Measure Theory and Integration

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Lebesque measures

Lebesque measure λ^n

1. The Lebesque measues is invariant under translation.
   If $r(B) = \lambda^n(x + B)$ with r a measure on B.
   $r(\cup_{i=1}^{\infty} B_i) = \lambda(x + \cup B_i) = \lambda(\bigcup(x + B_i)) = \sum_{i=1}^{\infty} r(B_i) = \sum_{i=1}^{\infty} \lambda(x + B_i)$.
   Take a rectangle $I = [0, b)$,
   $r(I) = \lambda(x + I) = \lambda([x + a, x + b)) = b - a = \lambda I$.

Lebesque measures

Lebesque measure λ^n

2. If μ on B is invariant under translations, and $\mu([0,1]^n) = k < \infty$, then $\mu = k\lambda^n$. $(\mu(A) = k\lambda^n(A))$
   If I is a rectangle, subdivide it in intervals of length $\frac{1}{M}$ with $M \in \mathbb{N}$.
   Claim: $\mu(I) = k(I)\mu([0, \frac{1}{M})^n)$, and $\lambda^n(I) = k(I)\lambda^n([0, \frac{1}{M})^n)$.
   Special case: $\mu([0,1)^n) = M^n\mu([0, \frac{1}{M})^n)$ and $\lambda([0,1)^n) = M^n\lambda^n([0, \frac{1}{M})^n)$.
   So $\mu(I) = \frac{k(I)}{M^n}\mu([0,1)^n)$ and $\lambda^n(I) = \frac{k(I)}{M^n}$
   $\mu(I) = \lambda^n(I)\mu([0,1)^n)$

Outer measure

Theorem

$\mathcal{S}$ a semi-ring, μ countably additive in $\mathcal{S}$.
If $S_i \in \mathcal{S}$ disjoint, $\cup S_i = S$, then $\mu(S) = \sum_{i=1}^{\infty} \mu(S_i)$.
Goal: extend μ to $\sigma(\mathcal{S})$ (as a measure).

Outer-measure

$A \subseteq X$
$C(A) = \{(S_j)_{i=1}^{\infty} | A \subseteq \bigcup_{j=1}^{\infty} S_j, S_j \in \mathcal{S}\}$.
$\mu^*(A) = inf\{\sum_{j=1}^{\infty} \mu(S_j) | (S_i)_{j=1}^{\infty} \in C(A)\}$ is the outer measure of A relative to $\mathcal{S}$.

Properties

1. $\mu^*(\emptyset) = 0$
2. $A \subseteq B \Rightarrow \mu^*(A) \subseteq \mu^*(B)$
3. $\mu^*(\bigcup_{i=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$ (countable subadditivity)

Outer measure

Proof

Find $\epsilon > 0$ such that for all $(S_k^j)_{k\in\mathbb{N}} \in C(A_j)$ we have that
$\sum_{i=1}^{\infty} \mu(S_k^j) \leq \mu^*(A_j) + \frac{\epsilon}{2^j}$.
$\mu^*(A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j) + \epsilon$

Claim

If $S \in \mathcal{S}$ then $\mu * (S) = \mu(S)$ so check:

- $\mu * (S) \leq \mu(S)$ (trivial)
- $\mu * (S) \geq \mu(S)$ (difficult)

Proof sketch:
$S_1, S_2 \in \mathcal{S}$, $S_1 \cap S_2 = \emptyset$.
$\overline{\mu}(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$

$\mathcal{A}^*$ and σ algebra

Definition

$\mathcal{A}^* = \{A \subseteq X | \mu^*(A \cap Q) + \mu^*(Q \cap A^c) = \mu^*(Q) | \forall Q \in X\}$
Claim: $\mathcal{S} \subseteq A^*$.
(trival) Claim: A^* is a σ-algebra
Claim: μ^* is a measure on A^*

Semi-ring

Example

$X = \{0,1\}^{\mathbb{N}}$ = all ∞ sequences of 0's and 1's.
$s \in X$, $x = (x_0, x_1, x_2, \dots)$.
Let 1 have probability p and 0 probability $1-p$.
Cylinder: $c_1 = \{x \in X | x_0 = 0, x_1 = 1, x_2 = 0\}$
$c_2 = \{x \in X | x_0 = 1, x_1 = 0, x_2 = 0, x_3 = 0\}$
$\mu(c_1) = p(1-p)^2$
$\mu(c_2) = p(1-p)^3$
Cylinders closed under intersection, either empty set or one of the 2 sets.
$S, T \in \mathcal{S}$, $S \setminus T = \dot\cup_{i=1}^{M} S_i \in \mathcal{S}$.
$c_1 \setminus c_2 = \{x \in X | x_0 = 1, x_1 = 0, x_2 = 0, x_3 = 1\}$ again a cylinder.
If $c_1 \cap c_2 = \emptyset$ then if $c_1 \cup c_2$ is a cylinder, then $\mu(c_1) + \mu(c_2) = \mu(c_1 \cup c_2)$. (Analog to lebesque proof, if union of intervals is again an interval, then...)

Semi-ring

Cylinders

If $C = \dot\cup_{i=1} C_i$ then $\mu(C) = \sum_{i=1}^{\infty} \mu(C_i)$.
But $C = \dot\cup_{i=1} C_i$ can never be true, so statement always holds. So semi-additivity is proven.
So you can extend this to a σ algebra generated by the cylinders.