Math 697: Fall 2007 Final Exam

You can use your class-notes and the textbook but nothing else. Don't talk to each other, talk to me.

Exercise 1 (Discrete-time queueing model). Let us consider a Markov chain describing, for example, the following car inspection station. The inspector enters the inspection station hourly (for example at 8:00, 9:00, and so on). If there are some cars waiting he takes the first one and inspects it, the inspection lasting exactly one hour. If there are no cars in line the inspector leaves and comes back one hour later. The inflow of cars into the inspection station during a one-hour period is given by a random variable ξ. For $j = 0, 1, 2, \cdots$ we have

$$p_j \equiv P\{\xi = j\} \equiv P\{\text{ j cars enter the inspection station during a given hour}\}$$

with $p_j \ge 0$ and $\sum_{j=0}^{\infty} p_j = 1$

1. Let ξ_n, $n = 1, 2, \cdots$ be i.i.d. random variables which have the same distribution as ξ. Show that the length of the line at the inspection station, X_n, measured hourly is a Markov chain given by $X_{n+1} = \xi_{n+1}$ if $X_n = 0$ and $X_{n+1} = X_n + \xi_{n+1} - 1$ if $X_n \ge 1$. Show also that the transition probabilities are given by

$$P_{0j} = p_j \quad \text{and} \quad P_{ij} = p_{j-i+1} \;\text{ for } i \ge 1\,, j \ge i-1\,. \tag{1}$$

2. Show that if either $p_0 = 0$ or $p_0 + p_1 = 1$ the chain is not irreducible.

3. Show that if $p_0 > 0$ and $p_0 + p_1 < 1$ the chain is irreducible.

4. In the case where the Markov chain is not irreducible determine which states are absorbing, recurrent, transient. Distinguish the cases (i) $p_1 = 1$, (ii) $p_0 > 0$, $p_1 > 0$, and $p_0 + p_1 = 1$, (iii) $p_0 = 1$, (iv) $p_0 = 0$, $p_1 < 1$.

5. We denote by $\sigma_j = \inf\{n > 0, x_n = j\}$ be the first return time to the state j and we set $\rho_{ij} = P(\sigma_j < \infty \,|\, X_0 = i)$. Show the following identities which hold for general Markov chains:

$$\begin{aligned}
P\{\sigma_j = n+1 \,|\, X_0 = i\} &= \sum_{k \ne j} P_{ik} P\{\sigma_j = n \,|\, X_0 = k\}\,, \quad n \ge 1\,, \\
P\{\sigma_j \le n+1 \,|\, X_0 = i\} &= P_{ij} + \sum_{k \ne j} P_{ik} P\{\sigma_j \le n \,|\, X_0 = k\}\,, \quad n \ge 0\,, \\
\rho_{ij} &= P_{ij} + \sum_{k \ne j} P_{ik} \rho_{kj}
\end{aligned}$$

6. For the queueing Markov chain with transition probabilities (1) show that the identity

$$\rho_{00} = \rho_{10}$$

holds and show also that

$$\rho_{ii-1} = \rho_{i-1i-2} = \cdots = \rho_{10}\,,$$

i.e. that ρ_{ii-1} is independent of i.

7. Show that for the queueing chain

$$P\{\sigma_0 = m \mid X_0 = i\} = \sum_{k=1}^{m-1} P\{\sigma_{i-1} = k \mid X_0 = i\} P\{\sigma_0 = m-k \mid X_0 = i-1\}\,,$$

and deduce from this that

$$\rho_{i0} = \rho_{ii-1}\rho_{i-10}\,, \quad i \geq 2\,.$$

8. Conclude that ρ_{00} is a solution of the equation

$$\rho = \phi(\rho)$$

where $\phi(s)$ is the moment generating function of ξ, i.e $\phi(\xi) = E[s^\xi] = \sum_{n\geq 0} p_n s^n$.

As we have seen in the chapter on branching processes the equation $\rho = \phi(\rho)$ has always $\rho = 1$ as a solution. If $\mu = E[\xi] \leq 1$ then 1 is the only solution while if $\mu > 1$ then there exists another solution $0 < \rho < 1$ if $p_0 > 0$. As for branching processes (with the a very similar proof which you **do not** need to repeat here) one shows that the desired probability ρ_{00} is the smallest solution of $\rho = \Phi(\rho)$ and thus

$$X_n \text{ recurrent} \quad \text{iff} \quad \mu \leq 1\,.$$

Exercise 2 Consider a Yule process, i.e., a birth/death process with $\lambda_n = n\lambda$ and $\mu_n = 0$. Let T_i be the time it takes for a population of size i to reach $i+1$. In this problem we indicate a method to actually compute the transition probabilities $P_{mn}(t) = P\{X_t = n|X_0 = m\}$ which is not based on solving directly the backward of forward equation.

1. Argue that T_i is exponential with rate $i\lambda$.

2. Let $X_1, X_2, \ldots, X_n$ be n independent exponentials random variables with parameter λ. Show

$$\max(X_1, \ldots, X_n) = T_n + T_{n-1} + \ldots T_1 \tag{2}$$

Hint: Interpret X_i as time the failure of a component, T_n as the time of the first failure, T_{n-1} as the time between the first and second failure, etc....

3. Deduce from (b) that $P\{T_1 + \ldots + T_n < t\} \;=\; (1 - e^{-\lambda t})^n$.

4. Use 3. and 1. to obtain that

$$P_{1n}(t) \;=\; (1 - e^{-\lambda t})^{n-1} - (1 - e^{-\lambda t})^n \;=\; e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \tag{3}$$

and hence that $P\{X_t = n \mid X_0 = 1\}$ has a geometric distribution.

5. Deduce from 4. that for $n > m$

$$Pmn(t) \;=\; \binom{n-1}{m-1} (e^{-\lambda t})^m (1 - e^{-\lambda t})^{n-m} \tag{4}$$

Exercise 3 Consider a M/M/1 queue, i.e, the birth and death process X_t with $\lambda_n = \lambda$ and $\mu_n = \mu$ where X_t denotes the number of people in the system (either being served or waiting in line to be served). Assume that $\lambda < \mu$ so that X_t is positive recurrent and has the stationary distribution $\pi(n) = (1-\rho)\rho^n$ with $\rho = \lambda/\mu$ and $n = 0, 1, 2, \cdots$.

We denote by Y_t the load on the server at time t, i.e., the length of time the server would have to work to clear the number of people in the system if nobody were allowed to enter the system after time t.

1. Check that
$$Y_t = \begin{cases} 0 & \text{if } X_t = 0 \\ T_1' + T_2 + \cdots T_{X_t} & \text{if } X_t \geq 1 \end{cases}$$
where $T_2, T_3, \cdots$ are the service times of the people waiting in line to be served and T_1' is the residual service time of the person currently at the service station. What is the p.d.f of T_1'?

2. Show that for $x \geq 0$
$$\lim_{t\to\infty} P\{Y_t \leq x\} = 1 - \rho e^{-\mu(1-\rho)x} .$$
This is the cumulative distribution function of Y_t for very large t, i.e. the queuing system can be considered to be stationary.

3. Compute $\lim_{t\to\infty} E[Y_t]$ and $\lim_{t\to\infty} \mathrm{var}(Y_t)$. You have to be very careful: the random variable Y_t is neither a discrete nor a continuous random variable: $P\{Y_t = 0\} \neq 0$ and Y_t has density away from 0 in the region $(0, \infty)$.