Reliability

Resources

Reference books

Ang, A. H-S., and W. H. Tang. *Probability Concepts in Engineering Planning and Design,* vol. 1, *Basic Principles*. New York, NY: John Wiley & Sons, 1975. ISBN: 9780471032007.

Rausand, M., and A. Hoyland. *System Reliability Theory: Models, Statistical Methods, and Applications*. 2nd ed. New York, NY: John Wiley & Sons, 2003. ISBN: 9780471471332.

G. Habchi and C. Barthod, *An overall methodology for reliability prediction of mechatronic systems design with industrial application.* Reliability Engineering & System Safety, 155 (2016), 236-254.

Website

https://www.weibull.com.

Recall of Basic Probability Theory and Statistics

Important Terms

Probability density function (pdf) – A continuous representation of a histogram that shows how the number of component failures are distributed in time.

Cumulative distribution function (cdf) – Also called the probability of failure, represents the probability that a brand new component will fail at or before a specified time.

From probability and statistics, given a **continuous random variable** X, we denote:

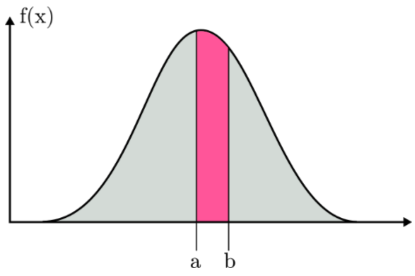
- The probability density ***PDF***, as $f(x)$.
- The cumulative distribution ***CDF***, as $F(x)$
- The *pdf* and *cdf* give a complete description of the probability distribution of a random variable.

Probability density function (pdf)

If X is a continuous random variable, then the probability density function, pdf, of X is a function $f(x)$ such that for two numbers, a and b with a ≤ b:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

In other words, the pdf defines the probability that X takes on a value in the interval [a,b] is the area under the density function from a to b. This is represented graphically in the following plot.



In reliability terms, this function gives us the probability that a failure occurs between time *a* and time *b*. This function completely describes the distribution, and is the basis for almost all of the familiar reliability and life data functions.

Properties of a density function:

- Must be positive. Obviously a probability cannot be a negative value
- Must be continuous by definition.
- The area under the entire curve must be equal to 1 (as the sum of the probabilities values of a law must be equal to1).

Example to better understand *PDF*

Take the example of heights of students in a class. Let’s say that the height of all the students is between 160 cm and 170 cm. It makes no sense to ask the probability that the height of the student is EXACTELY 165.84 cm; that probability is zero. We can, however, define the probability that the height of the student lies in the infinitesimal interval [163;166] cm for example. The function that gives this probability density is referred to as the probability density function or *pdf*.

Cumulative distribution function (cdf)

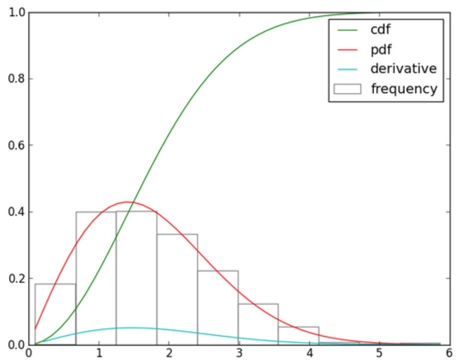
The cumulative distribution function, cdf, is a function $F(x)$ of a random variable X, and is defined for a number x by:

$$F(x) = P(X \leq x) = \int_{0,-\infty}^{x} f(s)\, ds$$

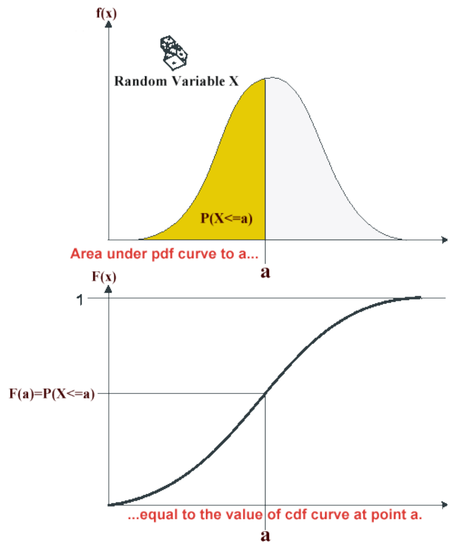
That is, for a given value x, $F(x)$ is the probability that the observed value of X will be at most x. Evaluated at x, is the probability that X will take a value less than or equal to x.
The cumulative distribution function, $F(x)$, denotes the area beneath the probability density function to the left of x.

***PDF* and *CDF* relationship**

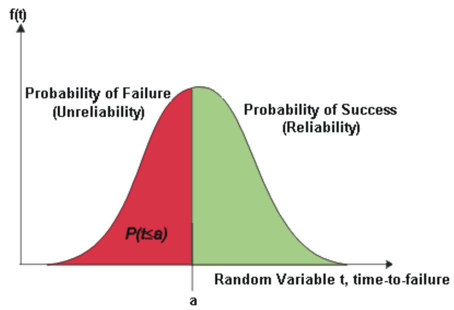
Probability looks at probability at one point, while Cumulative is the total probability of anything below it. As you can see in the diagram below, the cumulative is much greater than the just probability because it is the sum of many, and not just of one probabilities.



Following is a graphical representation of the relationship between the *PDF* and *CDF.*



The following figure illustrates the relationship between the reliability function and the *CDF*, or the unreliability function.



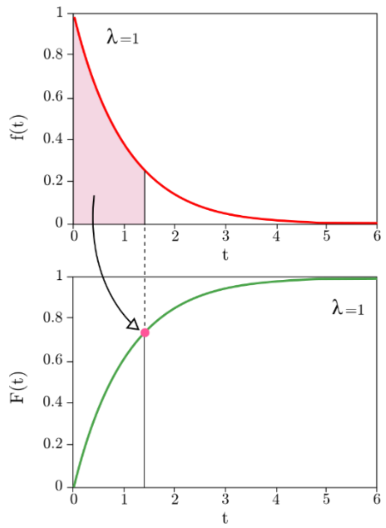
Example with the exponential distribution

The probability density function (*PDF*) of exponential distribution is:

$$f(t) = \lambda . e^{(-\lambda . t)}$$

The exponential cumulative distribution function can be derived from its *PDF* as follows :

$$F(t) = \int_0^t f(t)dt = \int_0^t \lambda e^{-\lambda t} dt = \lambda \left[\frac{e^{-\lambda t}}{-\lambda} \right]_0^t = \lambda \left[\frac{e^{-\lambda t}}{-\lambda} - \frac{1}{-\lambda} \right] = 1 - e^{-\lambda t}$$



F(t) in this particular case with the exponential distribution, is the Failure function of a product when the failure rate is constant. The failure density is linked to the failure rate by the following relation: $f(t) = \lambda(t).[1 - F(t)]$. Details are provided further.

Important Terms

Failure – The number of failures per unit time that can be expected to occur for the product.

Lifetime – A statistical measure (or estimate) of how long a product is expected to perform its intended functions under a specific set of environmental, electrical and mechanical conditions. Lifetime specifications can only describe the behaviour of a population; any single product may fail before or after the rated lifetime.

Mean Time Between Failures (MTBF) – The average time between failures during useful life for repairable or redundant systems.

Mean Time To Failure (MTTF) – The average time to failure during useful life for components or non-repairable systems.

Reliability – A statistical measure (or estimate) of the ability of a product to perform its intended functions under a specific set of environmental, electrical, and mechanical conditions, for a specific period of time. Reliability estimates for the entire useful life phase of a product are commonly reported using MTBF or MTTF.

