STAT 515-04: Beta random variables

Luc Rey-Bellet

University of Massachusetts Amherst

luc@math.umass.edu

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Beta Random Variables

Beta Random Variables

A random variable Y is an beta random variable with parameters $\alpha > 0$ and $\beta > 0$ if the PDF is

$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \quad \text{for } 0 \le y \le 1$$

where

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The normalization $B(\alpha,\beta)$ is called the beta function. (see `https://en.wikipedia.org/wiki/Beta_function` for a proof).
See also
`https://homepage.divms.uiowa.edu/~mbognar/applets/beta.html`
for a good online calculator

Mean and Variance of beta random variables

Mean and Variance

If Y is beta random variable with parameters $\alpha > 0$ and $\beta > 0$ then

$$E[Y] = \frac{\alpha}{\alpha+\beta} \qquad V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Using the normalization of the beta distribution

$$\begin{aligned}
E[Y] &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y y^{\alpha-1}(1-y)^{\beta-1} dy \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha+1-1}(1-y)^{\beta-1} dy \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

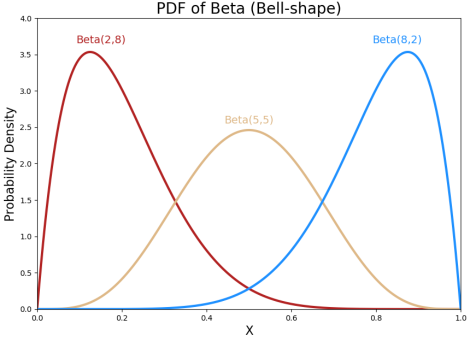
Examples

- Suppose that the random variable X has a $\mathrm{Beta}(\alpha, \beta)$ distribution with parameters $\alpha = 2$ and $\beta = 3$.
  1. What is the pdf?
  2. Compute the mean $\mathbb{E}[X]$ and variance $\mathrm{Var}(X)$.
  3. Find the probability $P(X < 0.5)$.

What is the beta RV good for?

- The beta random variable is supported on the interval $[0, 1]$ so it is good to model random phenomena taking values on an interval. If a random variable Z takes value in the interval $[c, d]$ then $\frac{Z-c}{d-c}$ takes value in $[0, 1]$ so we can always renormalize the interval.
  - Special case $\alpha = \beta = 1$ then $f(y) = 1$ and so Y is uniform.
  - Special case $\alpha = 2, \beta = 1$ then $f(y) = \frac{1}{2}y$.
  - Special case $\alpha = 2, \beta = 1$ then $f(y) = y(1-y)$

- If $\alpha = \beta$ the distribution is symmetric around $1/2$.
- If $\alpha < \beta$ then f has a peak for small y.
- If $\alpha > \beta$ then f has a peak for large y (close to 1).
- $y_{max} = \frac{(\alpha-1)}{(\alpha-1)+(\beta-1)}$



Amazon marketplace

Since Y takes values in $[0,1]$, the beta random variable is good at describing random proportions or random probabilities

Example: Amazon seller marketplace rankings: You want to buy a certain item on Amazon Marketplace where you have several vendors.

- Vendor 1 has 18 positive rating and 2 negative rating (90%).
- Vendor 2 has 180 positive ratings and 20 negative ratings (90%)

Build a probabilistic model for $Y =$ rating:

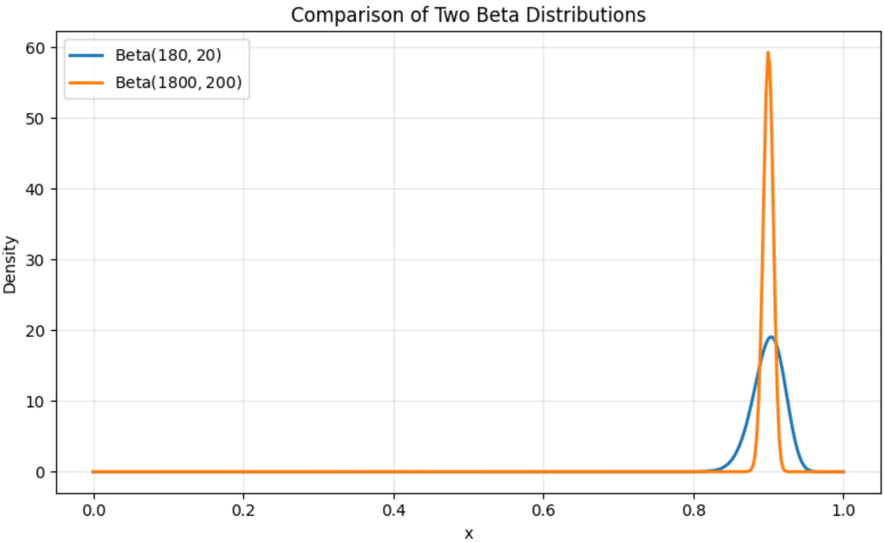
Vendor 1: $\alpha = 18$, $\beta = 2$. Then we have

$$E[Y_1] = \frac{\alpha}{\alpha+\beta} = \frac{9}{10} \qquad V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{9}{10}\frac{1}{10}\frac{1}{21}$$

Vendor 2: $\alpha = 180$, $\beta = 20$. Then we have

$$E[Y_2] = \frac{\alpha}{\alpha+\beta} = \frac{9}{10} \qquad V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{9}{10}\frac{1}{10}\frac{1}{201}$$

The variance of Y_2 is 10 times smaller!


Comparison of Two Beta Distributions
Beta(180, 20)
Beta(1800, 200)
Density
x
0
10
20
30
40
50
60
0.0
0.2
0.4
0.6
0.8
1.0