7 Sets

The language of sets has proved very convenient for expressing a wide variety of mathematical statements. It also aids mathematical thinking by allowing one to think of a collection of related things as a single entity, or set, which can then be given a name or symbol. (The mathematics student will already appreciate the importance of symbols.) The things in a set are called *members*, or *elements*. Sets may be defined by listing the elements in them. For example, $\{1, \pi, 107\}$ is the set consisting of the three numbers: 1, π, and 107. (The order in which the elements are listed is immaterial, so that $\{1, \pi, 107\} = \{107, 1, \pi\}$.) Almost always, however, we collect things together for a reason, that is, because they have some property in common. One speaks of "The set of Fs", meaning the set of all things which have the property F. This set is denoted by,

$$\{x \mid Fx\} \quad \text{or} \quad \{x : Fx\}$$

and read as,

"The set of xs such that Fx"

(There is no significance in choosing x; $\{y \mid Fy\}$ is the same set as $\{x \mid Fx\}$.)

Some sets have special names:

N is the set of natural numbers
Z is the set of integers
Q is the set of rational numbers
R is the set of real numbers
C is the set of complex numbers.

The symbol for "is a member of" is $\in$. Thus,

$$13 \in \{x \mid x \text{ is prime and } 10 < x < 20\}$$

So, "$y \in \{x \mid Fx\}$" is logically equivalent to "Fy".

Two sets are defined as the same, or equal, if they have the same members. In this respect, sets are unlike properties. For example, "having 3 angles equal" and, "having 3 sides equal" are different properties of triangles, but the sets,

$$\{x \mid x \text{ is a triangle with 3 equal angles}\}$$

and,

$$\{x \mid x \text{ is a triangle with 3 equal sides}\}$$

are the same set, since a triangle has 3 equal sides if and only if it has 3 equal angles. Thus for sets A and B, "A = B" means "$x \in A$ if and only if $x \in B$". Similarly, "$A \subset B$" (read as, "A is a subset of B") means that everything in A is also in B, that is, "if $x \in A$, then $x \in B$" or "All members of A are members of B" (the case A = B is not excluded).

Often, we define a subset of a given set by choosing those members that have some common property. Thus,

$$\{x \in \mathbf{R} \mid x^2 = 2\},$$

which is read as, "The set of x in **R** such that $x^2 = 2$", is the set of real numbers whose square is 2, that is,

$$\{\sqrt{2}, -\sqrt{2}\}.$$

Relations between sets are sometimes depicted by *Venn diagrams* (called after the nineteenth-century English logician, clergyman and historian John Venn, although actually invented by Leibniz). The elements of a set A are represented by points inside a circle. (If there are many sets, figures other than circles may be needed.) The points outside the circle represent those things not in the set A. (Figure 7.1.) The set of things not in set A is called the *complement* of A, and is denoted by $\bar{A}$ or $\tilde{A}$.

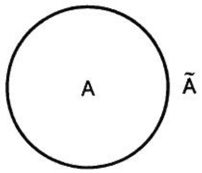

Figure 7.1

For any two sets A and B, the *intersection*, $A \cap B$, is the set of things that are in both A and B. (See Figure 7.2.) Thus,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

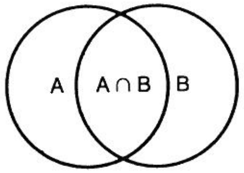

Figure 7.2

The *union* $A \cup B$ is the set of things which are in either A or B. (See Figure 7.3.) Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

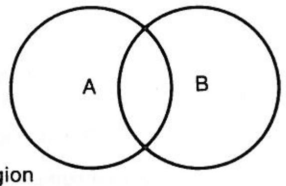

Figure 7.3 $A \cup B$ is the shaded region

Note that "or" is always used inclusively in mathematics, that is, to mean "and/or". Thus, "x is an F or x is a G" is true when x is an F, or a G, or both.

$A \subset B$ is represented in Figure 7.4.

