Theory Exercises 2425nj v 2024.09.11

1. Sets, Relations, Functions

Exercise 1.1

True or false:

a. 1 is an element of 1

b. 1 is an element of {1}

b. {1} is an element of {1}

c. {1} is a subset of {1}

d. 1 is a subset of {1}

Exercise 1.2

True or false:

a. $1 \in \{1\}$

b. $\{1\} \in \{1\}$

c. $\{1\} \subseteq \{1\}$

d. $1 \subseteq \{1\}$

Exercise 1.3

a. How many elements does {37, 1, 1, 37} have?

b. How many elements does { 2, 1+1, 5-3 } have?

c. How many elements does { 1/2, 2/4 } have ?

d. Which of these sets are equal to eachother?

{1, 2, 3} {3, 2, 1} {2, 1, 2, 3, 2} {2, 3, 1, 2, 1, 1} {3, 2, 3, 2}

e. Suppose x and y are natural numbers. How many elements does {x, y} have?

f. How many elements does { {0}, {0, 0} } have?

Exercise 1.4

Write each of the following sets in *roster notation*:

a. $A = \{ x \in \mathbb{N} \mid 3 < x < 9 \}$

b. $B = \{ x \in \mathbb{N} \mid x \text{ is even}, x < 11 \}$

c. $C = \{ x \in \mathbb{N} \mid 4 + x = 3 \}$

Exercise 1.5

Suppose A = {2, 3, 4, 5}

a. Explain why A is not a subset of $B = \{ x \in \mathbb{N} \mid x \text{ is even} \}$

b. Explain why A is a proper subset of C = {1, 2, 3, ..., 8, 9}

Exercise 1.6

Which of the following sets are equal to which others?

$A = \{ x \in \mathbb{R} \mid x^2 - 4x + 3 = 0 \}$

$B = \{ x \in \mathbb{R} \mid x^2 - 3x + 2 = 0 \}$

$C = \{ x \in \mathbb{N} \mid x < 3 \}$

$D = \{ x \in \mathbb{N} \mid x \text{ is odd}, x < 5\}$

Exercise 1.7
We define three sets as follows: P = {3, 6}, Q = {1, 3, 4, 5, 6, 8}, R = {3, 4, 8}
a. Draw a Venn diagram of these three sets (one diagram).
b. Which of the sets P, Q and R are proper subsets of each other?
c. "P and R are disjoint sets": true or false?

Exercise 1.8
We define the set A = { {0}, {0, 1}, {0, {0,1}} }
a. $|A|$ = ?
b. $|\wp(A)|$ = ?
c. Write $\wp(A)$ in roster notation
d. Is {0} a subset of A?
e. Is 0 an element of A?
f. Is there a largest subset of A? If yes, what is it?

Exercise 1.9
Set A is given by A = { 1, 0, {0,1,2}, {2} }
True of false (explain your answers) :
a. $\{0\} \in A$?
b. $\{0,1\} \subset A$?
c. $\{0,1,2\} \subseteq A$?
d. $\{1\} \in \wp(A)$?

Exercise 1.10
a. Does there exist a finite set A such that $A \cap \wp(A)$ is non-empty? Provide an example.
b. Is the empty set an element of $\wp(\wp(A)) \cap \wp(A)$?
c. Does there exist a finite set A such that $\wp(\wp(A)) \cap \wp(A)$ has more than one element? If yes, provide an example.
(Explain your answers.)

Exercise 1.11
Find x and y given (2x , x+y) = (6, 2).

Exercise 1.12
Define P = {1, 2, 3}, Q = {a, b, c}
Which of the following are functions from P to Q:
a. ({1, a}, {2, a}, {3 ,b})
b. ((1, a), (2, a), (3, b))
c. { {1, a}, {2, a}, {3, b} }
d. { (1, a), (2, a), (3, a) }
e. { (1, a), (2, b) }
f. { (1, a), (2, a), (2, b) }
g. { (1, b), (2, c), (3, a) }
h. { (1, b), (2, a), (3, a) }

Exercise 1.13

Define P = {1, 2}, Q = {a, b}

Which of the following are functions from P x P to Q:

a. { (1, b), (2, a), (2, a), (1, b) }

b. { (1, 1, a), (1, 2, a), (2, 1, b), (2, 2, b) }

c. { ((1, 1), a), ((1, 2), a), ((2, 1), b), ((2, 2), b) }

d. { ((1, a), 1), ((1, b), 1), ((2, a), 2), ((2, b), 2) }

Exercise 1.14

Define P = {1, 2, 3}, Q = {a, b, c}.