Serviceability – The ability of a product to be repaired by regular maintenance personnel, typically through replacement of a subsystem or one or more associated components

Reliability Functions

Reliability

The term “reliability” in engineering refers to the probability that a product, or system, will perform it’s designed functions under a given set of operating conditions for a specific period of time. It is also known as the “probability of survival”.

Reliability is defined as the probability that an individual unit of the product, operating under specified conditions, will work correctly for a specified length of time. Hence reliability is the probability of failure in the flat central part of the familiar bathtub curve shown below.

Several distributions defined by parameters are commonly used for reliability models, including:

- **Exponential**
- **Weibull**
- Gamma
- normal (Gaussian)
- log-normal
- log-logistic.

The choice of parametric distribution for a particular application can be made using graphical methods or using formal tests of fit.

The exponential law is well adapted when the **failure rate (λ) is constant**, (i.e. it is independent of time). The normal law represents behaviours where the lifetime of the population is homogeneous, the probability of failure is centred and symmetrical. It can be used to model maturity behaviour (without breakdowns) and then rapid wear and tear. The Weibull's law models each of the three phases of a material's life. It generalises the two previous laws but is more difficult to use and interpret.

Exponential model

The exponential distribution is most widely used distribution in reliability and risk.

Weibull model

Named after the Swedish professor Waloddi Weibull (1887-1979), the Weibull distribution is one of the most widely used life distributions in reliability analysis.

Weibull analysis is able to answer many life cycle engineering problems such as mean life estimation, reliability of products at any operational time and warranty cost estimation etc. The advantages of Weibull model in life data analysis can be extended to facilitate decision-making processes in many remanufacturing practices such as prediction of the number of cores returned for remanufacturing, estimation of spare parts or remedy resources needed for each failure mode, and so on.

The Weibull distribution has three parameters:

• Shape parameter β (beta)

• Scaling parameter η (eta)

• Position parameter γ (gamma)

Among the two parameters, the slope of the Weibull distribution, β, is very important as it determines which member of the family of Weibull failure distributions best fits or describes the data. It also indicates the class of failures in the “bathtub curve” failure modes.

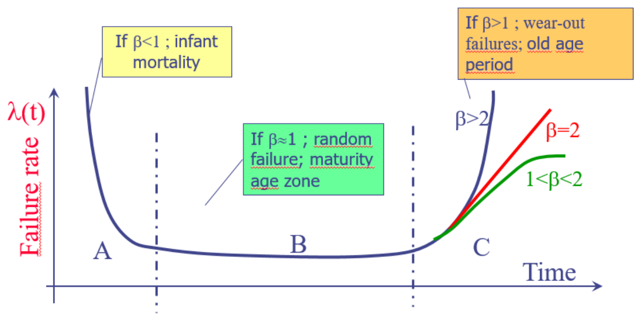
The Weibull shape parameter β indicates whether the failure rate is increasing, constant or decreasing.

• β < 1 indicates infant mortality;

• β = 1 means random failures (i.e. independent of time);

• β > 1 indicates wear-out failures.

The information about the β value is extremely useful for reliability centered maintenance planning and product life cycle management. This is because it can provide a clue to the physics of the failures and tell the analyst whether or not scheduled inspections and overhauls are needed. For instance, if β is less than or equal to one, overhauls are not cost effective. With β greater than one, the overhaul period or scheduled inspection interval can be read directly from the plot at an acceptable or allowable probability of failures. For wear-out failure modes, if the cost of an unplanned failure is much greater than the cost of a planned replacement, there will be an optimum replacement interval for minimum cost.



Scale parameter η

The scale parameter, or spread, η, sometimes also called the characteristic life, represents the typical time-to-failure in Weibull analysis. It is related to the Mean-Time-to-Failure (MTTF). In Weibull analysis, η is defined as the time at which 63.2% of the products will have failed

Location parameter γ

It is expressed in units of time. It indicates the date of the occurrence of the failure mode characterised by β.

