

Time Hybrids a new Generic Theory of Reality

This document is about the *Time Hybrids* book of *Fred Van Oystaeyen*.

Fred is an outstanding mathematician in such fields as *noncommutative algebraic geometry*, *virtual topology* and *functor geometry*.

I dedicate this document to Fred.

Fred told me about an old Chinese legend about a poet who chiselled a small poem on a small stone with a small chisel and threw that stone into the sea, hoping that, one day, someone would read the poem.

Fred compared his book with that stone.

I hope that this document will, somehow, increase the number of people that read his book.

How to read this document

This document does not present its content in a linear way.

Sections contain hyperlinks to sections that can be read by need.

Introduction

Warning

This document, especially its introduction, is a highly opinionated one.

Many sentences start with "I tend to think of", emphasizing the fact that I may be wrong.

I hope that my opinion about the book is more or less compatible with the opinion of Fred.

Specification and Implementations

A specification consists of declarations of features.

Declarations come with *laws*.

Together, features and laws are called *requirements*.

Implementations of a specification consist of *definitions* of declared features.

Definitions come with *proofs* of laws.

Note that a specification with *less* requirements allows for *more* implementations, so it is important to keep requirements as minimal as possible.

In a way you can think of a specification as a *description*.

I tend to think of a description as an implementation that is an *informal specification*.

For example, the picture on [The Treachery of Images](#) is a description (pun intended!) of a well known description of a pipe.

Frankly, if you would never have seen a pipe before, would you be able to make a pipe *only* from this description?

Generic Theory

I tend to think of a *generic theory* as a *theory consisting of specifications of theories*, a *framework theory of theories* or *template theory of theories*, where *theories fit into as implementations*.

Generic Theory of Reality

The *Time Hybrids* book describes a *Generic Theory of Reality*.

Hence I tend to think of a generic theory of reality as a *theory consisting of specifications of theories of reality*, a *framework theory of theories of reality* or *template theory of theories of reality*, where *theories of reality fit into as implementations*.

Quantum Theory and Relativity Theory

It is generally agreed upon that *quantum theory* and *relativity theory* are two *fundamental theories of reality*.

A *unifying theory* for quantum theory and relativity theory is yet to be agreed upon.

I tend to think of the generic theory of reality of the book as a *partially unifying theory* for quantum theory and relativity theory, concentrating on *common requirements*.

The book states that various phenomena of quantum theory and relativity theory, which, until now, have been considered counter-intuitive, can, within its generic theory of reality, be viewed in a more intuitive way.

The book also provides some basic insight in *how* to fit quantum theory and relativity theory into its partially unifying generic theory of reality.

Future Research

I tend to think that, providing *all details* on how to fit quantum theory and relativity theory into the partially unifying generic theory of reality, and comparing it with *observed reality*, is a challenging topic for future research.

For example, would it not be nice to be able to show that, somehow, the theory of Stephen Wolfram and Co, explained in

[https://www.ted.com/talks/stephen_wolfram_how_to_think_computationally_about_ai_the_universe_and_everything?language=en] can be seen as an implementation of the specification of Fred Van Oystaeyen?

If, one day, all those details would be provided, and if they all correspond to observed reality so far, then that would be an important scientific achievement.

If it is not possible to provide all those details, or if they do not all correspond to observed reality so far, then that would be useful as well.

The reason *why* may provide valuable insight into the generic theory of Fred and/or the theory of Stephen and/or quantum theory and/or relativity theory.

Reality and Compositionality

I tend to think of *compositionality* as an important aspect of reality.

Compositionality is about *components*.

Components can, starting from various *atomic components* be *composed* to *composed components* in various ways.

Category Theory

I tend to think of *category theory* as a *generic theory of mathematics*, a *theory consisting of specifications of theories of mathematics*, a *framework theory of theories of mathematics* or *template theory of theories of mathematics*, where *theories of mathematics* fit into as *implementations*.

Of course, as far as the book is concerned, the *theories of mathematics* involved are theories that are relevant for modeling reality.

Why Category Theory

Category theory is an *abstract* theory.

You could argue that using an abstract theory results in a steep learning curve.

But here is the thing: *abstraction* is about *simplification* and *simplicity is the ultimate sophistication*.

When I was a first year mathematics student, professor Gevers, my professor mechanics, said

- I am going to proceed slower in order to have covered more material at the end of the year.

Those were wise words.

Once you have built a solid foundation, agreed, first slowing down learning, you can, gradually, start accelerating, eventually speeding up learning.

Compositionality and Category Theory

Category theory is about *collections of objects* and *sets of morphisms*.

Collections are not necessarily sets.

Every morphism is one from a *source object* to a *target object*.

Category theory is compositional.

More precisely, compositionality of category theory is *sequential compositionality of morphisms*.

As is done in most documents about category theory, this document focuses on morphisms instead of on objects.

In general, it focusses on *pointfree concepts* rather than focussing on *pointful concepts*.

More precisely, it focusses on *pointfree, closed components* rather than on *pointful, open components*.

Programmatic notation

In this document I explain the content of the book using *programmatic notation*.

The programmatic notation defines a *domain specific language*, a.k.a. as *DSL*, that is a *library* that is written in the *Scala programming language*.

The DSL is a concise *formal language* that is syntactically verified by *Scala's* powerful type system.

Both the DSL library and the *Scala* language are briefly explained by need.

The *Scala* code is not really idiomatic code.

It is code that, i.m.h.o, is very suitable for learning purposes.

The DSL is both one the *domain of the book, reality*, being part of *physisc*, and one for the its *foundations*, being part of *mathematics*.

For the domain of the book, it uses verbose notation that, more or less, corresponds to the one used in physics.

For its foundations, it uses verbose notation that, more or less, corresponds to the one used in mathematics.

In what follows *Scala* is not explicitly mentioned any more.

Programmatic notation for specifications

Specifications are, programmatically, denoted as

- *value classes*, and,
- *type classes*,
- *unary type constructor classes*,
- *binary type constructor classes*,
-

More precisely, they are denoted as

- *traits without corresponding parameter* for value classes, and,
- *traits with corresponding parameter* for all other classes.

Programmatic notation for implementations

Implementations are, programmatically, denoted as

- *vals* for value classes, and,
- *givens* for all other classes.

Programmatic naming conventions

The programmatic notation uses names that are abbreviations using first letters of (sequences of (parts of)) words.

The programmatic notation uses names with letters of the [Greek alphabet](#) corresponding to first letters of words, according to the [Romanization of Greek alphabet](#).

The programmatic notation uses *symbolic operator names between backticks*.

Analogy between Physics and Computer Science

Lets consider, for a moment, the analogy between *physics* and *computer science*, in as far as the goal of physics is to understand what *reality* is all about, and the goal of computer science is to understand what *software* is all about.

Agreed, understanding reality is so much more difficult than understanding software: software is something we invented ourselves, while reality is, afaiik, still one great mystery.

It is now generally agreed upon that computer science benefits from category theory as a partially unifying theory of software theories into which *effectfree software theory* and *effectful software theory* fit as implementations.

Maybe, one day, it will be generally agreed upon that physics benefits from category theory as a partially unifying theory of physics theories into which *quantum theory* and *relativity theory* fit as implementations.

By the way, in a previous life, I was one of the researchers working on category theory as a partially unifying theory of effectfree software theory and effectful software theory using *monads*. I did most of my work as a "late at night hobby". I also worked two years at the University of Utrecht together with Erik Meijer, Graham Hutton and Doaitse Swierstra. I mainly wrote and taught courses, but I also did some research. Our group looked at monads as *computations* that are *computed*. Computations are generalizations of *expression* that are *evaluated*. Computations and expressions are *operational* components. Moreover, monads and expressions are *open*, pointful components.

Nowadays I look at *morphism* as *programs* that are run. Morphisms are generalizations of *functions*. Morphism and functions are *denotational* components. Moreover, morphism and functions are *closed*, pointfree components. You can look at my talk about *Program Description based Programming* on [flatMap, 8-9 May 2019, Oslo Norway](#). I am refactoring the work presented in Olso, upgrading from [Scala2](#) to [Scala3](#), and changing the paradigm name to *Program Specification based Programming* for reasons explained above.

In other words, I have been doing and I still am doing foundational work on software theories that is similar to the foundational work Fred is doing on physics theories.

Time Hybrids Domain: Specifications

Recently I attended a lecture of Fred on its book in Almeria. The lecture was accompanied with an informal paper.

Let's start with the first sentences of the abstract of the paper.