Which of the following are functions?

a. { (1, a), (2, b), (3, a) }

b. { (1, a), (2, b), (2, a) }

c. { (3, a), (2, b), (2, c) }

Exercise 1.15

Define P = {1, 2, 3}, Q = {a, b, c}.

Is this a function:

{ (3, a), (2, b), (2, c) }

a. What is the domain of this function?

b. What is the codomain of this function?

Exercise 1.16

Which of the following is a function? With what domain and range?

a. $\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1 \}$

b. $\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 3x^2 + 1 \}$

c. $\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x = 3y^2 + 1 \}$

d. $\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x = |y| \}$

e. $\{ (x, y) \in \mathbb{N} \times \mathbb{N} \mid x = |y| \}$

f. $\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x < 3y^2 \}$

g. $\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y^2 < 1 \}$

Exercise 1.17

Which of the following functions has an inverse?

a. $\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 3x + 1 \}$

b. $\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 3x^2 + 1 \}$

c. $\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y = 3x + 1 \}$

Exercise 1.18

Let R be the relation on $\mathbb{N}$ defined by $x + 3y = 12$, so $R = \{ (x, y) \in \mathbb{N} \times \mathbb{N} \mid x + 3y = 12 \}$

a. Write R in roster notation.

b. Find the domain and range of R.

c. Is R a function on its domain? If yes, determine the inverse of this function, if it exists.

Exercise 1.19 (*)

a. Is '$\cap$' a function? What is its domain?

b. Is $\wp$ a function? What is its domain?

2. Proposition Logic, Truth Tables, Proofs

Exercise 2.1

In this exercise, A,B, and C denote sets, P,Q, and R denote propositions, and x,y, and n denote numbers. Which of the following expressions are meaningful, and which are not?

a. $A \wedge P$

b. $A + n$

c. $B \vee C$

d. $|\ P\ | = n + 2x$

e. $P \cap Q$

Exercise 2.2

a. Is it true that $\neg P$ logically implies $\neg(P \wedge Q)$?

b. Is it true that $(P \vee Q)$ logically implies $((\neg Q) \vee P)$?

Exercise 2.3

a. Make a truth table for the proposition $(P \wedge Q) \vee (\neg P \vee \neg Q)$

b. Is this proposition a tautology, a contradiction or none of these?

Exercise 2.4

Check that the following logical equivalences are valid by making truth tables.

(≡ (unicode symbol u2261) is short for "is logically equivalent to")

$P \vee Q \equiv Q \vee P$	$P \wedge Q \equiv Q \wedge P$	(Commutativity)
$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	(Associativity)
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$		(Distributivity)
$P \vee \bot \equiv P$	$P \wedge \top \equiv P$	(Neutral Elements)
$P \vee \neg P \equiv \top$	$P \wedge \neg P \equiv \bot$	(LEM, EFQ)
$\neg(\neg P) = P$		(Double Negation)
$\neg\bot \equiv \top$	$\neg\top \equiv \bot$	
$P \vee P \equiv P$	$P \wedge P \equiv P$	(Idempotence)
$P \vee \top \equiv \top$	$P \wedge \bot \equiv \bot$	
$P \vee (P \wedge Q) \equiv P$	$P \wedge (P \vee Q) \equiv P$	(Absorption)
$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$	$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$	(De Morgan)
$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$	$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$	

Exercise 2.5

The truth table of the 'implies' connective is defined as follows:

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

a. Check the validity of each of the following logical equivalences using truth tables:
(≡ (unicode symbol u2261) is short for "is logically equivalent to")

