MATH 646: Homework 2

1. Let $M = (0\,,\infty)$ equipped with the Lebesgue measure and consider the map $T : M \to M$ given by

$$Tx = \begin{cases} \frac{1}{x} & \text{if} \quad x \in (0,1] \\ x-1 & \text{if} \quad x \in (1,\infty)\,. \end{cases}$$

What is the induced transformation T_A where $A = (0,1]$? Show that the measure (called Gauss measure) μ given by

$$\mu(B) = \frac{1}{\log 2} \int_B \frac{1}{1+x}\, dx\,,$$

is T_A invariant.

2. Let $\{a_n\}$ be a sequence of complex numbers. Show that

$$\lim_{n\to\infty} a_n = 0\,,$$

implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |a_j| = 0\,,$$

which in turn implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j = 0\,.$$

but that the reverse implications are wrong, in general.

3. Let T be an endomorphism of $(M, \mathcal{B}, \mu)$. Show that the following are equivalent

   (a) $\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$ for all $A, B \in \mathcal{B}$

   (b) $\lim_{n\to\infty} \langle U_T^n f\,,\, g\rangle = \langle f\,,\, 1\rangle\langle 1\,,\, g\rangle$ for all $f, g \in L^2(\mu)$.

   (c) $\lim_{n\to\infty} \langle U_T^n f\,,\, f\rangle = \langle f\,,\, 1\rangle\langle 1\,,\, f\rangle$ for all $f \in L^2(\mu)$.

   (d) $\lim_{n\to\infty} \langle U_T^n f\,,\, f\rangle = \langle f\,,\, 1\rangle\langle 1\,,\, f\rangle$ for all $f \in L^2(\mu)$ with $\langle f\,,\, 1\rangle = 0$.

4. Let T be an endomorphism of $(M, \mathcal{B}, \mu)$. Let ρ be a probability measure $(M, \mathcal{B})$ and let ρ^n be the measure define by $\rho^n(A) = \rho(T^{-n}A)$, $A \in \mathcal{B}$. Show that T is mixing if and only if $\lim_{n\to\infty} \rho^n(A) = \mu(A)$ for all $A \in \mathcal{B}$ and for all ρ which are absolutely continuous with respect to μ.

5. Let T be an endomorphism of $(M, \mathcal{B}, \mu)$ and let $\{f_j\}$ be a orthonormal basis for $L^2(\mu)$. Show that T is mixing if and only if $\lim_{n\to\infty}\langle U_T^n f_j\,,\, f_k\rangle = \langle f_j\,,\, 1\rangle\langle 1\,,\, f_k\rangle$ for all j, k.

6. **A weak mixing transformation which is not strong mixing.** Consider the measure space $([0,1],\mathcal{B},m)$ where $\mathcal{B}$ is the Borel σ-algebra and m Lebesgue measure. We define the map T by

$$Tx = \begin{cases} x+1/2 & \text{if} \quad x<1/2 \\ x-1/4 & \text{if} \quad 1/2<x<3/4 \\ x-5/8 & \text{if} \quad 3/4<x<7/8 \\ \vdots & \vdots \\ x-(1-1/2^n-1/2^{n+1}) & \text{if} \quad 1-1/2^n<x<1-1/2^{n+1} \\ \vdots & \vdots \end{cases}$$

Note that the map T is not defined for dyadic rationals $j/2^n$ which can be ignored in the sequel since they from a set of measure 0. Note further that, for any n, the map T permutes the dyadic intervals $(j/2^n\,,\,(j+1)/2^n)$, although not in accordance with the order on the intervals.

- Show that the map T is ergodic and has pure point spectrum: to do this let $K_n=(0\,,\,1/2^n)$ and show that

  $$f_n(x) \;=\; \sum_{j=0}^{2^n-1}\omega_n^{-j}\chi_{K_n}(T^jx)$$

  are eigenfunctions for the eigenvalues $\omega_n=\exp(2\pi i/2^n)$.

Let

$$I_0 \;=\; (0,1/2)\,,\; I_1 \;=\; (1/2,3/4)\,,\; I_2 \;=\; (3/4,7/8)\,,\;\cdots\,.$$

We have $[0,1]=M=\bigcup_{n=0}^{\infty}I_n \bmod 0$ and we set $A=\sum_{n=0}^{\infty}I_{2n}$. Let A' be a copy of A and set $M^A=M\cup A'$ (disjoint union) and define the map $T^A\,:\,M^A\to M^A$ by

$$\begin{aligned} T^Ax' &= Tx \quad \text{if} \quad x'\in A' \\ T^Ax &= x' \quad \text{if} \quad x\in A \\ T^Ax &= Tx \quad \text{if} \quad x'\in M\setminus A \end{aligned}$$

This is the integral automorphism corresponding to the function $f(x)=1+\chi_A(x)$ as defined in Homework 1.

- Show (directly) that the map T_A is ergodic.

Consider the function

$$S_n \;=\; \chi_A+\chi_A\circ T+\cdots+\chi_A\circ T^{2^n-1}$$

which is the number of visits x makes to A in the first 2^n iterations of T.

- Show that S_n assumes only two values, s_n and $s_n + 1$ and

$$m\{x \,:\, S_n(x) = s_n + 1\} \;=\; \frac{2}{3} \;\text{ or }\; \frac{1}{3}\,,$$

  depending whether n is even or odd. *Hint:* If $J_n = (1 - (1/2^n)\,,\,1)$, each of the iterations $TJ_n, \cdots T^{2^n-1}J_n$ is either contained in A or disjoint from A.

- Show that $(T^A)^{2^n+S_n(x)}(x) = T^{2^n}(x)$ for all $x \in M$ and that

$$(T^A)^{2^n+s_n+1}I_0 \subset I_0 \cup I_0'\,.$$

  Deduce that this excludes strong mixing.

Next we show that T^A is weak-mixing.

- Let f be an eigenfunction for U_{T^A} with eigenvalues $e^{2\pi i\alpha}$ $(0 \le \alpha < 1$ and $|f| = 1)$. Show that

$$|\int f(T^{2^n}x)\overline{f}(x)\,dm| \;=\; |\int e^{2\pi i\alpha S_n(x)}\,dm| \;<\; 1$$

  if $\alpha \neq 0$.

- Approximate f by linear combinations of characteristic functions of dyadic intervals and conclude that α must be 0.