Math 697 Fall 2014: Week 2

Exercise 1 Consider a normal random variable Z with $\mu = 0$ and $\sigma = 1$. Use Chernov bounds to show that for any $a > 0$

$$P(Z \geq a) \leq e^{-a^2/2}$$

Exercise 2 Consider a Poisson random variable N with mean λ. Use a Chernov bound to show that for $n > \lambda$

$$P\{N > n\} \leq e^{-\lambda} \frac{(e\lambda)^n}{n^n}$$

Exercise 3 On a friday night you enter a BBQ restaurant which promises that every customer is served within a minute. Unfortunately there are 30 customers in line and you have a date in in 40 minutes. Being a probabilist you immediately realize that the waiting time for each customer is exponential is mean 1. Estimate the probability that you will miss your date if you wait in line until you are served using (a) Chebyshev inequality, (b) The central limit theorem, (c) Chernov bounds.

Exercise 4 Chernov bounds for Binomial random variables. In this problem we work out Chernov bounds for Bernoulli and put them in a useful form for applications. We let X_i, $i = 1, 2, \cdots$ be independent Bernoulli random variables with parameters p and let $S_n = X_1 + \cdots + X_n$ so that S_n is Binomial with parameters n and p.

1. Show that for $0 \leq z \leq 1$ we have

$$\begin{aligned} P(S_n \geq zn) &\leq e^{-n \sup_{t>0}\left[zt - \ln((1-p)+pe^t)\right]} \\ P(S_n \leq zn) &\leq e^{-n \sup_{t<0}\left[zt - \ln((1-p)+pe^t)\right]} \end{aligned}$$

2. Consider the optimization problem

$$F(z) = \sup_{t \in \mathbf{R}} \left[zt - \ln((1-p)+pe^t)\right]$$

for $0 \leq z \leq 1$. Show that

$$F(z) = z \log\left(\frac{z}{p}\right) + (1-z)\log\left(\frac{1-z}{1-p}\right) .$$

3. Show that for $z \geq p$ we have

$$\begin{aligned} P(S_n \geq zn) &\leq e^{-nF(z)} \quad \text{for} \quad z > p \\ P(S_n \leq zn) &\leq e^{-nF(z)} \quad \text{for} \quad z < p \end{aligned}$$

4. Show that $0 < z < 1$ we have

$$F(z) - 2(z-p)^2 \geq 0\,.$$

   Hint: Differentiate twice.

5. Show that for any $\epsilon > 0$

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \epsilon\right) \;\leq\; 2e^{-2n\epsilon^2}\,.$$

Exercise 5 Confidence intervals for Binomial random variables. Suppose you are generating 10'000 Bernoulli random variable X_i with unknown mean μ and compute the empirical mean $S_{10'000} = X_1 + \cdots + X_{10'000}$ and find $S_{10'000} = 6543$.

1. Build a 99% confidence interval for the unknown value of μ using (a) the central limit theorem. (b) The Chernov bounds in Exercise 4.5

2. Compare the two results and discuss the pro's and con's of each approach.

Exercise 6 (One-sided Chebyshev inequality) Consider a random variable with mean μ and variance σ^2. Using the Chebyshev inequality we have that for any $a > 0$.

$$P(X - \mu > a) \leq P(|X - \mu| > a) \leq \frac{\sigma^2}{a^2}$$

But one can do better. Show that

$$P(X - \mu > a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Hint: Consider first the case $\mu = 0$ and apply Markov inequality to the random variable $(X + b)^2$.