Measure Theory and Integration

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Top 3 theorems

Theorems

1. Fatou: $u_i \geq 0$
   $\int \liminf_{i\to\infty} u_i d\mu \leq \liminf_{i\to\infty} \int u_i d\mu$
2. Monotone convergence (Beppu-lei)
   $0 \leq u_i \uparrow u \Rightarrow \int u_i d\mu \uparrow \int u d\mu$
3. Dominated convergence
   u_i integrable functions $u_i \to u$ (almost everywhere)
   Suppose $\exists w$ integrable: $|u_i| \leq w \ \forall i$ then:
   1. $\lim_{i\to\infty} \int |u_i - u| d\mu = 0$
   2. $\lim_{i\to\infty} \int u_i d\mu = \int u d\mu < \infty$

Top 3 theorems

Proof of 3

$|u_i| \leq w$, $|u| = \lim +i \to \infty |u_i| \leq w$
$|\int u_i d\mu - \int u d\mu| = |\int (u_i - u) d\mu| \leq \int \int |u_i - u| d\mu.$
Enough to proof 3.1. $|u_j - u| \leq |u_j| + |u| \leq 2w$, so
$2w - |u_i - u| \geq 0.$
Fatou: $\int 2w d\mu = \int \liminf_{i\to\infty}(2w - |u_i - u|) d\mu \leq$
$\liminf_{i\to\infty} \int (2w - |u_i - u|) d\mu = \int 2w - \limsup_{i\to\infty} \int |u_j - u| d\mu.$
So we have: $\int 2w d\mu \leq \int 2w - \limsup_{i\to\infty} \int |u_j - u| d\mu.$
Hence: $\limsup_{i\to\infty} \int |u_j - u| d\mu = 0$

Top 3 theorems

Example

Example to show that some condition is necessary.
$u_i = i1_{(0,\frac{1}{i}]}$, $i = 1, 2, \ldots$.
$\int u_i d\lambda = \frac{1}{i} i = 1$.
$\lim_{i\to\infty} \int u_i d\lambda = 1$, but $u_i \to u \equiv 0$.
So $\int u d\lambda = 0$.

Application

$(X, \mathcal{A}, \mu)$, $a \leq b$
$\mu_i : (a, b) x X \to \mathbb{R}$
$u(t, x) = u_t(x)$.
Take $v : (a, b) \to \mathbb{R}$ defined by $v(t) = \int_X u(t, x) d\mu(x)$.

Top 3 theorems

Theorem

Suppose:

- $\forall t \in (a, b)\ u_t(x) \in \mathcal{L}'(\mu)$
- $t \to u(t, x)$ is continuous, $\forall x \in X$
- $|u(t, x) \leq w(x) \in \mathcal{L}'(\mu)$

Then it is the case $v(t)$ is continuous.

Proof

$t_j \to t$, $u_j(x) = u(t_j, x) \to u(t, x)$ and $|u_j(x)| \leq w(x)$.
So $\lim_{j\to\infty} v(t_j) = \lim_{j\to\infty} \int u(t_j, x) d\mu = \int \lim_{j\to\infty} u(t_i, x) d\mu =$
$\int u(t, x) d\mu = v(t)$

Relation Riemann Lebesgue integrals

Lebesgue is an improvement, because it works on **any** space with a measure, but sometimes you might think Riemann is better:

Example

$f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} \frac{(-1)^n}{n+1} & x \in [n, n+1) \text{ with } n = 0, 1, 2, \ldots \\ 0 & x \in 0 \end{cases}.$$

Is this function integrable?

Let $f_k = |f| 1_{[0,k]}$, then $\lim_{k \to \infty} f_k \to |f|$.

But $\int |f| d\mu$ goes to infinity, so does not exist.

But Riemann integral gives: $\int_0^\infty f(x)dx = \sum_{n=0}^\infty \frac{(-1)^n}{n+1} \leq \infty$ does exists.

Lebesgue does not uses the minus sign.

Relation Riemann Lebesgue integrals

Theorem

$f : [a, b] \to \mathbb{R}$ bounded and measurable.

- f is Riemann integrable $\Leftrightarrow$ f is almost everywhere continuous, with respect to Lebesgue measure.
- Riemann integrable functions on $[a, b]$ are Lebesgue integrable, and the 2 integrals are the same.

