Math 623: Problem set 3

1. Let $I = [a, b]$ be a finite closed interval. The function f is said to be Lipschitz continuous on I if there exists a constant L such that for all $x, y \in I$ we have

$$|f(x) - f(y)| \leq L|x - y| .$$

   (a) Show that if f is Lipschitz continuous then f is continuous.

   (b) Show that if f is continuously differentiable on I then f is Lipschitz continuous.

   (c) Show that if $A \subset I$ has measure 0 and f is Lipschitz continuous then $f(A)$ has measure 0. *Hint:* If f is continuous then f maps a closed interval onto a closed interval. Use the definition of the exterior measure.

2. If a function f is integrable then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set A with $m(A) \leq \delta$ we have $\int_A |f| dx \leq \varepsilon$ (absolute continuity of the integral).

   We say that a sequence of functions $\{f_n\}_{n \geq 1}$ is **equi-integrable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set A with $m(A) \leq \delta$ we have $\int_A |f_n| dx \leq \varepsilon$ **for all** n. Prove the following

   Theorem *Let E is a set of finite measure, $m(E) < \infty$, and let $f_n : E \to \mathbf{R}$ be a sequence of functions which is equi-integrable. Show that if $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in E$ then $\lim_{n\to\infty} \int_E |f_n - f| dx = 0$.*
   Hint: Use Egorov Theorem as in the proof of DCT or BCT.

3. Prove the following

   Theorem *Let $-\infty < a < b < \infty$ and let $f : \mathbf{R}^d \times [a, b] \to \mathbf{R}$ be such that $f(x, t)$ is integrable for any $t \in [a, b]$. Let*

$$F(t) = \int f(x, t)\, dx .$$

   (a) Suppose that the map $t \mapsto f(x, \cdot)$ is continuous every x and that there exists an integrable function g such that $|f(x, t)| \leq g(x)$ for all x, t. Then the function $F(t)$ is continuous.

   (b) Suppose that $\frac{\partial f}{\partial t}(x, t)$ exist and that there exists and integrable function h such that $|\frac{\partial f}{\partial t}(x, t)| \leq h(x)$ for all x, t. Then the function $F(t)$ is differentiable and $F'(t) = \int \frac{\partial f}{\partial t}(x, t) dx$.

   Hint: Given t_0 let $\{t_n\}$ be an arbitrary sequence such that $\lim_n t_n = t_0$ and apply the Dominated Convergence Theorem.

4. In class proved first the bounded convergence theorem (using Egorov Theorem). We then proved Fatou's Lemma (using the Bounded Convergence theorem) and deduced from it the Monotone Convergence Theorem. Finally we proved the Dominated Convergence Theorem (using both the monotone Convergence Theorem and the Bounded Convergence Theorem).

   There are other ways to prove this sequence of results. For example

   (a) Deduce Fatou's Lemma from the Monotone Convergence Theorem by showing that for any sequence of measurable functions $\{f_n\}$ we have

   $$\int \liminf_n f_n \, dm \;\leq\; \liminf \int f_n \, dm$$

   Hint: Note that $\inf_{n\geq k} f_n \;\leq\; f_j$ for any $j \;\geq\; k$ and thus $\int \inf_{n\geq k} f_n \, dm \;\leq\; \inf_{j\geq k} \int f_j \, dm.$

   (b) Deduce the dominated Convergence Theorem from Fatou's Lemma. *Hint:* Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.

5. Exercise 6, p. 91

6. Exercise 9, p. 91

7. Exercise 11, p. 91

8. **(a)** Consider the functions $f_n(x) = \dfrac{n^2 x}{1 + n^3 x^2}$ defined on the interval $[0, 1]$. Show that the sequence $\{f_n\}$ is not uniformly bounded, i.e., there exists no constant M such that $|f_n(x)| \leq M$ for all $x \in [0, 1]$ and all n.

   (b) Find a nonnegative function $g(x)$ such that $\int_{[0,1]} g(x)dx < \infty$ and $f_n(x) \leq g(x)$ for all n and all $x \in [0, 1]$. *Hint: Fix x and maximize $f_n(x)$ over n by replacing the discrete variable n by a continuous one.*

   (c) Compute $\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx$.

9. Compute $\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1} \, dm$. Distinguish between $a > 0$, $a = 0$ and $a < 0$. Justify your computation carefully. *Hint:* Use the transformation of integrals under dilation.

10. Exercise 10, p. 91

11. (a) Suppose $\{f_n\}$ is a sequence of integrable functions such that $\sum_{n=1}^\infty \int |f_n|dx < \infty$. Show that $\sum_{n=1}^\infty f_n(x)$ converges a.e. to an integrable function and $\int \sum_n f_n = \sum_n \int f_n$. *Hint:* Use DCT.

    (b) Prove that for $a > -1$ we have $\int_{[0,1]} x^a(1 - x)^{-1} \log(x)dx \;=\; \sum_{n=1}^\infty \frac{1}{(a+n)^2}$