Measure Theory and Integration

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Repeat

Measurable functions

$f : (X\mathcal{A}) \to (X'\mathcal{A}')$
f is $(\mathcal{A}/\mathcal{A}')$ measurable if:
$f^{-1}(A') \in \mathcal{A}$ for all $A' \in \mathcal{A}$.
Special: $f(X, A) \to (\mathbb{R}, \mathbb{B})$
Take $f_1, f_2, \ldots$
$f(x) = \lim_{n\to\infty} f_n(x)$ if exists
$\liminf_{n\to\infty} f_n(x) = \sup_n(\inf_{k\geq n} f_k)$
$\limsup_{n\to\infty} f_n(x) = \inf_n(\sup_{k\geq n} f_k)$
If $f \geq 0$ then there are stepfunctions $0 \leq n_1, n_2, n_3 \uparrow f$.
Every stepfunction has the form $\sum_{i=0}^{n} y_i 1_{A_i}(x)$ (finitely many values)

Measurable functions

Claim

u, v are measurable functions

- $(u + v)(x) = u(x) + v(x)$
- $(uv)(x) = u(x)v(x)$
- $\max(u, v)(x) = \max u(x), v(x)$
- $\min(u, v)(x) = \min u(x), v(x)$

All these functions are measurable.
Why?
$\exists f_1, f_2, \ldots$ simple functions with $0 \leq f_i \uparrow u$
$\exists g_1, g_2, \ldots$ simple functions with $0 \leq g_i \uparrow u$
$f_i + g_i$ stepfunction hence, measurable.
$f_i + g_i \uparrow u + v$ so $u + v$ is measurable

Measurable functions

Corollary

u, v measurable with $(X, \mathcal{A}) \to (\mathbb{R}, \mathbb{B})$.
$\{x|u(x) \leq v(x)\} = \{x|(u - v)(x) \leq 0\} = (u - v)^{-1}((-\infty, 0]) \in \mathcal{A}$
Similarly for $u < v$, $u \neq v$, $u = v$.

Measurable function

$u = u^+ + u^-$.
$u^+ = \max\{0, u\}$ $u^- = \min\{0, u\}$

Integration

Step function

$f(x) = \sum_{j=1}^{m} y_j 1_{A_v}(x)$.
So $I_\mu(f) = \sum_{j=1}^{m} y_j \mu(A_j)$ is the integral of f with respect to μ.

Integration

Properties

- $I_\mu(1_A) = \mu(A)$
- $I_\mu(\lambda f) = \lambda I_\mu(f)$ if $\lambda \geq 0$
- $I_\mu(f+g) = I_\mu(f) + I_\mu(g)$
  $g(x) = \sum_{k=0}^{n} z_k 1_{B_k}(x)$
  $(f+g)(x) = \sum_{j=0}^{m} \sum_{k=0}^{n} (y_j + z_k) 1_{A_j \cap B_k}(x)$
  $I_\mu(f+g) = \sum_{j=0}^{m} \sum_{k=0}^{n} (y_j + z_k) \mu(A_j \cap B_k)$
  $= \sum_{j=0}^{m} y_j \sum_{k=0}^{n} \mu(A_j \cap B_k) + \sum_{k=0}^{n} z_k \sum_{j=0}^{m} \mu(A_j \cap B_k) =$
  $I_\mu(f) + I_\mu(g)$
- $0 \leq f \leq g$ gives $I_\mu(f) \leq I_\mu(g)$.
  $g = f - (g - f)$
  $I_\mu(g) = I_\mu(f) + I_\mu(g - f)$ with $I_\mu(g - f \geq 0)$.

Integration

Example

$f(x) = 1_{\mathbb{Q}}(x)$
$I_\lambda(f) = \lambda(\mathbb{Q}) = 0$

Definition of integral

$u : (X, \mathcal{A}, \mu) \to (\mathbb{R}^+, \mathbb{B})$ with u measurable.
$\int_X u d\mu = \int_X u(x) d\mu(x) = \int u d\mu = \sup\{I_\mu(g) | 0 \leq g \leq u, g \text{ simple}\}$

Integration

Proposition

if u is simple: $\int u d\mu = I_\mu(u)$.
To show:

- $I_\mu(x) \leq \int u d\mu$
  Supremum $\leq \int u d\mu$, because u is simple, so in the supremum.
- $I_\mu(x) \geq \int u d\mu$
  If $g \leq u \Rightarrow I_\mu(g) \leq I_\mu(u)$.

