Measure Theory and Integration

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Positive functions

Measure on natural numbers

$(\mathbb{N}, P(\mathbb{N}), \mu)$ with $\mu = \sum_{j=1}^{\infty} \alpha_j \delta_j$, $\alpha_j \geq 0$, δ_j point mass at j. This is a measure.
$u : \mathbb{N} \to \mathbb{R}^+$ with $u(k) = \sum_{j=1}^{\infty} u_j 1_{\{j\}}(k)$. Notation: u_k.

$$\begin{aligned}
\int_{\mathbb{N}} u d\mu &= \int_{\mathbb{N}} \sum_{j=1}^{\infty} \mu_j 1_{\{j\}} d\mu \\
&= \sum_{j=1}^{\infty} \int u_j 1_{\{j\}} d\mu \\
&= \sum_{j=1}^{\infty} u_j \mu(\{j\}) \\
&= \sum_{j=1}^{\infty} u_j \alpha_j
\end{aligned}$$

Positive functions

Fatou's lemma

$u_1, u_2, \ldots$ positive (measurable) functions.
$u = \liminf_{j\to\infty} u_j$.
Then $\int u d\mu \leq \liminf_{j\to\infty} \int u_j d\mu$

Proof

$u = \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j$.
Then $\int u d\mu = \int \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j d\mu$ but here $\inf_{j\geq k} u_j$ this is increasing.
Then $\int \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j d\mu = \sup_{k\in\mathbb{N}}(\int \inf_{j\geq k} u_j d\mu) \leq \sup_{k\in\mathbb{N}}(\inf_{j\geq k} \int u_j d\mu) = \liminf_{j\to\infty} \int u_j d\mu$. We use the monotonicity of the integrals.

General functions

Construction to general functions

Take the functions $u : (X, \mathcal{A}, \mu) \to (\mathbb{R}, \mathbb{B})$ with
$u = u^+ - u^- = \max 0, u - \min 0, u$. $\int_X u d\mu = \int u^+ d\mu - \int u^- d\mu$.
If $\int u^+ d\mu, \int u^- d\mu < \infty$, then u is **integrable**.
Write $u \in \mathcal{L}'(\mu)$.
Does this agree with previous definition?
If u is positive, then $\int u^- d\mu = 0$, so $\int u d\mu = \int u^+ d\mu$.

General functions

Properties

$u, v \in \mathcal{L}'(\mu)$

- $\alpha u \in \mathcal{L}'(\mu)$ and $\int (\alpha u) d\mu = \alpha \int u d\mu$.
  - $\alpha \geq 0$ then $(\alpha u)^+ = \alpha u^+$, $(\alpha u)^- = \alpha u^-$
    $\int \alpha u d\mu = \int (\alpha u)^+ d\mu - \int (\alpha u)^- d\mu =$
    $\alpha \int u^+ d\mu - \alpha \int u^- d\mu = \alpha \int u d\mu$
  - $\alpha < 0$ then $(\alpha u)^+ = -\alpha u^+$, $(\alpha u)^- = -\alpha u^-$
- $|u + v| \leq |u| + |v| = u^+ + u^- + v^+ + v^-$
  $(u + v)^+ - (u + v)^- = u + v = u^+ - u^- + v^+ - v^-$
  $\int (u + v)^+ + \int u^- + \int v^- = \int (u + v)^- + \int u^+ + \int v^+$
  $\int (u + v)^+ - \int (u + v)^- = \int u^+ - \int u^- + \int v^+ - \int v^-$
  $\int (u + v) d\mu = \int u d\mu + \int v d\mu$
- $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.
  $u \leq v \Rightarrow u^+ \leq v^+$, $u^- \geq v^-$.
  $\int u d\mu = \int u^+ d\mu - \int u^- d\mu \leq \int v^+ d\mu - \int v^- d\mu = \int v d\mu$

General functions

Properties (continued)

- $|\int u d\mu| \leq \int |u| d\mu$.
  $|\int u d\mu| = \max\{\int u d\mu, -\int u d\mu\} \leq$
  $\max\{\int |u| d\mu, -\int |u| d\mu\} = \int |u| d\mu$

Definition

$\int u d\mu = \int_X u d\mu$
Take $A \subseteq X$.
$\int_A u d\mu = \int_x 1_A u d\mu$.
$A, B \subseteq X$, $A \cap B \neq \emptyset$
$\int_{A \cup B} u d\mu = \int 1_{a \cap B} u d\mu = \int (1_A + 1_B) u d\mu = \int_A u d\mu + \int_B u d\mu$

General functions

Question

$u \geq 0$, $A \subseteq X$.
Let $r : A \mapsto \int_A u d\mu$, proof that r is a measure.

