Math 597/697: Homework 7

1. In a certain town each day is classified as either sunny, cloudy or rainy. There are never two sunny days in a row and the day after a sunny day is equally likely to be rainy or cloudy. If it is rainy or cloudy one, then with probability $1/2$ it will remains so the next day. If it changes then it is equally likely to be either of the two possibilities. Show that the Markov chain is reversible and compute the proportion of cloudy days, in the long run.

2. A knight is located at one of the four corners of an empty chessboard. It starts moving around the chessboard, using at each step any of its legal moves with equal probability. Compute the expected number of moves until the black knight returns to its initial position.

3. Suppose that the space state S is finite and that $P_{ij} > 0$ for all i, j. Show that the Markov chain X_n is reversible if and only if

$$P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji} \,.$$

4. Let X_i, $i = 1, 2, \cdots$ be i.i.d. random variables with mean μ and variance σ^2. Let $\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$ denote the *sample mean* of the X_i's. When σ^2 is unknown, its value can be estimated by the *sample variance*

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2 \,.$$

Show that the sample variance is an unbiased estimator of σ^2, i.e. verify that $E[S_n^2] = \sigma^2$.

5. Suppose that you wish to estimate the volume of a set B contained in the Euclidean space $\mathbf{R}^k$. You know that B is a subset of A and you know the volume of A. The "hit-or-miss" method consists in choosing n independent points uniformly at random in A and use the fraction of points which lands in B to get an estimate of the volume of B. (We used this method to compute the number π in class.)

Suppose now that D is a subset of A and that we know the volume of D and the volume of $D \cap B$. You decide to estimate the volume of B by choosing n points at random from $A \setminus D$ and counting how many land in B. What is the corresponding estimator of the volume of B for this second method? Show that this second method is better than the first one in the sense that the variance is smaller. How might this be used to improve the estimation of the number π.

6. Suppose f is a function on the interval $[0,1]$ with $0 < f(x) < 1$. Here are two ways to estimate $I = \int_0^1 f(x)dx$.

   (a) Use the "hit-or-miss" from the previous problem with $A = [0,1] \times [0,1]$ and $B = \{(x,y) \,:\, 0 \le x \le 1\,,\, 0 \le y \le f(x)\}$.

   (b) Let $U_1, U_2, \cdots$ be i.i.d. uniform random variables on $[0,1]$ and use the estimator

   $$\hat{I}_n \;=\; \frac{1}{n} f(U_i)\,.$$

   Show that that $\hat{I}_n$ has smaller variance than the estimator of (a).

7. (a) Consider the Markov chain X_n with state space $S = \{0, 1, \cdots, M\}$ and transition probabilities

   $$\begin{aligned} &P_{ii+1} = p_i\,, \quad P_{ii-1} = 1 - p_i\,, \quad i = 1, \cdots, M-1\,, \\ &P_{01} = p_0\,, \quad P_{00} = 1 - p_0\;, \\ &P_{MM} = p_M\,, \quad P_{MM-1} = 1 - p_M\;. \end{aligned}$$

   i. Show that X_n is time reversible and find the stationary distribution by solving the detailed balance equations

   $$\pi(i)P_{ij} = \pi(j)P_{ji}\,.$$

   ii. Consider the Ehrenfest urn model (see Homework 2) where $p_i = \frac{M-i}{M}$. Show, using (i), that $\pi(i) \;=\; \binom{M}{i}(\frac{1}{2})^M$

   (b) Consider the Markov chain with state space $S = \{0, 1, 2, \cdots\}$ and transition probabilities

   $$\begin{aligned} &P_{ii+1} = p_i\,, \quad P_{ii-1} = 1 - p_i\,, \quad i = 1, 2, \cdots \\ &P_{01} = p_0\,, \quad P_{00} = 1 - p_0\,. \end{aligned} \tag{1}$$

   i. Under what conditions on the p_i's does a stationary distribution exist?

   ii. Suppose you are given a probability distribution π on $S = \{0, 1, 2, \cdots\}$. Under what conditions on π is it possible to choose $p_0, p_1, p_2, \cdots$ so that π is a stationary distribution for the Markov chain with transition probabilities (1).

   iii. Find formulas for the p_i's in the case that π is a Poisson distribution with parameter λ.

8. **Normal random variable with the rejection method.** Let Z be a standard normal random variable with pdf $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. To generate Z, use the rejection method with the pdf $g(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$.

(a) Describe an algorithm using random numbers to generate the random variable with pdf $g(x)$.

(b) Describe the rejection algorithm.

(c) As a measure of the efficiency of the method compute the (average) number of random numbers needed to simulate Z.

9. **Generating a Poisson random variable**. Consider the following algorithm.

   - Generate random numbers U_1, U_2, $U_3, \cdots$ stopping at

   $$N + 1 = \min\{n : \prod_{i=1}^{n} U_i < e^{-\lambda}\}.$$

   Show that N is a Poisson random variable with parameter λ. *Hint:* Consider a Poisson process with rate 1.

10. Consider the following transition matrix Q for a Markov chain on the countable state space $S = \{0, 1, 2, \cdots\}$:

   $$Q_{0,1} = 1 \quad Q_{ii+1} = Q_{i,i-1} = \frac{1}{2}, \quad i \geq 1.$$

   Let π be the Poisson distribution with parameter $\lambda = 7.3$. Write the explicit transition probabilities of the Metropolis chain for π with proposal matrix Q.

11. The metropolis-Hasting algorithm is the following generalization to the case where the proposal matrix Q is not symmetric. Let π be a probability distribution on the state space S with $\pi(i) = 0$ for all i. Let Q be any transition matrix on S. For $i, j \in S$ with $i \neq j$ define

   $$t_{ij} = \frac{\pi(j)Q_{ji}}{\pi(i)Q_{ij}}.$$

   Let $A : [0, +\infty] \to [0, 1]$ be a function such that $A(z) = zA(1/z)$ for all $z \in [0, +\infty]$. Finally define

   $$P_{ij} = Q_{ij}A(t_{ij}), \quad i \neq j,$$

   and

   $$P_{ii} = 1 - \sum_{j \neq i} P_{ij}.$$

   Then $A(t_{ij})$ is the acceptance probability for a proposed transition from i to j.

   (a) Assume that $Q_{ij} > 0$ for every $i, j \in S$ and prove that P is the transition matrix of a Markov chain and that it is reversible with respect to π. Prove also that P is irreducible.

(b) Suppose that $Q_{ij} = 0$ for some values of i and j. Prove that P is still well-defined and reversible with respect to π. Show that P may not be irreducible in this case, even if Q is irreducible.

(c) Which functions of the form $A(z) = \frac{z^a}{1+z^b}$ can be used?

(d) Show that $A(z) = \min\{1, z\}$ can be used and that it leads to the the Metropolis algorithm if Q is symmetric. This choice of A is called "maximal". Why?