Math 645: Problem 5

1. Study the stability of the critical points of the equation

$$\begin{aligned} x_1' &= (x_1 - x_2)(1 - x_1 - x_2)/3 \,, \\ x_2' &= x_1(2 - x_2) \,. \end{aligned} \tag{1}$$

2. Consider the FitzHugh-Nagumo equation

$$\begin{aligned} x_1' &= f_1(x_1, x_2) = g(x_1) - x_2 \,, \\ x_2' &= f_2(x_1, x_2) = \sigma x_1 - \gamma x_2 \,, \end{aligned} \tag{2}$$

   where σ and γ are positive constants and the function g is given by $g(x) = -x(x - 1/2)(x - 1)$. Show that as the ratio σ/γ decreases the system undergoes a bifurcation from one equilibrium state to three equilibrium states. Compute the critical points and determine their stability properties. Some of the computations are lengthy and you might want to use a geometric argument to determine stability: just look at the directions of the vector field! It is also good idea to make a graph of the orbits before and after the bifurcation (use matlab or mathematica).

3. Let $x, y \in \mathbf{R}^n$ and consider the Hamiltonian $H(x, y) = \sum_{i=1}^n \frac{y_i^2}{2} + W(x)$ where $W(x)$ is of class $\mathcal{C}^2$. Assume that a is a nondegenerate critical point of W, i.e.

$$\nabla W(a) = 0 \qquad \text{and} \qquad \det\left(\frac{\partial^2 W}{\partial q_i \partial q_j}(a)\right) \neq 0 \,. \tag{3}$$

   and consider the Hamiltonian equation

$$x'' = -\nabla W(x) \,. \tag{4}$$

   Using *linearization* show that $(a, 0)$ is an unstable critical point if a is local maximum or a saddle point of W. *Hint:* Study the eigenvalues!

4. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be of class $\mathcal{C}^1$. Assume that the solutions of $x' = f(x)$ exists for all $t \in \mathbf{R}$ and denote by ϕ^t the corresponding flow $\phi^t(x) = x(t, 0, x)$.

   (a) Prove Liouville Theorem

$$\det\left(\frac{\partial \phi^t}{\partial x}\right) = \exp\left(\int_0^t \mathrm{div} f(\phi^s(x))\, ds\right) \,. \tag{5}$$

   where $\mathrm{div} f(x) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x)$. Hint: Use Liouville theorem for linear ODE and the variational equation.

   (b) Show that ϕ^t is volume preserving if and only if $\mathrm{div} f = 0$. *Hint*: A map $T : \mathbf{R}^n \to \mathbf{R}^n$ is volume preserving if $\mathrm{vol}(T(A)) = \mathrm{vol}(A)$ for all sets A compact for which ∂A is negligible (for Rieman integral) or Lebesgue measurable (if you prefer Lebesgue integral).

5. Consider the system of equations

$$\begin{aligned} x_1' &= -x_1 \,, \\ x_2' &= -x_2 + x_1^2 \,, \\ x_3' &= x_3 + x_2^2 \,. \end{aligned} \tag{6}$$

Compute the first four approximations $u^{(j)}(t,a)$ for the functions defining the stable manifold. Show that $u^{(3)}(t,a) = u^{(4)}(t,a)$ and thus $u(t,a) = u^{(3)}(t,a)$. Determine then the stable and unstable manifolds W^s and W^u.

6. Consider the equation (see the example in class)

$$\begin{aligned} x' &= x^2 \,, \\ y' &= -y \,. \end{aligned} \tag{7}$$

Show that for any $c \in \mathbf{R}$, the function

$$h_c(x) = \begin{cases} ce^{1/x} & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x \geq 0 \end{cases} \tag{8}$$

determines a center manifold for this system. Graph $h_c(x)$ for various c.

7. Using center manifolds, determine the qualitative behavior near the origin for the equation

$$\begin{aligned} x' &= xy \,, \\ y' &= -y - x^2 \,. \end{aligned} \tag{9}$$

8. Consider the equation

$$x' = A(t)x + g(t,x) \,, \tag{10}$$

where $A(t)$ is continuous and periodic of period p and g is continuous, satisfy a local Lipschitz condition, and

$$\lim_{\|x\| \to 0} \sup_{t > t_0} \frac{\|g(t,x)\|}{\|x\|} = 0 \tag{11}$$

Let R be the matrix given in Floquet Theorem. Show that 0 is stable if all the negative eigenvalues of R have negative real part and is unstable if at least one eigenvalue of R has positive real part. *Hint:* Consider the change of variables $x = P(t)y$ where $P(t)$ is the periodic matrix given in Floquet Theorem.

9. Determine the stability of the $(0,0)$ solution of

$$\begin{aligned} x_1' &= x_1 x_2^2 - 2x_2 \,, \\ x_2' &= x_1 - x_1^2 x_2 \,. \end{aligned}$$

10. Determine the stability of the $(0,0)$ solution of

$$\begin{aligned} x' &= -x + y + xy\,, \\ y' &= x - y - x^3 - y^3\,. \end{aligned}$$

11. Consider the equation

$$x' = Ax + f(x)\,, \tag{12}$$

where f is locally Lipschitz and satisfy the condition

$$\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0\,. \tag{13}$$

Assume that A is diagonalizable and that all its eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ are real. Show directly by using Liapunov functions, that

(a) If all the eigenvalues of A are negative, then 0 is asymptotically stable.

(b) If, for $0 < p < n$, $\lambda_1 \leq \cdots \lambda_p < 0$ and $0 < \lambda_{p+1}, \cdots < \lambda_n$ then 0 is unstable.

12. Consider the system

$$\begin{aligned} x' &= 2y(z-1)\,, \\ y' &= -x(z-1) \\ z' &= xy\,. \end{aligned} \tag{14}$$

(a) Show that $(0,0,1)$ is stable.

(b) Is it asymptotically stable?