1	$P \Rightarrow Q \quad \equiv \quad (\neg P) \vee Q$	
2	$P \Rightarrow Q \quad \equiv \quad \neg Q \Rightarrow \neg P$	("Contrapositive Law")
3	$R \Rightarrow (P \wedge Q) \quad \equiv \quad (R \Rightarrow P) \wedge (R \Rightarrow Q)$	(Conjunctive syllogism, 'and'-law)
4	$(P \vee Q) \Rightarrow R \quad \equiv \quad (P \Rightarrow R) \wedge (Q \Rightarrow R)$	(Disjunctive syllogism, aka Proof by Cases)
5	$\neg(P \Rightarrow Q) \quad \equiv \quad (P \wedge \neg Q)$	
6	$P \Rightarrow (Q \Rightarrow R) \quad \equiv \quad (P \wedge Q) \Rightarrow R$	
7	$\neg P \quad \equiv \quad (P \Rightarrow \bot)$	

b. Prove the equivalences 2 up to 7 by only using equivalence 1 and the laws of the previous exercise.

Exercise 2.6

a. Determine the truth table of the formula: $(P \Rightarrow Q) \vee (Q \Rightarrow P)$

b. Investigate by using truth tables whether or not the following two formulas are logically equivalent or not: $(P \Rightarrow (Q \vee R))$; $(P \Rightarrow Q) \vee (P \Rightarrow R)$

c. Are the following statements true for all sets A,B,C, or are there counterexamples?
- Either $A \subseteq B$ or $B \subseteq A$
- If $A \subseteq B \cup C$ then either $A \subseteq B$ or $A \subseteq C$

Exercise 2.7

For each of the following claims, determine if they are valid for all sets A, B, C or not. (And if not, provide a counterexample.)

a. $A \subseteq B$ iff $A \cup B = B$
b. $A \subseteq B$ iff $A \cap B = A$
c. $A \cup A = A \cap A = A$
d. $A \cap \emptyset = \emptyset$
e. $A \cup \emptyset = A$
f. If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$
g. If $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$
h. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$
i. If $A \cup B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$
j. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
k. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Exercise 2.8

What can you say about the following sets.

Can you formulate and prove a general law about them?

a. $(A \cap B) \cup A = ?$

b. $(A \cup B) \cap A = ?$

Exercise 2.9

Prove the following statements (or a sizable portion of them), for all sets A,B,C,D, or find a counterexample:

a. $A \setminus (A \setminus B) = A \cap B$

b. $(A \cap B) \cup (A \cap B^C) = A$

c. $A \cup (B \setminus A) = A \cup B$

d. $(A \setminus B) \setminus C = A \setminus (B \cup C)$

e. $(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) = (A \cup B \cup C) \setminus (A \cap B \cap C)$

f. $(A \setminus B)^C = A^C \cup B$

g. $(A \setminus (B \cup C)) \cup (A \setminus (B \cap C)) = (A \setminus B) \cup (A \setminus C)$

h. $(A \setminus (B \setminus C)) = (A \setminus B) \cup (A \cap C)$

i. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

j. $(A \setminus (B \cap C)) = (A \setminus B) \cup (A \setminus C)$

k. $(A \setminus (B \cup C)) = (A \setminus B) \cap (A \setminus C)$

l. $(A \cap B) \cup (A \cap C) \cup (B \cap C) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$

m. $(A \cup B \cup C) \cap (A \cup B^C \cup C) \cap (A \cup B \cup C^C) = (A \cup (B \cap C))$

n. $(A \cap B) \cup (A \cap B^C \cap C) = A \cap (B \cup C)$

o. $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B \cup C) \cap ((A \cap B) \cup (B \cap C) \cup (C \cap A))$

p. $(A \cap B) \cup (B \cap C) \cup (C \cap D) = (A \cup B \cup C) \cap (B \cup C \cup D)$

q. $(A \cap B) \cup (C \cap D) = (A \cup C) \cap (B \cup D)$ If $A = D$ en $B = C$

r. $(A \cap B) \cup (C \cap D) = (A \cup C) \cap (B \cup D)$ if $A = C$ en $B = D$

s. $(A \cup B) \cap (B \cup C) = (B \cup A) \cap (C \cup A)$

Exercise 2.10 (De Morgan Laws for sets)

