Measure Theory and Integration

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New theory for integration

Non-integratable function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$\int_0^1 f(x)dx$ is not integratable using the Riemann integral.

Why do we need a new way?

- More functions
- Abstraction!
  X some general space, $f : X \to \mathbb{R}$, calculate $\int_X f(x)dx$

Hopefully this new theory will extent the Riemann integral.

Idea

Look at the inverse of an interval in X. $f^{-1}((a,b])$. with $a, b \in X$.

Measure

Example

$\mathbb{R}$, $A, B \subseteq \mathbb{R}$. Let $m(A)$ be the 'measure of A'

Properties we want:

1. $m(A) \geq 0$
2. $m(\emptyset) = 0$
3. $m([a, b]) = b - a$
4. $m(A \cup B) = m(A) + m(B)$ if $A \cap B = \emptyset$
5. $m(A) = m(x + A) \forall x$

Suppose $A_1, A_2, A_3, \ldots$ are such that $A_i \cap A_j = \emptyset \; \forall i, j \in \mathbb{N}$

Does it follow that $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$? Assume it does.

These properties are inconsistent. There exists a set in $\mathbb{R}$ for which not all properties hold.

Measure

Proof 1 − 5 are inconsistent

$x \sim y$ if $x - y \in \mathbb{Q}$ (reflexive, symmetric, transitive)
$[0, 1]$ partitions into disjoint equivalence classes.
Let E be a set which contains exactly 1 point from each equivalence class.
$\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \dots\}$ (countable set)
$E_n = E + q_n$, $n = 1, 2, \dots$

Claim

$E_n \cap E_m = \emptyset$ for $n \neq m$

Measure

Proof

Let $z \in E_n \cap E_m$

$z = a_\alpha + q_n \; a_\alpha \in E$

$z = a_\beta + q_m \; a_\beta \in E$

$a_\alpha - a_\beta \in \mathbb{Q}$

So they are the same equivalence class.

Claim

$[0,1] \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq [-1,2]$

Proof $1-5$ are inconsistent (continued)

$1 \leq m(\bigcup_{n=1}^{\infty} E_n) \leq 3$

$1 \leq \sum_{n=1}^{\infty} m(E_n) \leq 3$

$m(E_n) = \gamma \; \forall n$

$\sum_{n=1}^{\infty} m(E_n) = 0$ or ∞

σ-algebrae

X a set, A a σ-algebra, $\mathcal{A}$ on X a collection of subsets of X which satisfies

- $X \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
- $A_i \in \mathcal{A},\ i = 1, 2, 3, \cdots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Properties

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
- $A \cap B = (A^c \cup B^c)^c$

Examples

- $\mathcal{A} = \{\emptyset, X\}$
- Power set of X
- $B \subset X$, $\{\emptyset, B, B^c, X\}$
- $\{A \subset X : A \text{countable or } A^c \text{countable}\}$

Claim

Let $\mathcal{A}_\alpha$, $\alpha \in I$ be a collection of σ-algebrae.
Then $\bigcap_\alpha \mathcal{A}_\alpha$ is again a σ-algebra

Proof

- $X \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A \in \mathcal{A}_\alpha \ \forall \alpha$
  So $A^c \in \mathcal{A}_\alpha \ \forall \alpha \Rightarrow A^c \in \bigcap_\alpha$
- Do yourself

Example

Let G be a collection of sets. $\sigma(G) = \bigcap \mathcal{A}$, $\mathcal{A}$ a σ algebra, $\mathcal{A} \geq G$. This is called the 'σ-algebra generated by G'.

Topological spaces

Consider $\mathbb{R}^n$, let O^n be the collection of open sets, C^n the collection of closed sets, K^n compact sets (bounded and closed).

Definition

$B(\mathbb{R}^n) = B^n$ is the **Borel** σ-algebra, and is defined as $B^n = \sigma(O^n)$

Theorem

$\sigma(O^n) = \sigma(C^n) = \sigma(K^n)$

Proof

$\sigma(C^n) \subset \sigma(O^n)$ and $\sigma(O^n) \subset \sigma(C^n)$
$\sigma(K^n) \subseteq \sigma(C^n)$
$C \in C^n$ with $C = \bigcup_{k=1}^{\infty}(C \cup B_k)$, B_k is a closed ball around 0 with radius k. And $C \in \sigma(K^n)$
So $\sigma(C^n) \subset \sigma(K^n)$

Definition

$J^n = \{[a_1, b_1) \times [a_2, b_2) \times [a_3, b_3) \times \dots\}$ a n-dimentional rectangle.
$J^{n,o} = \{(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \dots\}$ n-dimentional open rectangle. J^n_{rat} same as J^n with rational endpoints.
$J^{n,o}_{rat}$ same as $J^{n,o}$ but with rational endpoints.

Theorem

$B^n = \sigma(O^n) = \sigma(J^{n,o}) = \sigma(J^n) = \sigma(J^{n,o}_{rat}) = \sigma(J^n_{rat})$

Proof sketch

$O^n \supseteq J^{n,o} \supseteq J^{n,o}_{rat}$
So $\sigma(O^n) \supseteq \sigma(J^{n,o}) \supseteq \sigma(J^{n,o}_{rat})$
Remains to show that $\sigma(O^n) \subseteq \sigma(J^{n,o}_{rat})$.
Every open set can be written as the countable union of open rectangles, so this equality holds.

Proof sketch (continued)

We also can write $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$
And $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$