Modelling Exercises 2425nj

v 2024.09.26

Contents

Warming up

Getting started – part 1

Model the following questions with Z3, and let Z3 solve them.

a) Find boolean values for A,B,C,D and E such that the following formula becomes true:

$(\neg A \Rightarrow (B \wedge D)) \wedge (\neg B \vee \neg C \vee E) \wedge \neg (\neg C \Leftrightarrow (\neg A \wedge B)) \wedge \neg E \wedge D$

b) Find two integers, p and q, such that

p + q = 37 and p * q = 286

c) Solve

$x^2 + 115x + 3066 = 0$

Does it find both roots?

d) Is 104729 a prime number? Or 837149927, or 2778545904897799?

Getting started – part 2

a) We have booleans A, B, C, D and integer N.

How to express:

'N is equal to the number of booleans (from these 4) that are true?'

b) We have 5 integers X1, X2, X3, X4, X5.

How to express that these are 5 different numbers?

c) We have booleans A, B, C and D.

How to express the condition that exactly one of them is true?

Animals (*)

You must spend exactly 400 euro and buy exactly 100 animals. A dog cost 60 euro, a cat 4 euro and a mouse 1 euro each. You have to buy at least one of each. How many of each should you buy?

(source: a student from one of the previous semesters)

Square numbers (*)

Find three distinct natural numbers each pair of which sums up to a square (i.e. a number that is the square of some natural number).

(source: Ivar Moscovich 'Het Grote Breinbreker Boek' (Dutch))

Alphametic (*)

Let Z3 solve alphametic puzzles. For details and many examples, see e.g.:
http://www.tkcs-collins.com/truman/alphamet/alphamet.shtml
or https://www.pleacher.com/mp/puzzles/mcryp.html
or https://www.math.uni-bielefeld.de/~sillke/PUZZLES/ALPHAMETIC/alphametic-mike-keith.html

```
  EARTH
    AIR                                HAVE
   FIRE            SEND                 A
  WATER            MORE               GREAT
 ------+          -----+             ------+
 NATURE           MONEY              SUMMER
```

Lettered Dice 1 (*)

We have 3 4-sides dice, having (single) letters rather than numbers on their sides.
We want to be able to make each of the words CAT, SON, POD, RIG, PEG, TAP, DIN, APE by rolling all dice together and rearranging three of them in the right order.

Let Z3 determine the symbols on each dice.
(BTW, Z3 knows the type 'String'. But it isn't necessary to use that to solve this puzzle.)

(source: adapted from https://www.mathsisfun.com/puzzles/10-digit-number.html)

Lettered Dice 2 (*)

Same exercise as the previous one, but now with 4 3-sides dice, and we want to be able to make each word (of the same list of words as before) by throwing all 4 dice and then selecting 3 of them and rearranging them in the right order.

Trucks (**)

Six trucks have to deliver pallets of obscure building blocks to a magic factory.

- Every truck has a capacity of 8000 kg and can carry at most eight pallets.
- In total, the following has to be delivered:
  - Four pallets of nuzzles, each of weight 800 kg.
  - A number of pallets of prittles, each of weight 1300 kg.
  - Eight pallets of skipples, each of weight 1000 kg.
  - Eight pallets of crottles, each of weight 1500 kg.
  - Twelve pallets of dupples, each of weight 400 kg.
- Skipples need to be cooled; only two of the six trucks have the facility for cooling skipples.
- Nuzzles are very valuable; to distribute the risk of loss, no two pallets of nuzzles may be in the same truck.

Use Z3 to figure out a truckload distribution with the maximum number of pallets that can be delivered this way. Present Z3's answer in human readable form, for example in the form of a table.

Core

Longest Path (**)

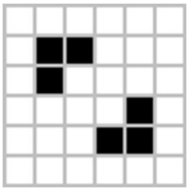
Eight points are positioned in a circle. We'll walk along the points, walking from one point to another, in such a way that:

- we never walk across the same point pair more than once, in either direction
- we never walk from a point to one of its two direct neighbouring points

Let Z3 find a walking path that satisfies the above constraints and is of maximal length (where length means the number of steps, not physical length).

