Chapter 10: Mixed strategies Nash equilibria, reaction curves and the equality of payoffs theorem

Nash equilibrium: The concept of Nash equilibrium can be extended in a natural manner to the mixed strategies introduced in Lecture 5. First we generalize the idea of a *best response* to a mixed strategy

Definition 1. *A mixed strategy $\widehat{\sigma}_R$ is a* best response *for R to some mixed strategy σ_C of C if we have*

$$\langle \widehat{\sigma}_R, P_R \sigma_C \rangle \geq \langle \sigma_R, P_R \sigma_C \rangle \qquad \text{for all } \sigma_R.$$

A mixed strategy $\widehat{\sigma}_C$ is a best response *for C to some strategy σ_R of R if we have*

$$\langle \sigma_R, P_C \widehat{\sigma}_C \rangle \geq \langle \sigma_R, P_C \sigma_C \rangle \qquad \text{for all } \sigma_C$$

We can then extend the definition to Nash equilbrium

Definition 2. *The mixed strategies $\widehat{\sigma}_R, \widehat{\sigma}_R$ are a Nash equilibrium for a two-player game with payoff matrices P_R and P_C if*

$$\widehat{\sigma}_R \text{ is a best response to } \widehat{\sigma}_C \qquad \text{and} \qquad \widehat{\sigma}_C \text{ is a best response to } \widehat{\sigma}_R$$

or in other words

$$\langle \widehat{\sigma}_R, P_R \widehat{\sigma}_C \rangle \geq \langle \sigma_R, P_R \widehat{\sigma}_C \rangle \qquad \text{for all } \sigma_R.$$
$$\langle \widehat{\sigma}_R, P_C \widehat{\sigma}_C \rangle \geq \langle \widehat{\sigma}_R, P_C \sigma_C \rangle \qquad \text{for all } \sigma_R$$

Reaction curves: For games with *two strategies* one can compute the best responses and the Nash equilbria in terms of the *reactions curves*. To explain the idea let us start with a example

Example: Matching Pennies The payoff matrices are given by

$$P_R = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad P_R = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and let us write the mixed strategies as

$$\sigma_R = (p, 1-p) \qquad \sigma_C = (q, 1-q)$$

To find the best response to σ_C we compute

$$
\begin{aligned}
\langle \sigma_R, P_R \sigma_C \rangle &= \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 2q-1 \\ 1-2q \end{pmatrix} \right\rangle \\
&= p(2q-1) + (1-p)(1-2q) = (2q-1) + p(4q-2)
\end{aligned}
$$

Since we are computing the best response for R to the strategy of C we consider the payoff

$$(2q - 1) + p(4q - 2)$$

for **fixed** q and **variable p** with $0 \leq p \leq 1$. This is a linear function of p and the maximum will depend on the slope of this function (here $4q - 2$), whether it is positive, negative, or 0.

Best response for R:

- $4q - 2 > 0$ (or $q < 1/2$) The slope is positive so the maximum is at $p = 1$.

- $4q - 2 < 0$ (or $q > 1/2$) The slope is negative so the maximum is at $p = 0$

- $4q - 2 = 0$ (or $q = 1/2$) The slope is 0 so the maximum is at any p between 0 and 1.

To find the best response for C we compute

$$
\langle \sigma_R, P_R \sigma_C \rangle = \left\langle \begin{pmatrix} p \\ 1 - p \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} q \\ 1 - q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p \\ 1 - p \end{pmatrix}, \begin{pmatrix} 1 - 2q \\ 2q - 1 \end{pmatrix} \right\rangle
$$
$$
= p(1 - 2q) + (1 - p)(2q - 1)
$$
$$
= (2p - 1) + q(2 - 4p)
$$

which we now consider has a function of the **variable** q and for **fixed** p. By maximizing over $0 \leq q \leq 1$ we find

Best response for C:

- $2 - 4p > 0$ (or $p > 1/2$) The slope is positive so the maximum is at $q = 1$.

