MATH 646: Homework 3

1. Let T be an endomorphism of $(M, \mathcal{B}, \mu)$. Show that if T^2 is ergodic, then T is ergodic. Show that T is mixing if and only if T^2 is mixing. Find an example which shows that T ergodic does not imply T^2 ergodic.

2. If T be an endomorphism of $(M, \mathcal{B}, \mu)$ the product $T \times T$ is an endomorphism of $(M \times M, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$. Show that the following are equivalent

   (a) T is weak-mixing.

   (b) $T \times T$ is ergodic.

   (c) $T \times T$ is weak-mixing.

   Hint: Show (a) implies (c) implies (b). For (b) implies (a) use the that weak mixing is equivalent to

   $$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(\mu(T^{-j}A \cap B) - \mu(A)\mu(B)\right)^2 \,=\, 0\,.$$

   Show also that T is mixing if and only if $T \times T$ is mixing.

3. (a) Let T be an irrational rotation of the circle, $Tx = x+\alpha$. Consider the trajectories $(\cdots, T^{-1}x, x, Tx, \cdots)$. Out of every distinct trajectory choose a particular point and let E denote the the set thus obtained. Check that the set $E + n\alpha(\mathrm{nod}1)$ is disjoint of $E + m\alpha(\mathrm{mod}1)$ for $n \neq m$. Conclude that E can not be a measurable set.
   Hint: $[0, 1] = \cup_{n=-\infty}^{\infty} E + n\alpha$

   (b) Define the function f as a follows, for any x, $f(x)$ is the intersection of the trajectory through x with the set E. Show that f satisfy $f(x) = f(x + \alpha)$ and conclude that f is not a measurable function.

   Remark: This set was described first by Lebesgue himself.

4. Let $A \in SL(d, \mathbf{Z})$, i.e., A has integer coefficients and $|\det(A)| = 1$. The linear map Ax from $\mathbf{R}^d$ to $\mathbf{R}^d$ induces an automorphism T of the d torus $\mathbf{K}^d = \mathbf{R}^d/\mathbf{Z}^d$ given by

   $$Tx \,=\, Ax\,(\mathrm{mod}1)\,.$$

   Show that T is ergodic and mixing if and only if A is hyperbolic (no eigenvalue has modulus 1).

5. Let P be irreducible and aperiodic transition probabilities and let $\mu = \mu_{p,P}$ be the corresponding invariant Markov measure for the shift σ. For two complex-valued functions functions ϕ , ψ on M, we define the correlation function

   $$C(n) \,=\, \int \phi(\sigma^n(x))\psi(x)d\mu(x) - \int \phi(x)d\mu(x) \int \psi(x)d\mu(x)\,.$$

Mixing means that $\lim_{n\to\infty} C(n) = 0$ for all ϕ, $\psi \in L^2(\mu)$. Suppose that

$$\phi(x) = \phi_i(x_0)\,, \quad \psi(x) = \psi(x_0)$$

i.e. ϕ and ψ depend only on the coordinate x_0. Show that in this case $C(n)$ decays exponentially fast, i.e., there exists constants $A > 0$ and $K < 1$ such that

$$|C(n)| \le AK^n (\sup_x |\phi|)(\sup_x |\psi|)\,.$$

Hint: The function ϕ can be identified with a column vector. Use this to rewrite $C(n)$ using the transition probabilities P and the invariant measure p.

6. Show that the maps

$$Tx = \begin{cases} 2|x - \frac{1}{4}| & 0 \le x \le \frac{1}{2} \\ 1 - 2|x - \frac{3}{4}| & \frac{1}{2} \le x \le 1 \end{cases}$$

and

$$Sx = \begin{cases} 1 - 2|x - \frac{1}{4}| & 0 \le x \le \frac{1}{2} \\ 2|x - \frac{3}{4}| & \frac{1}{2} \le x \le 1 \end{cases}$$

preserve Lebesgue measure and are isomorphic to a Markov shift. Determine the corresponding Markov measures and determine if they are ergodic or mixing.

7. The map

$$Tx = \begin{cases} 1 - \frac{3}{2}x & 0 \le x \le \frac{2}{3} \\ 2(x - \frac{2}{3}) & \frac{2}{3} \le x \le 1 \end{cases}$$

preserves a measure μ absolutely continuous with respect to Lebesgue measure and is isomorphic to a Markov endomorphism. Determine μ and the corresponding Markov measure.

8. Show that the Baker transformation

$$T(x,y) = \begin{cases} \left(2x(\mathrm{mod}1)\,,\, \frac{1}{2}y\right) & 0 \le x \le \frac{1}{2} \\ \left(2x(\mathrm{mod}1)\,,\, \frac{1}{2}(y+1)\right) & \frac{1}{2} \le x \le 1 \end{cases}$$

is isomorphic to the Bernouilli automorphism on $\prod_{i=-\infty}^{\infty}\{0,1\}$ with $p_0 = p_1 = \frac{1}{2}$.

9. Consider the map $Tx = \frac{1}{2}(x - \frac{1}{x})$ on $\mathbf{R}$.

   (a) Show that T is an endomorphism of $(\mathbf{R}, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra and $d\mu = \frac{1}{\pi(1+x^2)}\,dx$.

   (b) Show that the change of variables $x = \tan(t)$ carries T to a (normalized) Lebesgue measure preserving map S of $(-\frac{\pi}{2}\,,\, -\frac{\pi}{2})$.

   (c) Show that S is conjugated to the Bernouilli endomorphism on $\prod_{i=0}^{\infty}\{0,1\}$ with $p_0 = p_1 = \frac{1}{2}$.

(d) Find a map that carries T to the map $Vx = 4x(1-x)$ on $[0,1]$ which preserves the measure $\frac{1}{\pi\sqrt{x(1-x)}}\,dx$.

10. Suppose there are two urns containing 100 balls alltogether numbered 1 to 100. There is also a hat which contains 100 slips of paper numbered 1 to 100. Once every second we draw a slip of paper of the hat, read the number on it and replace it in the hat. We move the ball bearing that number from whichever urn it is into the other urn.

    (a) Describe the transition probabilities P_{ij} for this Markov chain and find the invariant measure p_i.

    (b) Suppose that initially one urns contains 100 balls and the other is empty. Let $N(n)$ be the expected number of balls in the first urn after drawing n slips of papers of the hat. Show that $\lim_{n\to\infty} \frac{1}{n} \sum_j = 0^{n-1} N(j) = \frac{1}{2}$.

    (c) Suppose that the system is in equilibrium, i.e., its distribution is described the Markov measure $\mu_{p,P}$ and that you peak into the first urn and it is empty. How long should you expect to wait to see this same urn empty again?