Prove the following statements, and illustrate them with Venn diagrams:

a. $(A \cap B)^C = A^C \cup B^C$

b. $(A \cup B)^C = A^C \cap B^C$

Exercise 2.11

Prove the following statement, or find a counterexample (recall that the notation |…| is used to denote the number of elements of a (finite) set :

If A and B are finite sets, then

$$| A \cup B | + | A \cap B | = |A| + |B|$$

Exercise 2.12

Prove the following statements, or find a counterexample:

a. $\wp(A \cup B) = \wp(A) \cup \wp(B)$

b. $\wp(A \cap B) = \wp(A) \cap \wp(B)$

Exercise 2.13

A two-way switch (Dutch: wisselschakelaar, hotelschakelaar) is a system with two switches controlling the same lightbulb (for instance). We will call the states of each switch 'up' and 'down' instead of 'on' and 'off'. (Why?)

Consider the following table.

Switch 1	Switch 2	Light
Up	Up	On
Up	Down	Off
Down	Up	Off
Down	Down	On

a. Does this table correctly represent the situation of a two-way switch?

b. Is it the only possible such table?

c. Which formula has a truth table similar to the table above?

d. What formula would correspond to a situation with 3 switches controlling the same lightbulb? And with 4 switches?

3. Quantifiers

Exercise 3.1

Define the set $U = \{2,3,4,5,6,7,8\}$.

Examine if the proposition is true or false:

a. $\exists x \in U\ [\ 2x > 10\ ]$

b. $\forall x \in U\ \forall y \in U\ [\ x + y = 4\ ]$

c. $\forall x \in U\ \exists y \in U\ [\ x < y\ ]$

Exercise 3.2

Write with one or more quantifiers in a symbolic form:

a. $x^2 > 2$ for every integer x

b. There exists an integer x such that $x^2 > x+10$

c. There exists an integer y such that $y^2 > 7$ is not true.

Which of these statements is true?

Exercise 3.3

Which of the following expressions are mathematically well-formed propositions? For those that are not well-formed, identify what was likely intended and provide a corrected version. For those that are well-formed, determine which statements are true and which are false.

a. $\forall n \in \mathbb{N}\ \exists\ n^2 \in \mathbb{N}$

b. $\forall x > 0\ [\ x^2 > 0\ ]$

c. $\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}\ [\ n > m\ ]$

d. $\exists n \in \mathbb{N}\ \forall m \in \mathbb{N}\ [\ n > m\ ]$

Exercise 3.4

For each of the following expressions, explain as clearly as possible why they are not well-formed propositions:

a. $\forall n \in \mathbb{N}\ [\ n^2\ ]$

b. $\exists x \in \mathbb{R}\ \forall x \in \mathbb{R}\ [\ x + x < 2x\ ]$

c. $\forall y \in \mathbb{R}\ \exists x \in \mathbb{N}\ [\ x = (y > 5)\ ]$

d. $(x + 1) = (y \land z)$

Exercise 3.5

Let $U = \{1,2,3\}$. All unqualified quantifiers are assumed to range over this set U.

True or false (and explain your answer):

a. $\exists n \in \mathbb{N}\ \forall x\ [\ x < n\ ]$

b. $\exists n\ \forall x\ [\ x < n\ ]$

Exercise 3.6

You read the following definition in a book:

$\{ (x - h)^2 \mid x < 2h \}$

Which of the following could be meant with this expression?

a. $\{ z \in \mathbb{R} \mid \exists x,h \in \mathbb{R} [z = (x - h)^2 \text{ and } x < 2h] \}$

b. $\{ z \in \mathbb{R} \mid \exists x \in \mathbb{R} [z = (x - h)^2 \text{ and } x < 2h] \}$

c. $\{ z \in \mathbb{R} \mid \exists h \in \mathbb{R} [z = (x - h)^2 \text{ and } x < 2h] \}$

d. Explain the differences between these three interpretations. Conclusion?

Exercise 3.7

Let P(x) be some propositional function (with x ranging over some domain of discourse, and all unspecified quantifiers are assumed to range over this domain of discourse).

