Measure Theory and Integration

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Product measures

Integration over multiple variables

$f(x, y) = \ldots$
$\int_A f(\overline{x})d\overline{x}$
$\int_{(0,1)^2} xydxdy = \int_0^1 x \int_0^1 ydx\ dx$ iterative integration of single variables.
$(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, r)$ are measure spaces.
Can we find $(X \times Y, ?, \rho)$.
If $A \in \mathcal{A}$, $C \in \mathcal{C}$, we want $A \times C$ to be measurable and $\rho(A \times C) = \mu(A)r(C)$.
Assume $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, r)$ are σ-finite.
$\exists A_1, A_2, \dots \in \mathcal{A}\ A_i \uparrow X$ such that $\mu(A_i) < \infty\ \forall i$.
$\mathcal{A} \times \mathcal{C} = \{A \times C | A \in \mathcal{A}, C \times \mathcal{C}\}$ is in general not a σ-algebra. It is a semi-ring.

Product measures

Definition

$A \otimes C = \sigma(A \times C)$

Lemma

If $\mathcal{A} = \sigma(\mathcal{F})$, $\mathcal{C} = \sigma(\mathcal{G})$ and $\exists F_i \in \mathcal{F}$, $F_i \uparrow X$ and $\exists G_i \in \mathcal{G}$, $G_i \uparrow Y$ then $\sigma(\mathcal{F} \times \mathcal{G}) = \mathcal{A} \otimes \mathcal{C}$.

Product measures

Proof

$\sigma(\mathcal{F} \times \mathcal{G}) \subseteq \sigma(\mathcal{A} \times \mathcal{C}) = \mathcal{A} \otimes \mathcal{C}$
$\sigma = \{A \in \mathcal{A} | A \times G \in \sigma(\mathcal{F} \times \mathcal{G}) \ \forall G \in \mathcal{G}\}$.
Now Σ is a σ-algebra.
First condition:
$X \times G = \bigcup_{i=1}^{\infty} F_i \times G = \bigcup_{i=1}^{\infty}(F_i \times G) \in \sigma(\mathcal{F} \times \mathcal{G})$.
Check the rest yourselves.
$\mathcal{F} \subseteq \Sigma$, $\sigma(\mathcal{F}) = \mathcal{A}$, $\mathcal{A} \subseteq \Sigma$, $\Sigma \subseteq \mathcal{A}$, so $\Sigma = \mathcal{A}$.
So $\mathcal{A} \times \mathcal{G} \subseteq \sigma(\mathcal{F} \times \mathcal{G})$.
Similarly, $\mathcal{F} \times \mathcal{C} \subseteq \sigma(\mathcal{F} \times \mathcal{G})$.
So $A \times C = (A \times Y) \cap (X \times C)$
$= \bigcup_{j,k}(A \times G_k) \cap (F_j \times C) \in \sigma(\mathcal{F} \times \mathcal{G})$ because
$(A \times G_k) \in \sigma(\mathcal{F} \times \mathcal{G})$ and $(F_j \times C) \in \sigma(\mathcal{F} \times \mathcal{G})$

Product measures

Product measure functions

$(X \times Y, \mathcal{A} \otimes \mathcal{C}, ?)$ ρ should be a measure on $\mathcal{A} \otimes \mathcal{C}$.
$\rho(A \times C) = \mu(A)r(C)$.
Even better: $\mathcal{A} = \sigma(\mathcal{F})$, $C = \sigma(\mathcal{G})$, only $\rho(F \times G) = \mu(F)r(G)$ with $F \in \mathcal{F}$, $G \in \mathcal{G}$.

Intersection stable

If $\mathcal{F}$ and $\mathcal{G}$ are intersection stable, then $\mathcal{F} \times \mathcal{G}$ are also intersection stable.

Uniqueness of measures (Repetition)

$(X, \mathcal{A})$, $\mathcal{A} = \sigma(\mathcal{F})$, $\mathcal{F}$ intersection stable.
$\exists F_i \uparrow X$, $F_i \in \mathcal{F}$ $\forall i$.
So the product measure is unique.

Product measures

Finding the product measure function

$\mathcal{A} \otimes \mathcal{C} = \sigma(\mathcal{F} \times \mathcal{G})$

Exisence of measure

$\rho(A \times C) = \mu(A)r(C) = \int_Y \int_X 1_{A\times C}(x,y)d\mu(x)dr(y) = \int_X \int_Y 1_{A\times C}(x,y)dr(y)d\mu(x) = \int_C \mu(A)dr(y) = \mu(A)\int_C dr(y) = \mu(A)r(C)$.

Suggestion: if $E \in \mathcal{A} \otimes C$ then

$\rho(E) = \int_Y \int_X 1_E(x,y)d\mu(x)dr(y) = \int_X \int_Y 1_E(x,y)dr(y)d\mu(x)$.

To check:

- $1_E(x,y)$ is measurable with y fixed with respect to x.
- Also $\int_X 1_E(x,y)d\mu(x)$ is measurable with respect to y.

Product measures

Theorem

$(X, \mathcal{A}, \mu)$, $(Y, \mathcal{C}, r)$ σ-finite measure spaces.
$\rho : \mathcal{A} \times \mathcal{C} \to [0, \infty)$
$\rho(A \times C) \mapsto \mu(A)r(C)$
ρ extents (uniquely) to a measure on $(X \times Y, \mathcal{A} \otimes \mathcal{C})$ so that
$\forall E \in \mathcal{A} \otimes \mathcal{C}$,
$p(E) = \int_Y \int_X 1_E(x, y) d\mu(x) dr(y) = \int_X \int_Y 1_E(x, y) dr(y) d\mu(x)$

Integration over product spaces

Integration

$(X, \mathcal{A}, \mu)$, $(Y, \mathcal{C}, r)$
$(X \times Y, \mathcal{A} \otimes \mathcal{C}, \rho)$
$u : X \times Y \to [0, \infty)$ which is $\mathcal{A} \otimes \mathcal{C}/\mathcal{B}(\mathbb{R})$ measurable
Then
$\int_{X \times Y} u d\rho = \int_Y \int_X u(x,y) d\mu(x) dr(y) = \int_X \int_Y u(x,y) dr(y) d\mu(x)$.
This is Tonelli's theorem.

Proof

Take $0 \le f_j \uparrow u$ f_j simple.
$f_j(x,y) = \sum_{k=0}^{N_j} \alpha_k 1_{E_k}(x,y)$.
We need the following functions:
$x \to u(x,y)$, $y \to u(x,y)$ which are measurable.
Claim:
$\int_{X\times Y} f_j d\rho = \int_Y \int_X f_j(x,y) d\mu(x) dr(y) = \int_X \int_Y f_j(x,y) dr(y) d\mu(x)$
then if $j \to \infty$ we have according to the monotone convergence theorem:
$\int_{X\times Y} u d\rho = \int_Y \int_X u(x,y) d\mu(x) dr(y) = \int_X \int_Y u(x,y) dr(y) d\mu(x)$

Integration over product spaces

Example

$f(x,y) = xy^2$
$\int_{[0,1]^2} f d\lambda^2 = \int_{[0,1]} \int_{[0,1]} f(x,y) d\lambda(x)\lambda(y) =$
$\int_{[0,1]} \int_{[0,1]} xy^2 d\lambda(x)\lambda(y) = \int_{[0,1]} y^2 \int_{[0,1]} x d\lambda(x)\lambda(y)$
We can use different measures, for example the counting measure.