(source: Ivar Moscovich 'Het Grote Breinbreker Boek' (Dutch))

Conway's Game of Life (***)



Consider a 2 dimensional grid of arbitrary size, each node of which is either 'alive' or 'dead', changing from one timestep to the next according to the following rules:

1. Any live cell with two or three live neighbours survives.
2. Any dead cell with three live neighbours becomes a live cell.
3. All other live cells die in the next generation. Similarly, all other dead cells stay dead.

Assignment:
Let Z3 find a (finite) pattern that repeats itself after some number of steps (aka 'oscillator'). (Such patterns exist already within a 6x6 grid, with period 2 or 4.)

(see: https://en.wikipedia.org/wiki/Conway%27s_Game_of_Life, and more generally https://en.wikipedia.org/wiki/Cellular_automaton. Other source: https://conwaylife.com/wiki).

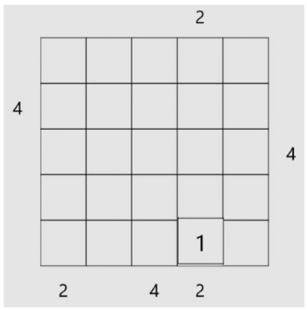
Coin denominations (**)

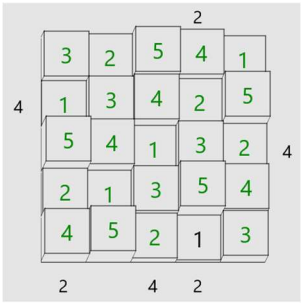
A country mints 3 denominations of coins, in whole numbers of centos. It requires 3 of these coins to make 20 centos, or 23 centos, or 29 centos. What are the denominations of coins? (There's only one solution.)

(source: https://erich-friedman.github.io/puzzle/npr/)

Skyscrapers (aka Towers) (***)

Find a 'general' way to solve puzzles of the following kind:



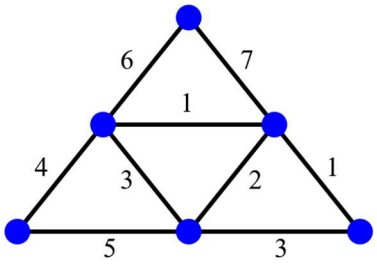


For the precise rules, variants and other details, see:
https://brainbashers.com/skyscrapers.asp.
See also "Simon Tatham's Puzzles".

Your solution should be able to solve at least 'hard' 4x4 and 5x5 puzzles.

Traveling Salesman (***)

Make use of Z3 to find the cheapest path to visit all six 'cities' in this picture. (The numbers along the edges indicate the travelcost between the cities).



Josephus Puzzle (***)

We have arranged 40 people in a circle. The people are numbered from 1 to 40, and they are arranged in that order (hence person 40 is between person 39 and person 1).
At the start we give a flag to person 1. Then they perform the following process: 1 tells 2 to leave the circle and hands over the flag to 3, who then tells 4 to leave and hands over the flag to 5, etc. When the circle is completed, this process continues, skipping the people who have already left (we assume all people obey the request to leave when it's given to them).
That is: everyone with the flag asks 'the next one that's still left over' to leave and then hands over the flag to 'the next one after that'.
This process continues until there's only one person left over.
Model this process in SMT, and let Z3 determine (the number of) that last person.

(source: https://en.wikipedia.org/wiki/Josephus_problem)

Medicine testing A (6,3,2) (**)

We want to test 6 medicines, but the tests are very expensive. In order to save costs, and yet to make sure that enough combinations of medicines will be tested, we want to set up a cost-effective scheme, testing the medicines in a number of rounds.
Let Z3 find out whether or not a schema can be found to test the medicines according to the following conditions, and if it can be done, present Z3's answer in a nice, human-readable format.

- There are 10 test rounds
- In each round, 3 (different) medicines are tested
- Every medicine is tested 5 times (i.e. in 5 different rounds
- Every pair of (two) medicines occurs together in 2 and no more than 2 rounds

(This is called a (6,3,2)-design, see https://en.wikipedia.org/wiki/Block_design)

Medicine testing B (**)

We want to test 7 medicines, and we are especially interested in the risks when using combinations of them at the same time, but the tests are very expensive to perform. Moreover, if too many medicines are tested at the same time, the results become unreliable. Hence, we want to design a schema of test rounds, satisfying the following constraints..

