# Measure Theory and Integration

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# Measurable mappings

## General setup

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be a measurable space.
Consider the mapping $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$.
If for all $A' \in \mathcal{A}'$, $T^{-1}(A') \in \mathcal{A}$, then we call $T$, $\mathcal{A}/\mathcal{A}'$ measurable.

## Example

$T : (\mathbb{R}, \mathbb{B}) \to (\mathbb{R}, \mathbb{B})$
Map horizontal to vertical intervals.

## Theorem

If $\mathcal{A}' = \sigma(\mathcal{G}')$, then if $T^{-1}(\mathcal{G}') \in \mathcal{A}$ for all $G' \in \mathcal{G}'$, then $T$ is $\mathcal{A}/\mathcal{A}'$ measureable.

# Measurable mappings

## Proof

$\Sigma' = \{A' \in \mathcal{A}' | T^{-1}(A') \in \mathcal{A}\}$.
By assumption:

- $G' \in \Sigma'$
- $\Sigma'$ is a sigma algebra.

From this it follows that $\sigma(\mathcal{G}') = \mathcal{A}' \subseteq \Sigma'$.
If $A' \in \Sigma'$, then $T^{-1}(A'^c) = (T^{-1}(A'))^c \in \mathcal{A}$

# Measurable mappings

## Example

$T : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathbb{R}^n$.
Suppose $T$ is continuous. Use topological definition: ($f^{-1}(O)$ is open for alle $O$ open sets).
$O \in \mathcal{O}$(open sets).
$T^{-1}(O)$ is open and hence in $\mathbb{B}^n$, so $T$ is measurable.

## Example

$(X_1, \mathcal{A}_1) \to^T (X_2, \mathcal{A}_2) \to^S (X_3, \mathcal{A}_3)$.
$T$ is $\mathcal{A}_1/\mathcal{A}_2$ measurable.
$S$ is $\mathcal{A}_2/\mathcal{A}_3$ measurable.
$(S \circ T)^{-1}(A_3 \in \mathcal{A}_3) = T^{-1}(S^{-1}(A_3)) \in \mathcal{A}_1$ because $S^{-1}(A_3) \in \mathcal{A}_2$, so $S \circ T$ is $\mathcal{A}_1/\mathcal{A}_3$ measurable.

# Measurable mappings

## Definition

Let $T : X \to (X', \mathcal{A}')$.
$\sigma(T)$ is the smallest $\sigma$-algebra $\mathcal{A}$ on $X$ which makes $T$ $\mathcal{A}/\mathcal{A}'$ measurable

## Example

$$T(x) = \begin{cases} 1 & x \in \mathcal{A} \\ 0 & x \notin \mathcal{A} \end{cases}$$

$\sigma(T) = \{\sigma, X, A, A^c\}$.
Check: $T^{-1}([2, \infty]) = \emptyset$.
$T^{-1}([-1, 2]) = X$.
$T^{-1}([\frac{1}{2}, \frac{3}{2}]) = A$

# Measurable mappings

## Example

$$T(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

We can now integrate this.

## Measure

$T : (X, \mathcal{A}) \to (X', \mathcal{A}')$.
$T$ is measurable. Let $\mu$ be a measure on $X$.
Let $\mu'(A') = \mu(T^{-1}(A'))$.
Claim: $\mu'$ is a measure on $\mathcal{A}'$.
If $A'_1, A'_2, \dots$ disjoint, show $\mu'(\cup_{i=1}^{\infty} A'_i) = \sum_{i=1}^{\infty} \mu'(A'_i)$.
$\mu'(\cup_{i=1}^{\infty} A'_i) = \mu(T^{-1}(\cup_{i=1}^{\infty} A'_i)) = \mu(\cup_{i=1}^{\infty} T^{-1}(A'_i)) =$
$\sum_{i=1}^{\infty} \mu(T^{-1}(A'_i)) = \sum_{i=1}^{\infty} \mu'(A'_i)$.
Notation: $\mu' = T\mu = T(\mu) = \mu T^{-1}$.

# Measurable mappings

## Example

$X = \{(i,j) | 1 \leq i,j \leq 6\}$.
$\mu((i,j)) = \frac{1}{36}$
$\mu(X) = 1$. This is the probability of throwing 2 dice.
$\xi : X \to \mathbb{R}$ with $\xi(i,j) = i + j$.
$\xi\mu(\{2\}) = \frac{1}{36} = \mu(\xi^{-1}(\{2\}))$.
$\xi\mu(\{7\}) = \frac{1}{6}$.
$\xi\mu$ is the distribution of $\xi$.