We introduce a generic model for space-time where time is just a totally ordered set ordering the states of the universe at moments where over (not in) each state we define potentials or pre-things which are going to

evolve via correspondences between the momentary potentials to existing things. Existing takes time and observing takes more time.

The following concepts are involved

- time moments,
- universe states,
- pre-things,
- space,

and,

- things.

Moreover

- pre-things are momentary (they do not really exist),
- things exist (which takes time) and,
- observing takes more time.

This document is work in progress.

For now only time moments, universe states and pre-things are dealt with.

Let's continue with another sentence of the content of the paper.

We can define a "place map", $p(t): PS(t) \rightarrow U(t)$ where some $A(t)$ is taken to an element $pA(t)$ of the nc-lattice $L(t)$ giving the topology of $U(t)$ such that $p(t)$ respects the (inclusion) partial orders on $PS(t)$ and $U(t)$.

This sentence uses notation that needs some explanation.

- Time moments are denoted as t .
- The universe state at time moment t is denoted as $U(t)$.
- The collection of sets of pre-things at time moment t is denoted as $PS(t)$.
- Sets of pre-things are denoted as $A(t)$.
- The "place map" $p(t)$ maps $A(t)$ to $pA(t)$.
- The *non-commutative lattice* on $U(t)$ is denoted as $L(t)$.

The non-commutative $L(t)$ lattice defines a *virtual topology* on $U(t)$.

$p(t)$ respects the non-commutative lattice order on $PS(t)$ and the subset order on $U(t)$.

Note that the statements above can be seen as requirements.

In what follows we denote them using programmatic notation.

Time

```
package timehybrids.specification

import specification.{Arbitrary, Ordered}
```

```
trait Time[Moment: Arbitrary: Ordered]:  
  
  // ...
```

`Time` is a type class for parameter `Moment`.

The requirements for `Moment` to be a `Time` type are

- `Moment` is an `Arbitrary` type,
- `Moment` is a `Ordered` type.

`Arbitrary` is fully explained in [Arbitrary](#).

`Ordered` is fully explained in [Ordered](#).

A type *implicitly* denotes a *set* and a *value* of a type *implicitly* denotes an *element* of a set.

Recall that we are building a DSL for reality and its foundations.

The requirements above allow us to, programmatically,

- write statements involving arbitrary time moments,
- state that one time moment is before another one.

Time moments are also simply called *moments*.

Recall that the code is not really the most idiomatic one, it is idiomatic for the DSL we are defining.

Instead of using the "*is-a*" *Object Oriented Programming* idiom we use the "*has-a*" *Functional Programming* idiom.

In other words, instead of using *inheritance* we use *delegation*.

Moreover, delegation is *implicit*.

Delegates need to be defined *explicitly* by *summoning* them.

They can then be *imported* by need and made available as *givens* by need.

Agreed, all this is somewhat verbose, but, the beauty of programmatic notation like this comes from the combination of *compactness* and *conciseness* of `trait declarations` like

- `trait Time[Moment: Arbitrary: Ordered],`

stating that, for `Moment` to be a `Time` type, `Moment` is required to be an `Arbitrary` type and an `Ordered` type.

```
// ...  
  
val ma: Arbitrary[Moment] = summon[Arbitrary[Moment]]  
  
val mo: Ordered[Moment] = summon[Ordered[Moment]]
```

```
// ...
```

Time related foundational delegates are defined.

```
// ...

val am: Moment = ma.arbitrary
```

Time related foundational members using members of **Time** related foundational delegates are defined.

You may wonder why similar members using **mo** are not defined.

This is because those members are **extensions** that are globally available.

Universe

```
package timehybrids.specification

import specification.{
  VirtualTopology,
  Sets,
  Category,
  ActionUponFunction,
  Functor
}

trait Universe[
  Set[_]: Sets,
  Morphism[_, _]: Category: ActionUponFunction,
  Moment: Time,
  State: [_] =>> VirtualTopology[Set, State]: [_] =>> Functor[
    [_, _] =>> Tuple2[Moment, Moment],
    Morphism,
    [_] =>> State
  ]
]:

// ...
```

Using the **type** definitions below you can read the **Universe** definition above as

```
trait Universe[
  Set[_]: Sets,
  Morphism[_, _]: Category: ActionUponFunction,
  Moment: Time,
  State: [_] =>> VirtualTopology[Set, State]: [_] =>> Functor[
```



```

    [_] =>> MomentMorphism,
    Morphism,
    [_] =>> State
  ]
]:

```

`Universe` is a type class for parameter `State`.

`Universe` also has a foundational parameter `Set` that is required to be a `Sets` unary type constructor.

`Sets` is fully explained in [Sets](#).

`Sets` *explicitly* denotes *the realm of all sets*.

Mathematically this is not a set.

Programmatically it is a constructive set (recall that, programmatically, a set is implicitly denoted by a type).

`Universe` also has a foundational parameter `Morphism` that is required to be a `Category` binary type constructor and a `ActionUponFunction` binary type constructor.

`Category` is fully explained in [Category](#).

`ActionUponFunction` is fully explained in [ActionUponFunction](#).

`Universe` also has a domain parameter `Moment` that is required to be a `Time` type.

Type `Tuple2[T, T]`, somewhat abusively, denotes an ordered set implicitly denoted by type `T`.

- A value `(l, r)`, somewhat abusively, denotes $\{ l \leq r \} = \{ \text{true} \}$.

Using the `type` definitions below the requirements for `State` to be a `Universe` type are

- `State` is a `VirtualTopology[Set, State]` type.
- `State` is a `□ =>> State` type,

`VirtualTopology` is fully explained in [VirtualTopology](#).

`Functor` is fully explained in [Functor](#).

```

// ...

val cs: Sets[Set] = summon[Sets[Set]]

val mc: Category[Morphism] = summon[Category[Morphism]]

val mauf: ActionUponFunction[Morphism] =
  summon[ActionUponFunction[Morphism]]

// ...

```

Foundational delegates are defined.

```
// ...

val mt: Time[Moment] = summon[Time[Moment]]

// ...
```

Time related domain delegates are defined.

```
// ...

type MomentMorphism = Tuple2[Moment, Moment]

// ...
```

Time related domain types are defined.

```
// ...

// `Universe` related foundational delegates are defined.

val svt: VirtualTopology[Set, State] =
  summon[VirtualTopology[Set, State]]

val mmΦsm: Functor[[_] =>> MomentMorphism, Morphism, [_] =>> State] =
  summon[Functor[[_] =>> MomentMorphism, Morphism, [_] =>> State]]

// ...
```

Universe related foundational delegates are defined.

```
// ...

type StateMorphism = Morphism[State, State]

// ...
```

Universe related domain types are defined.

```
// ...

val mmφsm: Function[MomentMorphism, StateMorphism] = mmΦsm.φ

val svts: Function[Set[State], State] = svt.sup
```

```
// ...
```

Universe related foundational members using members of **Universe** related foundational delegates are defined.

Composition2

```
package types

import specification.{Sets}

enum Composition2[Set[_]: Sets, Z]:
  case Atomic[Set[_]: Sets, Z](z: Z) extends Composition2[Set, Z]
  case Composed[Set[_]: Sets, Z](cs2: Set[Composition2[Set, Z]])
    extends Composition2[Set, Z]

import Composition2.{Composed}

def composition2[Set[_]: Sets, Z]
  : Set[Composition2[Set, Z]] => Composition2[Set, Z] =
  Composed.apply

def decomposition2[Set[_]: Sets, Z]
  : Composition2[Set, Z] => Set[Composition2[Set, Z]] =

  val sets: Sets[Set] = summon[Sets[Set]]

  import sets.{set2}

  c => set2 apply (c, c)
```

Composition2 is an example of *structural compositionality*.

Note that the **Set[Composition2[Set, Z]]** is a **Set2[Composition2[Set, Z]]**.

PreThings

```
package timehybrids.specification

import types.{Composition2, composition2, decomposition2}

import specification.{
  Arbitrary,
  Ordered,
  Sets,
  Category,
  ActionUponFunction,
  Functor,
```

```

Transformation
}

import implementation.{
  orderedCategory,
  functionCategory,
  functionValuedFunctor2
}

trait PreThings[
  Set[_],
  Morphism[_],
  Moment,
  State: [_] =>> Universe[Set, Morphism, Moment, State],
  PreObject: [_] =>> Arbitrary[
    Set[Set[Composition2[Set, PreObject]]]
]: [_] =>> Functor[
  [_] =>> Tuple2[Moment, Moment],
  Function,
  [_] =>> Set[Composition2[Set, PreObject]]
]: [_] =>> Transformation[
  [_] =>> Tuple2[Moment, Moment],
  Function,
  [_] =>> Set[
    Set[
      Composition2[Set, PreObject]
    ]
  ],
  [_] =>> Set[Set[Composition2[Set, PreObject]]]
]: [_] =>> Function[Set[Composition2[Set, PreObject]], State]
]:

// ...

