Math 597/697: Homework 4

1. The machine 1 is currently working and machine 2 will be put in use at a time t from now. If the lifetimes of the machines 1 and 2 are exponential random variables with parameters λ_1 and λ_2, what is the probability that machine 1 is the first machine to fail?

2. A radioactive source emits particles according to a Poisson process with rate $\lambda = 2$ particles per minute.

   (a) What is the probability to have 4 particles emitted between the first and second minute?

   (b) What is the probability that the first particle emitted is between the second and third minutes?

   (c) What is the probability that the fourth particles is emitted less than two minutes after the second one?

   (d) What is the expected time between the emission of the third particle and the emission of the seventh particle?

   (e) What is the conditional probability that 3 particles are emitted in the first minute given that 5 particle are emitted in the first two minutes?

   (f) What is the expected number of particles emitted between the third and fourth minutes given that 5 particle were emitted in the first two minutes?

   (g) What is the expectation of the arrival time of the fifth particle given that $N_t = 3$?

3. Consider a two-server system in which a customer is first served by server 1, then by server 2 and then departs. The service times at server i are exponential random variables with parameter μ_i. $i = 1, 2$. When you enter the system you find server 1 free and two customers at server 2, customer A in service and customer B waiting in line.

   (a) Find the probability P_A that A is still in service when you move over to server 2.

   (b) Find the probability P_B that B is still in service when you move over to server 2.

   (c) Compute $E[T]$, where T is the total time you spend in the system. *Hint:* Write $T = S_1 + S_2 + W_A + W_B$ where S_i is your service time at server i, W_A the amount of time you wait in queue when while A is being served, and W_B the amount of time you wait in queue when while B is being served.

4. Let N_t be a Poisson process with rate λ and let $0 < s < t$. Compute
   (a) $P\{N_t = n + k | N_s = k\}$
   (b) $P\{N_s = k | N_t = n + k\}$
   (c) $E[N_t N_s]$

5. Cars cross a certain point on the highway according to a Poisson process with a rate $\lambda = 3$ per min. Running blindly across the highway, what is the probability that you will be injured if it takes you S seconds to cross the road. (You should assume that you are injured whenever a car passes by)? What is the probability to be injured if you suppose that you are agile enough to avoid one car, but that you will be injured if you encounter two or more cars while crossing the highway?

6. Let N_t be a Poisson process with rate λ.
   (a) Suppose it is known that that $N_1 = n$. Determine the expected value of the first arrival S_1
   (b) Suppose it is known that $N_1 = 2$. Determine the expected value of $S_1 S_2$, the product of the two first arrival times.
   (c) Determine the conditional expectation $E[S_1 | N_t = 2]$

7. A certain radioactive material emits particle according to a Poisson process with rate λ. Each particle exists for a random duration and is then annihilated. The lifetimes S_i of distinct particles are independent and have a common cumulative distribution function $P\{S_i < s\} = G(s)$ and expectation $E[S_i] = \mu$. Let X_t denote the number of particles existing at time t (i.e. the number of particles which have been emitted but not been annihilated). Express $\lim_{t\to\infty} E[X_t]$ in terms of μ. What is the distribution of X_t for large t?

8. (a) Let $S_1 \leq S_2 \leq \ldots \leq S_n$ the arrival times of a Poisson process with rate λ and let $f(s)$ be a function. Show that

$$E\left[\sum_{i=1}^{N_t} f(S_i)\right] = \lambda \int_0^t f(s)ds \qquad (1)$$

   Hint: Use the order statistics property of the random variables S_i.

(b) Electrical pulses with iid amplitudes $X_1, X_2, \ldots$ arrive at a detector at random times $S_1, S_2, S_3, \ldots$ according to a Poisson process with rate λ. The detector output for the k-th pulse at time t is

$$\theta_k(t) = \begin{cases} 0 & t < S_k \\ X_k \exp(-\alpha(t - S_k)) & t \geq S_k \end{cases} \tag{2}$$

This means that the amplitude measured by the detector start with X_k and then decreases exponentially at rate α. The total amplitude measured by the detector is then

$$Z_t = \sum_{j=1}^{N_t} \theta_k(t) \,. \tag{3}$$

Assuming that X_i is independent of $\{N_t, t \geq 0\}$, compute the average amplitude $E[Z_t]$ measured by the detector

9. *A model for the price of a security*: Let $S(t)$ denote the price of a security at time $t \geq 0$. The price remains unchanged until a "shock" occurs, at which time the price of the stock is multiplied by a random factor. If the initial price of the security is S_0, then the price after the first shock will be $S_0 X_1$, the price after the second shock will be $S_0 X_1 X_2$, and so on... Assume that the shocks occurs according to a Poisson process N_t with rate λ, that the multiplicative factors X_0, X_1, etc.. are i.i.d exponential random variables with parameter μ and that $N_t, t \geq 0$ is independent of X_i. Compute the expected value and the variance of the security price $S(t)$ at time t.

10. A store opens at 8 am and closes at 5pm. The number of customers entering the store is a non homogeneous Poisson process. The rate at which customers enters increases linearly from 0 to 10 per hours between 8 am and 11 am. Between 11am and 3pm the rate is constant and equal to 10 and then between 3 pm and 5 pm it decreases linearly from 10 to 5 customers per hour. Determine the probability distribution of the numbers of customers that enter the store on this day.