# Measurable mappings

## Example

$\lambda^n$: $n$ dimentional Lebesque measure.
$T : \mathbb{R}^n \to \mathbb{R}^n$ linear.
$\mu = \lambda^n T^{-1}$.
$\mu(x + B) = \lambda^n T^{-1}(x + B) = \lambda^n(T^{-1}(x) + T^{-1}(B)) = \lambda^n(T^{-1}(B)) = \mu(B)$.
If $T$ is orthogonal (preserves angles and distances) $T^t T = id$.
$\mu = \kappa\lambda^n$.
$\lambda^n(B_1(0)) = \lambda^n(T^{-1}(B_1(0))) = \mu(B_1(0)) = \kappa\lambda^n(B_1(0))$ with $B_1(0)$ the unit ball. So $\kappa = 1$.

# Measurable functions

## Definition

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

## Measurable function

$1_A(x) : (X, \mathcal{A}) \to (\mathbb{R}, \mathbb{B})$.

$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$.

The indicator function of $A$.

## Definition

An elementary (simple, step) function is a function that takes only finitely many values.

### Example

Suppose the function $f$ takes values $y_1, y_2, \ldots, y_M$. And let $f^{-1}(\{y_i\}) = A_i$
Then $f(x) = \sum_{i=1}^{M} y_i 1_{A_i}(x) = \sum_{i=0}^{M} y_i 1_{A_i}(x)$ with $y_0 = 0$.
Every simple function has such a representation.

### Example

$f, g$ elementary with $g(x) = \sum_{i=0}^{N} z_i 1_{B_i}(x)$ and $f$ as previous example. Then $(f+g)(x) = \sum_{i=0}^{M} \sum_{j=0}^{N} (y_i + z_j) 1_{A_i \cap B_j}(x)$ .

# Measurable functions

**Theorem**

$u : (X, \mathcal{A}) \to (\mathbb{R}, \mathbb{B})$ is the pointwise limit of stepfunction $u(x) = \lim_{j\to\infty} f_j(x)$ with $|f_j| \leq |u|$.
Idea: with integration take the bars from above and below the function to obtain the integral.

# Measurable functions

## Proof

Take $u \geq 0$.
$j \in \mathbb{N}$. Divide $y$ axis of plot into pieces of size $2^{-j}$. We need $j2^j$ pieces to obtain value $j$ on the $y$ axis.
$f_j(x) = \sum_{k=0}^{j2^j} k2^{-j} 1_{A_k^j}(x)$ where $A_k^j = \{k2^{-j} \leq u < (k+1)2^{-j}\}$.
Difference between $f_k$ and $u$ is at most $2^{-j}$, for $j \to \infty$ this goes to 0, so pointwise convergence.
This was only for positive functions.
For all functions, define: $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$.
Now if we have $f_j \to u^+$ and $h_j \to u^-$ we have that $f_j - h_j \to u^+ - u^- = u$. So we can approximate general functions with stepfunctions.

# Measurable functions

## Measurable function

Suppose $u_1, u_2, \dots$ measurable and suppose $u_i(x) \to u(x)\ \forall x$, then $u$ is also measurable.
$\liminf_{j\to\infty} u_j(x) = \lim_{k\to\infty}(\inf_{j>k} u_j(x))$.
$\limsup_{j\to\infty} u_j(x) = \lim_{k\to\infty}(\sup_{j>k} u_j(x))$.
So we need to show that $\lim_{j\to\infty}\inf u_j(x)$ and $\lim_{j\to\infty}\sup u_j(x)$ are measurable.
$\lim_{k\to\infty}(\inf_{j>k} u_j(x)) = \sup_{k\in\mathbb{N}}(\inf_{j>k} u_j(x))$.
$\lim_{k\to\infty}(\sup_{j>k} u_j(x)) = \inf_{k\in\mathbb{N}}(\sup_{j>k} u_j(x))$.
Claim: $\{\sup_{j\in\mathbb{N}} u_j > a\} = \bigcup_{j\in\mathbb{N}}\{u_j > a\}$ but this is a countable union of measurable functions, so this must be measurable.
Same holds for other limit.
So $\liminf$ and $\limsup$ are measurable, hence $\lim$ is also measurable