Which of the following is a correct formulation of the sentence

"There exists one and only one x such that P(x) is true"

a. $\exists x [P(x)] \wedge \forall y [y = x]$

b. $\exists x [P(x)] \wedge \forall y [P(y) \Rightarrow x]$

c. $\exists x [P(x) \wedge \forall y [P(y) \Rightarrow x]]$

d. $\exists x [P(x) \wedge \forall y [P(y) \Rightarrow y = x]]$

e. $\exists x [P(x)] \wedge \forall x_1, x_2 [P(x_1) \wedge P(x_2) \Rightarrow x_1 = x_2]$

f. $\exists x [P(x)] \wedge \forall y [y \neq x \Rightarrow \neg P(y)]$

g. $\exists x [P(x) \wedge \forall y [y \neq x \Rightarrow \neg P(y)]]$

Exercise 3.8

In the spirit of the previous exercise, find correct formulations for:

a. "There exist *at least* two elements of V that have property P(..)"

b. "There exist *at most* two elements of V that have property P(..)"

c. "There exist *precisely* two elements of V that have property P(..)"

Exercise 3.9

a. For which values for s (in $\mathbb{R}$) is the following statement true:

"there exists an $x \in \mathbb{R}$ such that $x^2 < s - 37$"

(Draw a picture!)

b. Give an alternative description of the set below that does not contain any quantifier, nor any reference to x :

$\{ s \in \mathbb{R} \mid \exists x \in \mathbb{R} [x^2 < s - 37] \}$

Exercise 3.10

Write each proposition without negation:

a. $\neg\forall x\in\mathbb{Z}\ \exists n\in\mathbb{N}\ [\ x + n > 0\ ]$
b. $\neg\exists x\in\mathbb{Z}\ \exists y\in\mathbb{Z}\ [\ xy < 0\ ]$
c. $\neg\forall x\in\mathbb{N}\ \exists y\in\mathbb{N}\ [\ (x = 3) \wedge (y = 5)\ ]$
d. $\neg\forall x\in\mathbb{N}\ [\ x > 5\ ]$
e. $\neg\exists x\in\mathbb{N}\ [\ (x < 2) \Rightarrow (x = 5)\ ]$

Which of these statements is true?

Exercise 3.11

a. Prove or provide a counterexample:

$$\neg\exists x\in A\ [\ P(x) \wedge \neg Q(x)\ ] \Leftrightarrow \forall x\in A\ [\ P(x) \Rightarrow Q(x)\ ]$$

b. Write without negation:

$$\neg\exists a,b\in A\ [\ \neg(a = b) \wedge (P(a) \wedge P(b))\ ]$$

c. Express the formula from b. with a short English phrase.

Exercise 3.12

We have a sequence (2, 1, 3, 7, 3) = (f(1), f(2), f(3), f(4), f(5)) of integers.
All quantifiers range over the set {1,2,3,4,5}.
Examine whether the next statements are true or false – explain your answers:

a. $\forall i\ \exists j\ [\ f(i) \text{ is even} \Rightarrow f(j) = i\ ]$
b. $\forall i\ [\ f(i) \text{ is even} \Rightarrow \exists j\ [\ f(j) = i\ ]\ ]$
c. $\exists i\ \forall j\ [\ f(i) \leq f(j)\ ]$
d. $\forall i\ \exists j\ [\ f(i) \leq f(j)\ ]$
e. $\forall i\ \exists j\ [\ f(i) < f(j)\ ]$
f. $\forall i\ \exists j\ [\ f(i) \text{ is even} \Rightarrow f(i) < f(j)\ ]$

Exercise 3.13

We have a sequence $(a_1, a_2, \dots, a_5)$ of integers, satisfying the property

$$\forall i\ [\ a_i = 2i\ ]$$

(All quantifiers are assumed to range over the set {1,..,5}.)
Give the elements $(a_1, \dots, a_5)$.
Examine if the following statements are true or false – explain your answers:

a. $\forall i\ [\ a_i > 2\ ]$
b. $\forall i\ [\ i \leq 4 \Rightarrow a_{i+1} > 2.a_i\ ]$
c. $\exists i\ \exists j\ [\ a_i = a_j\ ]$
d. $\exists i\ \exists j\ [\ i \neq j \wedge a_i = a_j\ ]$
e. $\exists i\ \forall j\ [\ i \neq j \Rightarrow a_i > a_j\ ]$

