Math 624: Problem set 3

1. Let $[0,1]$ equipped with $\mathcal{M}$, the σ-algebra of Lebesgue measurable subsets of $[0,1]$. Let m denote the Lebesgue measure on $[0,1]$ and let μ denote the counting measure on $[0,1]$, i.e., for $E \subset [0,1]$, $\mu(E)$ is the number of elements in E. Let $D = \{(x,x)\,,\, x \in [0,1]\}$ denote the diagonal in $[0,1] \times [0,1]$. Show that $\int\int \chi_D dm d\mu$, $\int\int \chi_D d\mu dm$, and $\int \chi_D d(m \times \mu)$ are all unequal. Why does this not contradict Fubini Theorem?

   Hint: To compute $\int \chi_D d(m \times \mu)$ go back to the definition of $m \times \mu$.

2. Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space (not necessarily σ-finite). Let Y be a countable set, with the σ-algebra $\mathcal{N}$ consisting of all subsets of Y and let ν be any σ-finite measure on Y, e.g., the counting measure. Prove Fubini-Tonelli Theorem for this case.

3. Exercise 8, p.313.

4. Exercise 10 p. 314

5. Exercise 11, p. 315

6. Suppose ν is a σ-finite signed measure and μ and λ are σ-finite (positive) measures on $(X, \mathcal{M})$.

   (a) Suppose that $\nu \ll \mu$. Show that if $g \in L^1(\nu)$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and

   $$\int g\, d\nu \;=\; \int g \frac{d\nu}{d\mu} d\mu$$

   (b) Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Show that $\nu \ll \lambda$ and

   $$\frac{d\nu}{d\lambda} \;=\; \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\,.$$

   Hint: Use (a). This equality is called the *chain rule* (why?).

   (c) Suppose $\mu \ll \lambda$ and $\lambda \ll \mu$. Show that

   $$\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} \;=\; 1$$

   for μ or λ a.e. x.
   Hint: Use (b).

7. Suppose ν_i are σ-finite signed measures and μ_i a σ-finite (positive) measures on the the measure space $(X_i, \mathcal{M}_i)$, for $i = 1, 2$. Show that if $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2) .$$

8. Let $[0,1]$ equipped with $\mathcal{M}$, the σ-algebra of Lebesgue measurable subsets of $[0,1]$. Let m denote the Lebesgue measure on $[0,1]$ and let μ denote the counting measure on $[0,1]$. Show that $m \ll \mu$ but that there exists no f such that $dm = f d\mu$.

9. (a) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure and let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$. Let $\nu = \mu|_{\mathcal{N}}$ be the restriction of μ to $\mathcal{N}$. Show that if $f \in L^1(\mu)$ there exists $g \in L^1(\nu)$ (in particular g is $\mathcal{N}$ measurable) such that

$$\int f d\mu = \int g d\nu$$

Prove that such g is uniquely determined, almost everywhere. In probability g is called the conditional expectation of f.

   (b) Consider the measure space $([0,1], \mathcal{B}, m)$ where $\mathcal{B}$ is the Borel σ-algebra and m is the Lebesgue measure. Let $\mathcal{C}$ be the smallest σ-algebra which contains all intervals of the form $(-a, a)$. Let $f(x) = 3x^2 + 2x$. Compute $g(x)$ given in part (a).
   Hint: What does it mean for a function be $\mathcal{C}$ measurable?

10. Suppose that F and G are complex-valued function of bounded variations on $[a, b]$. Show the following integration by parts formula for Lebesgue-Stieljes integrals: If at least one of F and G is continuous then

$$\int_{(a,b]} F dG + \int_{(a,b]} G dF = F(b)G(b) - F(a)G(a)$$

Hint: Without restriction of generality you can assume that F and G are increasing and that G is continuous. Let $\Omega = \{(x, y) : a < x \leq y \leq b\}$ and compute $\mu_F \times \mu_G(\Omega)$ (in two ways) using Fubini.