- $2 - 4p < 0$ (or $p < 1/2$) The slope is negative so the maximum is at $q = 0$

- $2 - 4p = 0$ (or $p = 1/2$) The slope is 0 so the maximum is at any q between 0 and 1.

To find the Nash equilibria we argue as follows.

- If $q > 1/2$ then the best response is $p = 0$ but the best response to $p = 0$ is $q = 0$ which contradicts $q > 1/2$ and this does not lead to a Nash equilbrium.

- If $q < 1/2$ then the best response is $p = 1$ but the best response to $p = 1$ is $q = 1$ and again this does not lead to a Nash equilbrium.

- If $q = 1/2$ then the best response is any p and so if we choose $p = 1/2$ then the best response to $p = 1/2$ is any q, in particular $q = 1/2$.
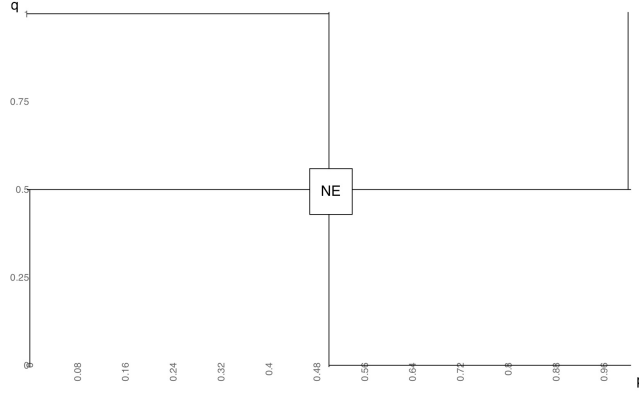
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Figure 1: Reaction curves for the matching pennies game

The conclusion is that there exists one Nash equilibrium

$$\sigma_R = (1/2, 1/2) , \quad \sigma_C = (1/2, 1/2) ,$$

A convenient way to put all this information together and to find the Nash equilibria in a a graphical manner is to draw the reaction curves which exhibit the best responses. For the matching pennies they are show in figure ??

Example: Battle of the sexes. Writing the payoff as $\sigma_R = (p, 1-p), \sigma_C = (q, 1-q)$ we have for the best response from R to C

$$
\begin{aligned}
\langle \sigma_R, P_R \sigma_C \rangle &= \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} 3-2q \\ 2q \end{pmatrix} \right\rangle \\
&= p(3-2q) + (1-p)2q = 2q + p(3-4q)
\end{aligned}
\tag{1}
$$

Since the fame is symmetric we obtain the payoff for C by exchanging p and q

$$\langle \sigma_R, P_C \sigma_C \rangle = 2p + q(3-4p)$$

We obtain **Best response for R:**

- $3 - 4q > 0$ (or $q < 3/4$) The slope is positive so the maximum is at $p = 1$.

- $3 - 4q < 0$ (or $q > 3/4$) The slope is negative so the maximum is at $p = 0$

- $3 - 4q = 0$ (or $q = 3/4$) The slope is 0 so the maximum is at any p between 0 and 1.

and **Best response for R:**

- $3 - 4p > 0$ (or $p < 3/4$) The slope is positive so the maximum is at $q = 1$.
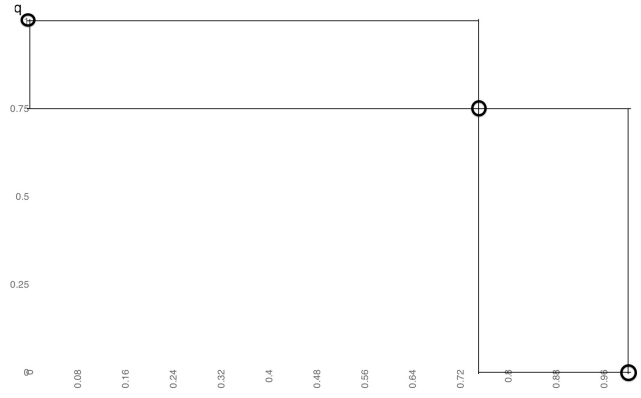
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Figure 2: Reaction curves for the battle of the sexes game

- $3 - 4p < 0$ (or $p > 3/4$) The slope is negative so the maximum is at $q = 0$

- $3 - 4p = 0$ (or $p = 3/4$) The slope is 0 so the maximum is at any q between 0 and 1.