```

Using the `type` definitions below you can read the `PreThings` definition above as

```

trait PreThings[
  Set[_],
  Morphism[_],
  Moment,
  State: [_] =>> Universe[Set, Morphism, Moment, State],
  PreObject: [_] =>> Arbitrary[
    PreInteractionsSet
]: [_] =>> Functor[
  [_] =>> MomentMorphism,
  Function,
  [_] =>> PreThingsSet
]: [_] =>> Transformation[
  [_] =>> MomentTransition,
  Function,
  [_] =>> Set2[PreThingsSet],
  [_] =>> PreInteractionsSet
]

```

```
]: [_] =>> Function[PreThingsSet, State]
]:
```

`PreThings` is a type class for parameter `PreObject`.

`PreThings` also has a foundational parameter `Set`.

`PreThings` also has a foundational parameter `Morphism`.

`PreThings` also has a domain parameter `Moment`.

`PreThings` also has a domain parameter `State` that is required to be a `Universe[Set, Morphism, Moment, State]` type.

Using the `type` definitions below the requirements for `PreObject` to be a `PreThings` type are

- `PreObject` is an `Arbitrary[PreInteractionsSet]` type.
- `PreObject` is a `☐ =>> PreThingsSet` type.
- `PreObject` is a `Transformation[[_], _] =>> MomentMorphism, Function, ☐ =>> Set2[PreThingsSet], ☐ =>> PreInteractionsSet` type.

`Transformation` is fully explained in [Transformation](#).

`functionValuedFunctor2`, is fully explained in [functionValuedFunctor2](#).

`orderedCategory`, is fully explained in [orderedCategory](#).

`functionCategory`, is fully explained in [functionCategory](#).

```
// ..

val su: Universe[Set, Morphism, Moment, State] =
  summon[Universe[Set, Morphism, Moment, State]]

// ...
```

`Universe` related domain delegates are defined.

```
// ...

import su.{cs}

import cs.{Set2}

type PreThing = Composition2[Set, PreObject]

type PreThingsSet = Set[PreThing]
```

```

type PreInteraction = Set2[PreThing]

type PreInteractionsSet = Set[PreInteraction]

// ...

```

PreThings related domain types are defined.

```

import su.{MomentMorphism}

val pisa: Arbitrary[PreInteractionsSet] =
  summon[Arbitrary[PreInteractionsSet]]

val mmϕptsf: Functor[
  [_, _] =>> MomentMorphism,
  Function,
  [_] =>> PreThingsSet
] =
  summon[
    Functor[
      [_, _] =>> MomentMorphism,
      Function,
      [_] =>> PreThingsSet
    ]
  ]

val ptss2Tpis: Transformation[
  [_, _] =>> MomentMorphism,
  Function,
  [_] =>> Set2[PreThingsSet],
  [_] =>> PreInteractionsSet
] = summon[
  Transformation[
    [_, _] =>> MomentMorphism,
    Function,
    [_] =>> Set2[PreThingsSet],
    [_] =>> PreInteractionsSet
  ]
]

val ptsϕs: Function[PreThingsSet, State] =
  summon[Function[PreThingsSet, State]]

// ...

```

PreThings related foundational delegates are defined.

```

// ...

```

```

val apis: Set[PreInteraction] = pisa.arbitrary

val mmϕptsf: Function[
  MomentMorphism,
  Function[PreThingsSet, PreThingsSet]
] = mmϕptsf.ϕ

val ptss2ϕtpis: Function[
  Set2[PreThingsSet],
  PreInteractionsSet
] = ptss2Tpis.τ

// ...

```

PreThings related foundational members using members of **PreThings** related foundational delegates are defined.

```

import su.{mc, mauf}

given Sets[Set] = cs

given Category[Morphism] = mc

given ActionUponFunction[Morphism] = mauf

```

Foundational **givens** using **imported** foundational delegates are defined.

```

// ...

import su.{mt, mmϕsm}

import mt.{ma, mo}

given Arbitrary[Moment] = ma

given Ordered[Moment] = mo

// ...

```

Time related foundational **givens** are defined.

```

given Functor[[_], _] =>> MomentMorphism, Morphism, [_] =>> State] = mmϕsm

// ...

```

Universe related foundational **givens** are defined.

```
// ...

given mmφptss2f: Functor[
  [_ , _] =>> MomentMorphism,
  Function,
  [_] =>> Set2[PreThingsSet]
] = functionValuedFunctor2[
  Set,
  [_ , _] =>> MomentMorphism,
  [_] =>> PreThingsSet
]

// ...
```

PreThings related foundational **givens** are defined.

```
// ...

val mmφptss2f: Function[
  MomentMorphism,
  Function[
    Set2[PreThingsSet],
    Set2[PreThingsSet]
  ]
] = mmφptss2f.φ

// ...
```

PreThings related foundational members using **PreThings** related foundational **givens** are defined

```
// ...

val pisφpts: Function[PreInteractionsSet, PreThingsSet] =
  pic =>
    for {
      pi <- pic
    } yield {
      composition2 apply pi
    }

val ptsφpis: Function[PreThingsSet, PreInteractionsSet] =
  pts =>
    for {
      pt <- pts
    } yield {
      decomposition2 apply pt
    }
```



```

val mmøpisf: Function[
  MomentMorphism,
  Function[PreInteractionsSet, PreInteractionsSet]
] = mm => ptsøpis `o` mmøptsf(mm) `o` pisøpts

// ...

```

Composed [PreThings](#) related foundational members using [PreThings](#) related foundational members using [PreThings](#) related foundational [givens](#) are defined.

The laws of [PreThingsRealityLaws](#), [PreThingsRealityLaws](#) are fully explained in [PreThingsRealityLaws](#).

Time Hybrids Domain: Laws

PreThingsLaws

Back to [PreThings](#)

```

// ...

// laws

import specification.{Law}

import implementation.{composedFunctor}

trait PreThingsFunctorCompositionLaws[L[_]: Law]:

  given støptcf: Functor[Morphism, Function, [_] =>> PreThingsSet]

  val composition2: L[
    Functor[
      [_, _] =>> MomentMorphism,
      Function,
      [_] =>> PreThingsSet
    ]
  ] = {
    mmøptsf
  } `=` {
    composedFunctor[
      [_, _] =>> MomentMorphism,
      Morphism,
      Function,
      [_] =>> State,
      [_] =>> PreThingsSet
    ]
  }

// ...

```

This law refers to the following excerpts from the paper.

Thus the total order of Time is just the total order of the states of the universe and we may think of U as a book with pages $U(t)$ indexed by elements of T .

The pages will be glued together by some maps $f(t,t')$ for $t < t'$ in T , where $<$ is the order of T

We let $s(t,t')$ denote a map from $PS(t)$ to $PS(t')$ which may be viewed as a correspondence from $S(t)$ to $S(t')$, this mathematical concept is just a map from subsets of $S(t)$ to subsets of $S(t')$

Although it is nowhere mentioned in the book or paper, this optional law relating $f(t,t')$ and $s(t,t')$ looks natural to me.

```
// ...

import implementation.{functionTargetOrdered, setOrdered}

trait PreThingsLaws[L[_]: Law]:

  val preInteractionAsPreThingComposition
    : L[PreInteraction => PreInteraction] = {
    decomposition2 `o` composition2
  } `=` {
    identity
  }

  val preThingAsPreInteractionDecomposition: L[PreThing => PreThing] = {
    composition2 `o` decomposition2
  } `=` {
    identity
  }

// ...
```

This law refers to the following, slightly adapted, excerpt from the book (it is taken for granted in the paper).

In fact, the difference between pre-things and pre-interactions is one of language only, we may just as well call a pre-thing $A(t)$ a pre-interaction $i(A(t), A(t))$.

```
// ...

val noPreThingFromNothing: MomentMorphism => L[PreThingsSet] =
  mm =>
    import cs.{set0}
    {
      mmφptsf(mm)(set0)
    } `=` {
      set0
    }
```

```
// ...
```

This law refers to the following, slightly adapted, excerpt from the paper.

