Measure Theory and Integration

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4 December 2017

Product measures

Existence of product measures

$(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, r)$ σ-finite measure spaces.
$\rho : \mathcal{A} \times \mathcal{C} \to [0, \infty)$.
$\rho(A \times C) = \mu(A)r(C)$.
ρ extents to a measure on $(X \times Y, \mathcal{A} \otimes \mathcal{C})$
$\rho(E) = \int_Y \int_X 1_E(x, y) d\mu(x) dr(y) = \int_X \int_Y 1_E(x, y) dr(y) d\mu(x)$

Product measures

Proof

$A_i \in \mathcal{A} \uparrow X$ $\mu(A_i) < \infty$ because σ-finite.
$C_i \in \mathcal{C} \uparrow Y$ $r(C_i) < \infty$.
$E_i = A_i \times C_i \uparrow X \times Y$.
Let D_j be the collection of all sets $D \in X \times Y$ such that
$\int_Y \int_X 1_{D \cap E_i}(x,y) d\mu(x) dr(y) = \int_X \int_Y 1_{D \cap E_i}(x,y) dr(y) d\mu(x)$.
Suppose we can show that each D_j is a Dynkin-system.
Then $\mathcal{A} \times \mathcal{C}$ is closed under intersections so
$\mathcal{A} \otimes \mathcal{C} = \sigma(\mathcal{A} \times \mathcal{C}) = \delta(\mathcal{A} \times \mathcal{C})$.
So $\mathcal{A} \otimes \mathcal{C} \subseteq D_j$ by some previous theorem.

Product measures

Proof (continued)

To proof:

- $A \times C \subseteq D_j$
  $\int \int 1_{(A\times C)\cap E}(x,y)d\mu(x)dr(y) =$
  $\int \int 1_{A_j\cap A}(x)1_{C_j\cap C}(y)d\mu(x)dr(y) =$
  $\int 1_{C_j\cap C}(y)\int 1_{A_j\cap A}(x)d\mu(x)dr(y) = \mu(A\cap A_j)r(C_j\cap C)$
- $D \in D_j$ then $D^c \in D_j$
  We have that $D^c \cap E_j = E_j \setminus (E_j \cap D)$.
  $\int_Y \int_X 1_{D^c\cap E}(x,y)d\mu(x)dr(y) =$
  $\int_Y \int_X 1_{E_j}(x,y)d\mu(x)dr(y) - \int_Y \int_X 1_{E_j\cap D}(x,y)d\mu(x)dr(y)$.
  Allowed because of σ-finiteness.
  $= \int_X \int_Y 1_{E_j}(x,y)dr(y)d\mu(x) - \int_X \int_Y 1_{E_j\cap D}(x,y)dr(y)d\mu(x)$
  by using the previous proof and work backwards.

Product measures

Proof (continued)

To proof:

- $D = \bigcup_k D_k$ disjoint union with $D_k \in D_j$ $\forall k$.
  Write $\int_Y \int_X 1_{D \cap E_j}(x, y) d\mu(x) dr(y) =$
  $\int_Y \sum_{k=1}^{\infty} \left(\int_X 1_{D_k \cap E_j}(x, y) d\mu(x) \right) dr(y) =$
  $\sum_{k=1}^{\infty} \int_Y \int_X 1_{D_k \cap E_j}(x, y) d\mu(x) dr(y) =$
  $\sum_{k=1}^{\infty} \int_X \int_Y 1_{D_k \cap E_j}(x, y) dr(y) d\mu(x)$ and work backwards.

Also need to proof that ρ is countable additive, but that is implied by the last proof of the dynkin-system.
Is $\int \int 1_{(A \times C) \cap E}(x, y) d\mu(x) dr(y)$ measurable?
Yes, because $1_{A \times C) \cap E_j} = 0$ or 1 on a measurable set.

Product measures

Tonelli's theorem

$u : X \times Y \to [0, \infty)$ $\mathcal{A} \otimes \mathcal{C}$-measurable.
$\int_{X \times Y} u d\rho = \int_Y \int_X u(x, y) d\mu(x) dr(y) = \int_X \int_Y u(x, y) dr(y) d\mu(x)$.

Fubini's theorem

$u : X \times Y \to \mathbb{R}$ $\mathcal{A} \otimes \mathcal{C}$-measurable.

1. $\int_{X \times Y} |u| d\rho$
2. $\int_Y \int_X |u(x, y)| d\mu(x) dr(y)$
3. $\int_X \int_Y |u(x, y)| dr(y) d\mu(x)$

- If one is finite $\Rightarrow$ all are finite.
- In that case Tonelli's theorem holds again.
- $x \to u(x, y)$ is $\mathcal{L}'(\mu)$ for r almost all y.
  $x \to \int_Y u(x, y) dr(y)$ is in $\mathcal{L}'(\mu)$.

Product measures

Explanation

$x \to u^{\pm}(x, y)$
$y \to \int_X u^{\pm}(x, y) d\mu(x)$
$u^{\pm} \leq |u| = u^+ + u^-$
$\int_X u^{\pm} d\mu(x) \leq \int_X |u(x, y)| d\mu(x) < \infty$ for r almost all y.
Also $\int_Y \int_X u^{\pm}(x, y) d\mu(x) dr(y) \leq \int_Y \int_X |u(x, y)| d\mu(x) dr(y) < \infty$.

Product measures

Theorem 13.11

$(X, \mathcal{A}, \mu)$ is a σ-finite measure space.
$u : X \to [0, \infty)$ measurable.
Then $\int_X u d\mu = \int_{(0,\infty)} \mu(u \geq t) d\lambda(t)$

Proof

$\int_X u(x) d\mu(x) = \int_X \int_{(0,\infty)} 1_{(0,u(x))}(t) d\lambda(t) d\mu(x) =$
$\int_X \int_{(0,\infty)} 1_E(x,t) d\lambda(t) d\mu(x)$ with $E = \{(x,t) | u(x) \geq t\}$.
With Tonelli:
$= \int_{(0,\infty)} \int_X 1_E(x,t) d\mu(x) d\lambda(t) = \int_{(0,\infty)} \int_X \mu(u \geq t) d\lambda(t)$

Product measures

Special case

$\phi : [0, \infty) \to [0, \infty)$ increasing and $\phi(0) = 0$ and continuous differentiable.
$\int_X (\phi \circ u)(x) d\mu(x) = \int_{(0,\infty)} \mu(\phi(u(x)) \geq t) d\lambda(t)$.
Is this equal to: $\int_0^\infty \mu(\phi(u(x)) \geq t) dt$?
Take $\phi(s) = t$ for some s.
Then we have that
$\int_0^\infty \phi'(s)\mu(\phi(u(x)) \geq \phi(x)) ds = \int_0^\infty \phi'(s)\mu(u(x) \geq s) ds$