Statistics

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Standard tests

In general, a test statistic Γ is based on an estimator. Often this yields a suitable test.

Tests

- Gauss test: $\hat{\mu} = \bar{X}$
- $T = \sqrt{n}\frac{\bar{X}-\mu_0}{\sigma}$
- Exponential (shifted): $\hat{\theta} = X_{(1)}$
- $T = X_{(1)}$

Here : several test that are based on normality (due to Central Limit Theorem which warrants normality for large n)

Two new distributions

Definition

The random variable has the χ^2_n distribution. (n = degrees of freedom) if W is identically distributed to $\sum_{i=1}^{n} Z_i^2$ $Z_i \sim N(0,1)$ independent identically distributed. For small n non-symmetric, symmetric for large n.

Definition

The random variable has the t_n distribution if T is identically distributed to $\frac{Z_0}{\sqrt{\frac{W}{n}}}$ with $Z_0 \sim N(0,1)$. and $W \sim \chi^2_n$ and Z_0, W independent.

Theorem

let $X_1, \ldots X_n$ independent identically distributed. Then:

1. $\bar{X} \sim N(\mu, \sigma^2)$
2. $n - 1/\sigma^2 * S_x^2 \sim \chi^2_{n-1}$
3. $\bar{X}$ and S_x^2 are mutually independent
4. $\sqrt{n}\frac{\bar{X}-\mu}{S_x} \sim t_{n-1}$

Proof of 4

$$\sqrt{n}*\frac{\bar{X}-\mu}{S_x} \sim t_{n-1} = (\sqrt{n}*(\bar{X}-\mu)/\sigma)/((n-1)S_x^2/(\sigma^2(n-1))) \sim \chi^2_n$$

T-test

Example

Suppose $X_1 \dots X_n \sim N(\mu, \sigma^2)$ with μ, σ^2 unknown.
$H_0 : \mu \leq \mu_0$ vs $H_a : \mu > \mu_0$
Test statistic: $T = \sqrt{n} * \frac{\bar{X}-\mu}{S_x}$ and $K_t = [c_{a_0}, \infty)$.
Determine c_{a_0}.
$\alpha_0 = sup_{\mu \leq \mu_0, \sigma^2 > \sigma} P(T \in K_t) = sup_{\mu = \mu_0, \sigma^2 > \sigma} P(T \in K_t) =$
$sup_{\mu = \mu_0, \sigma^2 > \sigma} P(T \geq c_{a_0})$
As $T \sim t_{n-1}$, take $c_{a_0} = t_{n-1, 1-\alpha_0}$

But $t_{n-1,1-\alpha_0} \geq \xi_{1-\alpha_0}$: critical region has shrunken.
Reason: t-test takes into account uncertainty of σ. For $n \to \infty$, then $t_{n,\alpha} \to \xi_\alpha$ and $S^2 \to \sigma^2$ thus both tests are 'asymptotically equivalent'.

Remark

The t-test is a test on the location, not on the spread. Hence, not $H_0 : \sigma^2 \geq \sigma_0^2$ vs $H_a : \sigma^2 < \sigma_0^2$ Use $(n-1)S_x^2/\sigma^2 \sim \chi^2_{n-1}$: reject if $S_x^2 \geq \sigma^2 * \chi^2_{n-1,1-\alpha/n-1}$

Sign test

Let $X_1 \ldots X_n$ continuous random variables from some distribution.
Interest centers around the median v. Wish to test whether v is larger then some v_0 $H_0 : v \leq v_0$ vs $H_a : v > v_0$.
Use a test statistic $T = \#\{X_i \text{ such that} X_i > v\}$.
Or, how many positive $X_i - v_i$ difference?
It suffices to study the signs of the differences, but this is the binomial test applied to the signs.
Parameters n and $P_v(X_i > v_0)$.

Sign test

If indeed $v = v_0$ then $P_{v_0} = 1/2$ corresponds to H_0.
Testing $H_0 : v \leq v_0$ is equivalent to $H_0 : p \leq p_0 = 1/2$.

Remark

Effectively, the data have been dichotomized. A considerable loss of information. Consequence: if normality is reasonable, the t-test has far superior power.

Sign test

Example

Suppose we observe $(X_1, Y_1) \ldots (X_n, Y_n)$ a paired sample.
Still we wish to test for a location difference $E(X_i) = E(Y_i)$?
Test statistic based $\bar{Z} = \bar{X} - \bar{Y}$ with $Z_i = X_i - Y_i$.