Exercise 3.14

We have three sequences of integers:

(37, 256, 88, 1, -1, 0)

(9, 8, 7, 6, 5, 4, 3, 2, 1, 0)

(1, -1, 2, -2, 3, -3, 4, -4)

Examine for each property below which of the above three sequences satisfies the property, all quantifiers are assumed to range over the set of indices in each case:

a. $\forall i\ \exists j\ [\ f(j) = -f(i)\ ]$

b. $\exists i\ \exists j\ [\ f(j) = -f(i)\ ]$

c. $\forall i\ \exists j\ [\ f(i) \text{ is odd } \Rightarrow f(i) = -f(j)\ ]$

d. $\forall i\ [\ f(i) > 40 \Rightarrow f(i) \text{ is even }]$

e. $\exists i\ [\ f(i) < 0\ ]$

f. $\exists i\ [\ f(i) > 0 \Rightarrow \forall j[f(j) > 0]\ ]$

Exercise 3.15

Give a sequence of 5 integers (x_1, …., x_5) such that all of the following conditions hold:

a. $\neg\forall i\ [\ x_i = i - 5\ ]$

b. $\exists! i\ [\ x_i = i\ ]$

c. $\forall i\ [\ x_i < x_{i+1}\ ]$

d. $\#\{\ i \mid x_i < 0\ \} = 2$

Exercise 3.16

In each case below, make a (finite) sequence of integers that *has* the property (or prove that no such sequence exists) and one that does *not* have the property (or prove that no such sequence exists):

a. $\forall i\ [\ (1 \le i \le 5 \Rightarrow f(i) > f(i-1)) \wedge (6 \le i \le 10 \Rightarrow f(i) < f(i-1))\ ]$

b. $\forall i\ [\ (1 \le i \le 5 \wedge f(i) > f(i-1)) \vee (6 \le i \le 10 \wedge f(i) < f(i-1))\ ]$

Does there exist a sequence that satisfies a. but not b.? Or vice versa?

Exercise 3.17 (*)

Make a (finite) sequence of integers from {0,1} that has the following property, or prove that such a sequence does not exist:

$$\forall i\ [\ f(i) = 1 \Leftrightarrow \#\{\ j \mid i \neq j \wedge f(j) = 1\ \} \text{ is odd }]$$

Can you find a simpler condition that is equivalent to this property?

Exercise 3.18

Make a (finite) sequence that does **not** have the property from 3.14.f, or prove that such a sequence does not exist.

Exercise 3.19

Consider the following matrix:

3	5	7	-2
-1	0	2	1
0	-1	6	0

We'll use A(i, j) to denote the entry at the j'th row (counting from the top) in the i'th column (counting from the left).

Which of the following statements is true (all quantifiers range over the appropriate set of indices):

a. $\forall i\ \exists j\ [\ A(i, j) > 1\ ]$

b. $\forall i\ \exists j\ [\ A(i, j)$ is odd $]$

c. $\exists j\ \forall i\ [\ A(i, j)$ is odd $]$

d. $\exists j\ \forall i\ \forall i'\ [\ (A(i, j) = 0\ \wedge\ A(i', j) = 0)\ \Rightarrow\ (i = i')\ ]$

Exercise 3.20

For each of the following properties, come up with a matrix that has the property (or prove that no such matrix exists) as well as a matrix that does not have the property (or prove that no such matrix exists):

a. $\forall i\ \exists j\ [\ A(i, j)$ is even $] \wedge \neg\exists j\ \forall i\ [\ A(i, j)$ is even $]$

b. $\exists j\ \forall i\ [\ A(i, j)$ is even $] \wedge\ \neg\forall i\ \exists j\ [\ A(i, j)$ is even $]$

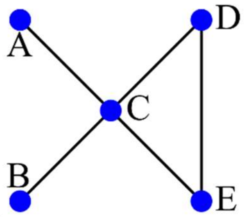
c. $\exists i\ \forall i'\ [\ i \neq i'\ \Leftrightarrow\ \exists j\ [A(i, j) \geq 2\ ]$

d. $\exists i\ \forall j\ \forall j'\ [\ (A(i, j) \geq 2\ \wedge\ A(i, j') \geq 2)\ \Rightarrow\ (j = j')\ ]$

e. $\exists i\ \neg\exists j, j'\ [\ (j \neq j') \wedge (A(i, j) \geq 2\ \wedge A(i, j') \geq 2)]$

Exercise 3.21

Take a look at the following graph:



We'll use the notation E(p,q) to mean "there is an edge between the vertices p and q", and we assume that all quantifiers are restricted to the vertices of the graph.

a. For each sentence below, indicate whether they are true or false. Explain your answers.

