

Statistics

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In general, a test statistic Γ is based on an estimator. Often this yields a suitable test.

Tests

- Gauss test: $\hat{\mu} = \bar{X}$
- $T = \sqrt{n} \frac{\bar{X} - \mu_0}{\sigma}$
- Exponential (shifted): $\hat{\theta} = X_{(1)}$
- $T = X_{(1)}$

Here : several test that are based on normality (due to Central Limit Theorem which warrants normality for large n)

Two new distributions

Definition

The random variable has the χ_n^2 distribution. (n = degrees of freedom) if W is identically distributed to $\sum_{i=1}^n Z_i^2$ $Z_i \sim N(0, 1)$ independent identically distributed. For small n non-symmetric, symmetric for large n .

Definition

The random variable has the t_n distribution if T is identically distributed to $\frac{Z_0}{\sqrt{\frac{W}{n}}}$ with $Z_0 \sim N(0, 1)$. and $W \sim \chi_n^2$ and Z_0, W independent.

Theorem

let X_1, \dots, X_n independent identically distributed. Then:

- ① $\bar{X} \sim N(\mu, \sigma^2)$
- ② $(n-1)S_x^2 / \sigma^2 \sim \chi_{n-1}^2$
- ③ \bar{X} and S_x^2 are mutually independent
- ④ $\sqrt{n} \frac{\bar{X} - \mu}{S_x} \sim t_{n-1}$

Proof of 4

$$\sqrt{n} * \frac{\bar{X} - \mu}{S_x} \sim t_{n-1} = (\sqrt{n} * (\bar{X} - \mu) / \sigma) / ((n-1)S_x^2 / (\sigma^2(n-1))) \sim \chi_n^2$$

Example

Suppose $X_1 \dots X_n \sim N(\mu, \sigma^2)$ with μ, σ^2 unknown.

$H_0 : \mu \leq \mu_0$ vs $H_a : \mu > \mu_0$

Test statistic: $T = \sqrt{n} * \frac{\bar{X} - \mu}{S_x}$ and $K_t = [c_{a_0}, \infty)$.

Determine c_{a_0} .

$$\alpha_0 = \sup_{\mu \leq \mu_0, \sigma^2 > \sigma} P(T \in K_t) = \sup_{\mu = \mu_0, \sigma^2 > \sigma} P(T \in K_t) = \sup_{\mu = \mu_0, \sigma^2 > \sigma} P(T \geq c_{a_0})$$

As $T \sim t_{n-1}$, take $c_{a_0} = t_{n-1, 1-\alpha_0}$

But $t_{n-1, 1-\alpha_0} \geq \xi_{1-\alpha_0}$: critical region has shrunk.

Reason: t-test takes into account uncertainty of σ . For $n \rightarrow \infty$, then $t_{n, \alpha} \rightarrow \xi_\alpha$ and $S^2 \rightarrow \sigma^2$ thus both tests are 'asymptotically equivalent'.

Remark

The t-test is a test on the location, not on the spread. Hence, not $H_0 : \sigma^2 \geq \sigma_0^2$ vs $H_a : \sigma^2 < \sigma_0^2$ Use $(n-1)S_x^2/\sigma^2 \sim \chi_{n-1}^2$: reject if $S_x^2 \geq \sigma^2 * \chi_{n-1, 1-\alpha/n-1}^2$

Sign test

Let $X_1 \dots X_n$ continuous random variables from some distribution.

Interest centers around the median v . Wish to test whether v is larger then some v_0 $H_0 : v \leq v_0$ vs $H_a : v > v_0$.

Use a test statistic $T = \#\{X_i \text{ such that } X_i > v\}$.

Or, how many positive $X_i - v_i$ difference?

It suffices to study the signs of the differences, but this is the binomial test applied to the signs.

Parameters n and $P_v(X_i > v_0)$.

If indeed $v = v_0$ then $P_{v_0} = 1/2$ corresponds to H_0 .
Testing $H_0 : v \leq v_0$ is equivalent to $H_0 : p \leq p_0 = 1/2$.

Remark

Effectively, the data have been dichotomized. A considerable loss of information. Consequence: if normality is reasonable, the t-test has far superior power.

Sign test

Example

Suppose we observe $(X_1, Y_1) \dots (X_n, Y_n)$ a paired sample.
Still we wish to test for a location difference $E(X_i) = E(Y_i)$?
Test statistic based $\bar{Z} = \bar{X} - \bar{Y}$ with $Z_i = X_i - Y_i$.