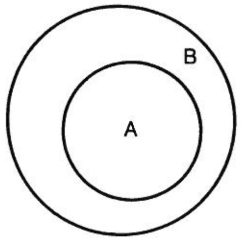

Figure 7.4

This diagram helps to clarify the reason why "All As are Bs" is logically equivalent to its contrapositive, "All not-Bs are not-As". If all points in A are in B, then obviously all points outside B are outside A, and vice versa. Any "all" statement can be represented by this diagram, since "All As are Bs" is logically equivalent to "The set of all As is a subset of the set of all Bs". Venn diagrams are not usually used in representing "some" statements.

Since "$A \subset B$" means, "All members of A are members of B" or, "If $x \in A$ then $x \in B$", the proof of a statement of the form "$A \subset B$" must look like this:

Let $x \in A$.

Therefore, $x \in B$.

Similarly, proving "A = B" involves proving an "if and only if" statement, so the proof of "A = B" will usually look like this:

Let $x \in A$.

Therefore $x \in B$.

Then, let $x \in B$.

Therefore, $x \in A$.

Example 1

Prove that the set of multiples of 4 is a subset of the set of even numbers.

Proof

Let x be an element of the set of multiples of 4.

That is, x is a multiple of 4.

So,

$$x = 4k$$

for some integer k,

$$= 2(2k)$$

Therefore, x is a multiple of 2, that is, x is even. So x is in the set of even numbers. This has shown that any member of the set of multiples of 4 is a member of the set of even numbers. So the set of multiples of 4 is a subset of the set of even numbers. ▮

Note

The example is artificially simple, in that the result to be proved does not really say any more than, "All multiples of 4 are even"; the reference to sets adds little in this case. But the example is useful for illustrating something important that happens in general: when proving an "$A \subset B$" statement using,

Let $x \in A$.

Therefore, $x \in B$.

The line after "Let $x \in A$" will usually be an explanation of what "$x \in A$" means. In the above case, "x is in the set of multiples of 4" means just "x is a multiple of 4".

The next example shows a general result about sets, rather than a result about a particular set (such as the set of even numbers). Nevertheless, the techniques of proof are the same as those described above.

Example 2

Prove that $(A \cap B) \cap C = A \cap (B \cap C)$ for any sets A, B, C.

Proof

Let,

$$x \in (A \cap B) \cap C$$

So,

$$x \in A \cap B \quad \text{and} \quad x \in C$$

So,

$$x \in A \quad \text{and} \quad x \in B \quad \text{and} \quad x \in C$$

So,

$$x \in A \quad \text{and} \quad x \in B \cap C$$

So,

$$x \in A \cap (B \cap C)$$

Therefore,

$$(A \cap B) \cap C \subset A \cap (B \cap C)$$

Now let,

$$x \in A \cap (B \cap C)$$

So,

$$x \in A \quad \text{and} \quad x \in B \cap C$$

So,

$$x \in A \quad \text{and} \quad x \in B \quad \text{and} \quad x \in C$$

So,

$$x \in A \cap B \quad \text{and} \quad x \in C$$

So,

$$x \in (A \cap B) \cap C$$

Therefore,

$$A \cap (B \cap C) \subset (A \cap B) \cap C$$

So we have shown that,

$$(A \cap B) \cap C = A \cap (B \cap C)$$ ▮

Diagrams are not considered adequate as a proof of such results, though they are often helpful. Diagrams would be impossible to understand when working with more than about four or five sets.

Example 3

For any integer n, the *ideal* $n\mathbf{Z}$ is defined to be the set,

$$\{x \in \mathbf{Z} \mid x = ny \text{ for some } y \in \mathbf{Z}\}$$

(i.e. the set of all multiples of n). Prove that $n\mathbf{Z} \subset m\mathbf{Z}$ if and only if n is a multiple of m.

Proof

The proof will look like this:

Let $n\mathbf{Z} \subset m\mathbf{Z}$.

Therefore, n is a multiple of m.

Let n be a multiple of m.

Therefore, $n\mathbf{Z} \subset m\mathbf{Z}$.

Inside the second half, there must be,

Let, $x \in n\mathbf{Z}$.

