Math 624: Problem set 4

1. Exercise 2, p.380
2. Exercise 4, p.381.
3. Exercise 8, p. 381
4. Suppose f is a convex function on an open interval I. Show that if $[a,b] \subset I$ then we have
$$f(b) - f(a) = \int_{[a,b]} f'(x+)\,dx = \int_{[a,b]} f'(x-)\,dx\,.$$
5. Prove that $L^\infty(X, \mathcal{M}, \mu)$ is a Banach space.
6. Suppose $1 \le p < q < r \le \infty$.
   (a) Show that $L^p \cap L^r$ is a Banach space with with norm $\|f\| \equiv \|f\|_p + \|f\|_r$ and that the inclusion $L^p \cap L^r \to L^q$ is a continuous map.
   (b) Show that $L^p + L^r$ is a Banach space with with norm $\|f\| \equiv \inf\{\|g\|_p + \|h\|_r\,;\, f = g + h\}$ and that the inclusion $L^q \to L^p + L^r \to L^q$ is a continuous map.
7. Let m be Lebesgue measure on $\mathbf{R}^d$.
   (a) Show that $L^p(\mathbf{R}^d, m)$ and l^p are separable if $1 \le p < \infty$.
   (b) Show that $L^\infty(\mathbf{R}^d, m)$ and l^∞ are *not* separable.
8. **Generalized Hölder's inequality.** Let $1 \le p_j \le \infty$ for $j = 1, \cdots, n$ and suppose
$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \le 1\,.$$
Show that if $f_j \in L^{p_j}$ then $\prod_{j=1}^n f_j \in L^r$ and
$$\left\| \prod_{j=1} f_{p_j} \right\|_r \le \prod_{j=1}^{n} \|f_j\|_{p_j}\,.$$
9. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f, g be nonnegative functions. Suppose that $0 < p < 1$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that
$$\int fg d\mu \ge \left(\int f^p\, d\mu \right)^{1/p} \left(\int g^q\, d\mu \right)^{1/q}\,.$$
Hint: use Hölder's inequality for suitable functions.

10. Let $(X, \mathcal{M}, \mu)$ and suppose that $f \in L^1(\mu) \cap L^2(\mu)$. Prove that

$$\lim_{p \to 1+} \|f\|_p = \|f\|_1 .$$

11. (Hölder's inequality should be called Roger's inequality or H"older-Roger's inequality). Let f, g be positive measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Let $0 < t < r < m < \infty$.

    (a) Show that if the integrals on the right are finite then the following holds (Roger's inequality)

    $$\left(\int f g^r \, d\mu \right)^{m-t} \leq \left(\int f g^t \, d\mu \right)^{m-r} \left(\int f g^m \, d\mu \right)^{r-t}$$

    Hint: Use Hölder's inequality.

    (b) Show conversely how the Hölder inequality follows from the Roger's inequality. *Hint:* let $t = 1$ amd $m = 2$.