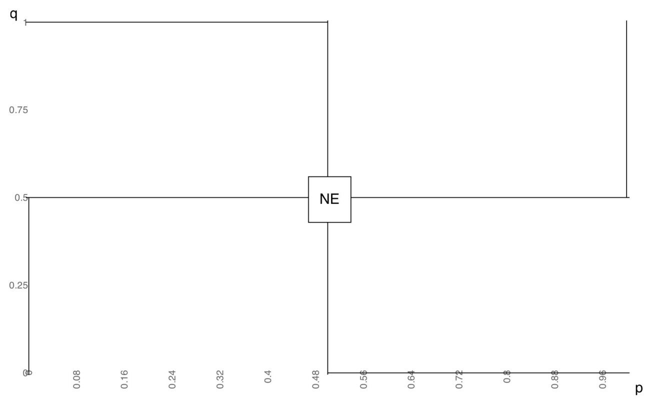

Figure 1: Reaction curves for the matching pennies game

The conclusion is that there exists one Nash equilibrium

$$\sigma_R = (1/2, 1/2) \ , \quad \sigma_C = (1/2, 1/2) \ ,$$

A convenient way to put all this information together and to find the Nash equilibria in a a graphical manner is to draw the reaction curves which exhibit the best responses. For the matching pennies they are show in figure

Example: Battle of the sexes. Writing the payoff as $\sigma_R = (p, 1-p), \sigma_C = (q, 1-q)$ we have for the best response from R to C

$$\begin{aligned}
\langle \sigma_R, P_R \sigma_C \rangle &= \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 3-2q \\ 2q \end{pmatrix} \right\rangle \\
&= p(3-2q) + (1-p)2q = 2q + p(3-4q) \qquad (1)
\end{aligned}$$

Since the fame is symmetric we obtain the payoff for C by exchanging p and q

$$\langle \sigma_R, P_C \sigma_C \rangle = 2p + q(3-4p)$$

We obtain **Best response for R:**

- $3 - 4q > 0$ (or $q < 3/4$) The slope is positive so the maximum is at $p = 1$.
- $3 - 4q < 0$ (or $q > 3/4$) The slope is negative so the maximum is at $p = 0$
- $3 - 4q = 0$ (or $q = 3/4$) The slope is 0 so the maximum is at any p between 0 and 1.

and **Best response for R:**

- $3 - 4p > 0$ (or $p < 3/4$) The slope is positive so the maximum is at $q = 1$.

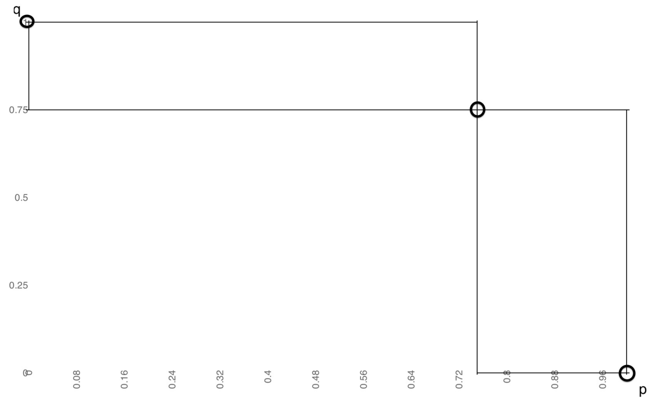

Figure 2: Reaction curves for the battle of the sexes game

- $3 - 4p < 0$ (or $p > 3/4$) The slope is negative so the maximum is at $q = 0$
- $3 - 4p = 0$ (or $p = 3/4$) The slope is 0 so the maximum is at any q between 0 and 1.

The best response curves are given in Figure 2

Exercise 1: Consider a *symmetric* two-strategies game with payoff matrices

$$P_R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P_R = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Exercises:

Exercise 1: Compute all the Nash equilibria for the Ultimatum game.

Exercise 2: Compute all the Nash equilibria for the Snowdrift game.

Exercise 3: The Samaritan dilemma: This dilemma occurs when deciding whether to provide help to a needy person and balancing whether the benefit providing help might be an incentive to being unproductive.

Mr and Mrs Roberts have a very lazy son (or daughter) named Collin (or Collette). They would like to help Collin financially but they do not want to contribute to his distress by allowing him to loaf around so they announce they might help Collin if he does not find a job. Collin however seeks work only he cannot depend on his parents for support and also he may not find work even if he searches for one.

1. Explain why the payoff table captures all the aspects of this the strategic situation

		Collin	
		Seek work	Loaf around
Mr and Mrs Roberts	Help Son	2 3	3 -1
	Tough love	1 -1	0 0

2. Compute all the Nash equilibria (pure and mixed) for this game. Draw the reaction curve diagram.

3. A similar strategic situation occurs for example in welfare where providing aid is sometimes argues as offering a disincentive to work. Or in foreign aid to poor country. Does the Nash equilibria shed some light on this dilemma?

Exercise 4: The IRS game The goal to the IRS is to either prevent and caught tax cheaters. We assume that the benefit to the IRS for preventing or catching cheating is 4 , the cost of auditing is $C < 4$. On the other hand for the tax payer the cost of complying and obeying the law is 1 (i.e. the payoff will be -1) while the penalty for cheating is $F > 1$ is he is getting caught.

The strategies for the IRS are to either audit or trust while for the tax payer they are to cheat or comply. Write down the payoff matrix and compute the Nash equilbria.

Exercise 5: Two strategies symmetric games: Consider a *symmetric* two-strategies game with payoff matrices

$$P_R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P_C = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

We assume that the game is generic, that is we do not have $a = c$ nor $b = d$ (see the discussion at the end of lecture 3).

1. Draw all possible reaction curves diagram (there are 4 of them)

2. From the point of view of Nash equilibria how many different types of game are there? (this is less than 4) Describe them (pure NE, mixed NE, how many?, etc...) and identify which Nash equilibria are symmetric.

Exercise 6: Two-strategies games: Consider a two-strategies game with payoff matrices

$$P_R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P_C = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

We assume that the game is generic, that is we do not have $a = c$ nor $b = d$ nor $A = C$ nor $B = D$.

1. Draw all possible reaction curves diagram (there are $4 \times 4 = 16$ of them).
2. From the point of view of Nash equilibria how many different types of game are there? (much much less than 16...). Describe them all (pure NE?, mixed NE?, etc..)