1) $\exists p\ \forall q\ [\ E(p,q)\ ]$
2) $\exists p\ \forall q\ [\ p \neq q \Rightarrow E(p,q)\ ]$
3) $\forall p\ \exists q\ [\ p \neq q \wedge E(p,q)\ ]$
4) $\exists p\ [\ E(p,p)\ ]$

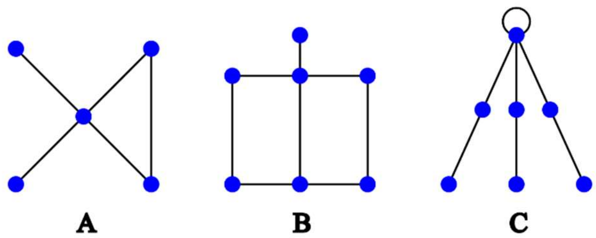
b. Point C can be characterized as the only point satisfying the property:

$$\forall q\ [\ x \neq q \Rightarrow E(x, q)\ ]$$

in the sense that this statement is true whenever x = C, and false for all other nodes.
Give a property using E(,), equality/inequality, quantifiers (∀, ∃) and connectives (∧ ∨ ⇒ ¬) that is true precisely when x is either D or E. (Of course your formula is not allowed to directly refer to the names of individual nodes.)

Exercise 3.22

Consider the following graphs:



Which of these graphs satisfy the property below?

$$\exists p\ \exists q\ \exists r\ [\ E(p, q) \wedge E(q, r) \wedge E(r, p)\ ]$$

Exercise 3.23

a. Come up with a graph that has the following property:

$$\exists p\ \exists q\ [\ (p \neq q) \wedge \forall x[\ x \neq p \Rightarrow E(p,x)\ ] \wedge \forall x[\ x \neq q \Rightarrow E(q,x)\ ]$$

b. Come up with a graph and a formula (a property of 2 nodes) that shows that the order of quantifiers matters.

Exercise 3.24 (*)

A function f: $\mathbb{R} \rightarrow \mathbb{R}$ is called *convex* if every line segment between two points on its graph lie above the graph of f.

Formulate this property using quantifiers:

f is convex if and only if … ?

Exercise 3.25 (*)

If A and B are sets, we use the notation B^A to denote the set of all functions from A to B.

Consider the statement:

"If E is a finite set, then every infinite sequence in E will eventually enumerate an element that it has already enumerated before."

a. Formulate this statement using quantifiers.

b. Is the statement true?

Exercise 3.26 (*)

The 'Pigeonhole principle' (https://en.wikipedia.org/wiki/Pigeonhole_principle) states that if *n* items are put into *m* containers, with $n > m$, then at least one container must contain more than one item.

a. Reformulate the Pigeonhole as a statement about sets and functions.

b. Use the pigeon hole principle to prove the following theorem:

There exists no set of 10 distinct natural numbers between 1 and 100 such that:

no two subsets (of this set of 10 numbers) have the same sum.

c. Find a set of 7 numbers between 1 and 100 such that no two of its subsets have the same sum.

4. Mathematical Induction Exercises

<u>Exercise 4.1</u>

For each of the following statements:

- Write down what this statement amounts to, for n = 0, 1, 2, and 3.
- Prove the statement using mathematical induction.

a. (for $n \geq 0$): $1 + 2 + 3 + \ldots + n = n(n+1) / 2$

b. (for $n \geq 1$): $1 + 4 + 7 + \ldots + (3n-2) = n(3n-1) / 2$

c. (for $n \geq 0$): $1 + 2 + 2^2 + 2^3 + \ldots\ldots + 2^n = 2^{n+1} - 1$

d. (for $n \geq 1$): $1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = 1/3\ (n(n+1)(n+2))$

e. (for $n \geq 1$): $1.2.3 + 2.3.4 + 3.4.5 + \ldots + n(n+1)(n+2) = 1/4\ (n(n+1)(n+2)(n+3))$

