Math 697 Fall 2014: Week 7

Exercise 1 On a chessboard compute the expected number of plays it takes a knight, starting in one of the four corners, to return to its initial position if we assume that at each each play it is equally likely to make any of its legal moves.

Exercise 2 A total of m white balls and m black balls are distributed among two urns, each of which contains m balls. At each stage a ball is randomly selected from each urn and the two selected balls are interchanged. Let X_n be the number of black balls in urn 1 after n stages.

1. Give the transition probabilities of the Markov chain X_n.
2. Without any computation find the stationary distribution.
3. Find the stationary distribution (and show that the Markov chain is reversible).

Exercise 3 ***(The time reversed chain)*** Let X_n be an irreducible Markov chain with transition probabilities $P(i,j)$ and stationary distribution $\pi(i)$. Define

$$\bar{P}(i,j) \;=\; \frac{\pi(j)}{\pi(i)} P(j,i)\,.$$

1. Show that $\bar{P}(i,j)$ is a stochastic matrix and that π is a stationary distribution for the Markov chain $\bar{X}_n$ with transition probabilities $\bar{P}(i,j)$.
2. Show that

   $$P\left\{\bar{X}_0 = i_0, \bar{X}_1 = i_1, \cdots, \bar{X}_n = i_n\right\} \;=\; P\left\{X_n = i_0, X_{n-1} = i_1, \cdots X_0 = i_n\right\}\,.$$

   The Markov chain $\bar{X}_n$ is called the *time reversed* chain for the Markov chain X_n.

Exercise 4 ***(Cycle condition for detailed balance)*** Suppose X_n an irreducible Markov chain with stationary distribution π.

1. Assume that for all pair i,j we have $P(i,j) > 0$ iff $P(j,i) > 0$.

   Show that X_n satisfies detailed balance if and only if for any n and any sequence of state $i_0, i_1, i_2, \cdots i_n$ we have

   $$P(i_0,i_1)P(i_1,i_2)\cdots P(i_{n-1},i_n)P(i_n,i_0) \;=\; P(i_0,i_n)P(i_n,i_{n-1})\cdots P(i_2,i_1)P(i_1,i_0)$$

   Hint: For the "if" part use the convergence to stationary distribution.
2. Assume that $P(i,j) > 0$ for all $i,j \in S$. Show that X_n satisfies detailed balance if and only if for any n and any sequence of state (i_0, i_1, i_2) we have

   $$P(i_0,i_1)P(i_1,i_2)P(i_2,i_0) \;=\; P(i_0,i_2)P(i_2,i_1)P(i_1,i_0)$$

that is it is enough to consider cycles of length 3.

Exercise 5 Given any graph (E, V) and using a Metropolis algorithm construct a Markov chain whose stationary distribution is the uniform distribution on all vertices. Your Markov chain chain should use only the local topology of your graph, like the random walk on the graph.

Exercise 6 (Metropolis-Hastings algorithm) Let $\pi(i) > 0$ be a probability distribution on the finite state space S. Let $Q(i, j)$ be a transition probability matrix (not necessarily symmetric). Set

$$T(i, j) = \frac{\pi(j)Q(j, i)}{\pi(i)Q(i, j)}$$

and suppose $A : [0, \infty] \to [0, 1]$ be a function such that $A(z) = zA(1/z)$ for all $z \in [0, \infty]$. Finally we define for $i \neq j$

$$P(i, j) = Q(i, j)A(T(i, j))$$

and $P(i, i) = 1 - \sum_{j \neq i} P(i, j)$. You can think of $A(T(i, j))$ has the acceptance probability for the proposed transition from i to j.

(a) Suppose $Q(i, j) > 0$ for all i, j. Prove that P is the transition matrix of a reversible Markov chain which satisfies detailed balance and has stationary distribution π. Prove also that P is irreducible.

(b) Suppose now that $Q(i, j)$ might be 0 for some values of i, j. Prove that P is still well-defined and satisfies detailed balance but that P might not be irreducible even if Q is irreducible.

(c) Find the values of a and b for which $A(z) = \frac{z^a}{1+z^b}$ can be used.

(d) Show that $A(z) = \min\{1, z\}$ can be used and that it leads to the Metropolis algorithm when Q is symmetric. In which sense is that A the "best choice of A?