Relation Riemann Lebesgue integrals

Proof

Take $P = \{a_i | a = a_0 < a_1 < \cdots < a_n = b\}$ a partition of $[a, b]$.
$\Delta_i = a_i - a_{i-1}$.
$I_i = [a_{i-1}, a_i)$.
$M_i = \sup\{f(x) | x \in I_i\}$
$m_i = \inf\{f(x) | x \in I_i\}$
Upper sum $U_p = \sum_{i=1}^n M_i \Delta_i$
Lower sum $L_p = \sum_{i=1}^n m_i \Delta_i$
Note $U_p = \int u_p d\lambda$ where $u_p = \sum_{i=1}^n M_i 1_{I_i}$.
And $L_p = \int l_p d\lambda$ where $l_p = \sum_{i=1}^n m_i 1_{I_i}$.
Refining the partition means adding points on the interval.
$P_1, P_2, \ldots$ with $U_i = U_{P_i}$ and $U_i = L_{P_i}$ and $u_i = u_{p_1}$ and $l_i = l_{p_i}$.
We have that $l_n \leq f \leq u_n \ \forall n$.

Relation Riemann Lebesgue integrals

Proof (continued)

We see that $l_1 \leq l_2 \leq \cdots \leq f \leq \cdots \leq u_2 \leq u_1$ because of definition of supremum and infimum.
$u = \lim_{i\to\infty} u_i$ and $l = \lim_{i\to\infty} l_i$ exist. Claim:
$\lim_{n\to\infty} \int_{[a,b]} u_n d\lambda = \int_{[a,b]} u d\lambda$.
$\lim_{n\to\infty} \int_{[a,b]} l_n d\lambda = \int_{[a,b]} l d\lambda$.
Dominant $w(x) = M$ on $[a, b]$ $M = \sup_{x\in X} |f(x)|$

Intermediate proof

Suppose x is not an endpoint of any of the P_n. (Excludes at most countably many).
For such x we have the following:
f is continuous at $x \Leftrightarrow u(x) = f(x) = l(x)$.

Relation Riemann Lebesgue integrals

Proof (continued)

Suppose f is Riemann integrable $\Rightarrow \int_{[a,b]} u d\lambda = \int_{[a,b]} l d\lambda$
With $(u - l) \geq 0$.
$\int_{[a,b]}(u - l) d\lambda = 0 \Rightarrow u = l$ almost everywhere $\Rightarrow u = l = f$ almost everywhere $\Rightarrow f$ is continuous almost everywhere.
Suppose f is almost everywhere continuous $\Rightarrow u(x) = f(x) = l(x)$ almost everywhere.
$\int_{[a,b]} u d\lambda = \int_{[a,b]} l d\lambda = \int_{[a,b]} f d\lambda$ so f is Riemann integrable because of the limits in the claim previously.

Proof

Riemann integrable means that
$\int u d\lambda = \int l d\lambda = \int_{[a,b]} f d\lambda = \int_a^b f(x) dx$. (In previous proof).

Improper Riemann integrals

Theorem

$f \geq 0$ $\lim_{a \to -\infty, b \to \infty} \int_a^b f(x)dx = \int_{-\infty}^{\infty} f(x)dx$ exists.
Then $\int_{\mathbb{R}} f d\lambda$ exists and has to be the same.

Proof

$f_n = f 1_{[-n,n]} \uparrow f$, and we proofed that $\int_{\mathbb{R}} f_n d\lambda = \int_{-n}^{n} f(x)dx$.
Now we have $\int_{-n}^{n} f(x)dx \to \int_{-\infty}^{\infty} f(x)dx$ by assumption.
And $\int_{\mathbb{R}} f_n d\lambda \to \int_{\mathbb{R}} f d\lambda$ by the monotone convergence theorem.
So $\int_{\mathbb{R}} f d\lambda = \int_{-\infty}^{\infty} f(x)dx$.

Assigment tips

Computation

$f_\alpha(x) = x^\alpha$ with $x > 0$.
$\int_{(0,1)} x^\alpha d\lambda =^{MCT} \lim_{j\to\infty} \int x^\alpha 1_{[\frac{1}{j},1)}(x) d\lambda = \lim_{j\to\infty} \frac{1}{\frac{1}{j}} x^\alpha dx =$
$\lim_{j\to\infty} [\frac{x^{\alpha+1}}{\alpha+1}]^1_{\frac{1}{j}} = \lim_{j\to\infty} (\frac{1}{\alpha+1} - \frac{1}{j^{\alpha+1}(\alpha+1)})$. Only $< \infty$ if and only if $\alpha > -1$.