- In each round, at most 3 (different) medicines are tested
- Every pair of (two) medicines occurs together in at least 2 rounds
- We want a scheme with as few tests done as possible

Use Z3 to find out an optimal schema, and present Z3's answer in a nice, human-readable format.

Euler Square (*)

(A, a)	(B, c)	(C, b)
(B, b)	(C, a)	(A, c)
(C, c)	(A, b)	(B, a)

In the square above, the first elements and second elements of each pair form two Latin squares (i.e. each element occurs exactly once in each row and exactly once in each column). Moreover, each of the 9 pairs occurs exactly once.
Such a 'matrix-of-pairs' is called an Euler square. In this case it has size 3x3.
Let Z3 find an Euler square of size 4x4.

(source: https://en.wikipedia.org/wiki/Mutually_orthogonal_Latin_squares)

Waterjugs (***)

We have 3 jugs of water, capacity: 8, 5, 3 liters resp. Initially there are 8 liters of water in the first jug, the other two jugs are empty.
Wanted end-situation: 4 liters in the first jug, 4 liters in the second jug, 3rd jug empty.

The jugs are irregularly shaped and unmarked; no water can be spilled; each step pouring water from one jug to another stops when either the source jug is empty or the destination jug is full, whichever happens first.

Use Z3 to find a solution (and present the answer in human readable form).

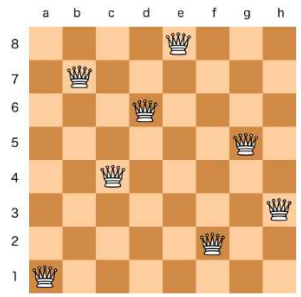
(Hints:

- Proposal for a model: (declare-fun C (Int Int) Int)
  where (C j t) represents the contents of jug number j at time t
- The data from the puzzle allows you to conclude that the total amount of water remains constant, and at each step, one of the jugs remains unaltered

)

8 Queens (*)

Let Z3 solve the famous Eight Queens Puzzle:
https://en.wikipedia.org/wiki/Eight_queens_puzzle



10 Queens (***)

Let Z3 place 5 white queens and 5 black queens on a 6x6 board such that no two queens of different colors can attack each other.

5 Queens (***)

Let Z3 place 5 queens on an 8x8 chessboard such that no two queens attack eachother and each field of the board is attacked by one of the queens (or is occupied by one).

(source: Steinhaus, Mathematical Snapshots, 3rd edition, 1983)

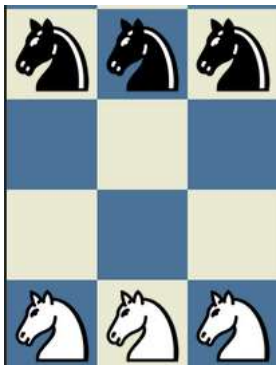
Knight's Tour (***)

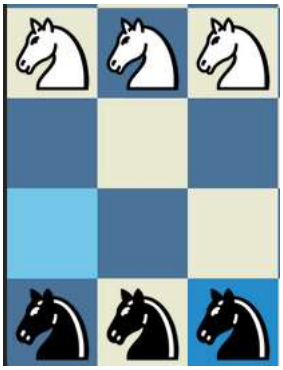
Let Z3 find a path for the knight to visit every field on the chess board without ever visiting the same field twice. (If Z3 is too slow to find a path that covers all 64 fields, a shorter path will also do, as long as your solution could 'in principle' find the complete 'tour'.)

(See: https://en.wikipedia.org/wiki/Knight's_tour)

Knight Swap Puzzle (***)

Let Z3 figure out how to move the knights from the first to the second situation, on this 3x4 board (white and black do not need to move in turns):





(source: https://www.chess.com/forum/view/fun-with-chess/knight-swap-puzzle)

Handshaking (***)

Five married couples (old school, husband and wife) came together at a party, making a total of ten people. As people arrived, a certain amount of handshaking took place in an unpredictable way, subject only to two obvious conditions: nobody shook his or her own hand and no husband shook hands with his wife. When it was all over, one of them (a husband) became curious and went around the party asking each person: "With how many people did you shake hands?. . . And you? . . . And you?" He asked nine people (everybody, including his own wife), and he got nine different answers.

