Math 597/697: Solution 5

1. The transition between the the different states is governed by the transition matrix

$$P = \begin{pmatrix} 0 & .1 & .6 & .3 \\ .6 & 0 & .2 & .2 \\ .4 & .1 & 0 & .5 \\ .5 & .1 & .4 & 0 \end{pmatrix}, \tag{1}$$

and $v_0 = 1/4$, $v_1 = 5$, $v_2 = 1/3$, $v_3 = 1/5$. Hence the generator is given by

$$A = \begin{pmatrix} -\frac{1}{4} & \frac{1}{40} & \frac{3}{20} & \frac{3}{40} \\ 3 & -5 & 1 & 1 \\ \frac{2}{15} & \frac{1}{30} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{10} & \frac{1}{50} & \frac{2}{25} & -\frac{1}{5} \end{pmatrix}. \tag{2}$$

2. The state space consists of all pairs (m, n) of nonnegative integers. The time until a given pair of male and female produces an offspring is an exponential R.V. with parameter λ. If the population at time consists of m males and n females there are nm possible pairs and hence we obtain

$$v_{(m,n)} = \lambda mn, \tag{3}$$

$$q_{(m,n)(m+1,n)} = p\lambda mn, \quad q_{(m,n)(m,n+1)} = (1-p)\lambda mn \tag{4}$$

$$P_{(m,n)(m+1,n)} = p, \quad P_{(m,n)(m,n+1)} = (1-p) \tag{5}$$

3. (a) The number of computers X_t in operating condition at time t is either $0, 1, 2$ and the number of computers operating is either 0 if $X_t = 0$ or 1 if $X_t = 1$ or 2 . The rate of failure of the operating computer is μ and the rate of repair is λ. There is only one repair facility so that at most one computer can be repaired at a time. We find

$$v_0 = \lambda, \quad v_1 = \lambda + \mu, \quad v_2 = \mu. \tag{6}$$

and

$$P_{01} = 1, \quad P_{10} = \frac{\mu}{\lambda+\mu}, \quad P_{12} = \frac{\lambda}{\lambda+\mu}, \quad P_{21} = 1. \tag{7}$$

Hence the generator is

$$A = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{pmatrix}. \tag{8}$$

The backward equations are, for $j = 0, 1, 2$,

$$\begin{aligned} \frac{d}{dt}P_{0j} &= -\lambda(P_{0j} - P_{1j})\,, && (9) \\ \frac{d}{dt}P_{1j} &= \mu(P_{0j} - P_{1j}) - \lambda(P_{1j} - P_{2j})\,, && (10) \\ \frac{d}{dt}P_{2j} &= \mu(P_{1j} - P_{2j})\,, && (11) \end{aligned}$$

and the forward equations are

$$\begin{aligned} \frac{d}{dt}P_{j0} &= -\lambda P_{j0} + \mu P_{j1}\,, && (12) \\ \frac{d}{dt}P_{j1} &= \lambda P_{j0} - (\lambda+\mu)P_{j1} + \mu P_{j2}\,, && (13) \\ \frac{d}{dt}P_{j2} &= \lambda P_{j1} - \mu P_{j2} && (14) \end{aligned}$$

The stationary distribution is

$$\pi_j = \frac{(\lambda/\mu)^j}{1 + (\lambda/\mu) + (\lambda/\mu)^2} \tag{15}$$

It is also a limiting distribution since the state space is finite and the chain is irreducible. In the long run, the proportion of time when the system is operating (i.e. in state 1 or 2) is

$$\frac{(\lambda/\mu) + (\lambda/\mu)^2}{1 + (\lambda/\mu) + (\lambda/\mu)^2} \tag{16}$$

(b) The difference with (a) is that the two computers can be in operation at the same time and so the rate of failures when $X_t = 2$ is twice the rate of (a).

$$v_0 = \lambda\,, \quad v_1 = \lambda + \mu\,, \quad v_2 = 2\mu\,. \tag{17}$$

Then

$$A = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}, \tag{18}$$

and the stationary distribution is in that case

$$\pi_j = \frac{\frac{1}{j!}(\lambda/\mu)^j}{1 + (\lambda/\mu) + \frac{1}{2}(\lambda/\mu)^2} \tag{19}$$

and so, in the long run, the proportion of time when the system is operating (i.e. in state 1 or 2) is

$$\frac{(\lambda/\mu) + \frac{1}{2}(\lambda/\mu)^2}{1 + (\lambda/\mu) + \frac{1}{2}(\lambda/\mu)^2} \tag{20}$$