Notion

Take $x \in X$ and $\Pi(x)$ a property of x.
Π holds **almost everywhere** means $N = \{x | \Pi(x) \text{ fails}\}$ has $\mu(N) = 0$. N is a 'null set'
For example: x is rational, we know $\mu(\mathbb{Q}) = 0$.

Theorem

- $\int_N u d\mu = 0$ for all N such that $\mu(N) = 0$.
- $\int |u| d\mu = 0 \Leftrightarrow |u| = 0$ almost everywhere.

General functions

Proof (1)

Let $u_j = \min\{|u|, j\} \uparrow |u|$ as $j \to \infty$.
$|\int_N u d\mu| = |\int (1_N u) d\mu| \leq \int (1_N |u|) d\mu = \sup_j \int (1_N u_j) d\mu \leq \sup_j \int j 1_N d\mu = \sup_j j \int 1_N d\mu = \sup_j (j\mu(N)) = 0$

Proof (2)

'$\Leftarrow$' $\int |u| d\mu = \int_{\{|u| \neq 0\}} |u| d\mu + \int_{\{|u|=0\}} |u| d\mu = 0 + 0$.
First integral equals 0, because $\{|u| \neq 0\}$ is a null set by assumption, second integral equals 0 because we integrate a null function.

General functions

Markov inequality

$$\frac{1}{j}\mu\{|u| \geq \frac{1}{j}\} \leq \int |u| d\mu$$

Intuition: $\frac{1}{j}\mu\{|u| \geq \frac{1}{j}\}$ is the x axis where the 'height' $\geq \frac{1}{j}$, times the height $\frac{1}{j}$ is always an area under the curve, hence smaller than the intregral.

Proof (2) continued

'$\Rightarrow$' $\mu(|u| > 0) = \mu(\bigcup_{j=1}^{\infty} |u| \geq \frac{1}{j}) \leq \sum_{j=1}^{\infty} \mu(|u| \geq \frac{1}{j}) \leq \sum_{j=1}^{\infty}(j \int |u| d\mu) = 0$ by assumtion of the integral.

General functions

Theorem

Suppose $u = v$ almost everywhere. Then we have:

- $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$
- $u \in \mathcal{L}'(\mu) \Rightarrow v \in \mathcal{L}'(\mu)$ and $\int u d\mu = \int v d\mu$

Proof

$N = \{x \in X | u \neq v\}$ is a null set by assumption.
Then $\int_N u d\mu = 0 = \int_N v d\mu$
$\int u d\mu = \int_N u d\mu + \int_{N^c} u d\mu = \int_N v d\mu + \int_{N^c} v d\mu = \int v d\mu$.

General functions

Statements

- u has property Π almost everywhere
- u is almost everywhere equal to a r which satisfies Π everywhere

Those statements are not the same!
$1_{\mathbb{Q}}(x)$ is never continuous, but almost everywhere equal to 0.
Contradiction.

General functions

Theorem

$\mathcal{L}'_{\mathbb{R}}(\mu)$ does not let values reach $\pm\infty$, but $\mathcal{L}'_{\overline{\mathbb{R}}}(\mu)$ does let functions reach $\pm\infty$
$u : X \to \overline{\mathbb{R}}$
$u \in \mathcal{L}'_{\overline{\mathbb{R}}}(\mu)$.
Then there is a function $\widetilde{u} \in \mathcal{L}'_{\mathbb{R}}(\mu)$ such that $\widetilde{u} = u$ almost everywhere and $\int \widetilde{u} d\mu = \int u d\mu$.

Proof

$N = \{|u| = \infty\}$ then $N = \{\bigcap_{j=1}^{\infty}(|u| \geq j)\}$
$\mu(N) = \lim_{i\to\infty} \mu(|u| > j) \leq \lim_{j\to\infty}(\frac{1}{j}\int |u| d\mu) = 0$