The best response curves are given in Figure **??**

The equality of payoffs theorem The method used above to compute the Nash equilibria works well if there are 2 strategies but is not very useful if three or more strategies are used since the optimization problems become much more complicated. We present here a method which, in principle, allows to compute, all Nash equilibria of a game. But the reader should be warned that computations become quickly quite lengthy and involved.

Some games do not have a Nash equilibrium in pure strategies (like rock-paper-scissors) or matching pennies but there is always one (and often many) if we consider mixed strategies.

Theorem 3. (Nash Theorem) *A game (with a finite number of strategies) always has at least one Nash equilibrium $(\widehat{\sigma}_R, \widehat{\sigma}_C)$ in mixed strategies.*

Many interesting examples of games are symmetric.

Theorem 4. (Nash Theorem for symmetric games) *For a symmetric game we have*

$$(\widehat{\sigma}_R, \widehat{\sigma}_C) \text{ is a NE} \iff (\widehat{\sigma}_C, \widehat{\sigma}_R) \text{ is a NE}$$

Moroever there always exists at least one symmetric NE

$$(\widehat{\sigma}_R, \widehat{\sigma}_C) = (\widehat{\sigma}, \widehat{\sigma})$$

The computation of Nash equilibria is based on the simple observation.

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Theorem 5. (Equality of payoff theorems) *Suppose $\widehat{\sigma}_R$ is a best response to σ_C. Then we have*

1. *If the strategy i and j are played with positive probability for R (that is we have $\widehat{\sigma}_R(i) > 0$ and $\widehat{\sigma}_R(j) > 0$) then the payoff to play i and j against σ_C are identical.*

$$\widehat{\sigma}_R(i) > 0 \text{ and } \widehat{\sigma}_R(j) > 0 \implies P_R\sigma_C(i) = P_R\sigma_C(j)$$

2. *If the strategy i is played with positive probability (that is we have $\widehat{\sigma}_R(i) > 0$) but the strategy j is played with probability 0 (that is $\widehat{\sigma}_R(j) = 0$) then the payoff for R to play j against σ_C is less than or equal to the payoff to play i against σ_C.*

$$\widehat{\sigma}_R(i) > 0 \text{ and } \widehat{\sigma}_R(j) = 0 \implies P_R\sigma_C(i) \geq P_R\sigma_C(j)$$

Proof. (i) It is best to argue by contradiction. Suppose that the strategies i and j are played with positive probability for R (that is we have $\widehat{\sigma}_R(i) > 0$ and $\widehat{\sigma}_R(j) > 0$) but the payoffs $P_R\sigma_C(i)$ and $P_R\sigma_C(j)$ are not equal. Let us say for example that we have $P_R\sigma_C(j) > P_R\sigma_C(i)$. Then we argue that $\widehat{\sigma}_R$ cannot be a best response to σ_C: since it it more favorable to play j than to play i if you choose a new mixed strategy where you play j with greater probability than $\widehat{\sigma}_R(i)$ and j with a smaller probability that $\widehat{\sigma}_R(j)$ and all the other strategies with the unchanged probabilities then your payoff in this new strategy against σ_C will strictly increase. So $\widehat{\sigma}_R$ was not a best response.