(c) Let us consider first the case (a). We compute the expected time to reach 0 starting from 2. We write $T = T_{21} + T_{10}$ where T_{21} is the first time the chain reach state 1 stating from state 2 and T_{10} is the first time the chain reach state 0 starting from 1. Clearly T_{21} is an exponential R.V. with mean $\frac{1}{\mu}$. To compute $E[T_{10}]$ we condition on the event A where the first transition is to 0 (probability $\mu/(\lambda+\mu)$ or the event B where the first transition is to 2 (probability $\lambda/(\lambda+\mu)$. Since the first transition occurs in average after a time $1/(\lambda+\mu)$ we find

$$E[T_{10}|A] = \frac{1}{\lambda+\mu}\,, \tag{21}$$

$$E[T_{10}|B] = \frac{1}{\lambda+\mu} + E[T_{21}] + E[T_{10}] \tag{22}$$

and so,

$$E[T_{10}] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}(E[T_{21}] + E[T_{10}]) \tag{23}$$

$$= \frac{1}{\lambda+\mu}(1+\frac{\lambda}{\mu}) + \frac{\lambda}{\lambda+\mu}E[T_{10}])\,. \tag{24}$$

Solving gives $E[T_{10}] = \frac{1+\frac{\lambda}{\mu}}{\mu}$ and $E[T] = \frac{2+\frac{\lambda}{\mu}}{\mu}$. In case (b) one finds $\frac{\frac{3}{2}+\frac{\lambda}{2\mu}}{\mu}$ instead.

4. Each component is an on/off system with generator

$$\begin{pmatrix} -\lambda_i & \lambda_i \\ \mu_i & -\mu_i \end{pmatrix}. \tag{25}$$

For each component the stationary and limiting distribution is for $i = A, B$,

$$\pi_0^{(i)} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad \pi_1^{(i)} = \frac{\lambda_i}{\lambda_i + \mu_i}. \tag{26}$$

By independence the stationary distribution of the two component-system is $\pi_i^{(A)}\pi_j^{(B)}$.

(a) If the components are working in parallel the is operating if at least of the components is in state 0, hence the desired probability is

$$1 - \pi_1^{(A)}\pi_1^{(B)} = 1 - \frac{\lambda_A\lambda_B}{(\lambda_A + \mu_A)(\lambda_B + \mu_B)}. \tag{27}$$

(b) If the components are working in series, both must work for the system to be operating, hence the desired probability is

$$\pi_0^{(A)}\pi_0^{(B)} = \frac{\mu_A\mu_B}{(\lambda_A + \mu_A)(\lambda_B + \mu_B)}. \tag{28}$$

5. Consider a continuous time branching process defined as follows. A organism lifetime is exponential with parameter λ and upon death, it leaves k offspring with probability p_k, $k \geq 0$. The organisms act independently of each other. We assume that $p_1 = 0$ so that, at death, there are no transition from a state into itself. Let X_t be the population at time t. A transition occurs when an organism dies and if there are n organisms the average time until a death is $1/n\lambda$ and so $v_n = n\lambda$. When an organism dies it is replaced by $0, 2, 3, 4, \ldots$ organisms and therefore we have

$$P_{nn-1} = p_0 \quad P_{nn+1} = p_2, \quad \ldots \quad P_{nn+k} = p_{k+1} \tag{29}$$

Therefore the generator is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ \lambda p_0 & -\lambda & \lambda p_2 & \lambda p_3 & \ldots \\ 0 & 2\lambda p_0 & -2\lambda & 2\lambda p_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{30}$$