Moreover the correspondence acting on the empty set is always the empty set; thus no pre-things arise as the result of a correspondence of the empty set! No pre-thing comes from nothing!

```
// ...
```

```
val unionOfSingletonPreInteractions
  : Set2[PreThingsSet] => L[PreInteractionsSet] =
  ptss2 =>
    import cs.{tuple2, set1, set2, union}
    tuple2(ptss2) match
      case (lpts, rpts) =>
        {
          union {
            for {
              lpt <- lpts
              rpt <- rpts
            } yield {
              ptss2φtpis(set2(set1(lpt), set1(rpt)))
            }
          }
        } `=` {
          ptss2φtpis(ptss2)
        }
    }
```

```
// ...
```

This law refers to the following excerpt from the paper, where [1] refers to the book.

The pre-interaction between $A(t)$ and $B(t)$ in $S(t)$ is written as $I(A,B)(t)$, in [1] I put $I(A,B)(t)$ equal to $\vee\{i(a(t),b(t))$ for $a(t)$ in $A(t), b(t)$ in $B(t)\}$.

```
// ...
```

```
val naturePreservingPreInteraction: MomentMorphism => L[Boolean] =
  mm =>
    {
      { mmφpisf(mm) `o` ptss2φtpis } `<=` {
        ptss2φtpis `o` mmφptss2f(mm)
      }
    }
    `=` {
      true
    }
  }
```

```
// ...
```

This law refers to the following excerpt from the paper (I changed $<$ to \leq).

However there is then a logical assumption, namely that $s(t,t')(I(A,B)(t)) \leq I(A,B)(t')$ for $t < t'$, meaning that the correspondences $s(t,t')$ do not change the nature of the later realisation as an interaction. Hence the $s(t,t')$ respect the dichotomy between pre-interactions and other potentials we will call pre-objects, both together are pre-things.

This is. i.m.h.o., really the most fundamental law of the realm of pre-things.

```
// ...

trait PreThingsPlacesLaws[L[_]: Law]:

  val orderPreserving
    : PreThingsSet => PreThingsSet => L[Boolean] =
    lpts =>
      rpts =>
        import su.{svt}
        import svt.{`<=<`}
        {
          lpts `<=<` rpts `=<` true
        } `=<` {
          ptsφs(lpts) `<=<` ptsφs(rpts) `=<` true
        }

// ...
```

This law refers to the following excerpt from the paper (already mentioned in the introduction).

We can define a “place map”, $p(t): PS(t) \dashrightarrow U(t)$ where some $A(t)$ is taken to an element $pA(t)$ of the nc-lattice $L(t)$ giving the topology of $U(t)$ such that $p(t)$ respects the (inclusion) partial orders on $PS(t)$ and $U(t)$.

```
// ...

val supremumOfAllSingletonPlaces: PreThingsSet => L[State] =
  pts =>
    import cs.{set1}
    import su.{svts}
    {
      svts {
        for {
          pt <- pts
        } yield ptsφs(set1(pt))
      }
    } `=<` {
      ptsφs(pts)
    }
```

```
}

// ...
```

This law refers to the following excerpt from the paper, where [2] refers to the "Virtual topology and functor geometry book" of Fred

In [2], I took $pA(t) = V\{p(\{a(t)\}), a(t) \in A(t)\}$ – we then say p is basic - which is harmless and seems logical for the notion of “place” yet we do not use that here.

```
// ...

val immobileAfter
  : MomentMorphism => L[Function[PreThingsSet, State]] =
  mm =>
    import su.{mmφsm}
    {
      ptsφs `o` mmφptsf(mm)
    } `=` {
      mmφsm(mm) `a` ptsφs
    }

val immobileOnInterval: MomentMorphism => PreThingsSet => L[State] =
  case (bm, em) =>
    import cs.{Interval, interval, all}
    import su.{mmφsm}
    val mi: Interval[Moment] = interval apply ((bm, em))
    pts =>
      all apply {
        for {
          m <- mi
        } yield {
          {
            (ptsφs `o` mmφptsf((bm, m)))(pts)
          } `=` {
            (mmφsm((bm, m)) `a` ptsφs)(pts)
          }
        }
      }
    }
```

This law refers to the following, slightly adapted, excerpt from the paper.

A string of correspondences, over an interval $I=[t,t']$, starting with $A(t)$ then yields places $p(A(t''))$ with t'' in I . If $f(t,t'')p(A(t))=p(A(t''))$ for all t'' in I , then we say the pre-thing A is immobile on I .

`composedFunctor`, is fully explained in `composedFunctor`.

`functionTargetOrdered`, is fully explained in `functionCategory`.

`setOrdered`, is fully explained in `functionValuedFunctor2`.

Back to [PreThings](#)

Mathematical Foundations Domain: Specifications

Types

Back to [TripleLaws](#)

Back to [CompositionNaturalTransformation](#)

Back to [UnitNaturalTransformation](#)

```
package types

type `o` = [G[_], F[_]] =>> [T] =>> G[F[T]]

type U = [T] =>> T
```

``o`` defines unary type constructor composition.

`U` defines the unary type constructor unit.

Back to [UnitNaturalTransformation](#)

Back to [CompositionNaturalTransformation](#)

Back to [TripleLaws](#)

Arbitrary

Back to [Time](#)

```
package specification

trait Arbitrary[T]:

  // ...
```

`Arbitrary` is a type class for parameter `T`.

```
// ...

// declared

def arbitrary: T
```

`Arbitrary` features are declared:

Arbitrary does not come with laws.

Recall that we are building a DSL for reality and its foundations.

The features above allow us to, programmatically,

- write statements involving *arbitrary elements*.

Note that **arbitrary** is declared as a **def**.

Two **arbitrary** values are not necessarily equal.

Back to [Time](#)

Ordered

Back to [Time](#)

Back to [VirtualTopology](#)

```
package specification

trait Ordered[T] extends Equality[T]:

  // ...
```

Ordered is a type class for parameter **T**.

Equality is fully explained in [Equality](#).

```
// ...

// declared

extension (lt: T) def `<`(rt: T): Boolean

// ...
```

Ordered features are declared.

```
// ...

// defined

extension (lt: T) def `<=`(rt: T): Boolean = lt `<` rt || lt `=` rt

// ...
```

Extra **Ordered** features are defined.

Recall that we are building a DSL for reality and its foundations.

The features above allow us to, programmatically,

- state that *one element is less than another one*,
- state that *one element is less than or equal to another one*.

The laws of **Ordered**, **OrderedLaws** resp. **TotallyOrderedLaws** are fully explained in **OrderedLaws**, resp. **TotallyOrderedLaws**.

Back to **VirtualTopology**.

Back to **Time**

Equality

Back to **Ordered**.

```
package specification

trait Equality[T]:

  // ...
```

Equality is a type class for parameter **T**.

```
// ...

// declared

extension (lt: T) def `=`(rt: T): Boolean

// ...
```

Equality features are declared.

Recall that we are building a DSL for reality and its foundations.

The features above allow us to, programmatically,

- state that *one element is equal to another one*.

The laws of **Equality**, **EqualityLaws** are fully explained in **EqualityLaws**.

Back to **Ordered**.

VirtualTopology

Back to [Universe](#)

```
package specification

trait VirtualTopology[Set[_]: Sets, T]
  extends Ordered[T],
    Meet[T],
    Join[T],
    Supremum[Set, T]
```

`VirtualTopology` is a type class for parameter `T`.

`Sets` is fully explained in [Sets](#).

`Ordered` is fully explained in [Ordered](#).

`Meet` is fully explained in [Meet](#).

`Join` is fully explained in [Join](#).

`Supremum` is fully explained in [Supremum](#).

Back to [Universe](#).

Meet

Back to [VirtualTopology](#)

```
package specification

trait Meet[T: Ordered: Arbitrary]:

  // ...
```

`Meet` is a type class for parameter `T`.

```
// ...

// declared

extension (lt: T) def ^ (rt: T): T

// ...
```

`Meet` features are declared.

The laws of `Meet`, `MeetLaws` are fully explained in [MeetLaws](#).

Back to [VirtualTopology](#)

Join

Back to [VirtualTopology](#)

```
package specification

trait Join[T: Ordered: Arbitrary]:

  // ...
```

Join is a type class for parameter **T**.

```
// ...

// declared

extension (lt: T) def v(rt: T): T

// ...
```

Join features are declared.

The laws of **Join**, **JoinLaws** are fully explained in [JoinLaws](#).

Back to [VirtualTopology](#)

Supremum

Back to [VirtualTopology](#)

```
package specification

trait Supremum[Set[_]: Sets, T: Ordered: Arbitrary]:
```

Supremum is a type class for parameter **T**.

Sets is fully explained in [Sets](#).

```
//

val sup: Function[Set[T], T]

// ...
```

Supremum features are declared.