(ii) Argue again by contradiction. Suppose that $\widehat{\sigma}_R(i) > 0$ and $\widehat{\sigma}_R(j) = 0$ but that $P_R\sigma_C(j) > P_R\sigma_C(i)$. Then $\widehat{\sigma}_R$ cannot be a best response to σ_C. To see this change your strategy $\widehat{\sigma}_R$ into a new strategy where you play now j with positive probability and i with probability 0. In this new strategy the payoff against σ_C is greater than for $\widehat{\sigma}_R$ and so $\widehat{\sigma}_R$ was not a best response. ∎

It is useful to give a name for the strategies which are played with positive probabilities:

Definition 6. *The* support *of a mixed strategies σ is the set of pure strategies which are played with positive probability. We denote the support of σ by $S(\sigma)$:*

$$S(\sigma) = \{i : \sigma(i) > 0)\}$$

For example if $\sigma = (1/7, 2/7, 0, 0, 4/7)$ then $S(\sigma) = \{1, 2, 5\}$ that is the mixed strategy σ the strategies played with positive probability are 1, 2, and 5.

Using the equality of payoff theorem we can devise a method to compute all Nash equilibria:

Algorithm to compute Nash equilibria

- Pick a support for both σ_R and σ_C.

- Compute the payoff for R, i.e., compute the vector $P_R\sigma_C$ and set the payoffs equal to each other if they are in the support of σ_R. Solve the equation for σ_C. Do the same for the payoff for C, i.e. compute $P_C^T\sigma_R$ and set the payoffs equal to each other if they are in the support of σ_C. Solve the equation for σ_R.

- Check that the strategies which are not in the support of σ_R and σ_C do not offer a greater payoff.

- Pick another support for both σ_R and σ_C and repeat the exercise.

At each step one tries to solve a system of linear equations which is not easy but the complexity of the algorithm grows very quickly since one needs to check *all* possible supports. For example if R has 2 strategies and C has 3 strategies there are 3 choices of support for R and 7 for C........

It is a bit less tedious to compute symmetric Nash equilibria since one can use the symmetry of the problem.

Algorithm to compute symmetric Nash equilibria $(\widehat{\sigma}, \widehat{\sigma})$

- Pick a support σ.

- Set all the payoff $P_R\sigma(i)$ equal for $i \in S(\sigma)$ and try to solve for σ.

- Check that for $j \notin S(\sigma)$ the payoff $P_R\sigma(j)$ is smaller than the payoff for $i \in S(\sigma)$.

- Pick another support for $\widehat{\sigma}$ and repeat.

Let us illustrate the techniques with a number of examples:

Example: Consider the game with payoff matrices

$$P_R = \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix}, \quad P_C = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

We try to find a NE in mixed strategies. We denote $\sigma_C = (p, 1-p)$ and $\sigma_R = (q, -1q)$. The payoff for R are given by

$$P_R\sigma_C = \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} 3-5p \\ 8p-5 \end{pmatrix}$$

while the payoff for C are given by

$$P_C^T \sigma_R = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 - q \end{pmatrix} = \begin{pmatrix} 2 - q \\ 1 + 2q \end{pmatrix}$$

Setting the payoff for R to be equal we find

$$3 - 5p = 8p - 5 \quad \Rightarrow \quad p = 8/13$$

while setting the payoff for C to be equal gives

$$2 - q = 1 + 2q \quad \Rightarrow \quad q = 1/3$$

So we have a NE

$$(\widehat{\sigma}_R, \widehat{\sigma}_C) = ((1/3, 2/3), (8/13, 5/13)).$$

Example: Symmetric 2-strategies game Consider the symmetric game with payoff matrices

$$P_R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P_C = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

We already know that the game as 1 pure strategy NE is $a > c, b > d$ or $a < c, b < d$ and 2 pure strategy NE if $a > c, b < d$ or $a < c, b > d$. To find when this game as a mixed Nash equilibrium we use the equality of payoff theorem. Denote $\sigma_C = (p, 1 - p)$, the strategy for C, then the payoff for R is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ 1 - p \end{pmatrix} = \begin{pmatrix} b + p(a - b) \\ d + p(c - d) \end{pmatrix}$$

