

☒ Laboratory 2

System identification of a synthetic model using the PEM method

Data generation. Consider the actual model

$$\mathbf{y}(t) = \mathcal{F}_0(z)u(t) + \mathcal{G}_0(z)\mathbf{e}_0(t), \quad t \in \mathbb{Z} \quad (9.7)$$

where

- $\mathcal{F}_0(z) = \frac{2.99z^{-1}-0.2z^{-2}}{1-0.96z^{-1}+0.97z^{-2}};$
- $\mathcal{G}_0(z) = \frac{1}{1-0.96z^{-1}+0.97z^{-2}};$
- \mathbf{e}_0 is WGN with standard deviation $\sigma_0 = 4.6$;
- the input u is a stationary deterministic signal and defined in the following way

$$u(t) = 4 \sum_{l=1}^5 \sin(\omega_l t), \quad \omega_l \in \left[\frac{\pi}{8}, \frac{\pi}{2} \right] \quad t \in \mathbb{Z}.$$

Generate a dataset (y^N, u^N) with $N = 200$ using model (9.7). Recall that

$$\begin{aligned} u^N &= [u(1) \ \dots \ u(N)]^T \\ y^N &= [y(1) \ \dots \ y(N)]^T. \end{aligned}$$

In particular, use the function `idinput` (of the System Identification Toolbox) to generate u^N .

PEM estimation. Compute the PEM estimate $\hat{\theta}_{PEM}(y^N, u^N)$ using the model structures containing the following models:

Model	Degrees
$\mathcal{M}_1(\theta) : A_\theta(z)\mathbf{y}(t) = B_\theta(z)u(t-1) + \mathbf{e}(t)$	$n_A = 2, n_B = 2$
$\mathcal{M}_2(\theta) : A_\theta(z)\mathbf{y}(t) = B_\theta(z)u(t-1) + C_\theta(z)\mathbf{e}(t)$	$n_A = 2, n_B = 2$ $n_C = 1$
$\mathcal{M}_3(\theta) : \mathbf{y}(t) = \frac{B_\theta(z)}{F_\theta(z)}u(t-1) + \mathbf{e}(t)$	$n_B = 2, n_F = 1$
$\mathcal{M}_4(\theta) : \mathbf{y}(t) = \frac{B_\theta(z)}{F_\theta(z)}u(t-1) + \frac{C_\theta(z)}{D_\theta(z)}\mathbf{e}(t)$	$n_B = 2, n_C = 1$ $n_D = 1, n_F = 2$

Let $\hat{\mathcal{F}}_j(z)$, $j = 1 \dots 4$, be the PEM estimates of $\mathcal{F}_0(z)$ using the previous model structures. Plot the Bode diagram of the transfer functions $\mathcal{F}_0(z)$ and $\hat{\mathcal{F}}_j(z)$, $j = 1 \dots 4$ using the function `bodeplot` of the System Identification Toolbox (e.g. `bodeplot(mt,me)` where `mt` is the true model and `me` is the estimated model).

Asymptotic analysis. Run the previous algorithm for computing the PEM estimates with $N = 8000$.

Question 1: What conclusions can you draw comparing these Bode plots for $N = 8000$? Motivate the answer (before run a couple of times the code).

Question 2: What conclusions can you draw from the Bode plots of the cases $N = 200$ and $N = 8000$? Motivate the answer (before run a couple of times the code).

Confidence intervals. Observe that the actual model (9.7) is $\mathcal{M}_1(\theta_0)$ where $\theta_0 = [a_{1,0} \ a_{2,0} \ b_{0,0} \ b_{1,0}]^T$ with $a_{1,0} = -0.96$, $a_{2,0} = 0.97$, $b_{0,0} = 2.99$, $b_{1,0} = -0.2$. Consider the PEM estimator using \mathcal{M}_1 . We compute the confidence intervals of the estimated coefficients in

$$\hat{\theta}_{PEM} = [\hat{a}_{1,PEM} \ \hat{a}_{2,PEM} \ \hat{b}_{0,PEM} \ \hat{b}_{1,PEM}]^T$$

as follows. Let $\hat{\mathbf{y}}_\theta(t|t-1)$ be the one-step ahead predictor using $\mathcal{M}_1(\theta)$. It is not difficult to see that

$$\Psi_\theta(t) = \frac{\partial \hat{\mathbf{y}}_\theta(t|t-1)}{\partial \theta} = [-\mathbf{y}(t-1) \ -\mathbf{y}(t-2) \ \mathbf{u}(t-1) \ \mathbf{u}(t-2)]^T.$$

Let $P = \sigma_0^2 \left(\bar{\mathbb{E}}_{\theta_0, \sigma_0^2} [\Psi_{\theta_0}(t) \Psi_{\theta_0}(t)^T] \right)^{-1}$. Compute an estimate of P as follows

$$\hat{P}_N = \hat{\sigma}_{PEM}^2 \left(\frac{1}{N} \sum_{t=1}^N \Psi_{\hat{\theta}_{PEM}}(t) \Psi_{\hat{\theta}_{PEM}}(t)^T \right)^{-1}.$$

If Theorem 5.2 can be applied, it is not difficult to show (see Example 5.11) that the following “certificates” of performance hold with high probability (and N sufficiently large):

$$|\hat{a}_{1,PEM} - a_{1,0}| \leq 1.96 \sqrt{\frac{(\hat{P}_N)_{1,1}}{N}},$$

$$|\hat{a}_{2,PEM} - a_{2,0}| \leq 1.96 \sqrt{\frac{(\hat{P}_N)_{2,2}}{N}},$$

$$|\hat{b}_{0,PEM} - b_{0,0}| \leq 1.96 \sqrt{\frac{(\hat{P}_N)_{3,3}}{N}},$$

$$|\hat{b}_{1,PEM} - b_{1,0}| \leq 1.96 \sqrt{\frac{(\hat{P}_N)_{4,4}}{N}}.$$

Compute the approximation errors (left hand side of the above inequalities) and the corresponding confidence intervals (whose extremes are characterized by the right hand side of the above inequalities) for $N = 200$ and $N = 8000$.

Question 3: Does θ_0 belong to the confidence interval? What conclusions can you draw? Motivate the answer.