The laws of **Supremum**, **SupremumLaws** are fully explained in **SupremumLaws**.

Back to **VirtualTopology**

Sets

Back to **Universe**

Back to **VirtualTopology**

Back to **Supremum**

```
package specification

trait Sets[UTC[_]] extends MonadPlus[UTC]:

  // ...
```

Sets is a unary type constructor class for parameter **UTC**.

MonadPlus is fully explained in **MonadPlus**.

```
// ...

// types

type Set0 = [Z] =>> Set[Z]

type Set1 = [Z] =>> Set[Z]

type Set2 = [Z] =>> Set[Z]

type Interval = [Z] =>> Set[Z]

// ...
```

Sets related types are defined.

Set is just a convenient type synonym of **UTC**.

```
// ...

// declared

extension [Z](lc: Set[Z]) def `=s=`(rc: Set[Z]): Boolean

extension [Z](lc: Set[Z]) def `

```

```
def tuple2[Z]: Set2[Z] => Tuple2[Z, Z]

def interval[Z: Ordered]: Option[Tuple2[Z, Z]] => Set[Z]

def all[Z, L[_]: Law]: Set[L[Z]] => L[Z]

// ...
```

Sets features are declared.

```
// ...

// defined

def set0[Z]: Set0[Z] = ζ

def set1[Z]: Z => Set1[Z] = v

extension [Z](ls: Set[Z]) def u(rs: Set[Z]): Set[Z] = ls `+` rs

def set2[Z]: Tuple2[Z, Z] => Set2[Z] = (l, r) => set1(l) u set1(r)

def union[Z]: Set[Set[Z]] => Set[Z] = μ

extension [Z](ls: Set[Z])
  def `<=s<=`(rs: Set[Z]): Boolean = ls `<s<` rs || ls `=s=` rs

// ...
```

Sets members are defined.

The laws of **Sets**, **SetsLaws**, are fully explained in **SetsLaws**.

Back to [Supremum](#)

Back to [VirtualTopology](#)

Back to [Universe](#)

Functor

Back to [Universe](#)

Back to [ActionUpon](#)

Back to [Triple](#)

Back to [NaturalTransformation](#)

Back to [ActionUponNaturalTransformation](#)

```
package specification

trait Functor[FBTC[_]: Category, TBTC[_]: Category, UTC[_]]:
```

Functor is a unary type constructor class for parameter **UTC**.

Functor has two parameters **FBTC** and **TBTC** that are required to be **Category** binary type constructors.

Category is fully explained in [Category](#).

```
// ...

// declared

def φ[Z, Y]: Function[FBTC[Z, Y], TBTC[UTC[Z], UTC[Y]]]
```

Functor features are declared.

The laws of **Functor**, **FunctorLaws** are fully defined in [FunctorLaws](#).

Back to [ActionUponNaturalTransformation](#)

Back to [NaturalTransformation](#)

Back to [Triple](#)

Back to [ActionUpon](#)

Back to [Universe](#)

MonadPlus

Back to [Sets](#)

```
package specification

trait MonadPlus[UTC[_]] extends Monad[UTC], Plus[UTC]:

// ...
```

MonadPlus is a unary type constructor class for **UTC**.

Monad is fully explained in [Monad](#).

Plus is fully explained in [Plus](#).

The laws of **MonadPlus**, **MonadPlusLaws**, are fully explained in [MonadPlusLaws](#).

Back to [Sets](#)

Monad

Back to [MonadPlus](#)

```
package specification

trait Monad[UTC[_]] extends Triple[Function, UTC]:

  // ...
```

Monad is a unary type constructor class for **UTC**.

Triple is fully explained in [Triple](#).

```
// ...

// defined

extension [Z, Y](utcz: UTC[Z])
  def map(zφy: Function[Z, Y]): UTC[Y] = φ(zφy)(utcz)

extension [Z, Y](utcz: UTC[Z])
  def flatMap(zφutcy: Function[Z, UTC[Y]]): UTC[Y] = μ(utcz map zφutcy)

// ...
```

Monad members are defined.

map and **flatMap** support *powerful for-iteration notation*.

See [flatmapthatshit](#).

Back to [MonadPlus](#)

Plus

Back to [MonadPlus](#)

```
package specification

trait Plus[UTC[_]] extends UtcComposition[UTC], UtcUnit[UTC]:

  // ...
```

Plus is a unary type constructor class for **UTC**.

[UtcComposition](#) is fully explained in [UtcComposition](#).

[UtcUnit](#) is fully explained in [UtcUnit](#).

The laws of [Plus](#), [PlusLaws](#) are fully explained in [PlusLaws](#).

Back to [MonadPlus](#)

Triple

Back to [Monad](#)

```
package specification

trait Triple[BTC[_], _]: Category, UTC[_]
  extends Functor[BTC, BTC, UTC],
    CompositionNaturalTransformation[BTC, UTC],
    UnitNaturalTransformation[BTC, UTC]:

  // ...
```

[Triple](#) is a unary type constructor class for [UTC](#).

[Triple](#) has a parameter [BTC](#) that is required to be a [Category](#) binary type constructor.

[Category](#) is fully explained in [Category](#).

[Functor](#) is fully explained in [Functor](#).

[CompositionNaturalTransformation](#) is fully explained in [CompositionNaturalTransformation](#).

[UnitNaturalTransformation](#) is fully explained in [UnitNaturalTransformation](#).

```
// ...

// delegates

val c: Category[BTC] = summon[Category[BTC]]

// ...
```

[Triple](#) delegates are defined.

The laws of [Triple](#), [TripleLaws](#) are fully explained in [TripleLaws](#).

Back to [Monad](#)

UtcComposition

Back to [Plus](#)

```
package specification

trait UtcComposition[UTC[_]]:

  // ...
```

`UtcComposition` is a unary type constructor class for `UTC`.

```
// ...

// declared

extension [Z](lutc: UTC[Z]) def `+`(rutc: UTC[Z]): UTC[Z]

// ...
```

`UtcComposition` features are declared.

The laws of `UtcComposition`, `UtcCompositionLaws` are fully explained in `UtcCompositionLaws`.

Back to [Plus](#)

UtcUnit

Back to [Plus](#)

```
package specification

trait UtcUnit[UTC[_]]:

  // ...
```

`UtcUnit` is a unary type constructor class for `UTC`.

```
// ...

// declared

def ζ[Z]: UTC[Z]
```

`UtcUnit` features are declared.

Back to [Plus](#)

Category

[Back to Universe](#)[Back to Functor](#)[Back to ActionUpon](#)[Back to Triple](#)[Back to NaturalTransformation](#)[Back to ActionUponNaturalTransformation](#)

```
package specification

trait Category[BTC[_], _] extends BtcComposition[BTC], BtcUnit[BTC]:
  // ...
```

Category is a binary type constructor class for parameter **BTC**.

BtcComposition is fully explained in [BtcComposition](#).

BtcUnit is fully explained in [BtcUnit](#).

Category *explicitly* denotes *the realm of all morphisms* of categories whose objects are sets that are implicitly denoted as types.

The laws of **Category**, **CategoryLaws** are fully defined in [CategoryLaws](#).

[Back to ActionUponNaturalTransformation](#)[Back to NaturalTransformation](#)[Back to Triple](#)[Back to ActionUpon](#)[Back to Functor](#)[Back to Universe](#)

BtcComposition

[Back to Category](#)

```
package specification

trait BtcComposition[BTC[_], _]:
```

BtcComposition is a binary type constructor class for parameter **BTC**.

```
// declared

extension [Z, Y, X](yμx: BTC[Y, X]) def `o` (zμy: BTC[Z, Y]): BTC[Z, X]
```

BtcComposition features are declared.

The laws of **BtcComposition**, **BtcCompositionLaws** are fully defined in **BtcCompositionLaws**.

Back to [Category](#)

BtcUnit

Back to [Category](#)

```
package specification

trait BtcUnit[BTC[_], _]:

  // ...
```

BtcUnit is a binary type constructor class for parameter **BTC**.

```
// ...

// declared

def ι[Z]: BTC[Z, Z]
```

BtcUnit features are declared.

Back to [Category](#)

ActionUponFunction

Back to [Universe](#)

```
package specification

type ActionUponFunction = [BTC[_], _] =>> ActionUpon[Function, BTC]
```

ActionUpon is fully explained in [ActionUpon](#).

Back to [Universe](#)

ActionUpon

Back to [ActionUponFunction](#)

Back to [ActionUponNaturalTransformation](#)

```
package specification

trait ActionUpon[LBTC[_], _], RBTC[_], _]: Category]:

  // ...
```

ActionUpon is a binary type constructor class for **LBTC**.