Assumption

$Z_1 \ldots Z_n \sim N(\mu, \sigma^2)$ with μ, σ^2 unknown. Or differences are identically distributed.
Then a t-test on the Z_i with $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$ uses
$T = \sqrt{n} * \frac{\bar{Z}-0}{S_z} \sim t_n$ under H_0 and
$K_t = (-\infty, t_{n-1,\alpha_0/2}] \cup [t_{n-1,\alpha_0}, \infty)$.

Paired sign test: apply to $X_i - Y_i > 0$.

Sign test

Note

$Var(Z_i) = Var(X_i) + Var(Y_i) - 2Cov(X_i, Y_i) \leq Var(X_i) + Var(Y_i)$
If X_i and Y_i positively correlated, we gain by pairing.

Example

Let $X_i \ldots X_m$ and $Y_1 \ldots Y_n$ be two unpaired samples.
Assume $X_i \sim N(\mu, \sigma^2)$ and $Y_j \sim N(v, \sigma)$ with μ, v, σ^2 unknown but variances are equal.
$H_0 : \mu - v =\geq 0$ vs $H_a : \mu - v < 0$.
Base test statistic on $\bar{X} - \bar{Y}$.
Then $Var(\bar{X} - \bar{Y}) = (\frac{1}{m} + \frac{1}{n}) * \sigma^2$

$T = \frac{\bar{X}-\bar{Y}}{S_{x,y} * \sqrt{\frac{1}{m}+\frac{1}{n}}}$ with $S_{x,y} = \frac{1}{n+m-2} * (\sum_{i=1}^{m}(x_i - \bar{X}) + \sum_{i=1}^{n}(y_i - \bar{Y}))$
as estimator of σ^2?

Note

$\bar{X} - \bar{Y} \sim N(0,1)$ with appropriate normalization.
χ^2 in denominator (sum of 'independent χ^2 are χ^2')
Under $\mu = v$, then $T \sim t_{m+n-2}$, $K_t = (-\infty, t_{m+n-2,\alpha_0}]$

Example

Measure the content (grams) of 20 packages of de Ruyter chocolate sprinkles and 18 packages of Venz chocolate sprinkles. Both packages say 300 grams.
Observe: $\bar{x} = 299.04, \bar{y} = 299.77, s_x^2 = 104, s_y^2 = 0.92$
Question 1: Enough sprinkles in de Ruyter packages?
Question 2: Equal amount of sprinkels in packes from both producers?

Answer 1

$X_1 \dots X_{20}$ content de Ruyter, $Y_1 \dots Y_{18}$ observations of Venz.
$X_i \sim N(\mu, \sigma^2)$ independent identically distributed $\mu, \sigma^2 > 0$ unknown.
$H_0 : \mu \geq 300$ and $H_a : \mu < 300$ and $\alpha_0 = 0.05$
Test statistic: $T = \sqrt{n} * \frac{\bar{x} - 300}{s_x} \sim t_{n-1}$. Substitute values gives:
$T_{ruyt} = \sqrt{20} * \frac{299.04 - 300}{\sqrt{1.04}} = -4.20$
$K_t = (-\infty, t_{n-1,\alpha_0}] = (-\infty, -1.73]$.
$T_{ruyt} \in K_t$ so reject H_0, de Ruyter does not pack enough.

Answer 2

$X_i \sim N(\mu, \sigma^2)$ and $Y_i \sim N(v, \sigma^2)$ and μ, v, σ^2 unknown.
$H_0 : \mu = v$ vs $H_a : \mu \neq v$ under $\alpha = 0.05$
$T = \frac{\bar{X} - \bar{Y}}{S_{x,y}\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}$ gives $K_t =$
$(-\infty, t_{n+m-2,\alpha/2}] \cup [t_{n+m-2,\alpha/2}, \infty) = (-\infty, -2.03] \cup [2.03, \infty)$
$S^2_{x,y} = 1/36(19 * 1.04 + 17 * 0.92) = 0.98$
$T = \frac{299.04 - 299.77}{\sqrt{0.98} * \sqrt{\frac{1}{20} + \frac{1}{18}}} = -2.77 \in K_t$
Conclusion: Ruyter and Venz do not provide equal amount.

1-sample

- Gauss-test $T = \sqrt{n}\frac{\bar{x}-\mu_0}{\sigma} \sim N(0,1)$
- t-test $T = \sqrt{n}\frac{\bar{x}-\mu}{S_x} \sim t_{n-1}$
- sign-test

2-sample

- paired t-test
- unpaired t-test
- paired sign-test