<u>Exercise 4.2</u>

Prove the following statement:

(for $n \in \mathbb{N}$, $n \geq 0$, $s \in \mathbb{R}$, $s \neq 1$):

$$1 + s + s^2 + \ldots + s^n = (s^{n+1} - 1) / (s - 1)$$

<u>Exercise 4.3</u>

Prove the following statement ("Bernoulli's Inequality") :

(for $n \in \mathbb{N}$, $n \geq 0$, $x \in \mathbb{R}$, $x \geq 0$):

$$(1 + x)^n \geq (1 + nx)$$

<u>Exercise 4.4</u>

According to wikipedia, mathematical induction is:

> *"a method for proving that a statement P(n) is true for every natural number, that is, that the infinitely many cases P(0), P(1), P(2), P(3), …all hold."*
> *(https://en.wikipedia.org/wiki/Mathematical_induction)*

However, in the previous two exercises, the statement to be proved involves both a natural number and a real number. Explain why (i.e. in what way, precisely) it's nevertheless possible to use mathematical induction to prove these statements?

<u>Exercise 4.5</u>

Prove the second one of the following two statements:

a. for n = 0, 1, 2, …: $1 + 1/4 + 1/9 + \ldots + 1/n^2 \leq 2 - 1/n$

b. for n = 0, 1, 2, …: $1 + 1/4 + 1/9 + \ldots + 1/n^2 \leq 2$

<u>Exercise 4.6</u>

For which $k \in \mathbb{N}$ is $2^k > k^2$? (Prove your answer!)

Exercise 4.7

a. Prove the following statement:

$$1/\sqrt{1} + \dots + 1/\sqrt{n} \geq \sqrt{n} \qquad (n \in \mathbb{N},\ n \geq 1)$$

(Where $\sqrt{x}$ denotes the square root of x.)

b. For which n is the inequality strict? (Prove your answer!)

Exercise 4.8

In every finite tree, the number of edges is one less than the number of nodes.
Prove this statement by induction to the size of the tree.

Exercise 4.9

Strong induction is the principle which allows you to conclude $\forall n \in \mathbb{N}$ [P(n)] from:

a) P(0) and

b) $\forall n \in \mathbb{N}$ [P(0) $\wedge \dots \wedge$ P(n) $\Rightarrow$ P(n+1)]

That is, in the induction step, you are allowed to use that the wanted property is true for *all* natural numbers *upto n*, not just for n.
How does strong induction follow from weak induction?

Exercise 4.10

Prove the following theorem:

Let f: $\mathbb{R} \to \mathbb{R}$ be a function such that

for all x,y in $\mathbb{R}$: f(x+y) = f(x) + f(y)

Then there exists an s in $\mathbb{R}$ such that for all x in $\mathbb{Q}$:

f(x) = s.x

Exercise 4.11

Given a function f: $\mathbb{N} \to \mathbb{N}$, define a new function g: $\mathbb{N} \to \mathbb{N}$ as follows:

$$g(n) := f(0) + f(1) + \dots + f(n)$$

Suppose that g(n) = 0 for every n.
Prove that f(n) = 0 for every n.

Exercise 4.12

On the grid $\mathbb{N} \times \mathbb{N}$, we define a 'good walk' to be a path that starts in (0,0) and then 'moves' in steps that are either: one step to the right or: one step up, as in the picture:

Given n,m $\in \mathbb{N}$, we define C(n, m) as the number of good walks ending in (n, m).
It's easy to see that:

for every n $\in \mathbb{N}$: C(n, 0) = 1 (why?)

for every n $\in \mathbb{N}$: C(0, n) = 1 (why?)

for every n,m $\in \mathbb{N}$ with n ≥ 1 and m ≥ 1: C(n, m) = C(n-1, m) + C(n, m-1) (why?)

Prove that (for every n, m $\in \mathbb{N}$)

C(n+1, m+1) = C(0,m) + C(1,m) + … + C(n,m)