So, $x \in m\mathbf{Z}$,

in order to derive, $n\mathbf{Z} \subset m\mathbf{Z}$.

The full proof is:

Let,

$$n\mathbf{Z} \subset m\mathbf{Z}$$

Now,

$$n \in n\mathbf{Z}$$

So,

$$n \in m\mathbf{Z}$$

Therefore,

$$n = my$$

for some $y \in \mathbf{Z}$ (from the definition of $m\mathbf{Z}$).

That is, n is a multiple of m.

So we have proved that if $n\mathbf{Z} \subset m\mathbf{Z}$ then n is a multiple of m.

Now suppose n is a multiple of m,

So,

$$n = mz$$

for some $z \in \mathbf{Z}$.

Let,

$$x \in n\mathbf{Z}$$

So,

$$x = ny$$

for some $y \in m\mathbf{Z}$

$$= mzy$$

since $n = mz$

That is, x is a multiple of m.

So,

$$x \in m\mathbf{Z}$$

Thus,

$$n\mathbf{Z} \subset m\mathbf{Z}$$

So we have proved that if n is a multiple of m, then $n\mathbf{Z} \subset m\mathbf{Z}$.

Altogether we have proved that $n\mathbf{Z} \subset m\mathbf{Z}$ if and only if n is a multiple of m. ∎

Notes

1. The step:

$$n\mathbf{Z} \subset m\mathbf{Z}$$

Now,

$$n \in n\mathbf{Z}$$

So,

$$n \in m\mathbf{Z}$$

is an example of the more general argument form,

All-As are Bs
x is an A
therefore, x is a B.

That is, what is true of all As is true of any particular one. There is another example of this type of argument in Exercise 14(b) below.

2. The set with no members is called the *empty set*, and is denoted $\varnothing$. (Does the empty set really exist? Philosophically inclined readers may like to regard it as a useful fiction, like the number zero.) The empty set is considered to be a subset of *every* set. This has some technical advantages, such as making the number of subsets of a set of n elements equal to 2^n. (See Example 5 in Chapter 8.) Now,

$$\varnothing \subset A$$

is equivalent to the "if" statement "If $x \in \varnothing$ then $x \in A$". However, $x \in \varnothing$ is false (for any x), so we must make a convention about statements "If p then q" in cases where p is false. The convention is to count all such statements as true. This is consistent with counting $\varnothing \subset A$ as true. Similarly,

$$\varnothing \subset A$$

is equivalent to the "all" statement,

"All members of $\varnothing$ are members of A"

but there are no members of $\varnothing$. The convention is to count "All As are Bs" as true if there are no As.

These rather odd conventions are technically useful, for example in starting inductions where the case $n = 1$ (or $n = 0$) means an empty set is being considered. The point of the conventions is to separate the question "Does being a B follow from being an A?", from the question "Are there any As?".

Linear algebra

The *span* of a set of vectors is the set of all linear combinations of them. We write this in symbols thus:

If,

$$S = \{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n\}$$

then,

$$\text{span } S = \{\underset{\sim}{v} : \underset{\sim}{v} = a_1\underset{\sim}{v}_1 + \ldots + a_n\underset{\sim}{v}_n \text{ for some scalars } a_1, \ldots, a_n\}$$

For example, in $\mathbf{R}^3$,

$$\text{span } \{(1, 0, 0), (0, 1, 0)\}$$

is the xy-plane, since,

$$\begin{aligned}\text{span } \{(1, 0, 0), (0, 1, 0)\} &= \{v : v = a(1, 0, 0) + b(0, 1, 0)\}\\ &= \{(a, b, 0)\}\end{aligned}$$

that is, the set of all vectors with the z-coordinate zero, which is the xy-plane.

This example gives some sense to the word "span". The two vectors are a kind of "framework" that is enough to "support" the whole plane, in the sense that every point in the plane can be got by extending out from the two vectors (1, 0, 0) and (0, 1, 0). (See Figure 7.5.)

