Math 623: Problem set 2

1. Exercise 26, p.43
2. Suppose that A is a measurable set in $\mathbf{R}^d$ with $m(A) > 0$. Show that for any $q < m(A)$ there exist a measurable set $B \subset A$ with $m(B) = q$. *Hint:* Prove it first for the case that $m(A) = p < \infty$.
3. Exercise 28, p.43
4. Exercise 32, p.43
5. Exercise 33, p.43
6. Show that if $f : \mathbf{R}^d \to \mathbf{R}$ is measurable, then $|f|$ is measurable. Show that the converse is not always true.
7. Suppose $f : \mathbf{R}^d \to \mathbf{R}$ is finite-valued. Show that f is measurable if and only if $f^{-1}(A)$ is measurable for every Borel set A.
8. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable. Show that f and f' are measurable functions.
9. (a) Suppose $f : \mathbf{R} \to \mathbf{R}$ is a monotone function. Show that $f^{-1}(A)$ is a Borel set for every Borel set A. In particular f is measurable.

   (b) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a one to one continuous function. Show that f maps Borel sets onto Borel sets.
10. (a) Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ and a measurable set A such that $f(A)$ is not measurable.

    (b) Give an example of a function $g : \mathbf{R} \to \mathbf{R}$ and a measurable set A such that $g^{-1}(A)$ is not measurable.

    (c) Give an example of a measurable set such which is not a Borel set.

    (d) Give an example of a continuous function g and a measurable function h such that $h \circ g$ is not measurable.

    Hint: Let $F : [0, 1] \to [0, 1]$ be the Cantor Lebesgue function constructed in Exercise 2, chapter 1, and extend it to $\mathbf{R}$ by setting $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$. Finally consider

    $$f(x) = x + F(x)\,.$$

    Use 9(b) to show that if C is the middle third cantor set then $m(f(C)) = 1$and thus f maps a set of measure 0 onto a set of positive measure.

    Using this fact, problem 4 (Exercise 32 (b)), and problem 9 again, you can deduce (a), (b), (c), (d).