ActionUpon has a parameter that is required to be a **Category** binary type constructor.

Category is fully explained in [Category](#).

```
// ...

// declared

def actionFunctor[Z]: Functor[RBTC, Function, [Y] =>> LBTC[Z, Y]]

// ...
```

ActionUpon features are declared.

```
// ...

// defined

extension [Z, Y, X](lyμx: RBTC[Y, X])
  def `a` (rzμy: LBTC[Z, Y]): LBTC[Z, X] = actionFunctor.φ(lyμx)(rzμy)

// ...
```

ActionUpon members are defined.

Functor is fully explained in [Functor](#).

Back to [ActionUponNaturalTransformation](#).

Back to [ActionUponFunction](#).

CompositionNaturalTransformation

Back to [Triple](#)

```
package specification

import types.`o`

trait CompositionNaturalTransformation[
  BTC[_ , _]: Category,
  UTC[_]: [_[_]] =>> Functor[BTC, BTC, UTC]
]:

  // ...
```

``o`` is fully explained in [Types](#).

`CompositionNaturalTransformation` is a unary type constructor class for `UTC`.

```
// ...

// declared

val compositionNaturalTransformation: NaturalTransformation[
  BTC,
  BTC,
  UTC `o` UTC,
  UTC
]

// ...
```

`CompositionNaturalTransformation` features are declared.

`NaturalTransformation` is fully explained in [NaturalTransformation](#).

```
// ...

// defined

def  $\mu$ [Z]: BTC[(UTC `o` UTC)[Z], UTC[Z]] =
  compositionNaturalTransformation. $\tau$ 
```

`CompositionNaturalTransformation` members are defined.

Back to [Triple](#)

UnitNaturalTransformation

Back to [Triple](#)

```
package specification

import types.{U}

trait UnitNaturalTransformation[
  BTC[_ , _]: Category,
  UTC[_]: [_[_]] =>> Functor[BTC, BTC, UTC]
]:
```

U is fully explained in [Types](#).

UnitNaturalTransformation is a unary type constructor class for **UTC**.

```
// ...

// declared

val unitNaturalTransformation: NaturalTransformation[BTC, BTC, U, UTC]

// ...
```

UnitNaturalTransformation features are declared.

NaturalTransformation is fully explained in [NaturalTransformation](#).

```
// ...

// defined

def v[Z]: BTC[U[Z], UTC[Z]] = unitNaturalTransformation.τ
```

UnitNaturalTransformation members are defined.

Back to [Triple](#)

NaturalTransformation

Back to [CompositionNaturalTransformation](#)

Back to [UnitNaturalTransformation](#)

```
package specification

trait NaturalTransformation[
  FBTC[_ , _],
  TBTC[_ , _]: Category,
```

```

    FUTC[_]: [_[_]] =>> Functor[FBTC, TBTC, FUTC],
    TUTC[_]: [_[_]] =>> Functor[FBTC, TBTC, TUTC]
] extends Transformation[FBTC, TBTC, FUTC, TUTC]:

// ...

```

`NaturalTransformation` is a value class.

`NaturalTransformation` has two parameters, `FBTC` and `TBTC` that are required to be `Category` binary type constructors.

`NaturalTransformation` has two parameters, `FUTC` and `TUTC` that are required to be `Functor` unary type constructors.

`Category` is fully explained in [Category](#).

`Functor` is fully explained in [Functor](#).

`Transformation` is fully explained in [Transformation](#).

The laws of `NaturalTransformation`, `NaturalTransformationLaws` are fully defined in [NaturalTransformationLaws](#).

Back to [UnitNaturalTransformation](#).

Back to [CompositionNaturalTransformation](#).

ActionUponNaturalTransformation

```

package specification

trait ActionUponNaturalTransformation[
  FBTC[_]: Category: [_[_], _]] =>> ActionUpon[FBTC, TBTC],
  TBTC[_]: Category,
  FUTC[_]: [_[_]] =>> Functor[TBTC, FBTC, FUTC],
  TUTC[_]: [_[_]] =>> Functor[TBTC, TBTC, TUTC]
] extends Transformation[TBTC, FBTC, FUTC, TUTC]:

// ...

```

`ActionUponNaturalTransformation` is a value class.

`ActionUponNaturalTransformation` has a parameter, `FBTC` that is required to be `Category` binary type constructor and an `ActionUpon` binary type constructor.

`ActionUponNaturalTransformation` has a parameter, `TBTC` that is required to be `Category` binary type constructor

`NaturalTransformation` has two parameters, `FUTC` and `TUTC` that are required to be `Functor` unary type constructors.

[Category](#) is fully explained in [Category](#).

[ActionUpon](#) is fully explained in [ActionUpon](#).

[Functor](#) is fully explained in [Functor](#).

[Transformation](#) is fully explained in [Transformation](#).

The laws of [ActedUponNaturalTransformation](#), [ActionUponNaturalTransformationLaws](#) are fully defined in [ActionUponNaturalTransformationLaws](#).

Transformation

Back to [PreThings](#)

Back to [NaturalTransformation](#)

Back to [ActionUponNaturalTransformation](#)

```
package specification

trait Transformation[FBTC[_], TBTC[_], FUTC[_], TUTC[_]]:
```

[Transformation](#) is a value class.

```
// declared

def τ[Z]: TBTC[FUTC[Z], TUTC[Z]]
```

[Transformation](#) features are declared.

Back to [ActionUponNaturalTransformation](#)

Back to [NaturalTransformation](#)

Back to [PreThings](#)

Mathematical Foundations Domain: Laws

Law

Back to [OrderedLaws](#)

Back to [TotallyOrderedLaws](#)

Back to [EqualityLaws](#)

```
package specification
```

```
trait Law[UTC[_]]:

  // ...
```

Law is a unary type constructor class for parameter **UTC**.

```
// ...

// declared

extension [Z, Y](lly: UTC[Y]) def `=>`(rlz: UTC[Z]): UTC[Z]

extension [Z](l: Z) def `=`(r: Z): UTC[Z]

extension [Z](ll: UTC[Z]) def `&`(r1: UTC[Z]): UTC[Z]

extension [Z](ll: UTC[Z]) def `|`(r1: UTC[Z]): UTC[Z]
```

Law features are declared.

Laws are *conditional equational* laws with *conjunction* and *disjunction*.

Back to [EqualityLaws](#)

Back to [TotallyOrderedLaws](#)

Back to [OrderedLaws](#)

OrderedLaws

Back to [Ordered](#)

```
// ...

// laws

trait OrderedLaws[L[_]: Law]:

  val reflexive: T => L[Boolean] = t =>
    {
      t `<=` t
    } `=` {
      true
    }

  val antiSymmetric: T => T => L[Boolean] = lt =>
    rt =>
    {
      lt `<=` rt `=` true & rt `<=` lt `=` true
    } `=> {
```



```

    lt `=` rt `=` true
  }

  val transitive: T => T => T => L[Boolean] = lt =>
    mm =>
      rt =>
        {
          lt `<=` mm `=` true `&` mm `<=` rt `=` true
        } `=>` {
          lt `<=` rt `=` true
        }
      }
    }
  }

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Partially ordered sets](#).

[Law](#) is fully explained in [Law](#).

Back to [Ordered](#)

TotallyOrderedLaws

Back to [Ordered](#)

```

// ...

// laws

trait TotallyOrderedLaws[L[_]: Law]:

  val stronglyConnected: T => T => L[Boolean] = lt =>
    rt =>
      {
        lt `<=` rt `|` rt `<=` lt
      } `=`` {
        true
      }
    }
  }

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Total order](#) for more details.

[Law](#) is fully explained in [Law](#).

Back to [Ordered](#)

EqualityLaws

Back to [Equality](#)

```
// ...

// laws

trait EqualityLaws[L[_]: Law]:

  val reflexive: T => L[Boolean] = t =>
    {
      t `==` t
    } `==` {
      true
    }

  val symmetric: T => T => L[Boolean] = lt =>
    rt =>
      {
        lt `==` rt `==` true
      } `=>` {
        rt `==` lt `==` true
      }

  val transitive: T => T => T => L[Boolean] = lt =>
    mm =>
      rt =>
        {
          lt `==` mm `==` true & mm `==` rt `==` true
        } `=>` {
          lt `==` rt `==` true
        }
      }
    }
```

Hopefully the laws do not surprise you.

[Law](#) is fully explained in [Law](#).

