Economics of Financial Markets

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Contents

Asset Allocation

In this first part of the exam we will start with market data and we will perform various asset allocation exercises in Python, based on the concepts proposed by Markowitz in a 1952[1] essay on Modern Portfolio Theory. In particular, the idea behind MPT is to construct a portfolio of assets in such a way as to maximise the expected return and minimise risk through diversification. As we will see later, MPT assumes risk aversion of investors and thus a rational investor will not invest in portfolio X if there is a portfolio Y in which there are higher returns for the same risk or lower risk for the same return. In our case, this exercise will be carried out on a sample of 12 securities, selected according to precise criteria that will be defined in the following paragraphs. Then, with the support of the Security Market Line, we will be able to study the average risk/return ratio of the market, so that we can compare our securities and portfolio with a benchmark.

In the following step we will implement the Black-Littermann approach, which allows us to include views (reflecting the investor's experience and expectations) in the asset selection phase and combines them with market expectations (prior) to produce an expected return (to be estimated) that takes both factors into account (posterior).

Next, we will implement a standard Bayesian Asset Allocation, considering different priors and variance-covariance matrix.

Lastly, the Global Minimum Portfolio Variance approach will allow us to obtain the portfolio on the frontier that minimises the standard deviation (and thus the variance).

In the conclusion part, detailed comparisons will be made, and appropriate explanations given regarding the similarities and differences of the various portfolios. At the end, we will define an average portfolio that takes into account the statistics of all portfolios, so that all assumptions defined for each model can be taken into account (even if in a small portion).

1.1 Returns analysis

In order to perform an asset allocation exercise, for the vast majority of cases, only one piece of input data is needed: the closing price of securities and consequently the yield of securities.

[1] Markowitz, H.M. (March 1952). "Portfolio Selection". The Journal of Finance. 7 (1): 77–91. doi:10.2307/2975974. JSTOR 2975974

We therefore first asked ourselves why this was true. We can certainly state that return analysis is essential for several reasons, including:

- Performance evaluation: the study of returns allows one to understand whether a certain investment is generating profits or losses and allows one to determine how much weight individual factors contribute to the outcome.
- Identification of trends and patterns: the analysis of stock returns can reveal trends and patterns in the behaviour of stock prices over time. This second interpretation is especially the basis of statistical arbitrage strategies, i.e. those strategies that allow a 'free lunch' in the market, without taking any risk.
- Comparison with the market: as mentioned in the introduction, having a benchmark is necessary for the study of one's portfolio.
- Risk management: by analysing past returns, investors can assess the volatility and variance of the returns of a stock or portfolio, as well as the correlation with other financial assets.

Starting from closing prices, it is possible to move on to percentage returns (whether daily or monthly) on the basis of the Holding Period Return formula:

$$\overline{R}_i = \frac{P_1^i - P_0^i + D^i}{P_0^i}$$

where D^i represents the dividend (or coupon in the case of bonds) and the prices are the final and initial prices respectively.

In our analysis, we focused on the mean, standard deviation, variance, skewness and kurtosis. From a statistical point of view, this can be identified as the study of moments of returns and thus the distribution of returns.

In fact, the sample mean can be approximated with the Central Limit Theorem to the population mean (E[X]) and thus to the first moment. Furthermore, we know that there is the following relationship: $Var(X) = E[X^2] - E[X]^2$ which through the second moment allows us to calculate the variance and thus, by taking the square root, the standard deviation, used in practice as volatility. It is also crucial to understand how symmetrical the distribution is with respect to the mean and how heavy-tailed or light-tailed it is. The latter two characteristics of the distribution are indicated by skewness and kurtosis, respectively. In formulas:

Moment number	Name	Measure of	Formula
1	Mean	Central tendency	$\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$
2	Variance (Volatility)	Dispersion	$\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})^2}{N}$
3	Skewness	Symmetry (Positive or Negative)	$Skew = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{(X_i - \bar{X})}{\sigma}\right]^3$
4	Kurtosis	Shape (Tall or flat)	$Kurt = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{(X_i - \bar{X})}{\sigma}\right]^4$

Where X is a random variable having N observations (i = 1,2,…,N).

Figure 1: Mean, Variance, Skewness and Kurtosis

The calculations were carried out identically for daily and monthly[2] input data. Clearly, the analysis carried out on returns with monthly timeframes shows greater price fluctuations and therefore, although the mean is not in all cases greater, the standard deviation always is, as is reasonable to expect since daily variations incorporate smaller fluctuations in most cases. As for skewness and kurtosis, the same cannot be said.