Setting the payoff to be equal we find

$$b + p(a - b) = d + p(c - d)$$

or

$$p[(a - c) + (d - b)] = d - b.$$

So we find

$$p = \frac{(d - b)}{(a - c) + (d - b)}.$$

Since the game is symmetric if we denote that strategy for R by $\sigma_R = (q, 1 - q)$ the payoff for C are

$$\begin{pmatrix} b + q(a - b) \\ d + q(c - d) \end{pmatrix}$$

and equating them gives the same solution $q = \frac{(d-b)}{(a-c)+(d-b)}$. Note that p and q should be between 0 and 1 and this occurs only if $a > c, b < d$ or $a < c, b > d$.

A symmetric two strategies game with matrix $P_R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $P_C = P_R^T$ has a mixed strategy NE if $a > c, b < d$ or $a < c, b > d$. The NE is symmetric and given by $(\widehat{\sigma}_R, \widehat{\sigma}_C) = (\widehat{\sigma}, \widehat{\sigma})$ with

$$\widehat{\sigma} = \left(\frac{(d-b)}{(a-c)+(d-b)}, \frac{(a-c)}{(a-c)+(d-b)} \right)$$

Example: Nash equilibria for Rock-scissors-papers The payoff matrices are

$$P_R = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad P_C = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Note that the game is a symmetric one so we should find a symmetric Nash equilibrium. The computation of Nash equilibria goes in several steps.

- Assume that *one of the player use all his three pure strategies*, for example take $\sigma_C = (p_1, p_2, 1 - p_1 - p_2)$. Then the payoffs for R against this mixed strategy are given by

$$P_R \sigma_C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 - p_1 - p_2 \end{pmatrix} = \begin{pmatrix} 2p_2 + p_1 - 1 \\ 1 - p_2 - 2p_1 \\ p_1 - p_2 \end{pmatrix}$$

  We set the payoffs to be equal and find two equation

$$2p_2 + p_1 - 1 = p_1 - p_2 \quad \rightarrow p_2 = 1/3$$
$$1 - p_2 - 2p_1 - 1 = p_1 - p_2 \quad \rightarrow p_1 = 1/3$$

  so we must have $\sigma_C = (1/3, 1/3, 1/3)$. Since the game is symmetric by reversing the roles of R and C we find then $\sigma_R = (1/3, 1/3, 1/3)$ and we have found a (symmetric) NE.

- We try next to find NE where one player plays only 2 of his strategies, say let us pick $\sigma_C = (p, 1 - p, 0)$. Then the payoff for R is

$$P_R \sigma_C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} p \\ 1 - p \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - p \\ -p \\ 0 \end{pmatrix}$$

If we try to set the payoff equal to each other we find $1 - p = -p$ which is impossible or $-p = 0$ which gives $p = 0$ or $1 - p = 0$ which gives $p = 1$. In both case it means that C is actually using only one of his strategies and not two as assumed. Note that the game is also symmetric in the strategies and so picking another 2 strategies for any of the player will lead to the same conclusion.

- Finally assume that C use only a pure strategy in his NE, for example his first one (Rock). Then we have

$$P_R \sigma_C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and the best response for R is then to play his third strategy (Paper, of course). But conversely Rock for C is not a best response to Paper for R and so this does not give a NE.

To conclude we find

$$\text{Unique NE} \quad \widehat{\sigma}_R = (1/3, 1/3, 1/3), \, \widehat{\sigma}_C = (1/3, 1/3, 1/3)$$

∎

Example: NE for coordination game: Let us consider the symmetric game where two players shows 1, 2, or 3 fingers and win the number of finger they show if both player show the same number of fingers and lose the number of finger they show if the numbers of fingers do not match. This game is a *coordination game* since it is advantageous for a player to play the same strategy as his opponent. Coordination have typically *many* Nash equilibria.