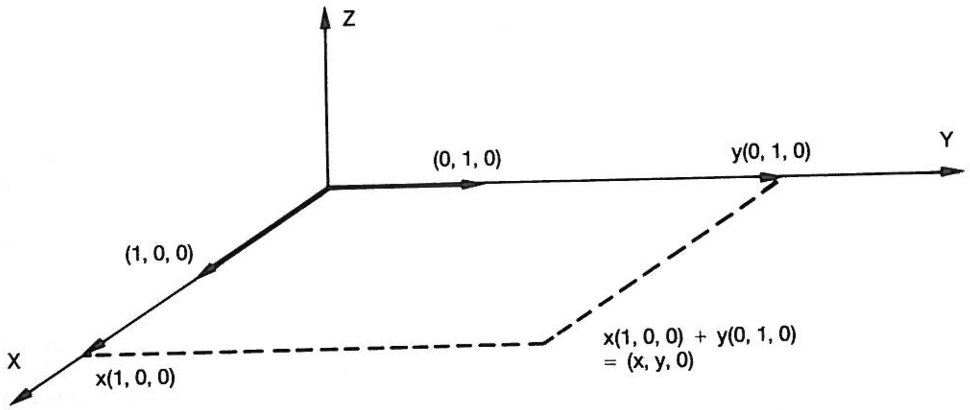

Figure 7.5

Example 4

Prove that the span of (1, 1) and (1, 2) consists of all vectors (a, b), (that is, span $\{(1, 1), (1, 2)\} = \mathbf{R}^2$).

This asks us to show that *every* point (a, b) is of the form $x(1, 1) + y(1, 2)$ for some $x, y \in \mathbf{R}$. This statement has two quantifiers, an "all" and a "some".

Proof

Suppose $(a, b) = x(1, 1) + y(1, 2)$. (We don't know that x, y exist yet; the aim is to show that they do.)

Then,

$$\begin{aligned}(a, b) &= (x, x) + (y, 2y)\\ &= (x + y, x + 2y)\end{aligned}$$

So,

$$a = x + y$$

and,

$$b = x + 2y$$

Subtracting the first equation from the second equation,

$$b - a = y$$

and substituting this in the first,

$$a = x + b - a$$

so,

$$2a - b = x$$

This shows that for *all* (a, b), we can find x and y (namely $x = 2a - b$ and $y = b - a$), such that,

$$(a, b) = x(1, 1) + y(1, 2)$$

Therefore every (a, b) is a linear combination of (1, 1) and (1, 2).

That is,

$$\mathbf{R}^2 \subset \text{span}\{(1, 1), (1, 2)\}$$

Also, it is obvious that,

$$\text{span}\{(1, 1), (1, 2)\} \subset \mathbf{R}^2$$

since every linear combination of (1, 1) and (1, 2) is a vector in $\mathbf{R}^2$.

Therefore span$\{(1, 1), (1, 2)\} = \mathbf{R}^2$. ▮

(We have applied the techniques above: To show two sets are equal, we show that everything in the first is in the second, and everything in the second is in the first.)

Notes

1. Usually the set S is a small finite set, as in Example 4, while span S is an infinite set. However, sense can still be given to the definition if S itself is infinite.
2. The following sentences all mean the same:
   "Span {(1, 0, 0), (0, 1, 0)} is the xy-plane";
   "The span of (1, 0, 0) and (0, 1, 0) is the xy-plane";
   "(1, 0, 0) and (0, 1, 0) span the xy-plane".
   Notice in the first two, "span" is a noun while in the last it is a verb.

The following sentences are meaningless:
"(1, 0, 0) and (0, 1, 0) span";
"S is a span";
"The xy-plane is a span".

Example 5

Prove that span$\{(2, 4), (1, 2)\} \neq \mathbf{R}^2$.

As in Example 4, it is obvious that, span $\{(2, 4), (1, 2)\} \subset \mathbf{R}^2$, so we have to show that $\mathbf{R}^2$ is not a subset of span$\{(2, 4), (1, 2)\}$. That is, we have to show that span$\{(2, 4), (1, 2)\}$ is not the whole of $\mathbf{R}^2$.