Clearly in the study of returns it is highly recommended to use both in order to understand how the stock has behaved, and thus how it might behave, in both a short-term and a long-term horizon, ideally giving prevalence to the time horizon of the investment.

We then moved on to the study of the variance-covariance matrix and the correlation matrix, a necessary step to analyse the degree of dependence of the variables. While the former using covariance gives us information on the linear relationship between two variables, indicating how they vary together over time, the correlation coefficient[3] measures the strength and direction of the linear relationship between two variables.

$$\rho_{X,Y} = corr(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

These matrices will be crucial in the optimisation of the portfolio and in the choice of the sample to be analysed.

[2] Appendices: 1A. Statistics of Return (daily).

[3] The Pearson correlation coefficient was used in the computations.

Below, as an example[4], we show the beginning of the variance-covariance matrix calculated from the daily data.

	LEONARDO	ECOSUNTEK	LANDI RENZO	PIRELLI & C	STELLANTIS	...
LEONARDO	0,000546	0,000075	0,00017	0,000207	0,000269	...
ECOSUNTEK	0,000075	0,001018	0,000068	0,000069	0,000086	...
LANDI RENZO	0,00017	0,000068	0,000944	0,000189	0,000179	...
PIRELLI & C	0,000207	0,000069	0,000189	0,000497	0,000283	...
STELLANTIS	0,000269	0,000086	0,000179	0,000283	0,000594	...
...	...	...	...	...	...	...

Figure 2: Variance-Covariance Matrix (daily)

We do the same for the correlation matrix calculated from daily data.

	LEONARDO	ECOSUNTEK	LANDI RENZO	PIRELLI & C	STELLANTIS	...
LEONARDO	1	0,100288	0,236294	0,380301	0,471567	...
ECOSUNTEK	0,100288	1	0,069465	0,101375	0,110058	...
LANDI RENZO	0,236294	0,069465	1	0,296525	0,239223	...
PIRELLI & C	0,380301	0,101375	0,296525	1	0,541273	...
STELLANTIS	0,471567	0,110058	0,239223	0,541273	1	...
...	...	...	...	...	...	...

Figure 3: Correlation Matrix (daily)

A much more interesting and intuitive way of looking at these matrices is with the help of heatmaps. Since, as explained in detail in the next section, the choice of our portfolio will be based on the correlation matrix, we have decided to show below both heatmap[5] with the corresponding considerations, first calculated with daily and then monthly data.

[4] The calculation of the four entire matrices (variance-covariance and correlation, daily and monthly), for reasons of size (88 rows × 88 columns), will not be attached in the appendix, but is nevertheless easily accessible in the section on the codes used.

[5] Please note that in the heatmap between the name of one security and the next there is another which is not shown for reasons of space.