Weibull functions

Reliability	$R(t) = e^{\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]}$
Failure	F(t)=1-R(t)= $1 - e^{\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]}$
Failure probability density	$f(t) = \frac{dF(t)}{dt} = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]$

Failure rate

A way to look at the failure behaviour in time is to examine the failure rate. Failure rate is function of time. For instance, over a year, a bearing used in a car might have a higher failure rate in summer than in winter.

Failure rates can be expressed using any measure of time, but **hours** is the most common unit in practice. Other units, such as km, revolutions, etc., can also be used in place of "time" units.

Mathematically, the failure rate function is a conditional form of the pdf, as seen in the following equation:

$$\lambda(t) = \frac{f(t)}{R(t)}$$

under the condition is that no failure has occurred before time *t*.

then

$$\lambda(t) = \frac{R(t_1) - R(t_2)}{(t_2 - t_1) \cdot R(t_1)} = \frac{R(t) - R(t+\Delta t)}{\Delta t \cdot R(t)}$$

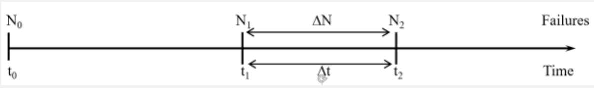
over a time interval $\Delta t = (t_2 - t_1)$ from t_1 (or t) to t_2.

This curve is usually called a bathtub curve after its characteristic shape. The failure rate is often high in the initial phase. This can be explained by the fact that there may be undiscovered defects (known as “infant mortality”) in the items; these soon show up when the items are activated. When the item has survived the infant mortality period, the failure rate often stabilizes at a level where it remains for a certain amount of time until it starts to increase as the items begin to wear out. From the shape of the bathtub curve, the lifetime of an item may be divided into three typical intervals: the burn-in period, the useful life period and the wear-out period. The useful life period is also called the chance failure period.

Constant Failure Rate Reliability Models

Important Terms

N – total number of products put into operation at time t0 (sample size)
N0 – number of failures at t0 (generally equal to 0)
N1 – number of failures at t1
N2 – number of failures at t2
DN – number of failures on the interval [t1, t2]
Dt – time interval [t1, t2]



If the failure rate is **constant** for all times, $\lambda(t) = \lambda_o$, often referred to as "failure rate **on average**", the corresponding reliability and probability density function are **exponential functions** given by:

$$R(t) = e^{(-\lambda_0 . t)}$$

$$f(t) = \lambda_0 . e^{(-\lambda_0 . t)}$$

The mean time to failure is readily obtained as:

$$\text{MTTF} = E(t) = \frac{1}{\lambda_0}$$

While MTBF can be a useful metric to estimate reliability, especially when comparing components from multiple vendors, the trap many people fall into is wrongly assuming that the MTBF figure directly equates to the expected life of the component.

Estimators

It is obviously not realistic to calculate failure rates by building many units and running them for many hours, under expected operating conditions. This is especially true for well-designed and properly built supplies, with extremely low failure rates, where the number of supplies and hours required to get valid results would be in the thousands. Instead based on representative samples of a population statistical analysis techniques can be used to estimate failure rates.

If you don't know the failure law on which your component is based, then use estimators.

Reliability function estimators depend on the value of N (number of products put into operation)

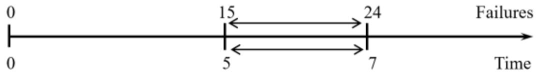
Estimator	N
Median ranks	1<N<=20
Average ranks	20<N<=50
Cumulative frequencies	N>50

Failure Estimators: F(t)

Point estimator at time t

Median ranks if 1<N<=20	$F(t) = \frac{N_t - 0.3}{N + 0.4}$
Average ranks if 20<N<=50	$F(t) = \frac{N_t}{N + 1}$
Cumulative frequencies if N>50	$F(t) = \frac{N_t}{N}$

