

MATH 646: Homework 1

1. (a) Let (S^1, \mathcal{B}, m) be the circle with the Borel σ -algebra \mathcal{B} and the Lebesgue measure m . Show that the maps $Tz = z^n$, where $n \geq 2$, is a positive integer are ergodic.
- (b) Show that Lebesgue almost all numbers in $[0, 1)$ are normal to base 2, i.e., the frequency of 1 in the binary expansion of x is $1/2$.
2. Let (K^n, \mathcal{B}, m) be the n -torus with the Borel σ -algebra \mathcal{B} and the Lebesgue measure m . The real numbers a_1, a_2, \dots, a_M are said to be *rationally independent* if $\sum_{m=1}^M k_m a_m = 0$, $k_i \in \mathbf{Z}$, implies that $k_1 = k_2 = \dots = k_M = 0$. Note that α is irrational if and only if α and 1 are rationally independent. For $\gamma = (\gamma_1, \dots, \gamma_n) \in K^n$ the translation on the n -torus K^n T_γ is given by

$$T_\gamma(x_1, \dots, x_n) = (x_1 + \gamma_1, \dots, x_n + \gamma_n) \pmod{1}. \quad (1)$$

- (a) Show T_γ is an endomorphism of K^n , for all γ .
- (b) Show that T_γ is ergodic if and only if $\gamma_1, \gamma_2, \dots, \gamma_n, 1$ are rationally independent.
- (c) If T_γ is not ergodic, construct an invariant function which is not a constant a.e.
3. Complete the second part of Birkhoff Theorem (see classnotes).
4. Consider the translation of the circle $Tx = x + \alpha \pmod{1}$. If α is irrational the map T is an ergodic endomorphism of (S^1, \mathcal{B}, m) where \mathcal{B} is the Borel σ -algebra and m is the Lebesgue measure. By Birkhoff ergodic theorem the ergodic averages converge m -a.e. for any function $f \in L^1$. A stronger version of this theorem can be proved in this particular case in an elementary way (Historically this is also one the first ergodic theorem ever to be proved by Bohl 1909 and Weyl 1916).
- (a) Prove that for any trigonometric function $f = e^{2\pi i n x}$ with $n \neq 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = 0,$$

uniformly in x .

- (b) Show that if f is continuous then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f dm \quad (2)$$

uniformly in x . *Hint:* Approximate f by a trigonometric polynomial.

(c) Show that for any closed interval $[a, b] \subset [0, 1]$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(T^j x) = m(A) \quad (3)$$

for all x . *Hint:* Approximate χ_A from below and from above by continuous function.

5. **Distribution of the first digits of 2^n .** Using the results of the previous problem to show the following.

(a) A sequence of number d_0, d_1, \dots with $a_j \in [0, 1]$ is said to be equidistributed if for any interval $[a, b] \subset [0, 1]$ we have

$$\lim_{n \rightarrow \infty} \frac{\#\{j; d_j \in [a, b], 0 \leq j \leq n-1\}}{n} = (b-a).$$

Show that if α is irrational then $\{n\alpha\}$ is equidistributed (here $\{x\}$ is the fractional part of x).

(b) Consider the first digit of 2^n , $n = 0, 1, 2, \dots$. Show that

$$\lim_{n \rightarrow \infty} \frac{\#\{j; \text{the first digit of } 2^j \text{ is } 7, 0 \leq j \leq n-1\}}{n} = \log_{10} \left(1 + \frac{1}{7}\right)$$

6. **Integral and induced automorphisms.**

(a) Let T be an automorphism of the (M, \mathcal{B}, μ) and let $A \in \mathcal{B}$ with $\mu(A) > 0$. We define the σ -algebra

$$\mathcal{B}_A = \{B \in \mathcal{B}, B \subset A\},$$

and the probability measure

$$\mu_A(B) = \frac{\mu(B)}{\mu(A)}, B \in \mathcal{B}_A.$$

Let $n_A(x)$ be the first return time

$$n_A(x) = \inf\{n \geq 1, T^n x \in A\}.$$

The map

$$T_A x = T^{n_A(x)} x$$

is called the *induced automorphism* by T and A . Show that T_A is an automorphism of $(A, \mathcal{B}_A, \mu_A)$ and that if T is ergodic then T_A is ergodic.

- (b) Let T be an automorphism of the (M, \mathcal{B}, μ) and let f be a positive, integer valued function in $L^1(\mu)$. We define a new measure space M^f , whose points have the form (x, i) , where $x \in M$ and $1 \leq i \leq f(x)$. A σ -algebra \mathcal{B}^f on M^f is constructed in an obvious way and a measure μ^f is defined as follows: for any subset of the form (A, i) , $A \in \mathcal{B}$, $A \in \mathcal{B}$ we set

$$\mu^f((A, i)) = \frac{\mu(A)}{\int_M f(x) d\mu},$$

and we define a map T^f by

$$T^f(x, i) = \begin{cases} (x, i+1) & \text{if } i+1 \leq f(x), \\ (Tx, 1) & \text{if } i+1 > f(x) \end{cases}.$$

The map T^f is called the *integral automorphism* for the automorphism T and the function f . One can visualize the space M^f as a "tower" whose foundation is M and which has $f(x)$ floors over the point $x \in M$. The maps T^f acts as follows, it lifts a point x up one floor if there is a floor above x and if this is not possible it maps x to Tx on ground floor.

- i. Show that T^f is an automorphism of $(M^f, \mathcal{B}^f, \mu^f)$.
- ii. Show that $T = (T^f)_M$, where M is identified with $(M, 1)$ (the ground floor).
- iii. Show that if T_A is the induced automorphism by T and A and $\bigcup_{n \geq 0} T^n A = M$ then T is the integral automorphism for T_A and the function $n_A(x)$, i.e. $T = (T_A)^{n_A}$.
- iv. Show that if T is ergodic then so is T^f .