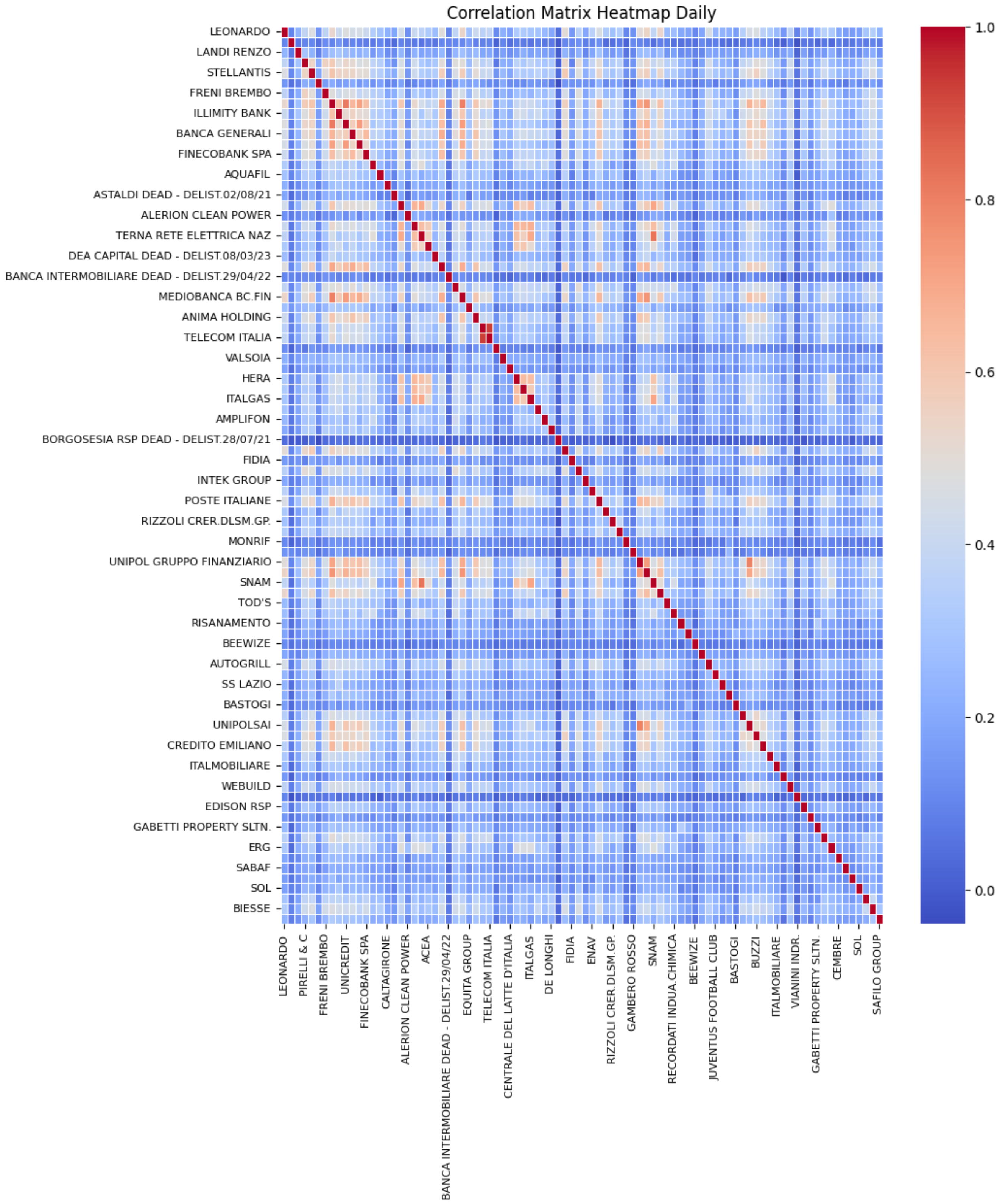


Figure 4: Correlation Matrix Heatmap (daily)

