Complexe Analyse

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Applications of the Residue Theorem

Integral

If $f : \mathbb{R} \to \mathbb{R}$ continuous, difine $\int_{\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$.

Theorem

Let $p.q$ polynomials such that

- $\deg(p) < \deg(q) - 1$.
- $x \in \mathbb{R} \Rightarrow q(x) \neq 0$.
- Every root of q is of order 1.
- $q(z) = 0 \Rightarrow p(z) \neq 0$.

Then $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{q(z_k)=0, Im(z_k)>0} \frac{p(z_k)}{q'(z_k)}$.

Applications of the Residue Theorem

Example

$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$. Now take $p(z) = 1$, $q(z) = z^2 + 1$.
$q(z) = 0 \Leftrightarrow z \in \{\pm i\}$. So our conditions are satisfied.
$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = 2\pi i \sum_{z_k \in \{i\}} \frac{p(z_k)}{q'(z_k)} = 2\pi i \frac{1}{2i} = \pi$.

Proof

Let $R > 0$ be big enough such that the half circle of radius R contains all zeroes of q. Then
$\int_{-R}^{R} \frac{p(x)}{q(x)} dx + \int_{C_R} \frac{p(z)}{q(z)} dz = 2\pi i \sum_{q(z_k)=0, Im(z_k)>0} Res_{z=z_k} \frac{p(z)}{q(z)}$.
The claim follows if $\int_{C_R} \frac{p(z)}{q(z)} dz \to_{R\to\infty} 0$.
$|\int_{C_R} \frac{p(z)}{q(z)} dz| \leq \pi R \max_{z \in C_R} |\frac{p(z)}{q(z)}| \leq c\pi R \frac{1}{R^2} \to_{R\to\infty} 0$. Because of condition $\deg(p) < \deg(q) - 1$. All zeroes of q of order 1, such that we write $Res_{z=z_k} \frac{p(z)}{q(z)} = \frac{p(z)}{q'(z)}$ (see last week).

Calculating Fourier Transform

Fourier transformation

$\mathcal{F} : Map(\mathbb{R}, \mathbb{R}) \to Map(\mathbb{R}, \mathbb{R})$ with $f \mapsto \mathcal{F}(f)$, with $\mathcal{F}(f)(a) = \int_{-\infty}^{\infty} f(x)\cos(ax)dx$ with $a \in \mathbb{R}$.
Let $f(z) = \frac{p(z)}{q(z)}$ be a rational function with real coefficients.
Assume $q(x) \neq 0$ for all $x \in \mathbb{R}$.
Goal: Calculate $\int_{-\infty}^{\infty} f(x)\cos(ax)dx$ and $\int_{-\infty}^{\infty} f(x)\sin(ax)dx$ with $a > 0$.

Idea

Since $\int_{-\infty}^{\infty} f(x)\cos(ax)dx = Re \int_{-\infty}^{\infty} f(x)e^{iax}dx$ and $\int_{-\infty}^{\infty} f(x)\sin(ax)dx = Im \int_{-\infty}^{\infty} f(x)e^{iax}dx$ we calculate $\int_{-\infty}^{\infty} f(x)e^{iax}dx$.

Calculating Fourier Transform

Theorem

Assume $\deg(p) \leq \deg(q) - 2$ with coefficients in $\mathbb{R}$, then $\int_{-\infty}^{\infty} \cos(ax)dx = Re(2\pi i \sum_{Im(z_k)>0, q(z_k)=0} Res_{z=z_k}(f(z)e^{iaz}))$ and $\int_{-\infty}^{\infty} \sin(ax)dx = Im(2\pi i \sum_{Im(z_k)>0, q(z_k)=0} Res_{z=z_k}(f(z)e^{iaz}))$.

Proof

$\int_{-R}^{R} f(x)e^{iax}dx =$
$2\pi i \sum_{Im(z_k)>0, q(z_k)=0} Res_{z=z_k}(f(z)e^{iaz}) - \int_{C_R} f(z)e^{iaz}dz$.
Now $|\int_{C_R} f(z)e^{iaz}dz| \leq \pi R \max_{z\in C_R} |f(z)e^{iaz}| \leq \pi R \max_{z\in C_R} |f(z)| \cdot \max_{z \in C_R} |e^{iaz}| \leq c\pi R \frac{1}{R^2}$. Because $\max_{z\in \mathbb{C}_R} |f(z)| \leq c\frac{1}{R^2}$ and $\max_{z\in C_R} |e^{iaz}| \leq 1$ because $|e^{iaz}| = |e^{ia(x+iy)}| = e^{-ay} < 1$ because $a > 0$ and $y > 0$.

Calculating Fourier Transform

Remark

Similarly for $a < 0$.

Example

$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2} \cos(2x)dx$, now $p(z) = 1$,
$q(z) = (z^2 + 4)^2 = (z - 2i)^2(z + 2i)^2$.
By the theorem we have that
$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2} \cos(2x)dx = Re(2\pi i Res_{z=2i} f(z)e^{i2z})$.
Now $f(z)e^{2iz} = \frac{g(z)}{(z-2i)^2}$ with $g(z) = \frac{1}{(z+2i)^2}e^{2iz}$. Hence
$Res_{z=z_k} f(z)e^{2iz} = g'(2i) = \frac{5}{32e^4 i}$ and
$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2} \cos(2x)dx = Re(2\pi i \frac{5}{32e^4 i}) = \frac{5\pi}{16e^4}$.

Question

What happens if q has roots on the real axis.

Calculating Fourier Transform

Theorem

Assume that q has simple roots $x_1, \dots, x_n$ along the real line and that $\int_{C_R} \frac{p(z)}{q(z)} dz \to_{R\to\infty} 0$.
Then $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx =$
$2\pi i \sum_{Im(z_k)>0, q(z_k)=0} Res_{z=z_k} \frac{p(z)}{q(z)} - \pi i \sum_{x \in \{x_1,\dots,x_n\}} Res_{z=x} \frac{p(z)}{q(z)}$.
Then
$\int_{-R}^{x_1-\rho} f(z)dz + \dots + \int_{x_n+\rho}^{R} + \sum_{i=1}^{n} \int_{C_\rho^i} f(z)dz + \int_{C_R} f(z)dz =$
$2\pi i \sum_{Im(z_k)>0, q(z_k)=0} Res_{z=z_k} f(z)$.
$\int_{C_R} f(z)dz \to_{R\to\infty} 0$
$\int_{-R}^{x_1-\rho} f(z)dz + \dots + \int_{x_n+\rho}^{R} \to \int_{-\infty}^{\infty} f(z)dz$ as $\rho \to 0$.
The claim follows if $\int_{C_\rho^i} f(z)dz \to \pi i Res_{z=x_i} f(z)$. But $f(z) = g_k(z) + \frac{B_k}{z-x_k}$ and with $B_k = Res_{z=x_k} f(z)$ and g_k analytic at x_k. So $\int_{C_\rho^i} f(z)dz = \int_{C_\rho^i} g_k(z) + B_k \int_{C_\rho^i} \frac{1}{z-x_k} = \pi i$ if $\rho \to 0$.