Use Z3 to find out how many hands his wife shook, and whether there is only one possible answer. Can you explain the results that Z3 finds?

(source: Paul Halmos, Problems for Mathematicians Young and Old, 1991)

4 weights (***)

Let Z3 solve the following puzzle:

You have a balance scale with four weights. With these four weights you must balance any whole number load from 1kg all the way up to 40kg. How much should each of the four weights weigh?

(You may place weights on both sides of the scale.)

(source: https://www.mathsisfun.com/puzzles/only-4-weights.html)

Deviant among 7 (***)

We have 7 people, one of which is contaminated with a disease, and we need to figure out which one. We have an unlimited amount of test-packages, and the tests are 100% reliable, but it takes 24 hrs before the test result is shown, and we need to find the culprit as soon as possible.
Fortunately, a test-package can be used on an unlimited number of people before it's 'used up', but of course, this way, the result of a test will only tell us that one of the people it was applied to is infected, but not which one of these people it was.
Let Z3 find a schema of tests to be performed on people, in such a way that the combination of test results will always uniquely identify the infected person, within 24 hrs.
Use Z3 to find out the minimum number of test-packages required for this. Can you explain the number that Z3 finds?

Reverse program (**)

Consider the following program:

```
for i := 1 to 10 do
   if a > b then
      b := 2b; a := a - 3
   else
      a := 2a; b := b - 5
   fi
od
```

Find initial values for a and b such that afterwards a = 1000 and b = 999.

Bridge crossing 1-2-5-10 (***)

Albert, Bob, Charlie and Dan want to cross a bridge.

- The bridge can carry at most two people at the same time.
- In order to cross the bridge safely, a torch is needed, and they have only one torch.
- The four men walk at different speeds. It takes Albert 10 minutes to cross the bridge, Bob 5 minutes, Charlie 2 minutes and Dan 1 minute.
- When two people cross the bridge together, the slowest determines the pace of both.

What is the fastest way for all four to cross the bridge?
(Use Z3, but also use your intelligence. Keep it simple!)

Missionaries and Cannibals (***)

Three missionaries and three cannibals want to get to the other side of a river. There is a small boat, which can fit only two. To prevent a tragedy, there can never be more cannibals than missionaries together.

(source: http://brainden.com/crossing-river.htm)

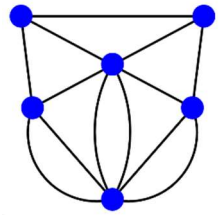
Three coaches (***)

Three are three Athletes (Alex, Brook and Chris) and their individual Coaches (Murphy, Newlyn and Oakley) standing on the shore. No Coach trusts their Athlete to be near any other Coach unless they are also with them. There is a boat that can hold a maximum of two persons. How can the six people get across the river?

(source: https://www.mathsisfun.com/puzzles/three-coaches-three-athletes-and-a-river.html)

Eulerian path (**)

Let Z3 find a path in the following graph that visits each edge once and only once:



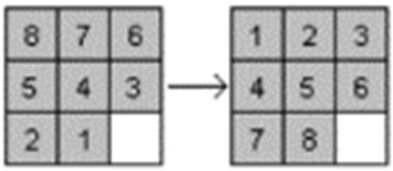
(See https://en.wikipedia.org/wiki/Eulerian_path)

Hanoi Towers (****)

Let Z3 find a solution to the famous Towers of Hanoi puzzle.
(See https://en.wikipedia.org/wiki/Tower_of_Hanoi)
(Remember the main point of this course: tell Z3 *what* to solve, not *how* to solve it.).