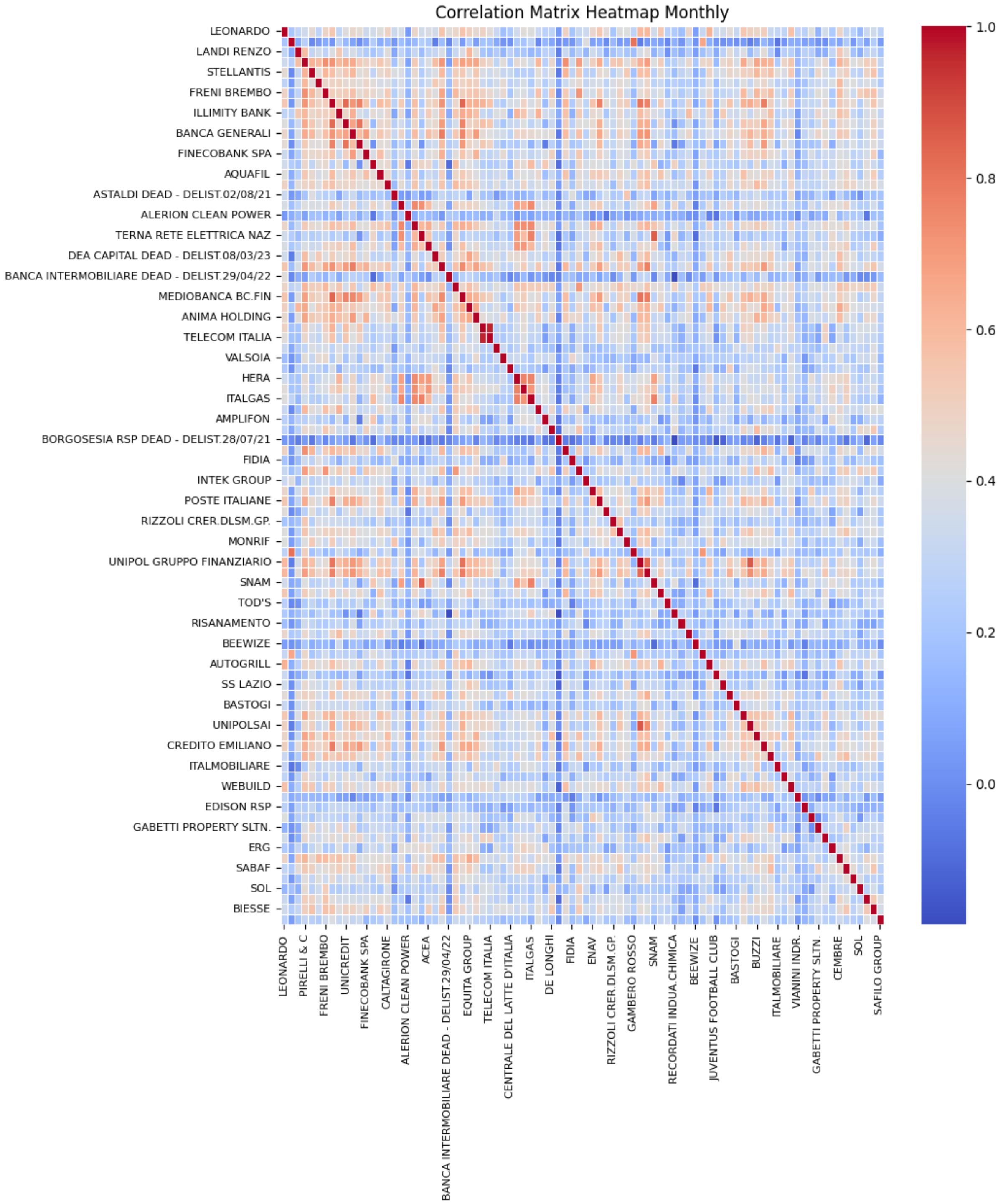


Figure 5: Correlation Matrix Heatmap (monthly)

It is not difficult to observe that when the timeframe becomes monthly, returns tend to be more correlated. This is due to the nature of data aggregation and the reduction in noise caused by daily fluctuations. When using daily data, share price fluctuations can be influenced by short-term factors such as news, market events and impulsive trades. This can cause higher volatility and lower correlation between daily stock returns.

On the other hand, the use of monthly data reduces the impact of short-term noise and provides a more stable view of stock performance and allows better identification of medium- to long-term trends.

1.2 The choice of the sample

The objective of this paragraph is to illustrate the process that led us to the choice of possible securities that we will use for the composition of our portfolio. We have chosen a method that also leaves a space (even if limited) for qualitative judgments, since, in our opinion, it can happen that quantitative data alone do not capture certain more descriptive aspects of securities (such as the choice of sector).

The main assumptions of our model are:

- imposition of correlation coefficients between securities lower than 0.6: we chose this value because we believe that higher coefficients would not leave room for the advantages of diversification, while we did not choose lower values because they are difficult to find for equal returns, since a (non-diversifiable) market risk determined, for example, by macroeconomic variables, could negatively affect returns (e.g. an increase in interest rates affects, in general negatively, all equities, even if with some exceptions).
- exclusion of delisted securities from the selection: these companies could lead to liquidity, not full disclosure or other problems (such as past or ongoing illicit proceedings) that could consequently adversely affect the returns of the companies in consideration;
- each company chosen must belong to a different sector: this is to avoid risks linked to the sector to which the company belongs that may not have occurred since 2015 (for which the results are positive), but which may cause serious problems in the future.

Therefore, we manually grouped the companies by sector, identifying 14 sectors:

Energy	Banking	Real estate, Holding and Mutual Fund	Industrial	
ECOSUNTEK	INTESA SANPAOLO	DEA CAPITAL DEAD - DELIST.08/03/23	AQUAFIL	RECORDATI INDUA.CHIMICA
ENEL	ILLIMITY BANK	TAMBURI INV.PARTNERS	CALTAGIRONE	CEMENTIR HOLDING
ALERION CLEAN POWER	UNICREDIT	EQUITA GROUP	EL EN	BUZZI
A2A	BANCA GENERALI	ANIMA HOLDING	AMPLIFON	DANIELI
TERNA RETE ELETTRICA NAZ	BPER BANCA	BORGOSESIA RSP DEAD - DELIST.28/07/21	DE LONGHI	WEBUILD
ACEA	FINECOBANK SPA	RISANAMENTO	CNH INDUSTRIAL	VIANINI INDR.
HERA	BANCA MEDIOLANUM	BRIOSCHI SVILUPPO IMMBL	FIDIA	SABAF
IREN	BANCA INTERMOBILIARE DEAD - DELIST.29/04/22	BASTOGI	INTERPUMP GROUP	SOL
ITALGAS	MEDIOBANCA BC.FIN	ITALMOBILIARE	INTEK GROUP	BIESSE
EDISON RSP	POSTE ITALIANE	GABETTI PROPERTY SLTN.	RIZZOLI CRER.DLSM.GP.	
ERG	CREDITO EMILIANO			

