Math 645: Problem 6

1. Consider the functions $H(x, y)$ with

   (a) $H(x, y) = \frac{1}{3}x^3 - x + y^2$,

   (b) $H(x, y) = x^2 - y^2$,

   (c) $H(x, y) = y\sin(x)$.

   Sketch the phase portraits (on the same phase plane) for both the equations

   $$x' = -\frac{\partial H}{\partial x}\,, \quad y' = -\frac{\partial H}{\partial y}\,,$$

   and

   $$x' = \frac{\partial H}{\partial y}\,, \quad y' = -\frac{\partial H}{\partial x}\,.$$

   How are they related? Determine the critical points and their stability prope‘qrties.

2. Consider the Hamiltonian system with Hamiltonian $H(x, y) = y^2 + W(x)$ and assume that x_0 is an inflection point for $W(x)$. Sketch the corresponding phase diagram in a nieghborhood of $(x_0, 0)$ (it is called a "cusp").

3. Sketch the phase portrait of a system in the plane having

   (a) An orbit γ with $\alpha(\gamma) = \omega(\gamma) = \{x_0\}$ but $\gamma \neq \{x_0\}$.

   (b) An orbit γ where $\omega(\gamma)$ consists of one limit orbit.

   (c) An orbit γ where $\omega(\gamma)$ consists of one limit orbit and one critical point.

   (d) An orbit γ where $\omega(\gamma)$ consists of two limit orbits and one critical point.

   (e) An orbit γ where $\omega(\gamma)$ consists of two limit orbits and two critical points.

   (f) An orbit γ where $\omega(\gamma)$ consists of four limit orbits and four critical points.

4. The system given in cylindrical coordinates by

   $$\begin{aligned} r' &= r(1-r)\,, \\ \theta' &= 1\,, \\ z' &= -z\,. \end{aligned} \tag{1}$$

   has exactly one periodic orbit. Determine this periodic orbit and compute the Poincaré map for the half-plane $y = 0, x > 0$ perpendicular to the periodic orbit. Show that this orbit is asymptotically stable.

5. Let $x_p(t)$ be a periodic solution of period p for the system $x' = f(x)$ with $x \in \mathbf{R}^2$. Show that $x_p(t)$ is a stable (resp. unstable) limit cycle if $\int_0^p \nabla \cdot f(x_p(t)) < 0$ (resp. > 0).

6. Show that $(2\cos(2t), \sin(2t))^T$ is periodic orbit for the system

$$\begin{aligned} x' &= -4y + x\left(1 - x^2/4 - y^2\right), \\ y' &= x + y\left(1 - x^2/4 - y^2\right), \end{aligned} \tag{2}$$

and show that it is stable. *Hint: Use the previous problem.*

7. Consider the system given, in polar coordinates, by

$$\begin{aligned} r' &= ar + r^3 - r^5, \\ \theta' &= 1. \end{aligned} \tag{3}$$

Determine the phase plane for representative values of a and describe the bifurcations of the systems.

8. Consider the system

$$\begin{aligned} x' &= x - rx - ry + xy, \\ y' &= y + rx - ry - x^2, \end{aligned} \tag{4}$$

where $r = \sqrt{x^2 + y^2}$. Show that this system can be written in polar coordinates as

$$\begin{aligned} r' &= r(1 - r), \\ \theta' &= r(1 - \cos\theta), \end{aligned} \tag{5}$$

Show that there are two critical points $(0, 0)$ (unstable source) and $(1, 0)$ (saddle node). Use this information and Poincaré-Bendixson Theorem to show that every solution $x(t)$ which does not pass through the origin satisfy $\lim_{t\to\infty} x(t) = (1, 0)$, but that $(1, 0)$ is not stable.

9. Consider the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2)\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \\ y' &= x + y(1 - x^2 - y^2)\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right). \end{aligned} \tag{6}$$

Show that this system has infinitely many periodic orbits (limit cycles). Determine which ones are stable.

10. For the system $x' = f(y)$, $y' = g(x) + y^k$, give a sufficent condition for the system to have no periodic orbit.

11. Consider the equation

$$x'' + (x^2 + x'^2 - 1)x' + x = 0. \tag{7}$$

Show that this system has a unique periodic orbit which is a stable limit cycle for every trajectory, except the one starting at the origin.