Slide puzzle (8 tiles) (****)

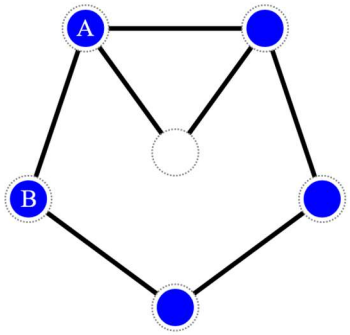
Let Z3 solve the following variation of the famous 15 puzzle (where you slide tiles to the empty space on the grid). The initial situation is on the left, the wanted end situation on the right:



(See https://en.wikipedia.org/wiki/15_puzzle, and http://www.mathematische-basteleien.de/15puzzle.htm ('The Eight Puzzle'))

Slide puzzle (pentagon+1) (***)

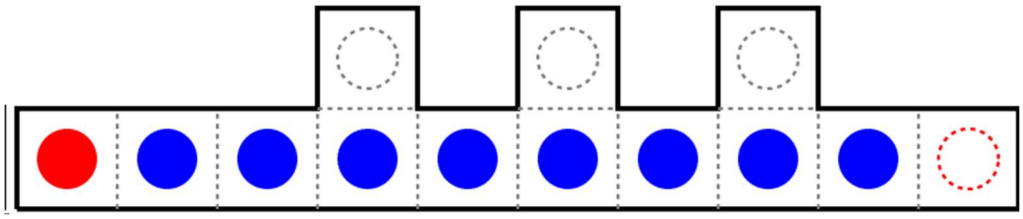
We have 5 coins positioned as in the picture below. One place is unoccupied. Moving one coin at a time (every position can contain only one coin at the same time, and of course only moves across black lines are allowed), we want coins A and B eventually to switch places. Let Z3 find a solution.



(source: Ivar Moscovich 'Het Grote Breinbreker Boek' (Dutch))

Slide puzzle (corridor) (****)

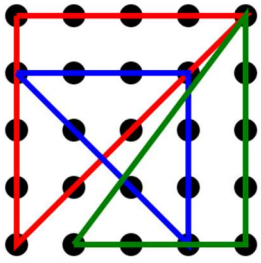
Slide coins to empty positions. The red coin should end up at the empty spot at the right end. Model this with SMT and let Z3 find a solution.



(source: Ivar Moscovich 'Het Grote Breinbreker Boek' (Dutch))

Triangles in a grid (****)

In the picture below you see 3 triangles that cover each integer point of a 5x5 grid, in such a way that all vertices of all triangles are themselves points of that grid (not just points with integer coefficients).



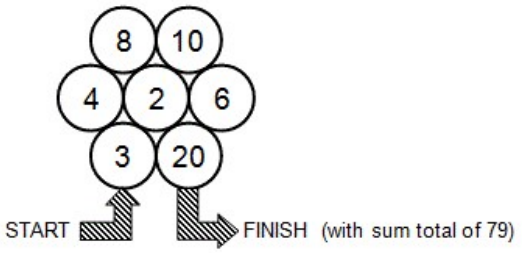
Use Z3 to find another set of triangles subject to the same conditions.

Intransitive Dice (***)

Let Z3 find a set of 3 *intransitive dice*.
See: https://en.wikipedia.org/wiki/Intransitive_dice

Prime number path (**)

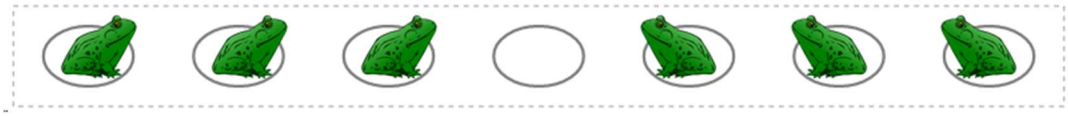
In the following Maths challenge, you travel from 'START' to 'FINISH' in no specific order or direction from one adjacent circle to another, keeping the accumulating sum total always equal to another 'Prime' number. Values in adjacent circles can be used more than once but 'ping-ponging' between circles (i.e. returning immediately to the circle that you have just left) is not allowed. The following challenge is to finish with a sum total of 79 (a prime number).



Created By Gordon R. Burgin. More at https://gordonburgin.com

Toads and Frogs (***)

All 6 frogs need to be moved to the other side. Every frog can either jump one place ahead or jump over one frog. Jumping backwards is not allowed.
Model this with SMT, and let Z3 find a solution.