Automotive	Food and Beverage	Publishing	Telecommunications	Sport
LANDI RENZO	DAVIDE CAMPARI MILANO	CAIRO COMMUNICATION	TELECOM ITALIA RSP	JUVENTUS FOOTBALL CLUB
STELLANTIS	AUTOGRILL	MONRIF	TELECOM ITALIA	SS LAZIO
PIRELLI & C	ENERVIT	GAMBERO ROSSO	MFE B	
PININFARINA	VALSOIA	CLASS EDITORI		
FRENI BREMBO	CENTRALE DEL LATTE D'ITALIA			

Insurance	Transport	Tech	Clothing	Oil
CATTOLICA ASSICURAZIONI DEAD - DELIST.	LEONARDO	BEEWIZE	TOD'S	SNAM
UNIPOL GRUPPO FINANZIARIO	ASTALDI DEAD - DELIST.02/08/21	EXPRIVIA	VINCENZO ZUCCHI	ENI
ASSICURAZIONI GENERALI	ENAV	BEGHELLI	RATTI	
UNIPOLSAI	CEMBRE	DATALOGIC	SAFILO GROUP	

Figure 6: The choice of the sample: splitting by sector

Once the companies within the same sector were identified, we chose the companies with the best risk-return (minimising risk and maximising return), giving more weight to lower standard deviation rather than higher returns. Once we had selected the 14 stocks that met our initial assumptions, we eliminated 2 of them, excluding those with the highest average correlation resulting from the monthly correlation matrix (we used this because it sometimes has a greater propensity to indicate trends, as explained above). We then defined our 12 stocks: ALERION CLEAN POWER, FINECOBANK SPA, STELLANTIS, AMPLIFON, EXPRIVI, CEMBRE, SNAM, DAVIDE CAMPARI MILANO, CAIRO COMMUNICATION, TELECOM ITALIA RSP, RATTI, JUVENTUS FOOTBALL CLUB.

The result of our choice can be further supported with an additional monthly correlation matrix, this time considering only the selected securities. The clear benefits of diversification, shown by significantly lower correlation coefficients, highlight the effectiveness of the method.