Proof

This is a "not all" statement: "Not all (a, b) are of the form x(2, 4) + y(1, 2)". To show this we exhibit a particular (a, b) which is not of the form x(2, 4) + y(1, 2). We can see that anything of the form x(2, 4) + y(1, 2) will have the second co-ordinate twice the first, so we choose a vector which does not have that property, say (1, 3). The proof is:

If,

$$(1, 3) = x(2, 4) + y(1, 2)$$

then,

$$\begin{aligned}(1, 3) &= (2x, 4x) + (y, 2y)\\ &= (2x + y, 4x + 2y)\end{aligned}$$

so,

$$1 = 2x + y$$

and,

$$\begin{aligned}3 &= 4x + 2y\\ &= 2(2x + y)\\ &= 2.1\\ &= 2\end{aligned}$$

but,

$$3 \neq 2$$

so (1, 3) cannot be of the form x(2, 4) + y(1, 2).

So, (1, 3) is not in the span of (2, 4) and (1, 2).

Therefore, the span of (2, 4) and (1, 2) does not include all points (a, b).

So span $\{(2, 4), (1, 2)\} \neq \mathbf{R}^2$. ▮

Calculus

The *general solution* of a differential equation is the set of all solutions. Thus, when we say, "The general solution of $\frac{df}{dx} = f$ is Ae^x", we mean that the set of all solutions of $df/dx = f$ is,

$$\{f : f = Ae^x \text{ for some real number } A\}$$

Example 6

If S is the set of all solutions of,

$$\frac{df}{dx} = f$$

and T is the set of all solutions of,

$$\frac{d^2f}{dx^2} = f$$

show that,

$$S \subset T$$

Method 1

Solving the differential equations, we find,

$$S = \{f : f = Ae^x \text{ for some real number } A\}$$

and,

$$T = \{f : f = Ae^x + Be^{-x} \text{ for some real numbers } A, B\}$$

and clearly,

$$S \subset T$$

Method 2

Let,

$$f \in S$$

So,

$$\frac{df}{dx} = f$$

Differentiating,

$$\frac{d^2f}{dx^2} = \frac{df}{dx}$$

but this is f, from the line above.

So f satisfies,

$$\frac{d^2f}{dx^2} = f$$

That is,

$$f \in T$$

Which method is better? See Exercise 23.

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. (a) Write down an "all" statement equivalent to, "$A \subset B$". Then write down your "all" statement in "if" form.
(b) Use this "if" statement to explain the form of the proof of the statement, "$A \subset B$".

*2. The truth of the relation, $B \subset (A \cup B)$, may be verified from a Venn diagram, but this does not constitute a proof. How would you prove this statement?

*3. Prove that, "If $A \subset B$ and $B \subset C$ then $A \subset C$" (without using a Venn diagram).

*4. (a) For any two sets A and B explain what is meant by:
(i) $A \cap B$
(ii) $A \cup B$
(b) Prove that, $(A \cap B) \subset B$

*5. (a) Write down an "if and only if" statement equivalent to, $A = B$.
(b) Use this to explain the form of the proof of the statement, $A = B$.

*6. (a) How would you prove the statement, $B \cup B = B$?
(b) Prove it.

*7. Prove that $(A \cap B) \subset (A \cup B)$

*8. (a) Prove that $B \cup \varnothing = B$.
(b) Simplify the following:
(i) $B \cup B$
(ii) $B \cap B$
(iii) $B \cap \bar{B}$
(iv) $B \cup \varnothing$
(v) $B \cap \varnothing$
(Check that you can prove your answers.)

*9. (a) Explain by the use of an appropriate Venn diagram that: If $A \subset B$ then $\bar{B} \subset \bar{A}$.

(b) Are the two statements, "All As are Bs", and its contrapositive, "All not-Bs are not-As" logically equivalent?

**10. Prove that, $A \cup (B \cup C) = (A \cup B) \cup C$.

**11. (a) Draw appropriate diagrams to verify the truth of the statement, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, then prove it.
(b) Prove that, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**12. De Morgan's laws state:

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$
$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

(a) Draw appropriate Venn diagrams to verify the truth of, $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.
(b) Prove, $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.
(c) Prove, $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

**13. Comment on the reasoning,

Most As are Bs
Most As are Cs

Therefore some Bs are Cs.