The backward equations are

$$\frac{d}{dt}P_{0j} = 0, \tag{31}$$

$$
\begin{aligned}
\frac{d}{dt}P_{1j} &= \lambda\left(p_0P_{0j}+p_2P_{2j}+p_3P_{3j}+\ldots-P_{1j}\right), && (32)\\
\vdots \quad & \quad \vdots && (33)\\
\frac{d}{dt}P_{nj} &= n\lambda\left(p_0P_{n-1j}+p_2P_{n+1j}+p_3P_{n+2j}+\ldots-P_{nj}\right) && (34)
\end{aligned}
$$

and the forward equations are

$$
\begin{aligned}
\frac{d}{dt}P_{j0} &= \lambda p_0 P_{j1}\,, && (35)\\
\frac{d}{dt}P_{j1} &= -\lambda P_{j1}+2\lambda p_0 P_{j2}\,, && (36)\\
\frac{d}{dt}P_{j2} &= \lambda p_2 P_{j1}+-2\lambda P_{j2}+3\lambda p_0 P_{j3}\,, && (37)\\
\frac{d}{dt}P_{j3} &= \lambda p_3 P_{j1}+2\lambda p_2 P_{j2}-3\lambda P_{j3}+4\lambda p_0\lambda P_{j3}\,, && (38)\\
\vdots \quad & \quad \vdots && (39)
\end{aligned}
$$

In case of binary splitting we have if the probability of death without offspring is p

$$
A = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ \lambda p & -\lambda & \lambda(1-p) & 0 & \ldots \\ 0 & 2\lambda p & -2\lambda & 2\lambda(1-p) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{40}
$$

As ones sees from the matrix A the Markov chain is not irreducible, 0 is an absorbing state and all other states are transient. The distribution $\pi = (1, 0, 0, 0, \ldots)$ is a stationary state, which describe the situation where nobody lives. Note that the transition matrix P_{ij} is given by a random walk, so one might expect that if $p < 1/2$ every initial distribution will converge to π while if $p \geq 1/2$ there will be a positive probability to escape to infinity.

6. Consider the $M/M/\infty$ queue with arrival rate λ and service rate μ.

   (a) The $M/M/\infty$ queue is a Birth and Death process with $\lambda_n = \lambda$

and $\mu_n = n\mu$. Hence the generator is

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -\lambda-\mu & \lambda & 0 & 0 & \dots \\ 0 & 2\mu & -\lambda-2\mu & \lambda & 0 & \dots \\ 0 & 0 & 3\mu & -\lambda-3\mu & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{41}$$

(b) Using the general solution for the stationary and limiting distribution for a Birth and Death process (the $M/M/\infty$ is irreducible) one finds

$$1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\dots\lambda_0}{\mu_n\dots\mu_1} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!\mu^n} = e^{(\lambda/\mu)} \tag{42}$$

and so

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-(\lambda/\mu)}. \tag{43}$$

The $M/M/\infty$ is positive recurrent and the stationary distribution is Poisson with parameter λ/μ.

(c) Since

$$P\{X_{t+h} = n+1 \,|\, X_t = n\} = \lambda h + o(h), \tag{44}$$
$$P\{X_{t+h} = n-1 \,|\, X_t = n\} = n\mu h + o(h), \tag{45}$$
$$P\{X_{t+h} = n \,|\, X_t = n\} = (1 - \lambda h - n\mu h) + o(h), \tag{46}$$

we have

$$E\left[X_{t+h} \,|\, X_t = n\right] \tag{47}$$
$$= (n+1)\lambda h + (n-1)n\mu h + n(1-\lambda h - n\mu h) + o(h) \tag{48}$$
$$= \lambda h - n\mu h + n + o(h). \tag{49}$$

and

$$E\left[X_{t+h}\right] = \lambda h - \mu h E\left[X_t\right] + E\left[X_t\right] + o(h) \tag{50}$$

Therefore

$$\frac{d}{dt}E\left[X_t\right] = \lambda - \mu E\left[X_t\right]. \tag{51}$$

and

$$E\left[X_t\right] = E\left[X_0\right] e^{-\mu t} + \frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right). \tag{52}$$

7. Consider a queuing system with one single server, arrival rate λ and serving rate μ.

   (a) When N customers are in the system, the arriving customers give up and do no enter the system. So the state space is $\{0, 1, \dots, N\}$, the generator is

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & \dots & 0 \\ \mu & -\lambda-\mu & \lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \mu & -\lambda-\mu & \lambda \\ 0 & \dots & \dots & 0 & \mu & -\mu \end{pmatrix}. \tag{53}$$