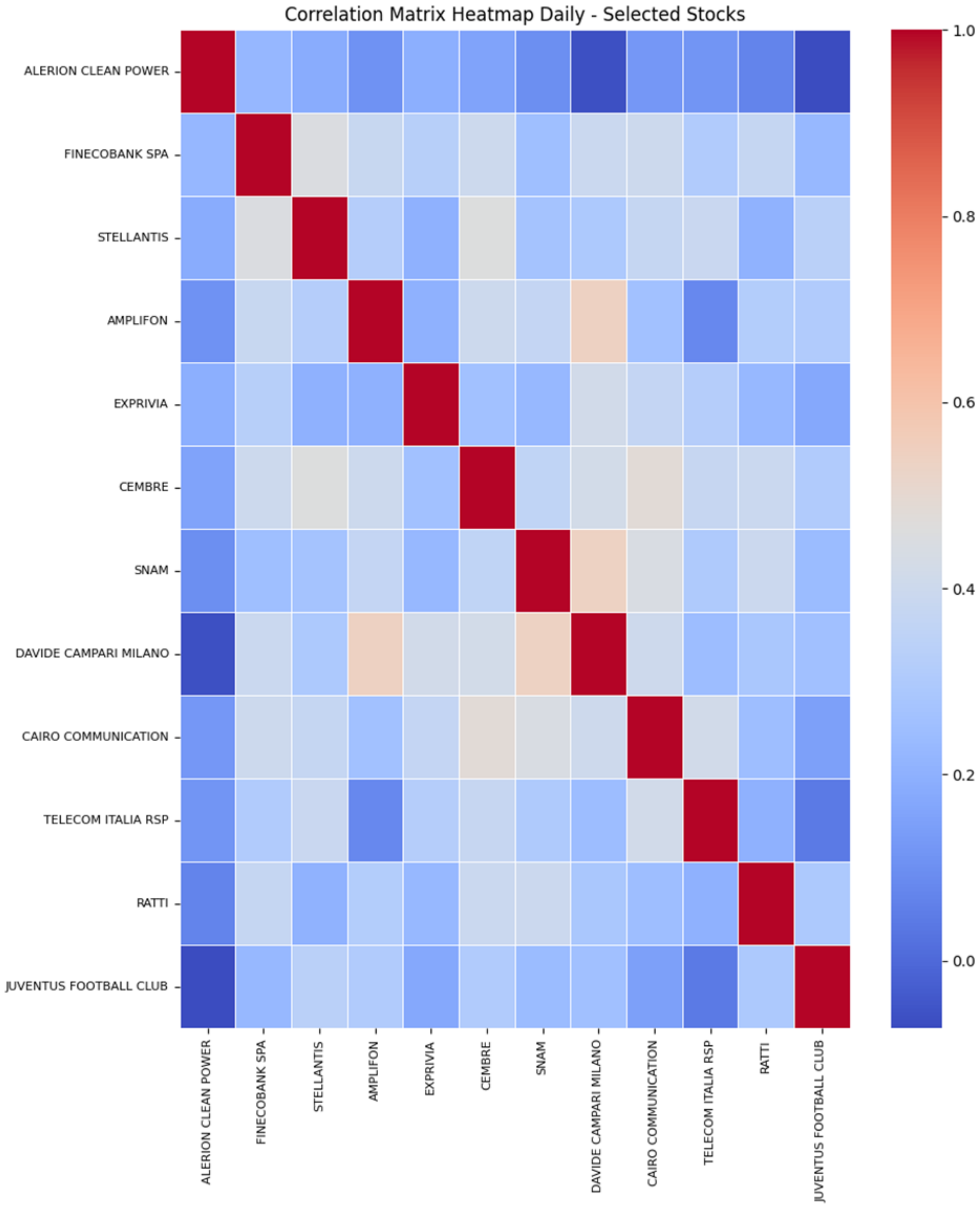


Figure 7: Correlation Matrix Heatmap (daily): Selected stocks

To conclude the return analysis paragraph, as required by point 4, we report below the graphs of the 12 shares of interest.

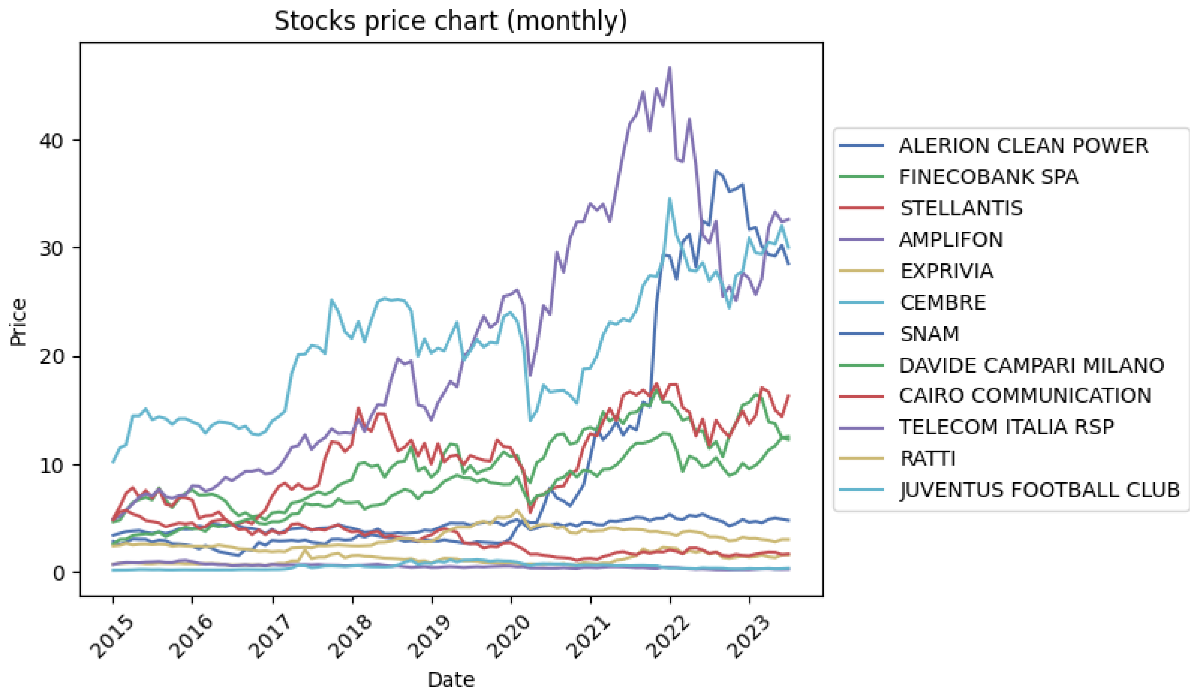


Figure 8: Stocks price chart (monthly)

Regarding common behaviour, it can be said that indicatively all stocks (although in different ways) were affected by a drop between mid-February 2020 and mid-March 2020 due to COVID-19. One can also observe a subsequent general rise in the subsequent phase, mainly due to expansionary monetary policy decisions and the "end" of the pandemic.