Example (N = 100) ➔ Cumulative frequencies



$$F(5) = \frac{15}{100} = 0.15 \qquad F(7) = \frac{24}{100} = 0.24$$

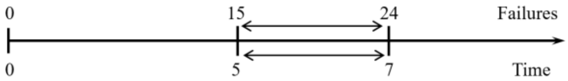
Reliability Estimators: R(t)

Point estimator at time t

Whatever the value of N, first compute the estimator of F and then apply the complementarity relation

$$R(t) = 1 - F(t)$$

Example (N = 100)



$$R(5) = 1 - 0.15 = 0.85 \qquad R(7) = 1 - 0.24 = 0.76$$

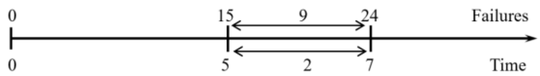
$$R(5) = \frac{100 - 15}{100} = 0.85 \qquad R(7) = \frac{100 - 24}{100} = 0.76$$

Density estimators: *f*(t)

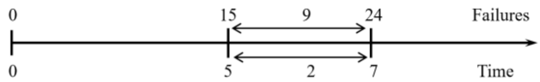
Estimator on **interval** estimator (failure/product and unit of time)

Median ranks if 1<N<=20	$f_{[t1;t2[} = \frac{\Delta N}{(N + 0.4) \times \Delta t}$
Average ranks if 20<N<=50	$f_{[t1;t2[} = \frac{\Delta N}{(N + 1) \times \Delta t}$
Cumulative frequencies if N>50	$f_{[t1;t2[} = \frac{\Delta N}{N \times \Delta t}$

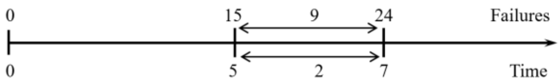
Example (with N = 100)



$$f_{[5;7[} = \frac{9}{100 \times 2} = 0.045 \ \textit{failures/ product} \ \& \ \textit{t.u.}$$



$$\lambda_{[5;7[} = \frac{9}{85 \times 2} = 0.053 \ \textit{failures/product} \ \& \ \textit{t.u.}$$



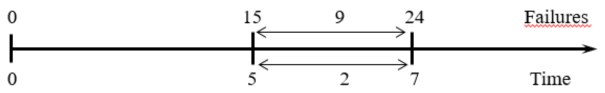
$$E(t)_{[5;7[} = \frac{1}{0.053} = \frac{85 \times 2}{9} = 18.89 \ \textit{u.t.}$$

Failure Rate Estimators: λ(t)

Estimator on **interval** estimator (failure/product and unit of time)

Median ranks if 1<N<=20	$\lambda_{[t1;t2[} = \dfrac{\Delta N}{(N+0.7-N_1)\times\Delta t}$
Average ranks if 20<N<=50	$\lambda_{[t1;t2[} = \dfrac{\Delta N}{(N+1-N_1)\times\Delta t}$
Cumulative frequencies if N>50	$\lambda_{[t1;t2[} = \dfrac{\Delta N}{(N-N_1)\times\Delta t}$

Example (with N = 100)



$$\lambda_{[5;7[} = \frac{9}{85\times 2} = 0.053\ \textit{failures/product \& t.u.}$$

Predictive Reliability

Predictive reliability (predicted or theoretical) is calculated on the basis of a mathematical model defined from the functional decomposition of the system into subsets, components, etc. and the estimated or predicted reliability of its components. The analysis and calculation steps are usually the following:

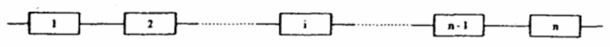
- Decompose the system into components (parts, subsystems ...) and establish the functional links between the components
- Identify component reliability models or collect reliability at a given time
- Search for a model of the system and calculate its reliability

