Measure Theory and Integration

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Semi-rings

Semi-ring $\mathcal{S}$

- $\emptyset \in \mathcal{S}$
- $S, T \in \mathcal{S} \Rightarrow S \cap T \in \mathcal{S}$
- $S, T \in \mathcal{S} \Rightarrow S \setminus R = \dot\cup_{j=1}^{m} S_j, S_j \in \mathcal{S}$

Example

Rectangles in $\mathbb{R}^n$, $[a_1, b_1) \times \cdots \times [a_n, b_n)$

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Theorem (aratheodory)

$\mathcal{S}$ a semi ring, with $\mu : \mathcal{S} \to [0, \infty]$ such that:

- $\mu(\emptyset) = 0$
- if $S_j \in \mathcal{S}$ disjoint and $\dot\cup_{j=1}^{\infty} S_j \in S$, then $\mu(\dot\cup_{j=1}^{\infty} S_j) = \sum_{j=1}^{\infty} \mu(S_j)$.
  THEN μ has an extension to a measure on $\sigma(\mathcal{S})$. By last weeks theorem, it is also unique.

Example

$\mathcal{S}$ rectangles.
$\mu([a_1, b_1) \times \cdots \times [a_n, b_n)]) = \prod_{i=1}^{\infty}(b_i - a_i)$ ('volume')

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Claim (1 dimension)

Suppose $\bigcup_{i=1}^{\infty}[a_i, b_i) = [a, b)$
With $[a_i, b_i) = I_i$, and $I_i \cap I_j = \emptyset \ \forall i \neq j$, show that
$\sum_{i=1}^{\infty} \mu(I_i) = \mu([a, b))$

Proof of claim

Watch out for ∞ in the sum and let $I_j^{O(\epsilon)} = (a_j - \frac{\epsilon}{2^j}, b_j)$.
Then we have $[a, b-\epsilon] \subseteq \bigcup_j I_j^{O(\epsilon)}$.
By properties of compactness (topology) there exists a finite open subcover.
$\exists N = N(\epsilon)$ such that $[a, b-\epsilon] \subseteq \bigcup_{j=1}^N I_j^{O(\epsilon)}$.
Note that $\bigcup_{j=1}^N I_j \subseteq [a, b)$.
$0 \leq \mu([a, b)) - \sum_{j=1}^N \mu(I_j) =$
$\mu([a, b-\epsilon)) + \epsilon - \sum_{j=1}^N \mu(I_j^{O(\epsilon)}) + \sum_{j=1}^N \frac{\epsilon}{2^j} \leq 2\epsilon$.
(Because $\mu([a, b-\epsilon)) \leq \sum_{j=1}^N \mu(I_j^{O(\epsilon)})$).
First let $N \to \infty$ and then $\epsilon \to 0$, because N depends on ϵ.
So $0 \leq \mu([a, b)) - \sum_{j=1}^\infty \mu(I_j) \leq 2\epsilon$ then
$0 \leq \mu([a, b)) - \sum_{j=1}^\infty \mu(I_j) \leq 0$.
$\mu([a, b)) = \sum_{j=1}^\infty \mu(I_j)$

Lebesque measure

A Lebesque measure on $\mathbb{B}(\mathbb{R}^n)$ is denoted λ^n and $\lambda^1 = \lambda$.
$[a, b) \subseteq [a, b] \subseteq [a, b + \epsilon)$
$\lambda([a, b)) \leq \lambda([a, b]) \leq \lambda([a, b + \epsilon))$
$b - a \leq \lambda([a, b]) \leq b - a + \epsilon$
So $\lambda([a, b]) = b - a$. $\lambda(\{3\}) = 0$.
$\lambda(\mathbb{Q}) = 0$, countable union of points.
All countable sets have Lebesque measure 0.

Example

Uncountable set C with $\lambda(C) = 0$.
Cantor set: iterative take out the range $\frac{1}{3}$ to $\frac{2}{3}$ from all intervals.
Then $C = \bigcap_{i=1}^{\infty} C_1$.
$\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} (\frac{2}{3})^{n-1} = 0$. (Using continuity from above because measure is finite on the sets.)
Cantor set is in $\mathbb{B}(\mathbb{R})$ because it is a intersection of closed intervals.

Cantor set uncountable

Suppose $C = \{x_1, x_2, x_3, \ldots\}$.
C_n has 2^n intervals. A decreasing sequence of closed intervals is non empty. Then there must be a point $y \in C$ not equal to any x_i, but this is not possible if $C = \{x_1, x_2, x_3, \ldots\}$. So C is uncountable.

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Proof

- Lebesque measure is invariant under translations. That is for all $B \in \mathbb{B}(\mathbb{R}^n)$, $\lambda^n(B) = \lambda^n(x + B)$.
- If μ is a measure on $\mathbb{B}(\mathbb{R}^n)$ which is invariant under translations, and $\mu((0,1)^n) = \kappa < \infty$ then $\mu(B) = \kappa\lambda^n(B)$