1.3 Mean Variance optimal portfolio allocation, Efficient Frontier and SML

In this section we will focus on the first asset allocation exercise, following the Mean Variance model, undoubtedly the best known of the ideas proposed by Modern Portfolio Theory.

As can easily be deduced from the title of the model, the idea of this approach focuses on the study of portfolio return and variance; in particular, the model assumes rationality and risk aversion (relative for each individual) of investors, who, consequently, for the same return will choose the portfolio with the lowest variance and for the same variance the one with the highest return.

It is therefore necessary to define: the expected returns $E(R_p)$ and the variance of the portfolio σ_p^2:

$$\mathrm{E}(R_p) = \sum_i w_i \, \mathrm{E}(R_i)$$

where R_i is the return of the individual asset and ω_i represents the weight of the asset within the portfolio (the sum of the weights must necessarily be one, but this does not necessarily imply positivity of the weights);

$$\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where σ_i is the standard deviation of the asset e ρ_{ij} represents the correlation coefficient.

This theory shows us that an investor can reduce the volatility of his or her portfolio (in the sense of standard deviation), simply by combining assets that are not perfectly correlated, in accordance with the above.

Indeed, from Fama (1976)[6] we know that the optimum number of securities in the portfolio can be around 15 (in our case we have 12 securities). Moreover, Statman[7] finds that risk does not fall further for $n > 100$.

Regarding optimisation, the ultimate goal is to find the optimal weights, i.e. the composition of the portfolio such that a certain objective is achieved. It is possible to consider the portfolio that maximises the Sharpe Ratio, the one that minimises the variance or many other alternatives. The portfolio that maximises the Sharpe Ratio will be represented in the efficient frontier paragraph and the GMVP in the last asset allocation method.

For this part, we decided to consider an approach that allows us to emphasise the concept of risk aversion. In fact, the exercise requires calculating optimal weights and portfolio statistics, considering two specific cases: in the first case, short selling is allowed, while in the second case it is not.

According to the literature, short selling is perceived (and in fact is) by investors as a riskier phenomenon and therefore we wanted to follow this approach, defining different aversion parameters for the two cases, equal to 7 in the first case (short selling allowed) and 5 in the second.

[6] Fama, E. F. (1976). Efficient Capital Markets: Reply. The Journal of Finance, 31(1), 143–145. https://doi.org/10.2307/2326404.
[7] Statman, M. (1987). How Many Stocks Make a Diversified Portfolio? The Journal of Financial and Quantitative Analysis, 22(3), 353–363. https://doi.org/10.2307/2330969.

More specifically, we based our choice on Liu and Xu (2010)[8] , according to which the function (objective) we want to maximize is:

U = $\boldsymbol{\mu} - \boldsymbol{\lambda\sigma}^2$,

where μ is the expected portfolio return, σ^2 is the portfolio variance and λ (λ ≥ 0) is the risk aversion risk aversion, through which we can define the relative importance of returns and risk. About the interpretation of λ, the larger the value of λ, the greater the risk to be penalised. Conversely, the smaller the λ, the less important will be the risk term in the target. In the extreme case where λ = 0, the risk term is ignored, and the return term dominates completely.

A typical "plain vanilla" mean-variance portfolio optimization problem has the following form:

$$\underset{\mathbf{h}}{\text{Maximize}}: \quad \mathbf{r}^{\mathrm{T}}\mathbf{h} - \lambda\, \mathbf{h}^{\mathrm{T}}\boldsymbol{\Sigma}\,\mathbf{h}$$
$$\text{Subject to}: \quad \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1$$

where r is the vector of asset excess returns, Σ is the variance - covariance matrix, h is the vector of portfolio weights, and e is a vector of 1's (which represents the constraint under which the sum of weights must be 1).