Reliability block diagram (RBD)

A reliability block diagram is a success-oriented network describing the function of the system. It shows the logical connections of (functioning) components needed to fulfill a specified system function

Series Structure

In a series configuration, a failure of any component results in the failure of the entire system. Since the components are **independent**, the corresponding reliability block diagram is:



Schematic illustration of a series system

In other words, for a pure series system, if there are n components in series, where the reliabilities are independent of the i-th component is denoted by R_i, the system reliability is:

$$R_s = R_1 \times R_2 \times \cdots \times R_n = \prod_{i=1}^{n} R_i$$

i.e. the system reliability is equal to the product of the reliabilities of its constituent components.

Parallel Structure

Example

An electronic system has 1000 components in reliability series. The reliability of each component is 0.999. Determine System reliability.

Solution

$$R_s = \prod_{i=1}^{1000} 0.999 = (0.999)^{1000} = 0.368$$

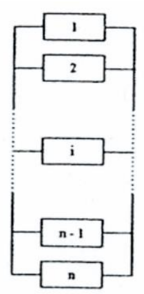
Commentary

1.Even though a component reliability of 0.999 sounds good, the sheer number of these components causes a low system reliability.

2.Even though 0.999~1.0, the difference is crucial. For high reliability values, the probability of failure often gives a clearer picture. For example, increasing the component reliability from 0.999 to 0.9999 re-quires a ten-fold reduction of the failure probability.

The reliability of a system containing independent subsystems can usually be increased by using subsystems in a redundant way.

If the components are in parallel, system performs if any one component remains operational. The corresponding reliability block diagram is:



Schematic illustration of a parallel system

If there are n components in parallel, where the probability of failure of the i-th component is denoted by F_i, the probability of system failure is:

$$F_s = F_1 \times F_2 \times \cdots \times F_n = \prod_{i=1}^{n} F_i$$

the system reliability is then:

$$R_s = 1 - \prod_{i=1}^{n} F_i = 1 - \prod_{i=1}^{n} (1 - R_i)$$

This model assumes that there is statistical independence among failures of the subsystems. This assumption is important because dependence between subsystems can have a significant effect on system reliability.

Redundancy

Redundancy is the surest method of preventing system failure. There are four kinds of redundancy: diverse, homogenous, active, and passive.

Active redundancy guards against both system and element failure. It also allows for element failure, repair, and substitution with minimal disruption of system performance.

Passive redundancy is ideal for noncritical elements, but it will result in system failure when used for elements critical to system operation. Passive redundancy is the simplest and most common kind of redundancy.

Diverse redundancy is resistant to a single cause of failure, but is complex to implement and maintain. For the train, a single cause is unlikely to result in a cascade failure in all three braking systems.

Homogenous redundancy is relatively simple to implement and maintain but is susceptible to single causes of failure — i.e., the type of cause that results in failure in one element can result in failure of other redundant elements.

Use diverse redundancy for critical systems when the probable causes of failure cannot be anticipated. Use homogenous redundancy when the probable causes of failure can be anticipated. Use active redundancy for critical systems that must maintain stable performance in the event of element failure or extreme changes in system load. Use passive redundancy for noncritical elements within systems, or systems in which performance interruptions are tolerable.

Important Terms

Redundancy – Use of more elements than necessary to maintain the performance of a system in the event of failure of one or more of the elements.

Active Redundancy – Application of redundant elements at all times (e.g., using multiple independent pillars to support a roof). All components are active from the start. Only one component working allows the functioning of the whole system.

Passive Redundancy – Application of redundant elements only when an active element fails (e.g., using a spare tire on a vehicle in the event of a flat tire). Only the first component is activated. If it fails, then the second component is activated. The whole system fails when the last component fails.

Diverse Redundancy – use of multiple elements of different types (e.g., high-speed trains often have diverse redundancy in their braking systems—one electric brake, one hydraulic brake, and one pneumatic brake)

Homogenous Redundancy – Use of multiple elements of a single type (e.g., use of multiple independent strands to compose a rope).