Theorem 9.6

$u_1 \leq u_2 \leq \ldots \uparrow u$, with $u = \sup_{i \in \mathbb{N}} u_i$ then $\int u d\mu = \sup_{i \in \mathbb{N}} \int u_i d\mu$

Corrolary

Suppose stepfunctions $f_1, f_2, \ldots \uparrow u$, then $\int u d\mu = \lim_{i \to \infty} \int f_i d\mu$.
(Sequence does not matter, we can take any sequence converging to the function)

Proof of Theorem 9.6

- $u \le v \Rightarrow \int u d\mu \le \int v d\mu$
- $\sup_j \int u_j d\mu \le \int \sup_j u_j d\mu$
  Why? $u_j \le u \Rightarrow \int u_j d\mu \le \int u d\mu$
  $\Rightarrow \sup_j \int u_j d\mu \le u d\mu$
- $f \le u$, f simple $\Rightarrow I_\mu(f) \le \sup_j \int u_j d\mu$
  $f = \sum_{k=0}^m y_k 1_{A_k}$
  $f \le u$ with $u = \sup_j u_j$
  $\alpha \in (0,1)$, then $\alpha f(x) \le u_j(x)\ \forall j \ge N(x, \alpha)\ (< \infty)$
  So $B_j = \{\alpha f \le u_j\}$ increases to X.
  $\alpha \sum_{k=0}^m y_k \mu(A_k \cap B_i) = I_\mu(\alpha 1_{B_i} f) \le \int u_j d\mu \le \sup_j \int u_j d\mu$
  because $\alpha 1_{B_i} f \le 1_{B_i} u_j \le u_j$.
  Now first take $\lim_{i \to \infty}$ so that $B_i \uparrow X$, and then $\lim_{\alpha \to 1}$.
  Then $\alpha \sum_{k=0}^m y_k \mu(A_k \cap B_i) \to \sum_{k=0}^m y_k \mu(A_k) = I_\mu(f)$.

Proof of Theorem 9.6 (continued)

- $\int 1_A d\mu = \mu(A)$
- $\int \alpha u d\mu = \alpha \int u d\mu \ \alpha \geq 0$
  Take $f_1, f_2, \ldots \uparrow u$ stepfunctions.
  $\int \alpha f_i d\mu = \alpha \int f_i d\mu$.
  For $i \to \infty$, we get $\int \alpha u d\mu = \alpha \int u d\mu$

Sums and intergrals

u_j positive measurable.
$\int \sum_{j=0}^{m} u_j d\mu = \sum_{j=0}^{m} \int u_j d\mu$
For $m \to \infty$, we get $\int \sum_{j=0}^{m} u_j d\mu = \int \sum_{j=0}^{\infty} u_j d\mu$ and we get $\sum_{j=0}^{m} \int u_j d\mu = \sum_{j=0}^{\infty} \int u_j d\mu$.
We have $= \int \sum_{j=0}^{\infty} u_j d\mu = \sum_{j=0}^{\infty} \int u_j d\mu$, so sums are interchangable.

Integrals

Example

$y \in X$ with $(X, \mathcal{A}, \delta_y)$ a measurable space with $\delta_y = \begin{cases} 1 & y \in A \\ 0 & y \notin A \end{cases}$.

Then $\int u d\delta_y = \mu(y)$.

Proof

Take stepfunction $f = \sum_{j=0}^{m} \phi_j 1_{A_j}$. We now have that $y \in A_{j_0}$.
This set is unique.
$\int f d\delta_y = \sum_{j=0}^{m} \phi_j \delta(A_j) = \phi_{j_0} = f(y)$.
Now take stepfunctions $f_k \uparrow u$, then
$\int u d\delta_y = \lim_{k\to\infty} \int f_k d\delta_y = \lim_{k\to\infty} f_k(y) = u(y)$.