

## Expectation Maximisation Algorithm

Maximum likelihood from incomplete data.

We can use EM for finding a local maximum of a gaussian mixture:

$$\log \text{likelihood} = \ln(p(X|\pi, \mu, \Sigma)) = \sum_{n=1}^N \ln \left( \sum_{k=1}^K \pi_k \mathcal{N}(x_n | \mu_k, \Sigma_k) \right)$$

There is no closed form for calculating the local maxima so we must use an iterative approach.

$$\mu_k = \frac{1}{N_k} \sum_{n=1}^N \gamma(z_{nk}) x_n \Sigma_k = \frac{1}{N_k} \sum_{n=1}^N \gamma(z_{nk}) (x_n - \mu_k)(x_n - \mu_k)^T \pi_k = \frac{N_k}{N} \text{ where } \gamma(z_{nk}) = p(z = 1 | x_n) \text{ and } N_k = \sum_n \gamma(z_{nk})$$

To initialise EM we choose our starting values for mu, sigma and pi as well as the number of gaussians k.

We can then compute responsibilities which is the probability that data point n comes from mixture K

$$\gamma(z_{nk}) = \frac{\pi_k \mathcal{N}(x_n | \mu_k, \Sigma_k)}{\sum_{j=1}^K (\pi_j \mathcal{N}(x_n | \mu_j, \Sigma_j))} = \text{Bayes theorem}$$

We then plug the new responsibilities into the previous formulae to calculate new constants for each gaussian.

**General Case:**

$$\ln p(X|\theta) = \ln \left\{ \sum_z p(X, Z|\theta) \right\} \quad Z \text{ are hidden variables (not observed) also called latent variables } \{X, Z\} \text{ is the}$$

$$\text{Let } q(Z) \text{ be the distribution over the hidden variables } \ln(p(X|\theta)) = l(q, \theta) + KL(q||p) \quad l(q, \theta) = \sum_Z q(Z) \ln \left( \frac{p(X|Z, \theta)}{q(Z)} \right)$$

The KL function above is the KL divergence between probability distributions p and q. Its a measure of how different those distributions are. KL divergence cannot become negative.

if p = q then KL(p,q) = 0.

Our aim to to maximise the l function above

- In the E-step, increase  $l$  by updating  $q$
- In the M-step, increase  $l$  by updating  $\theta$

In the E-step we want to maximise the  $l$  function, we can use the decomposition of the log likelihood above to realise that if we minimise KL then  $l$  must increase. So we can set  $p = q$ .

In the M-step we want to find parameters  $\theta$  that maximise the  $L$  function leaving  $q$  fixed. This will increase the log likelihood since  $KL = 0$ .

We can repeat these steps to constantly increase the log likelihood.