

# Probabilistic Graphical Models

## Conditional Independence:

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2) \quad P(x_3|x_1, x_2) = P(x_3|x_2)$$

Distribution P,  $x_3$  is independent of  $x_1$  conditional on  $x_2$ .

This relationship can be shown with a directed acyclic graph.

For a distribution with no conditional independence relations a suitable BN representation would be:



Figure 1: Untitled

An arrow from A to B means A is the parent of B. The set of parents of a node is  $pa_k$ .

$$p(X) = \prod_{k=1}^K p(x_k|pa_k) = \text{Joint Probability Distribution}$$

A given DAG represents a set of joint distributions, each distribution in the set corresponds to a choice of values for the conditional distributions. In a Bayesian approach, we have to define a prior probability distribution over parameters which represent our beliefs about their values prior to observing the data.

## Polynomial Regression Model:

$$p(t, W) = p(W) \prod_{n=1}^N p(t_n|W) \quad t = \text{data} \quad W = \text{parameters}$$

This assumes that our data is independent and identically distributed (conditional on  $W$ ). This can be represented by a bayesian network:

Dots here represent  $t_n$  that can't fit in the space, instead of showing it like this we can use plate notation:



Figure 2: Untitled



Figure 3: Untitled

The full model contains:

data:  $x = (x_1, \dots, x_N)^T$       output:  $t = (t_1, \dots, t_N)^T$       parameter vector:  $w$

hyperparameter:  $\alpha$       noise variance:  $\sigma^2$



Figure 4: Untitled

Shaded circles can be observed, non-shaded circles cannot be observed.

**Hierarchical Regression:**

$$P(\theta, y, \mu, \sigma^2) = P(\mu, \sigma^2) \prod_{i=1}^k P(y_i | \theta_i) P(\theta_i | \mu, \sigma^2)$$

**Conditional Independence:**

$$P(x, y | S) = P(x | S) P(y | S) \implies x \text{ is conditionally independent of } y$$

## Hierarchical regression (abbreviated)



Figure 5: Untitled

$$P(A, B|C) = \frac{P(A, B, C)}{P(C)} = \frac{P(A|C)P(B|C)P(C)}{P(C)} = P(A|C)P(B|C)$$

$$A \perp B | C \quad \text{A and B are independent given C}$$

A collider is a node on some path where both arrows point to the node and both arrows on the path.