Assumption

$Z_1 \dots Z_n \sim N(\mu, \sigma^2)$ with μ, σ^2 unknown. Or differences are identically distributed.

Then a t-test on the Z_i with $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$ uses

$$T = \sqrt{n} * \frac{\bar{Z} - 0}{S_z} \sim t_n \text{ under } H_0 \text{ and}$$

$$K_t = (-\infty, t_{n-1, \alpha_0/2}] \cup [t_{n-1, \alpha_0}, \infty).$$

Paired sign test: apply to $X_i - Y_i > 0$.

Sign test

Note

$Var(Z_i) = Var(X_i) + Var(Y_i) - 2Cov(X_i, Y_i) \leq Var(X_i) + Var(Y_i)$
If X_i and Y_i positively correlated, we gain by pairing.

Example

Let $X_1 \dots X_m$ and $Y_1 \dots Y_n$ be two unpaired samples.

Assume $X_i \sim N(\mu, \sigma^2)$ and $Y_j \sim N(\nu, \sigma^2)$ with μ, ν, σ^2 unknown but variances are equal.

$H_0 : \mu - \nu \geq 0$ vs $H_a : \mu - \nu < 0$.

Base test statistic on $\bar{X} - \bar{Y}$.

Then $Var(\bar{X} - \bar{Y}) = (\frac{1}{m} + \frac{1}{n}) * \sigma^2$

$T = \frac{\bar{X} - \bar{Y}}{S_{x,y} * \sqrt{\frac{1}{m} + \frac{1}{n}}}$ with $S_{x,y} = \frac{1}{n+m-2} * (\sum_{i=1}^m (x_i - \bar{X})^2 + \sum_{i=1}^n (y_i - \bar{Y})^2)$

as estimator of σ^2 ?

Note

$\bar{X} - \bar{Y} \sim N(0, 1)$ with appropriate normalization.

χ^2 in denominator (sum of 'independent χ^2 are χ^2 ')

Under $\mu = \nu$, then $T \sim t_{m+n-2}$, $K_t = (-\infty, t_{m+n-2, \alpha_0}]$

Example

Measure the content (grams) of 20 packages of de Ruyter chocolate sprinkles and 18 packages of Venz chocolate sprinkles. Both packages say 300 grams.

Observe: $\bar{x} = 299.04$, $\bar{y} = 299.77$, $s_x^2 = 104$, $s_y^2 = 0.92$

Question 1: Enough sprinkles in de Ruyter packages?

Question 2: Equal amount of sprinkles in packages from both producers?

Answer 1

$X_1 \dots X_{20}$ content de Ruyter, $Y_1 \dots Y_{18}$ observations of Venez.
 $X_i \sim N(\mu, \sigma^2)$ independent identically distributed $\mu, \sigma^2 > 0$
unknown.

$H_0 : \mu \geq 300$ and $H_a : \mu < 300$ and $\alpha_0 = 0.05$

Test statistic: $T = \sqrt{n} * \frac{\bar{x} - 300}{s_x} \sim t_{n-1}$. Substitute values gives:

$$T_{ruyt} = \sqrt{20} * \frac{299.04 - 300}{\sqrt{1.04}} = -4.20$$

$$K_t = (-\infty, t_{n-1, \alpha_0}] = (-\infty, -1.73].$$

$T_{ruyt} \in K_t$ so reject H_0 , de Ruyter does not pack enough.

Answer 2

$X_i \sim N(\mu, \sigma^2)$ and $Y_i \sim N(\nu, \sigma^2)$ and μ, ν, σ^2 unknown.

$H_0 : \mu = \nu$ vs $H_a : \mu \neq \nu$ under $\alpha = 0.05$

$$T = \frac{\bar{X} - \bar{Y}}{S_{x,y} \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2} \text{ gives } K_t =$$

$$(-\infty, t_{n+m-2, \alpha/2}] \cup [t_{n+m-2, \alpha/2}, \infty) = (-\infty, -2.03] \cup [2.03, \infty)$$

$$S_{x,y}^2 = 1/36(19 * 1.04 + 17 * 0.92) = 0.98$$

$$T = \frac{299.04 - 299.77}{\sqrt{0.98 * \sqrt{\frac{1}{20} + \frac{1}{18}}}} = -2.77 \in K_t$$

Conclusion: Ruyter and Venz do not provide equal amount.

1-sample

- Gauss-test $T = \sqrt{n}\text{something something} \sim N(0, 1)$
- t-test $T = \text{something something} \sim t_{n-1}$
- sign-test

2-sample

- paired t-test
- unpaired t-test
- paired sign-test