The payoff matrices are given by

$$P_R = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -2 \\ -3 & -3 & 3 \end{pmatrix}, \quad P_C = \begin{pmatrix} 1 & -2 & -3 \\ -1 & 2 & -3 \\ -1 & -2 & 3 \end{pmatrix}$$

The game is symmetric and we shall use this fact to simplify the computations. The computation of Nash equilibria goes in several steps.

- Assume player C use all his three strategies, for example $\sigma_C = (p_1, p_2, 1 - p_1 - p_2)$. Then the payoffs against this strategies are given by

$$P_R \sigma_C = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -2 \\ -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 - p_1 - p_2 \end{pmatrix} = \begin{pmatrix} 2p_1 - 1 \\ 4p_2 - 2 \\ 3 - 6p_1 - 6p_2 \end{pmatrix}$$

We set the payoffs to be equal and find

$$2p_1 - 1 = 3 - 6p_1 - 6p_2 \qquad 4p_2 - 2 = 3 - 6p_1 - 6p_2$$

or

$$8p_1 + 6p_2 = 4 \qquad 6p_1 + 10p_2 = 5$$

and after some algebra we find $\sigma_C = (5/22, 8/22, 9/22)$. Since the game is symmetric, reversing the role of R and C we also find $\sigma_R = (5/22, 8/22, 9/22)$.

- Let us assume that C uses only his first two strategies (one finger and two fingers) $\sigma_C = (p, 1 - p, 0)$. Then we have

$$P_R \sigma_C = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -2 \\ -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} p \\ 1 - p \\ 0 \end{pmatrix} = \begin{pmatrix} 2p - 1 \\ 2 - 4p \\ 0 \end{pmatrix}$$

  If we set the payoffs to be equal we find $2p - 1 = 2 - 4p$ or $p = 1/2$. For that choice the payoffs are then $(0, 0, 0)$. So we can assume that R is playing is first two strategies with positive probability. His third strategy yields a payoff which is not bigger. But now if R use his first two strategies using the symmetry of the game we find the same result for C. So we obtain a Nash equilibrium $\sigma_R = (1/2, 1/2, 0), \sigma_C = (1/2, 1/2, 0)$.

- Let us assume that C uses only his first and third strategies, $\sigma_C = (p, 1 - p, 0)$. Then we find $P_R \sigma_C = (2p - 1, 0, 3 - 6p)$ and arguing as in the previous case we find a Nash equilibrium $\sigma_R = (1/2, 0, 1/2), \sigma_C = (1/2, 0, 1/2)$.

- Let us assume that C uses only his second and third strategies, then we obtain in a similar way $\sigma_R = (0, 1/2, 1/2), \sigma_C = (0, 1/2, 1/2)$.

- Finally let us assume that C plays a pure strategy, say 1. Then we have

$$P_R \sigma_C = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -2 \\ -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

  and so the best response for R is to play 1. By symmetry we find the Nash equilibrium $\sigma_R = (1, 0, 0), \sigma_C = (1, 0, 0)$. One argues similarly with playing pure strategies 2 and 3 and one finds two more Nash equilbrium in pure strategies.

To summarize we have found 7 Nash equilibria, all of them symmetric, i.e., we have $(\widehat{\sigma}_R = \widehat{\sigma}_C) = (\widehat{\sigma}, \widehat{\sigma})$ with

$$\widehat{\sigma} = (5/22, 8/22, 9/22),$$

$$\widehat{\sigma} = (1/2, 1/2, 0), \quad \widehat{\sigma} = (1/2, 0, 1/2), \quad \widehat{\sigma} = (0, 1/2, 1/2)$$

$$\widehat{\sigma} = (1, 0, 0), \quad \widehat{\sigma} = (0, 1, 0), \quad \widehat{\sigma} = (0, 0, 1).$$