Back to [Equality](#)

MeetLaws

Back to [Meet](#)

```
// ...

// laws

trait MeetLaws[L[_]: Law]:

  val at = summon[Arbitrary[T]].arbitrary

  val greatestSmallerThanBoth: T => T => L[Boolean] =
    lt =>
      rt =>
```

```

{
  ((lt ∧ rt) `<=` lt) `=` true `&` ((lt ∧ rt) `<=` rt) `=` true
} `&` {
  (at `<=` lt) `=` true `&` (at `<=` rt) `=` true
} `=>` {
  at `<=` (lt ∧ rt) `=` true
}

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Meet](#).

Back to [Meet](#)

JoinLaws

```

// ...

// laws

trait JoinLaws[L[_]: Law]:

  val at = summon[Arbitrary[T]].arbitrary

  val smallestGreaterThanBoth: T => T => L[Boolean] =
    lt =>
      rt =>
        {
          (lt `<=` (lt v rt)) `=` true `&` (rt `<=` (lt v rt)) `=` true
        } `&` {
          (lt `<=` at) `=` true `&` (rt `<=` at) `=` true
        } `=>` {
          (lt v rt) `<=` at `=` true
        }

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Join](#).

Back to [Join](#)

SupremumLaws

Back to [Supremum](#)

```

// ...

// laws

trait SupremumLaws[L[_]: Law]:

```

```

val sets: Sets[Set] = summon[Sets[Set]]

import sets.{map, all}

val at = summon[Arbitrary[T]].arbitrary

val smallestGreaterThanAll: Set[T] => L[Boolean] =
  ts =>
    {
      all apply {
        for {
          t <- ts
        } yield t `<=` sup(ts) `=` true
      }
    } `&` {
      all apply {
        for {
          t <- ts
        } yield t `<=` at `=` true
      }
    } `=>` {
      sup(ts) `<=` at `=` true
    }

given join: Join[T]

val joinAsSupremum: Tuple2[T, T] => L[T] =
  val sets = summon[Sets[Set]]
  import sets.{set2}
  (lt, rt) =>
    {
      sup(set2(lt, rt))
    } `=` {
      lt v rt
    }

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Supremum](#).

Back to [Supremum](#)

SetsLaws

Back to [Sets](#)

```
// ...
```

```
// laws
```

```

trait SetsLaws[L[_]: Law]:

  trait Set2Related:

    def unordered[Z]: Tuple2[Z, Z] => L[Set2[Z]] =
      (l, r) =>
        {
          set2(l, r)
        } `=` {
          set2(r, l)
        }

    import implementation.{functionCategory}

    def iso2[Z]: L[Function[Tuple2[Z, Z], Tuple2[Z, Z]]] = {
      identity[Tuple2[Z, Z]]
    } `=` {
      tuple2 `o` set2
    }

    def osi2[Z]: L[Function[Set2[Z], Set2[Z]]] = {
      identity[Set2[Z]]
    } `=` {
      set2 `o` tuple2
    }

  trait EqualityRelated:

    def reflexive[Z]: Set[Z] => L[Boolean] = s =>
      {
        s `=s` s
      } `=` {
        true
      }

    def symmetric[Z]: Set[Z] => Set[Z] => L[Boolean] =
      l =>
        r =>
          {
            l `=s` r `=` true
          } `=>` {
            r `=s` l `=` true
          }

    def transitive[Z]: Set[Z] => Set[Z] => Set[Z] => L[Boolean] =
      l =>
        m =>
          r =>
            {
              { l `=s` m `=` true } `&` { m `=s` r `=` true }
            } `=>` {
              l `=s` r `=` true
            }

```

```

trait OrderedRelated:

  def reflexive[Z]: Set[Z] => L[Boolean] = s =>
    {
      s `<=s<=` s
    } `=` {
      true
    }

  def antiSymmetric[Z]: Set[Z] => Set[Z] => L[Boolean] =
    l =>
      r =>
        {
          { l `<=s<=` r } `=` true & { r `<=s<=` l } `=` true
        } `=> {
          l `=s` r `=` true
        }

  def transitive[Z]: Set[Z] => Set[Z] => Set[Z] => L[Boolean] =
    l =>
      m =>
        r =>
          {
            { l `<=s<=` m } `=` true & { m `<=s<=` r } `=` true
          } `=> {
            { l `<=s<=` r } `=` true
          }

```

Hopefully the laws do not surprise you.

[Law](#) is fully explained in [Law](#).

Back to [Sets](#)

FunctorLaws

Back to [Functor](#)

```

// ...

// laws

trait FunctorLaws[L[_]: Law]:

  def identity[Z]: L[TBTC[UTC[Z], UTC[Z]]] =
    val fbtc = summon[Category[FBTC]]
    val tbtc = summon[Category[TBTC]]
    φ(fbtc.ι[Z]) `=` tbtc.ι[UTC[Z]]

  def composition[Z, Y, X]
    : FBTC[Z, Y] => FBTC[Y, X] => L[TBTC[UTC[Z], UTC[X]]] =
    fzμy =>

```

```

fγμx =>
{
  φ(fγμx `o` fzμy)
} `=` {
  φ(fγμx) `o` φ(fzμy)
}

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Functor](#).

Back to [Functor](#)

MonadPlusLaws

Back to [MonadPlus](#)

```

// ...

// laws

trait MonadPlusLaws[L[_]: Law]:

  def mappingOverZero[Z, Y]: Function[Z, Y] => L[UTC[Y]] =
    zφy =>
    {
      for {
        z <- ζ[Z]
      } yield {
        zφy(z)
      }
    } `=` {
      ζ
    }

  def mappingOverPlus[Z, Y]: Function[Z, Y] => UTC[Z] => UTC[Z] =>
    L[UTC[Y]] =
    zφy =>
    lutc =>
    rutc =>
    {
      for {
        z <- lutc `+` rutc
      } yield {
        zφy(z)
      }
    } `=` {
      {
        for {
          lz <- lutc
        } yield {
          zφy(lz)

```

```

    }
  } `+` {
    for {
      lz <- rutc
    } yield {
      zφy(lz)
    }
  }
}

def flatMappingWithIdentityOverZero[Z]: L[UTC[Z]] = {
  for {
    utcz <- ζ[UTC[Z]]
    z <- utcz
  } yield {
    identity(z)
  }
} `=` {
  ζ
}

def flatMappingWithPlus[Z]: UTC[UTC[Z]] => UTC[UTC[Z]] =>
L[UTC[UTC[Z]]] =
  lutcutcz =>
  rutcutcz =>
  {
    for {
      lutcz <- lutcutcz
      rutc <- rutcutcz
    } yield {
      lutcz `+` rutc
    }
  } `=` {
    lutcutcz `+` rutcutcz
  }
}

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Monad](#)

Back to [MonadPlus](#)

PlusLaws

Back to [Plus](#)

```

// ...

// laws

trait PlusLaws[L[_]: Law]:

```



```
def leftZero[Z]: UTC[Z] => L[UTC[Z]] =
  utc =>
    {
      ζ `+` utc
    } `=` {
      utc
    }

def rightZero[Z]: UTC[Z] => L[UTC[Z]] =
  utc =>
    {
      utc `+` ζ
    } `=` {
      utc
    }
```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Monad](#)

Back to [Plus](#)

TripleLaws

Back to [Triple](#)

```
// ...

// laws

import specification.{Law}

trait TripleLaws[L[_]: Law]:

  import types.{`o`}

  import c.{ι}

  def associativity[Z]: L[BTC[(UTC `o` UTC `o` UTC)[Z], UTC[Z]]] = {
    μ `o` μ
  } `=` {
    μ `o` φ(μ)
  }

  def leftIdentity[Z]: L[BTC[UTC[Z], UTC[Z]]] = {
    μ `o` ν
  } `=` {
    ι
  }

  def rightIdentity[Z]: L[BTC[UTC[Z], UTC[Z]]] = {
    μ `o` φ(ν)
```

```

    } `=` {
      ι
    }
  }

```

`ι`o`` is fully explained in [Types](#).

Hopefully the requirements do not surprise you.

Anyway, see, for example, [Monad \(category theory\)](#) for more details.

Back to [Triple](#)

UtcCompositionLaws

Back to [UtcComposition](#)

```

// ...

// laws

trait UtcCompositionLaws[L[_]: Law]:

  def associativity[Z]: UTC[Z] => UTC[Z] => UTC[Z] => L[UTC[Z]] =
    lutc =>
      mutc =>
        rutc =>
          {
            (lutc `+` mutc) `+` rutc
          } `=` {
            lutc `+` (mutc `+` rutc)
          }

```

Back to [UtcComposition](#)

CategoryLaws

Back to [Category](#)

```

// ...

