Math 645: Homework 2

1. Show the following: If the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, where $f(t,x)$ is a continuous function, has a unique solution, then the Euler polygons converge to this solution.

2. Consider the Cauchy problem $x' = f(t,x)$, $x(0) = 0$, where

$$f(t,x) = \begin{cases} 4\text{sign}(x)\sqrt{|x|} & \text{if } \ |x| \geq t^2 \\ 4\text{sign}(x)\sqrt{|x|} + 4(t - \frac{|x|}{t})\cos(\pi\frac{\log t}{\log 2}) & \text{if } \ |x| < t^2 \end{cases} \tag{1}$$

The function f is continuous on $\mathbf{R}^2$. Consider the Euler polygons $x_h(t)$ with $h = 2^{-i}$, $i = 1,2,3,\cdots$. Show that $x_h(t)$ does not converge as $h \to 0$, compute its accumulation points, and show that they are solution of the Cauchy problem. *Hint: the solutions are* $\pm 4t^2$.

3. Consider the the Cauchy problem $x' = f(t,x)$, $x(0) = 0$ where f is given by

$$f(t,x) = \begin{cases} 0 & \text{if } \ t \leq 0, \ \ x \in \mathbf{R} \\ 2t & \text{if } \ t > 0, \ \ x \leq 0 \\ 2t - \frac{4x}{t} & \text{if } \ t > 0, \ \ 0 \leq x < t^2 \\ -2t & \text{if } \ t > 0, \ \ t^2 \leq x \end{cases} \tag{2}$$

   (a) Show that f is continuous. What does that imply for the Cauchy problem?

   (b) Show that f does not satisfy a Lipschitz condition in any neighborhood of the origin.

   (c) Apply Picard-Lindelöf iteration with $x_0(t) \equiv 0$. Are the accumulation points solutions?

   (d) Show that the Cauchy problem has a unique solution. What is the solution?

   Note that this problem shows that existence and uniqueness of the solution does not imply that the Picard-Lindelöf iteration converges to the unique solution.

4. Consider the Cauchy problem $x' = \lambda x$, $x(0) = 1$, with $\lambda > 0$ on the interval $[0,1]$. Compute the Euler polygons $x_h(t)$. Show that

$$\frac{\lambda}{1+\lambda h} x_h(t) \leq \frac{d}{dt} x_h(t) \leq \lambda x_h(t) \,. \tag{3}$$

and deduce from this the classical inequality

$$\left(1 + \frac{\lambda}{n}\right)^n \leq e^\lambda \leq \left(1 + \frac{\lambda}{n}\right)^{n+\lambda} \tag{4}$$

5. Let a, b, c, and d be positive constants. Consider the Predator-Prey equation $x' = x(a - by)$, $y' = y(cx - d)$ with positive initial conditions $x(t_0)$ and $y(t_0)$. Using the change of variables $p = \log(x)$ and $q = \log(y)$ express the equations as an Hamiltonian systems and deduce that the solution exist for all t.

6. Show that the following ODE's have global solutions (i.e., defined for all $t > t_0$).

   (a) $\begin{array}{l} x = 4y^3 + 2x \\ y' = -4x^3 - 2y - \cos(x) \end{array}$.

   (b) $x'' + x + x^3 = 0$.

   (c) $x'' + x' + x + x^3 = 0$.

   (d) $\begin{array}{l} x' = \frac{\sin(2t^2 x)x^3}{1+t^2+x^2+y^2} \\ y' = \frac{x^2 y}{1+x^2+y^2} \end{array}$.

   (e) $\begin{array}{l} x' = 5x - 2y - y^2 \\ y' = 2y + 6x + xy - y^3 \end{array}$.

7. Prove the following generalizations of Gronwall Lemma.

   - Let $a > 0$ be a positive constant and $g(t)$ and $h(t)$ be nonnegative continuous functions. Suppose that for any $t \in [0, T]$

     $$g(t) \le a + \int_0^t h(s)g(s)\,ds\,. \tag{5}$$

     Then, for any $t \in [0, T]$

     $$g(t) \le a e^{\int_0^t h(s)\,ds}\,. \tag{6}$$

   - Let $f(t) > 0$ be a positive function and $g(t)$ and $h(t)$ be nonnegative continuous functions. Suppose that for any $t \in [0, T]$

     $$g(t) \le f(t) + \int_0^t h(s)g(s)\,ds\,. \tag{7}$$

     Then, for any $t \in [0, T]$

     $$g(t) \le f(t) e^{\int_0^t h(s)\,ds}\,. \tag{8}$$

8. Consider the FitzHugh-Nagumo equation

   $$\begin{aligned} x_1' &= f_1(x_1, x_2) = g(x_1) - x_2\,, \\ x_2' &= f_2(x_1, x_2) = \sigma x_1 - \gamma x_2\,, \end{aligned} \tag{9}$$

   where σ and γ are positive constants and the function g is given by $g(x) = -x(x - 1/2)(x - 1)$.

   (a) In the x_1-x_2 plane draw the graph of the curves $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$.

   (b) Consider the rectangles $ABCD$ whose sides are parallel to the X_1 and x_2 axis with two opposite corners located on the $f_2(x_1, x_2) = 0$. Show that if the rectangle is taken sufficiently large, a solution which start inside the rectangle stays inside the rectangle forever. Deduce from this that the equations for any initial conditions x_0 have a unique solutions for all time $t > 0$.

9. Show that the solutions of

$$\begin{aligned} x_1' &= x_1(3 - 4x_1 - 2x_2)\,, \\ x_2' &= x_2(4 - 2x_1 - 3x_2)\,, \end{aligned} \tag{10}$$

have a unique solution for all $t \geq 0$, for any initial conditions x_{10} , x_{20} wich are nonnegative. *Hint*: A possibility is to use a similar procedure as in the previous exercise.

10. Let $A = \{(t,x) \in \mathbf{R} \times \mathbf{R}\,;\, 0 \leq t \leq a\,, |x| \leq b\}$ and let $f\,:\,A \to \mathbf{R}$ be a continuous function with $M = \sup_{(t,x)\in A} \|f(t,x)\|$. Consider the Cauchy problem

$$x' = f(t,x)\,, \quad x(0) = 0 \tag{11}$$

   (a) Assume that

      i. $f(t,x) \geq 0$.

      ii. f is an increasing function of x, i.e. $f(t,x_1) \leq f(t,x_2)$ if $x_1 \leq x_2$.

   Show that the Picard-Lindelöf iteration $x_n(t)$ converges for $t \in [0,\alpha]$ where $\alpha = \min(a, b/M)$.

   (b) Show that the limit $x(t) = \lim_{n\to\infty} x_n(t)$ in (b) is a solution of the Cauchy problem. Is it the unique solution?