Math 597/697: Homework 2

You are encouraged to work in groups. But please write your solutions yourself. If you have questions about the homework, please ask them in class, the other students will profit from them too.

1. Consider a markov chain with state space $\{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix} . \tag{1}$$

   Compute the following probabilities

   (a) If the chain starts in state 0 what is the probability that it is in state 1 at time 3?

   (b) If the chain starts in state 0 what is the probability that it is in state 1 at times 1 and 3?

   (c) At time 0 the chain is in state 0 with probability 1/4 and in state 1 with probability 3/4. What is the probability distribution of X_2?

2. Consider a Markov chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} .4 & .2 & .4 \\ .6 & 0 & .4 \\ .2 & .5 & .3 \end{pmatrix} . \tag{2}$$

   Compute a stationary distribution for this markov chain.

3. Consider a Markov chain with state space $\{0, \cdots, 5\}$ and transition matrix

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix} . \tag{3}$$

   What are the communication classes. Which ones are recurrent and which one are transient?

4. *The Ehrenfest urn model.* This is a model of a diffusion of molecules through a membrane. Imagine two containers, labeled A and B, contain a total of $2a$ balls. At each time unit a ball is selected at random from the totality of $2a$ balls and moved to the other container (think a molecule passing through the membrane). Each such selection generates a transition of the process. Clearly one would expect the balls to fluctuate with a averga drift from the urn with more balls towards the urn with less balls. Let Y_n denote the number of balls in urn A at time n and let us define $X_n = Y_n - a$.

   (a) What is the state space of X_n?.
   (b) Write down the transition probability matrix.
   (c) Give the communicability classes of the process. Are they recurrent? transient?

5. *An inventory model.* This models a situation where some commodity is stocked in order to satisfy a continuous demand. The replenishment of the stocks takes place at the end of periods denoted by $n = 0, 1, 2, \cdots$.

   During period n the total demand for a commodity is a random variable ξ_n:
$$P\{\xi_n = k\} \;=\; a_k \quad k = 0, 1, 2, 3, \cdots \tag{4}$$
   with $a_{k \geq 0}$ and $\sum_{k=0}^{\infty} a_k = 1$. One assumes that the random variables ξ_n are i.i.d.

   The strategy to replenish the stocks is specified by the two numbers s and S with $s < S$. If, at the end of the period the stock, the stock is no greater than s, then the stock is replenished to reach S. If it is greater than s, then no action is taken.

   Let X_n denote the number of items in stocks at the end of the n-th period. The state of the process X_n is taken to be $S, S-1, \cdots, +1, 0, -1, -2, \cdots$ where a negative value is to be interpreted as an unfilled demand wich is satisfied immediately upon restocking.

   (a) Express X_{n+1} in terms of X_n, ξ_n, S, and s. and give the transition probabilities
   (b) Suppose that $s = 0$ and $S = 2$ and
$$P\{\xi_n = 0\} \;=\; .5 \quad P\{\xi_n = 1\} \;=\; .4 \quad P\{\xi_n = 2\} \;=\; .1\,. \tag{5}$$
   In this case what are the possible values of X_n? Write down the transition matrix.

Maybe you want to do (b) before (a)

6. The weather is in two states: rainy or not rainy and it depends on the weather in the previous two days. If it was rainy yesterday and and the day before yesterday it will be rainy today with probability 0.8. If it was not rainy yesterday and the day before yesterday it will be rainy today with probability 0.2. If it was rainy yesterday but not rainy the day before yesterday, then the weather will be rainy today with probability 0.5. If it was not rainy yesterday but rainy the day before yesterday, then the weather will be rainy today with probability 0.4.

   (a) Explain why it is not reasonable to model the weather on a single day as a markov chain.

   (b) Show that one can construct a markov chain by taking as a state the weather in two consecutive days. Write the corresponding transition probabilities (it is a 4×4 matrix).

   (c) Compute the probability that it is going to rain tomorrow given that it rained yesterday but not the day before yesterday

7. Let i be a state of Markov chain. We define the following random variable

$$\tau_i = \min\{n \geq 1, X_n = i\} \tag{6}$$

   This is called a *hitting time*, τ_i is the first time that the Markov chain reaches the state i.

   (a) Show that recurrence and transience of a state can be expressed in terms of τ_i as follows:

$$i \text{ is recurrent if and only if } P\{\tau_i < \infty | X_0 = i\} = 1 \tag{7}$$

$$i \text{ is transient if and only if } P\{\tau_i = \infty | X_0 = i\} > 0 \tag{8}$$

   (b) Consider the Markov chain with transition probabilities

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ .25 & .25 & .25 & .25 \end{pmatrix}. \tag{9}$$

   Compute $P\{\tau_i = n | X_0 = i\}$ for $i = 0, 1, 2, 3$. (Don't forget $n = \infty$).

8. This homework should be done with the help of a computer, with maple, mathematica, or whatever program you prefer. It will illustrate the various long term behavior of Markov chains.

   Students who are not familiar with such programs are strongly encouraged to work with one of your colleagues who is not. And conversely.

   For the following four transition matrices you should computer P^n to a sufficiently high power n and interpret the results. Does $\lim_{n\to\infty} P^n_{ij}$ exists? Is $\lim_{n\to\infty} P_n^{ij} = \pi_j$ independent of of i? Interpret your results in terms of recurrence, transience, periodicity, irreducibility, invariant distributions, etc..

   (a)

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/8 & 2/3 & 5/24 \\ 0 & 1/6 & 5/6 \end{pmatrix}. \tag{10}$$

   (b) Random walk on $\{0,1,2,3,4\}$ with reflecting boundary conditions (Why the name?)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{11}$$

   (c) Random walk on $\{0,1,2,3,4\}$ with absorbing boundary conditions (Why the name?)

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

   (d)

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 1/8 & 2/3 & 5/24 \\ 0 & 0 & 0 & 1/6 & 5/6 \end{pmatrix}. \tag{13}$$