Measure Theory and Integration

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Measure

Story for next 2 weeks

- Definition
- Properties
- Examples (Lebesque measure)

Definition

X a space, $\mathcal{A}$ a σ-algebra
$(X, \mathcal{A})$ measure space
A measure on $\mathcal{A}$ is a map, $\mu : \mathcal{A} \to [0, \infty)$ which suffices

1. $\mu(\emptyset) = 0$
2. If $A_1, A_2, A_3, \dots \in \mathcal{A}$ pairwise disjoint, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ (countable additivity from last week)

The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Measure

Examples of measures

1. Dirac measure δ_x, $x \in X$, $\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin X \end{cases}$
   Also called 'point mass at x'
2. $(\mathbb{R}, \mathcal{A}, \gamma)$ with $\mathcal{A} = \{A | A \vee A^c \text{ countable}\}$
   $\gamma(A) = \begin{cases} 1 & A^c \text{ countable} \\ 0 & A \text{ countable} \end{cases}$
3. $(X, \mathcal{A})$, $|A|$ = the number of elements in A. 'Counting measure'
4. $\mathbb{N} = \{0, 1, 2, \dots\}$. Let P_k, $k = 0, 1, 2, , \dots$
   $P_k \geq 0$, $\sum_{k=1}^{\infty} P_k = 1$. $A \subseteq \mathbb{N}:\ P(A) = \sum_{k \in A} P_k$.
   'Probability'

Measure

Hope

$(\mathbb{R}, \mathbb{B})$ with $\mathbb{B}$ the Borel σ-algebra.
$\lambda : \mathbb{B} \to [0, \infty]$ such that $\lambda([a, b]) = b - a$ for $b \geq a$.

- Existence?
- Uniqueness?

Countable additivity

If $A_1, A_2, A_3, \dots \in \mathcal{A}$ pairwise disjoint, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. Countable or finite. Why not uncountable? Using λ, a single point has measure 0, but the uncountable union has measure 1. That does not work

Properties of a measure

$(X, \mathcal{A}, \mu)$ measure space.

1. $A \subseteq B \Rightarrow \mu(A) \geq \mu(B)$, because $B = A \uplus (B \setminus A)$ with $\uplus$ is the disjoint union.
   $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$
2. $A \subseteq B$, $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$
3. $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$
   $(\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B))$
   Proof: $A \cup B = (A \cap B) \uplus (A \setminus B) \uplus (B \setminus A)$
   $\mu(A \cup B) + \mu(A \cap B) = \mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B)$
   $\mu(B \setminus A) + \mu(A \cap B) = \mu(B)$
   $\mu(A \setminus B) + \mu(A \cap B) = \mu(A)$
4. $\mu(A \cup B) \leq \mu(A) + \mu(B)$ (sub additivity)
   Extend to $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Theorem

$(X, \mathcal{A})$ measureable space. $\mu : \mathcal{A} \to [0, \infty)$ is a measure if and only if:

1. $\mu(\emptyset) = 0$
2. $\mu(A \,\dot\cup\, B) = \mu(A) + \mu(B)$
3. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and $\bigcup_{i=1}^{\infty} A_i = A$, then $\mu(A) = \lim_{i\to\infty} \mu(A_i)$. 'Continuity from below'. Notation $A_i \uparrow A$. $A = \lim A_i$.
   (We can change the limit from inside to outside the measure)

Hance, countable additivity can be replaced with continuity from below.

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Why not continuity from above?

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$
$A = \bigcap_{i=1}^{\infty} A_i$
$\mu(A) \neq \lim_{i \to \infty} \mu(A_i)$

Counter example

$\mathbb{N}$ conting measure
$A_i = \{i, i+1, \dots\}$, $|A_i| = \infty$
$A_i \downarrow \infty$, so $|\infty| = 0$.

Proof of theorem

Assume μ is a measure.

1. Trivial
2. Trivial
3. Take $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \uparrow A$
   Let $B_1 = A_1$, $B_2 = A_2 \subsetneq A_1$, $\ldots$, $B_k = A_k \subsetneq A_{k-1}$
   B_i pairwise disjoint
   $A_k = \bigcup_{j=1}^{\infty} B_j$ so $\bigcup_{i=1}^{\infty} A_i = \bigcup_{j=1}^{\infty} B_j$
   So $\mu(A) = \mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j) = \lim_{k\to\infty} \sum_{j=1}^{k} \mu(B_j)$
   $\lim_{k\to\infty} \sum_{j=1}^{k} \mu(B_j) = \lim_{k\to\infty} = \mu(B_1 \cup\!\!\!\cdot \ldots \cup\!\!\!\cdot B_k) =$
   $\lim_{k\to\infty} \mu(A_k)$

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Proof (continued)

Converse: Suppose $1, 2$ and 3 are true.
We need to show that μ is countably additive.
So let $B_1, B_2, \dots$ be pairwise disjoint.
Let $A_k = B_1 \cup \cdots \cup B_k$.
$A_k \uparrow A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{j=1}^{\infty} B_j$
$\mu(A) = \lim_{k\to\infty} \mu(A_k) = \lim_{k\to\infty} \sum_{j=1}^{k} \mu(B_j) = \sum_{j=1}^{\infty} \mu(B_j)$
A is countable additive.

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Alternatives for statement 3

3. μ is continuous from above $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$, $A = \bigcap A_i$. $\mu(A) = \lim_{i\to\infty} \mu(A_i)$
3. Implied by last item. If $A_k \downarrow \emptyset$, then $\mu(A_k) \to \infty$

Addendum

If $\mu(A) < \infty \ \forall A \in \mathcal{A}$, then all 3 statements can be interchanged.

Proof sketch of addendum

We have:
Finite measure $\mu \leftrightarrow 1, 2, 3$ (shown)
$3' \rightarrow 3''$ (trivial)
Finite measure $\rightarrow 3'$
To show: $3'' \rightarrow 3$ **(1)**
Finite measure $\mu \leftrightarrow 3'$ **(2)**

1. Suppose $A_k \uparrow A$ then $A \setminus A_k \downarrow \emptyset$
   Then $\mu(A \setminus A_k) = \mu(A) - \mu(A_k) \rightarrow 0$ because μ is finite.
   So $\mu(A) = \mu(A_k)$
2. Suppose $A_k \downarrow A$ then $A_1 \setminus A_k \uparrow A_1 \setminus A$
   So $\mu(A_1 \setminus A_k) \rightarrow \mu(A_1 \setminus A)$
   $\mu(A_1) - \mu(A_k) \rightarrow \mu(A_1) - \mu(A)$
   So $\mu(A_k) \rightarrow \mu(A)$

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Theorem

μ a measure
$\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ (sub additivity)

Proof

$B_k = A_1 \cup \cdots \cup A_k$
$B_k \uparrow \bigcup_{i=1}^{\infty} A_i = A$
So
$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{k\to\infty} \mu(B_k) \leq \lim_{k\to\infty} \sum_{i=1}^{k} \mu(A_i) = \sum_{k=1}^{\infty} \mu(A_i)$