K-out-N redundancy system

A system that is functioning if and only if at least k of the n components are functioning

R_1, R_2, ... R_n are the elementary independent reliabilities of the components of the system at a given time.

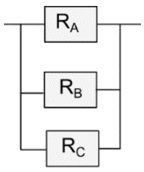
$$R_S = \sum_{i=k}^{n} \left(C_n^i . R^i . (1-R)^{n-i} \right)$$

n: total number of components
R: reliability of a component
k: number if components to operate simultaneously
with: $C_n^i = \frac{n!}{k!(n-k)!}$

⚠ This relation (using binomial law) is applicable only if the R are identical, otherwise, application of the truth table.

Truth table

R_A, R_B and R_C are different



(2/3)

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	F_A x R_B x Rc = (1-R_A) x R_B x Rc
1	0	0	0
1	0	1	R_A x F_B x Rc = R_A x (1-R_B) x Rc
1	1	0	R_A x R_B x Fc = R_A x R_B x (1-Fc)
1	1	1	R_A x R_B x Rc
$\sum$			Rs = (1-R_A) x R_B x Rc + R_A x (1-R_B) x Rc + R_A x R_B x (1-Fc) + R_A x R_B x Rc

Mixed Models

One system configuration that is often encountered is one in which subsystems are in series, but redundancy (active) is applied to a certain critical sub-system(s). A typical block diagram follows:

Application to the exponential model

$$\mathrm{R(t)} = \mathrm{e}^{(-\lambda_0 . t)}$$

F(t) = $1-\mathrm{e}^{(-\lambda . t)}$ with: λ = 1/E(t) (failure rate)

Serial system	$R(t) = e^{(-t\sum_{i=1}^{n}\lambda_i)} = e^{(-t\lambda_S)}$
The parallel system (n components)	$R(t) = 1 - \prod_{i=1}^{n}(1-e^{(-\lambda_i t)})$
The parallel system (2 components)	$R(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$

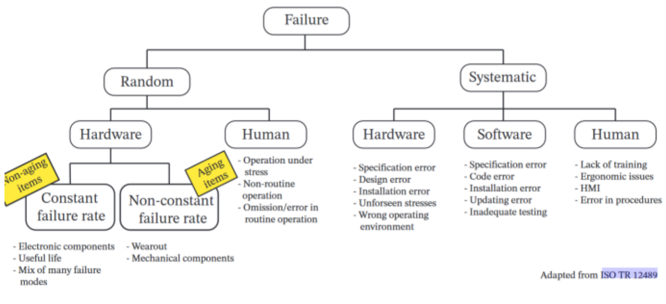
Classification of failures

Parial: the system has a degraded performance, but is still able to perform its essential functions.

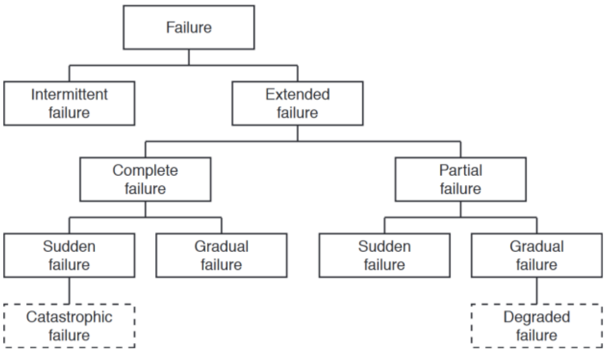
Complete: failure of one or several items cause immediate cessation of the system ability to perform the required function.

Failure Classification in ISO TR 12849

ISO TR 12489 adds to the category random failures, to capture that some systematic faults (induced by human errors) can be regarded as random



Other mean of failure classifications



Failure classification (adapted from Blanche and Shrivastava 1994).