(See: https://en.wikipedia.org/wiki/Toads_and_Frogs)

Tweedledum and Tweedledee (***)

See: https://erich-friedman.github.io/puzzle/logic/

Tweedledum lies on Monday, Tuesday, and Wednesday; Tweedledee lies on Thursday, Friday, and Saturday. They both tell the truth on Sunday. You come upon the two of them, and they make two statements. In each case, let Z3 determine who is who, and what day it is.

Let Z3 solve two of the puzzles found on the page mentioned above.

Tweedledum and Tweedledee Revisited (****)

Write an SMT solution that solves all 10 puzzles from the page mentioned in the previous puzzle with a generic formulation of the overall constraints of the puzzle.
(Hint: recall the (pop) and (push) instructions from the example files.)

Extra Challenging

Paranoid Pirate Puzzle (*****)

There are 5 Islands in a row.
Parker the Paranoid Pirate has a ton of gold hidden on one of them. During the day you have enough time to sail to any one of the islands to find the gold. But every night Parker moves all that gold exactly one island over (so if it was on island 4 it gets moved to either island 3 or 5, etc.)
Let Z3 find a (fixed) path of islands for you to visit, so that you are certain that you will eventually be on the island where the gold is located.

(source: slightly adapted from https://www.mathsisfun.com/puzzles/5-islands.html)

Batteries Puzzle (*****)

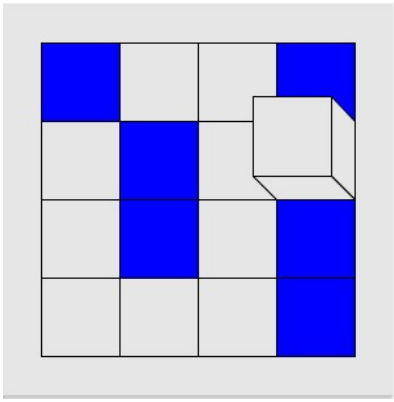
A flashlight requires 2 AA-type batteries to operate. Both these batteries need to be charged for the flashlight to provide light.
We happen to have 8 batteries, but 4 of them are charged while the other 4 are discharged (hence unusable), and they look all the same. So we would have to test them two at a time inside the flashlight.
Let Z3 find a sequence of 10 or fewer pairs which guarantees us to find a working pair, however unlucky we may be. What is the minimum number of pairs needed?
(source: https://puzzling.stackexchange.com/questions/108023/flashlight-and-8-batteries)

Counterfeit Coin (*****)

We have 12 coins, all having the same weight, except for one. That one is either heavier or lighter than the others. Let Z3 design a scheme of 3 weighings that is able to identify which coin is the phony one, and whether it's heavier or lighter.

Cube (*****)

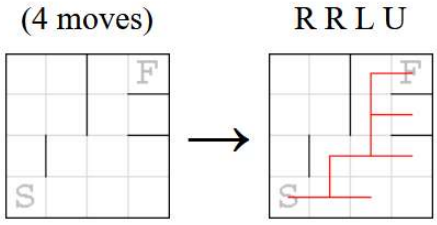
Find a Z3 solution for Cube puzzles as in:
https://www.chiark.greenend.org.uk/~sgtatham/puzzles/js/cube.html



Robot (*****)

See: https://erich-friedman.github.io/puzzle/robot/

You want to program a robot with a list of movement commands (Up, Down, Left, and Right) to get it from Start to Finish. The robot attempts to move one square at a time in the directions in its program. If the robot bumps into a wall, it does not move. When it runs out of instructions, it repeats the instructions it has in a cycle. Here's an example:



Write an SMT solution that can solve any 5x5 puzzle of the kind displayed on the page mentioned above.

Robozzle (*** ***)

Find an SMT solution for Robozzle puzzles, see http://www.robozzle.com/js/

Sokoban (*** ***)

Find an SMT solution for Sokoban puzzles, see https://en.wikipedia.org/wiki/Sokoban

Tic-Tac-Toe (*** ***)

Tic-Tac-Toe / Three-in-a-row is a well-known game for two players who take turns marking the spaces in a three-by-three grid with *X* or *O*. The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row is the winner.

Let Z3 show that Tic-Tac-Toe is a draw, that is, none of the two players has a strategy that is certain to win.

(see: https://en.wikipedia.org/wiki/Tic-tac-toe, https://en.wikipedia.org/wiki/Strategy_(game_theory))