**14. A set of real numbers is called *bounded* if it does not "go to infinity". More precisely, S is bounded if there exist real numbers M, N such that for all $s \in S$, $M < s < N$. (For example, the set of real numbers x such that $1 < x^3 < 2$ is bounded, since for all $x \in S$, $1 < x < 1.5$.)
(a) Give an example of a set which is not bounded.
(b) Prove that if S is bounded and $T \subset S$, then T is bounded.
(c) Prove that any finite set of real numbers is bounded.
(d) Prove that if T is not bounded and $T \subset S$, then S is not bounded.
(e) Prove that if S and T are bounded, then $S \cap T$ is bounded.
(f) If S and T are bounded, is $S \cup T$ always bounded? Prove your answer.
(g) Let S and T be bounded. Let,

$$U = \{u \in \mathbf{R}: u = s + t \text{ for some } s \in S, t \in T\}$$

Show that U is bounded. It might help to calculate some examples first, say,

$$S = [0, 1], T = [2, 3]$$

(The set U is sometimes denoted S + T, since it is the set of all sums of something in S with something in T.)

**15. A region in the plane is called *convex* if the line segment joining any two points in the region lies wholly inside the region. For example, an ellipse, a parallelogram, a triangle and a straight line are convex, but an annulus and a star-shaped region are not. In symbols, R is convex if, for all (x_1, y_1) and (x_2, y_2) in R, $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in R$ for all $\lambda \in [0, 1]$.
(a) Prove that if R and S are convex, then $R \cap S$ is convex.

(b) If R and S are convex, is $R \cup S$ always convex? Prove your answer.
(c) Prove that if R is convex, then the reflection of R in the x-axis is convex.
(d) If R is convex, is the set,

$$2R = \{(x, y):(x, y) = (2x', 2y') \text{ for some } (x', y') \in R\}$$

always convex? Prove your answer and illustrate with a diagram.

**16. Prove that $n\mathbf{Z} = m\mathbf{Z}$ if and only if $n = m$ or $n = -m$

***17. Fill in the blank and prove the theorem:

"$A \cap (B \cup C) = (A \cap B) \cup C$ if and only if ——"

Linear algebra

*18. (a) Show that $\mathbf{R}^2 = \text{span}\{(1, 1), (1, -1)\}$
(b) Show that $\mathbf{R}^2 \neq \text{span}\{(1, 1), (-3, -3)\}$

*19. Show that $(1, 2, 3)$ is not in the span of $(1, 1, 1)$ and $(1, 0, 2)$

*20. Show that any vector (a, b, c) such that $a + b + c = 0$ is in the span of $\{(1, 1, -2), (1, 0, -1)\}$

**21. Show that the span of any set of vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ in a vector space V is a subspace of V (that is, is closed under addition and scalar multiplication).

***22. Show that, $\text{span}\{(a, b), (c, d)\} = \mathbf{R}^2$ if and only if $ad - bc \neq 0$.

Calculus

*23. Let S be the set of all solutions of,

$$\frac{d^2f}{dx^2} = f$$

and let T be the set of all solutions of,

$$\frac{d^4f}{dx^4} = f$$

Show that $S \subset T$.

*24. Show that the set of solutions of,

$$\frac{df}{dx} = 2x$$

is infinite.

**25. The general solution of,

$$\frac{d^2y}{dx^2} = y$$

may be written as $Ae^x + Be^{-x}$.

(a) Check that $\sinh x$ and $\cosh x$ are solutions of this differential equation.
(b) Prove that $y = C \sinh x + D \cosh x$ is a solution, for any real numbers C and D.
(c) Is,

$$\{(y : y = C \sinh x + D \cosh x \text{ for some } C, D \in \mathbf{R}\}$$

equal to the set,

$$\{y : y = Ae^x + Be^{-x} \text{ for some } A, B \in \mathbf{R}\}?$$