Math 623 Fall 2011: Final exam

Problem 1 Suppose f_n is a sequence of non-negative measurable functions such that

- $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere
- $\lim_{n\to\infty} \int_{\mathbf{R}} f_n dx = \int_{\mathbf{R}} f dx$

Show that for any measurable set $E \subset \mathbf{R}$

$$\lim_{n\to\infty} \int_E f_n dx = \int_E f dx \,.$$

Hint: Use Fatou's lemma for $f_n\chi_E$ and $f_n\chi_{E^c}$.

Problem 2 Let f be a bounded measurable nonnegative function such that $\int_{\mathbf{R}} f dx = K$ for some $0 < K < \infty$. Compute

$$\lim_{n\to\infty} \int_{\mathbf{R}} n \log \left[1 + \left(\frac{f(x)}{n} \right)^{\alpha} \right] dx$$

for $\alpha > 0$ in the following three cases

1. $0 < \alpha < 1$.
2. $\alpha = 1$
3. $1 < \alpha < \infty$

Hint: Use Fatou's lemma in part 1.

Problem 3

1. Suppose $f : [a, b] \to \mathbf{R}$ is a function of bounded variation and $g : I \to \mathbf{R}$ is Lipschitz continuous where I is an interval which contains the range of f. Prove that $g \circ f$ is of bounded variation.
2. Suppose $f : [a, b] \to \mathbf{R}$ is a function of bounded variation such that there exists $\alpha > 0$ with $f(x) \geq \alpha$ for all $x \in [a, b]$. Prove that there exists two increasing functions g and h such that

   $$f(x) = \frac{g(x)}{h(x)} \,.$$

   Hint: use part 1.

Problem 4 For $f \in L^2([a, b])$ let us define

$$T(f)(x) = \int_a^x f(t)dt$$

1. Show that T defined a bounded linear operator on $L^2([a, b])$ and that $\|T\| \leq (b - a)$.
2. Compute the adjoint operator T^*.
3. Show that T does not have any eigenvalues. (*Hint:* Try and solve $T(f) = \lambda f$. Maybe you can treat the case $\lambda = 0$ and $\lambda \neq 0$ separately).