Analysis 1B

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Rijen

Notatie rij:

- (a_n)
- $\{a_n\}$

Reeks:

$n \to a_n = a(n)\ \mathbb{N} \to \mathbb{R}$

$\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n = S_n$

$S_1 = a_1,\ S_2 = a_1 + a_2.$

(S_n) is weer een rij.

Vraag

Convergeert een rij?

Rijen

Voorbeeld

$$a_n = \frac{1}{n}, c \in \mathbb{N}$$
$$\lim_{n\to\infty} = 0$$
$$S_n = \sum_{k=1}^{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Convergeert (S_n)? Neem S_{2^m}

$$\frac{1}{1} \geq \frac{1}{2^0} \geq \frac{1}{2} \quad \frac{1}{2} \geq \frac{1}{2^1} \geq \frac{1}{2}$$
$$\frac{1}{3} + \frac{1}{4} \geq \frac{2}{2^2} \geq \frac{1}{2} \quad \frac{1}{5} + \dots + \frac{1}{8} \geq \frac{4}{2^3} \geq \frac{1}{2} \dots$$

$S_{2^m} = 1 + \frac{m}{2}$. Dus divergeert.

Voorbeeld

$a_n = \frac{1}{n^2}$

$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \to_{n\to\infty} \frac{\pi^2}{6}$

Rij van partiele sommen: (S_n) heet de reeks

$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

Let op!

Als de rij convergent is hoeft de reeks nog niet convergent te zijn!

Voorbeeld

$a_n = \frac{1}{2^n}, n \in \mathbb{N}$

$\sum_{k=1}^{n} \frac{1}{2^k} = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$

De meetkundige reeks

$\sum_{k=1}^{n} r^k = 1 + r^1 + r^2 + \cdots + r^n$

$(1 + r^1 + r^2 + \cdots + r^n)(1 - r) = 1 - r^{n+1}$

$\sum_{k=1}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

$\sum_{k=1}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=1}^{n} r^k = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} =$

$\frac{1}{1-r} - \frac{1}{1-r} \lim_{n\to\infty} r^{n+1} = \frac{1}{1-r}$

Voorbeeld (vervolg)

$$\lim_{n\to\infty} r^n = 0 \text{ voor } |r| < 1$$

$$\lim_{n\to\infty} r^n = 1 \text{ voor } r = 1$$

$$\lim_{n\to\infty} r^n = ? \text{ voor } r < -1 \text{ of } r > 1$$

$$\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2$$

Rijen

Convergeert $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$?

$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} (\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{2} + \frac{1}{2} - \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} \to 1$

Want $\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} = \frac{(A+B)k+A}{k(k+1)}$

Stelling

Als $\sum_{n=1}^{\infty} a_n$ convergent is dan geldt:

$\lim_{n\to\infty} a_n = 0$

Let op! $\frac{1}{n} \to 0$ maar $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$

Rijen

$$\sum_{k=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$
$$S_n - S_{n-1} = a_n$$
$$R_n = S_{n-1} \ \lim_{n\to\infty}(S_n - R_n) = \lim_{n\to\infty} a_n = 0$$
$$\lim_{n\to\infty}(S_n - R_n) = \lim_{n\to\infty}(S_n) - \lim_{n\to\infty}(R_n) = S - S = 0$$

Stelling

$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ beiden convergeren, dan

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

Rijen

Stelling

Als $\sum_{n=1}^{\infty} |a_n| < \infty$ convergeert dan heet $\sum_{n=1}^{\infty} a_n$ absoluut convergent.

Stelling

Als $\sum_{n=1}^{\infty} a_n$ absoluut convergent is dan is hij ook convergent.

$$\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow -\infty < \sum_{n=1}^{\infty} a_n < \infty$$

Rijen

Voorbeeld stelling

Het werkt niet andersom!

$$\sum_{k=1}^{\infty} \frac{(-1)^{k=1}}{k} = ln(2)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

Bewijs

$$\sum_{k=1}^{n}(\frac{1}{2}(|a_k| + a_k)) - \sum_{k=1}^{n}(\frac{1}{2}(|a_k| - a_k))$$

$$0 \leq b_n = \frac{1}{2}(|a_n| - a_n) \leq |a_n|$$

$$0 \leq c_n = \frac{1}{2}(|a_n| - a_n) \leq |a_n|$$

Bewijs (vervolg)

$$B_n = \sum_{k=1}^{n} b_k, C_n = \sum_{k=1}^{n} c_k$$

$$S_n = \sum_{k=1}^{n} |a_k|,$$

$$B_n \leq S_n \leq S \quad C_n \leq S_n \leq S$$

$$\lim_{n\to\infty} B_n = B \quad \lim_{n\to\infty} C_n = C$$

$$\lim_{n\to\infty} (B_n - C_n) = B - C \quad B_n - C_n = \sum_{k=1}^{n} a_k \text{ Convergeert!}$$

$$\sum_{k=1}^{n} (\frac{1}{2}(|a_k| + a_k)) - \sum_{k=1}^{n} (\frac{1}{2}(|a_k| - a_k)) = \sum_{k=1}^{n} a_k$$

Rijen

$1.4323232323\cdots = 1.4\overline{32} \in \mathbb{Q}?$
$\frac{1}{3} = 0.\overline{3}$
$1.4\overline{32} = 1.4 + 0.0\overline{32} = 1.4 + 0.032 + 0.00\overline{32} =$
$1.4 + \frac{32}{1000} + \frac{32}{100000} + \dots$
$= 1.4 + \frac{32}{10^3} + \frac{32}{10^5} + \frac{32}{10^7} + \dots$
$= 1.4 + \frac{32}{1000}(1 + \frac{1}{100} + \frac{1}{10000} + \dots)$
$= 1.4 + \frac{32}{1000}(1 + 10^{-2} + 10^{-4} + \dots)$
$r = \frac{1}{100}$
$1.4 + \frac{32}{1000}\sum_{k=1}^{n}\frac{1}{100^k} = \frac{1}{1-\frac{1}{100}} = 1.4 + \frac{32}{1000}\frac{100}{99} = \frac{14.99+32}{10*99} = \frac{709}{495}$