Math 421 Final Exam Fall 2002

Name:__________________________

Solve problem 1 and 7 out of problems 2 to 9. If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number _____.

1. (16 points) a) Show that the Laurent series of $\frac{1}{\sin(z)}$, centered at 0, has the form

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots \text{terms of order at least five.} \tag{1}$$

(You can use equality (1) in the subsequent parts, even if you do not derive it).

b) Find the principal part at $z = 0$ of the function $f(z) = \dfrac{1-z}{z^5 \cdot \sin(z)}$

c) Find all the singularities of $f(z)$ (given in part b) in the disk $\{|z| < 4\}$ and determine their type (isolated, removable, pole of what order, essential).

d) Find the residue at each isolated singularity in D.

2. (12 points) a) Compute $\sin(\frac{\pi}{4} + i\ln(3))$. Simplify your answer as much as possible.

b) Find all solutions of the equation $\cos(z) = i$.

3. (12 points) Compute the integral $\int_C \frac{\sin(z)+1}{e^{3z}-e^z}dz$, where C is the circle $\{|z|=1\}$ traversed counterclockwise.

4. (12 points) a) Find the Taylor series of the function $f(z) = \dfrac{z+1}{z-1}$ centered at 0 and determine its radius of convergence. Justify your answer.

b) Find the Laurent series of the function $f(z)$, given in part a), valid in the domain $|z| > 1$.

5. (12 points) a) Use the definition of contour integrals, in order to prove the equality

$$\int_C e^{\bar{z}} dz = \int_C e^{4/z} dz, \tag{2}$$

where C is the circle $\{|z| = 2\}$, traversed counterclockwise.

Caution: The exponent of the integrand, on the left hand side, is the complex conjugate $\bar{z}$ of z.

b) Find the Laurent series of $e^{4/z}$ centered at zero and classify the type of singularity at $z = 0$.

c) Use the equality (2) in order to evaluate the integral $\int_C e^{\bar{z}} dz$.

6. (12 points) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin(\theta)}.$$

7. (12 points) Let C_A be the straight line segment from $A+iA$ to $-A+iA$, where A is a positive real number. Prove the inequality

$$\left|\int_{C_A} \frac{e^{iz}}{z^2+1}dz\right| \quad \le \quad \frac{2Ae^{-A}}{A^2-1}.$$

8. (12 points) Determine whether the following statements are true or false. Justify your answers!

   a) If $f(z)$ and $g(z)$ are analytic at a point z_0 and $g(z_0) = g'(z_0) = 0$, but both $f(z_0)$ and $g''(z_0)$ are non-zero, then

$$\mathrm{Res}_{z=z_0}\left(\frac{f}{g}\right) = 0.$$

   b) There exists an entire non-constant function $f(z)$ satisfying the inequality

$$|f(z)| \le |z|e^{-|z|}$$

c) If C is a simple closed contour, and z_0 does not belong to the domain D bounded by C, then there is a single valued branch of $\log(z - z_0)$, defined for all z in D.

d) There exists an entire function, whose real part is xe^y.

9. (12 points) Evaluate the improper integral

$$\int_0^\infty \frac{dx}{x^4+1}$$