Math 523H–Homework 6

1. (a) If f is a continuous function such that $f(x) = 0$ for every rational x, show that $f = 0$.

   (b) If f and g are two continuous functions such $f(x) = g(x)$ for every rational x, show that $f = g$.

   (c) Suppose that f is continuous function which satisfies $f(x+y) = f(x) + f(y)$ for all x and y. Show that $f(x) = ax$ for some constant a. *Hint:* Consider first $x = n$ an integer, then $x = \frac{1}{n}$, then x rational.

2. For the following functions determine if they are uniformly continuous on the given set. Justify your answer by using appropriate theorems.

   (a) $f(x) = x^3 + \sin(x)$ on $[0,2]$. (b) $f(x) = \frac{x^2+x-6}{x-2} + \cos(2x)$ on $(0,2)$.
   (c) $f(x) = \frac{1}{1-x}$ on $[0,1)$. (d) $f(x) = x^2 \sin(\frac{1}{x})$ on $(0,1]$.

3. Show by an ϵ-δ argument that $f(x) = x^2$ is uniformly continuous on $[0,3]$.

4. Show that the function $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to 0 on $[0,1]$. *Hint:* For each n compute the maximum and minimum of f_n on the interval $[0,1]$.

5. Consider the function $f_n(x) = \frac{x^n}{1+x^n}$ on $[0,\infty)$. Does f_n converge uniformly on $\mathbb{R}$?

6. Consider the function given by $f(x) = \sum_{k=1}^{\infty} \frac{x^n \cos(2nx)}{n^2 2^n}$ on the interval $[-2,2]$. Show that that the function f is continuous.

7. Consider the sequence of functions $f_n(x) = (n+1)x^n(1-x)$ on the interval $[0,1]$.

   (a) Compute $f(x) = \lim_{n\to\infty} f_n(x)$. Is f continuous?

   (b) Show that f_n does not converge uniformly to f.
   Hint: Find the maximum of $f_n(x)$.

8. Consider the series
$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{1+x^2}\left(\frac{1}{1+x^2}\right)^n.$$

   (a) Show that the series converges absolutely for all $x \in \mathbb{R}$.

   (b) Show that the series does not converges uniformly on $[-1,1]$.

   (c) Compute $f(x)$. Is is continuous?