   The limiting distribution can be found using the general formula for the birth and death process with $\lambda_n = 0$ for $n \geq N$. One finds for $0 \leq j \leq N$

$$\pi_j = \frac{(\lambda/\mu)^j}{1 + \sum_{k=1}^N (\lambda/\mu)^k} = \begin{cases} \frac{(\lambda/\mu)^j (1-(\lambda/\mu))}{1-(\lambda/\mu)^{N+1}} & \lambda \neq \mu \\ \frac{1}{N+1} & \lambda = \mu \end{cases}. \tag{54}$$

   (b) When n customers are in the system, an arriving customer will join the system with probability $1/(n+1)$. We have $\mu_n = \mu$ for $n \geq 1$ and $\lambda_n = \lambda/(n+1)$ for $0n \geq 0$. Then

$$1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \dots \lambda_0}{\mu_n \dots \mu_1} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n! \mu^n} = e^{(\lambda/\mu)} \tag{55}$$

   and the stationary distribution is Poisson with parameter λ/μ.

8. Consider a Yule process (pure birth with linear growth) and let T_i be the time it takes for a population of size i to reach $i+1$.

   (a) Since it is a pure birth process T_i is exponential with rate $i\lambda$.

   (b) Let us suppose we have n components with i.i.d exponential lifetime X_i. The R.V. $\max(X_1, \dots, X_n)$ is the time it takes for the n components to fail. Starting with n components, the time until the first failure is the minimum of n exponential R.V. with parameter λ which is an exponential R.V. T_n with parameter $n\lambda$. By the memoryless property of the exponential, the time between the first failure and the second failure is, again, the minimum of

$n-1$ exponential R.V. with parameter λ which is is an exponential R.V. T_{n-1} with parameter $(n-1)\lambda$. Hence

$$\max(X_1, \ldots, X_n) = T_n + T_{n-1} + \ldots + T_1 \tag{56}$$

(c) We have, by (b), and independence

$$\begin{aligned} P\{T_1 + T_2 + \ldots T_n < t\} &= P\{\max(X_1, \ldots, X_n) < t\} \\ &= P\{X_1 < t\} \ldots P\{X_n < t\} \\ &= (1 - e^{-\lambda t})^n \end{aligned} \tag{57}$$

(d) Since X_t is a pure birth process the Markov chain X_t only makes transition from j to $j+1$. So $T_1 + \ldots + T_{n-1}$ is the time necessary to reach n starting from 1 and $P\{T_1 + \ldots + T_{n-1} < t\}$ is the probability that $X_t \geq n$

$$\begin{aligned} P_{1n}(t) &= P\{X_t = n \,|X_0 = 1\} \\ &= P\{X_t \geq n \,|X_0 = 1\} - P\{X_t \geq n+1 \,|X_0 = 1\} \\ &= P\{T_1 + \ldots + T_{n-1} < t\} - P\{T_1 + \ldots + T_n < t\} \\ &= (1 - e^{-\lambda t})^{n-1} - (1 - e^{-\lambda t})^n \\ &= e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \end{aligned} \tag{58}$$

Hence $P\{X_t = n \,|\, X_0 = 1\}$ has a geometric distribution with parameter $e^{-\lambda t}$.

(e) By (d) X_t conditioned on $\{X_0 = 1\}$ is a geometric random variable. By definition of a Birth process, the individuals reproduce independently of each other, and therefore X_t conditioned on $\{X_0 = m\}$ is the sum of m geometric random variables.

The sum of m geometric R.V. with parameters p is called a negative binomial R.V with parameters (m, p) . A geometric R.V. is the number of independent trials necessary to obtain one success, so that a negative binomial R.V is the number of trials necessary to obtain exactly m successes. The probability distribution of the negative binomial R.V. is, for $k \geq m$, (at least m trials are necessary)

$$p(k) = \binom{k-1}{m-1} p^m (1-p)^{k-m} \,. \tag{59}$$

The combinatorial factor is obtained by noting that the last trial must be a success and by counting the number of ways of having the remaining $m-1$ successes.

Therefore we obtain, for $n \geq m$

$$P_{mn}(t) \;=\; \binom{n-1}{m-1}(e^{-\lambda t})^m(1-e^{-\lambda t})^{n-m} \tag{60}$$