// laws

trait CategoryLaws[L[_]: Law]:

  def leftIdentity[Z, Y]: BTC[Z, Y] => L[BTC[Z, Y]] = zμy =>
    {
      ι `o` zμy
    } `=` {
      zμy
    }

```

```
def rightIdentity[Z, Y]: BTC[Z, Y] => L[BTC[Z, Y]] = zμy =>
{
  zμy `o` 1
} `=` {
  zμy
}
```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Category](#).

Back to [Category](#)

BtcCompositionLaws

Back to [BtcComposition](#)

```
// ...

// laws

trait CompositionLaws[L[_]: Law]:

  def associativity[Z, Y, X, W]
    : BTC[X, W] => BTC[Y, X] => BTC[Z, Y] => L[BTC[Z, W]] =
    xμw =>
      yμx =>
        zμy =>
          {
            (xμw `o` yμx) `o` zμy
          } `=` {
            xμw `o` (yμx `o` zμy)
          }
```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Category](#).

Back to [BtcComposition](#)

NaturalTransformationLaws

Back to [NaturalTransformation](#)

```
// laws

trait EqualityNaturalTransformationLaws[L[_]: Law](
  transformation: Transformation[FBTC, TBTC, FUTC, TUTC]
):
```

```

val futc = summon[Functor[FBTC, TBTC, FUTC]]

val tutc = summon[Functor[FBTC, TBTC, TUTC]]

def natural[Z, Y](using
  equality: Equality[TBTC[FUTC[Z], TUTC[Y]]])
): FBTC[Z, Y] => L[Boolean] =
  fzφy =>
    {
      {
        transformation.τ `o` futc.φ(fzφy)
      } `=` {
        tutc.φ(fzφy) `o` transformation.τ
      }
    } `=` {
      true
    }

trait OrderedNaturalTransformationLaws[L[_]: Law](
  transformation: Transformation[FBTC, TBTC, FUTC, TUTC]
):

  val futc = summon[Functor[FBTC, TBTC, FUTC]]

  val tutc = summon[Functor[FBTC, TBTC, TUTC]]

  def natural[Z, Y](using
    ordered: Ordered[TBTC[FUTC[Z], TUTC[Y]]])
  ): FBTC[Z, Y] => L[Boolean] =
    fzφy =>
      {
        {
          transformation.τ `o` futc.φ(fzφy)
        }
      } `<=` {
        {
          tutc.φ(fzφy) `o` transformation.τ
        }
      } `=` {
        true
      }

```

Hopefully the laws do not surprise you.

Anyway, see, for example, [Natural Transformation](#).

Back to [BtcComposition](#)

Back to [NaturalTransformation](#)

ActionUponNaturalTransformationLaws

Back to [ActionUponNaturalTransformation](#)

```
// ...

// laws

trait ActionUponNaturalTransformationLaws[L[_]: Law](
  transformation: NaturalTransformation[RBTC, LBTC, FUTC, TUTC]
):

  val futc = summon[Functor[RBTC, LBTC, FUTC]]

  val tutc = summon[Functor[RBTC, RBTC, TUTC]]

  def natural[Z, Y]: RBTC[Z, Y] => L[LBTC[FUTC[Z], TUTC[Y]]] =
    fzφy =>
      {
        transformation.τ `o` futc.φ(fzφy)
      } `=` {
        tutc.φ(fzφy) `a` transformation.τ
      }
```

Back to [ActionUponNaturalTransformation](#)

Mathematical Foundations Domain: Implementations

functionValuedFunctor2

Back to [PreThings](#)

```
package implementation

import specification.{Sets, Category, Functor}

given functionValuedFunctor2[
  Set[_]: Sets,
  BTC[_]: Category,
  UTC1[_]: [_[_]] =>> Functor[BTC, Function, UTC1]
]: Functor[
  BTC,
  Function,
  [Z] =>> Set[UTC1[Z]]
] with

  val sets: Sets[Set] = summon[Sets[Set]]

  val btcToFunctionFunctor = summon[Functor[BTC, Function, UTC1]]

  import types.{`o`,`a`}
```

```
import sets.{Set2, tuple2, set2}

type UTC2 = [Z] =>> (Set2 `o` UTC1)[Z]
def φ[Z, Y]: BTC[Z, Y] => Function[UTC2[Z], UTC2[Y]] =
  zμy =>
    val zφy = btcToFunctionFunctor.φ(zμy)
    d =>
      tuple2(d) match
        case (l, r) => set2(zφy(l), zφy(r))
```

Back to [PreThings](#)

orderedCategory

Back to [PreThings](#)

```
package implementation

import specification.{Arbitrary, Ordered, Sets, Category}

given orderedCategory[Set[_]: Sets, T: Arbitrary: Ordered]
  : Category[[_], _] =>> Tuple2[T, T] with

  type BTC = [_], _] =>> Tuple2[T, T]

  extension [Z, Y, X](yμx: BTC[Y, X])
    def `o`(zμy: BTC[Z, Y]): BTC[Z, X] =
      (yμx, zμy) match
        case ((l1t, lrt), (rlt, rrt)) =>
          require(l1t `<=` lrt && lrt == rlt && rlt `<=` rrt)
          (l1t, rrt)

  def ι[Z]: BTC[Z, Z] =
    val at = summon[Arbitrary[T]].arbitrary
    (at, at)
```

Back to [PreThings](#)

functionCategory

Back to [PreThings](#)

```
package implementation

import specification.{Category, ActionUponFunction, Functor}

given functionCategory: Category[Function] with

  type BTC = [Z, Y] =>> Function[Z, Y]
```

```

extension [Z, Y, X](yμx: BTC[Y, X])
  def `o` (zμy: BTC[Z, Y]): BTC[Z, X] = z => yμx(zμy(z))

  def ι[Z]: BTC[Z, Z] = z => z

given functionFunctionActionUpon: ActionUponFunction[Function] with

type BTC = [Z, Y] =>> Function[Z, Y]

def actionFunctor[Z]: Functor[BTC, Function, [Y] =>> Function[Z, Y]] =
  new:
    def φ[Y, X]: Function[
      BTC[Y, X],
      Function[Function[Z, Y], Function[Z, X]]
    ] = yμx => zμy => yμx `o` zμy

```

Back to [PreThings](#)

composedFunctor

Back to [PreThingsLaws](#)

```

package implementation

import specification.{Category, Functor}

import types.{`o`}

given composedFunctor[
  FBTC[_]: Category,
  MBTC[_]: Category,
  TBTC[_]: Category,
  F2MUTC[_]: [_[_]] =>> Functor[FBTC, MBTC, F2MUTC],
  M2TUTC[_]: [_[_]] =>> Functor[MBTC, TBTC, M2TUTC]
]: Functor[FBTC, TBTC, M2TUTC `o` F2MUTC] with

type UTC = [Z] =>> (M2TUTC `o` F2MUTC)[Z]

def φ[Z, Y]: Function[FBTC[Z, Y], TBTC[UTC[Z], UTC[Y]]] =
  summon[Functor[MBTC, TBTC, M2TUTC]].φ `o`
  summon[Functor[FBTC, MBTC, F2MUTC]].φ

```

Back to [PreThingsLaws](#)

functionTargetOrdered

Back to [PreThingsLaws](#)

```

package implementation

import specification.{Arbitrary, Ordered}

given functionTargetOrdered[Z: Arbitrary, Y: Ordered]: Ordered[Function[Z, Y]]
with

  val az: Z = summon[Arbitrary[Z]].arbitrary

  val ordered: Ordered[Y] = summon[Ordered[Y]]

  import ordered.{`= ` => `=t=`, `<` => `<t<`}

  type T = Function[Z, Y]

  extension (lt: T) def `=`(rt: T): Boolean = lt(az) `=t=` rt(az)

  extension (lt: T) def `<`(rt: T): Boolean = lt(az) `<t<` rt(az)

```

Back to [PreThingsLaws](#)

setOrdered

Back to [PreThingsLaws](#)

```

package implementation

import specification.{Sets, Ordered}

given setOrdered[Z, Set[_]: Sets]: Ordered[Set[Z]] with

  val sets: Sets[Set] = summon[Sets[Set]]

  import sets.{`=s=`, `<s<`}

  type T = [Z] =>> Set[Z]

  extension (lt: T[Z]) def `=`(rt: T[Z]): Boolean = lt `=s=` rt

  extension (lt: T[Z]) def `<`(rt: T[Z]): Boolean = lt `<s<` rt

```

Back to [PreThingsLaws](#)