Please, note that the two maximization are equivalent, since

$$\mu = \mathbf{r}^{\mathrm{T}}\mathbf{h} \quad \text{and}$$
$$\sigma^2 = \mathbf{h}^{\mathrm{T}}\boldsymbol{\Sigma}\,\mathbf{h}$$

After implementing this optimisation in Python, we obtained the following results:

Portfolio (daily) without constraint	
Optimal weights	
ALERION CLEAN POWER	0,1798
FINECOBANK SPA	0,0431
STELLANTIS	0,0274
AMPLIFON	0,1443
EXPRIVIA	0,0538
CEMBRE	0,2184
SNAM	0,0478
DAVIDE CAMPARI MILANO	0,2986
CAIRO COMMUNICATION	-0,0876
TELECOM ITALIA RSP	-0,1059
RATTI	0,1225
JUVENTUS FOOTBALL CLUB	0,0579
Portfolio statistics	
Expected return	0,1012
Volatility	1,1795
Variance	1,3913
Skewness	-0,6194
Kurtosis	9,6695
Sharpe ratio	0,0604

Portfolio (daily) with constraint	
Optimal weights	
ALERION CLEAN POWER	0,2042
FINECOBANK SPA	0,0000
STELLANTIS	0,0015
AMPLIFON	0,1652
EXPRIVIA	0,0432
CEMBRE	0,1845
SNAM	0,0000
DAVIDE CAMPARI MILANO	0,2869
CAIRO COMMUNICATION	0,0000
TELECOM ITALIA RSP	0,0000
RATTI	0,0730
JUVENTUS FOOTBALL CLUB	0,0415
Portfolio statistics	
Expected return	0,0900
Volatility	1,1512
Variance	1,3253
Skewness	-0,8218
Kurtosis	12,0543
Sharpe ratio	0,0521

Portfolio (monthly) without constraint	
Optimal weights	
ALERION CLEAN POWER	0,2108
FINECOBANK SPA	0,0326
STELLANTIS	0,0376
AMPLIFON	0,0803
EXPRIVIA	-0,0201
CEMBRE	0,1082
SNAM	0,2006
DAVIDE CAMPARI MILANO	0,4379
CAIRO COMMUNICATION	-0,2085
TELECOM ITALIA RSP	-0,0381
RATTI	0,1415
JUVENTUS FOOTBALL CLUB	0,0172
Portfolio statistics	
Expected return	2,1417
Volatility	5,2194
Variance	27,2424
Skewness	-0,2291
Kurtosis	1,1482
Sharpe ratio	0,4046

Portfolio (monthly) with constraint	
Optimal weights	
ALERION CLEAN POWER	0,2465
FINECOBANK SPA	0,0000
STELLANTIS	0,0074
AMPLIFON	0,1528
EXPRIVIA	0,0000
CEMBRE	0,0200
SNAM	0,0000
DAVIDE CAMPARI MILANO	0,4535
CAIRO COMMUNICATION	0,0000
TELECOM ITALIA RSP	0,0000
RATTI	0,0773
JUVENTUS FOOTBALL CLUB	0,0425
Portfolio statistics	
Expected return	2,0387
Volatility	5,4593
Variance	29,8044
Skewness	-0,2672
Kurtosis	0,8799
Sharpe ratio	0,3679

Figure 9: Portfolio optimization (daily and monthly, without constraint and with constraint)

[8] Liu, Scott and Xu, Rong, The Effects of Risk Aversion on Optimization, February 2010 (February 22, 2010). MSCI Barra Research Paper No. 2010-06, Available at SSRN: https://ssrn.com/abstract=1601412.

We can then move on to the stage of defining the efficient frontier. It can be defined as the combination of optimal portfolios and is thus the set of portfolios with the highest level of returns for a given level of risk. The individual preferences of each investor determine which portfolio to choose on the efficient frontier. From a graphical point of view, the efficient frontier is a particular curve that combines the portfolio that maximises the Sharpe Ratio and the GMVP.

To simulate an efficient frontier, the prices (daily and monthly) for each security are necessary and sufficient: on the basis of this data, it is possible to compute the mean and the variance-covariance matrix.

We can define the portfolio that maximises the Sharpe ratio by solving the following optimisation problem:

$$max \left(\frac{\sum_{i=1}^{N} W_i.\mu_i - R_f}{\sqrt{\sum_i \sum_j W_i.W_j.\sigma_{ij}}} \right)$$

subject to

$$\sum_{i=1}^{N} W_i = 1$$

Where the numerator of the objective function denotes the excess returns of the investment over that of a risk-free asset and the denominator the risk of the investment. The objective is to maximize the Sharpe Ratio.

The basic constraint indicate that the investor wishes to have a fully invested portfolio. If required, the constraint of the non-negative weights can be added, i.e. $0 \leq W_i \leq 1$.

Let us finally represent the efficient frontier on a graph, indicating with X the portfolio that maximises the Sharpe Ratio.

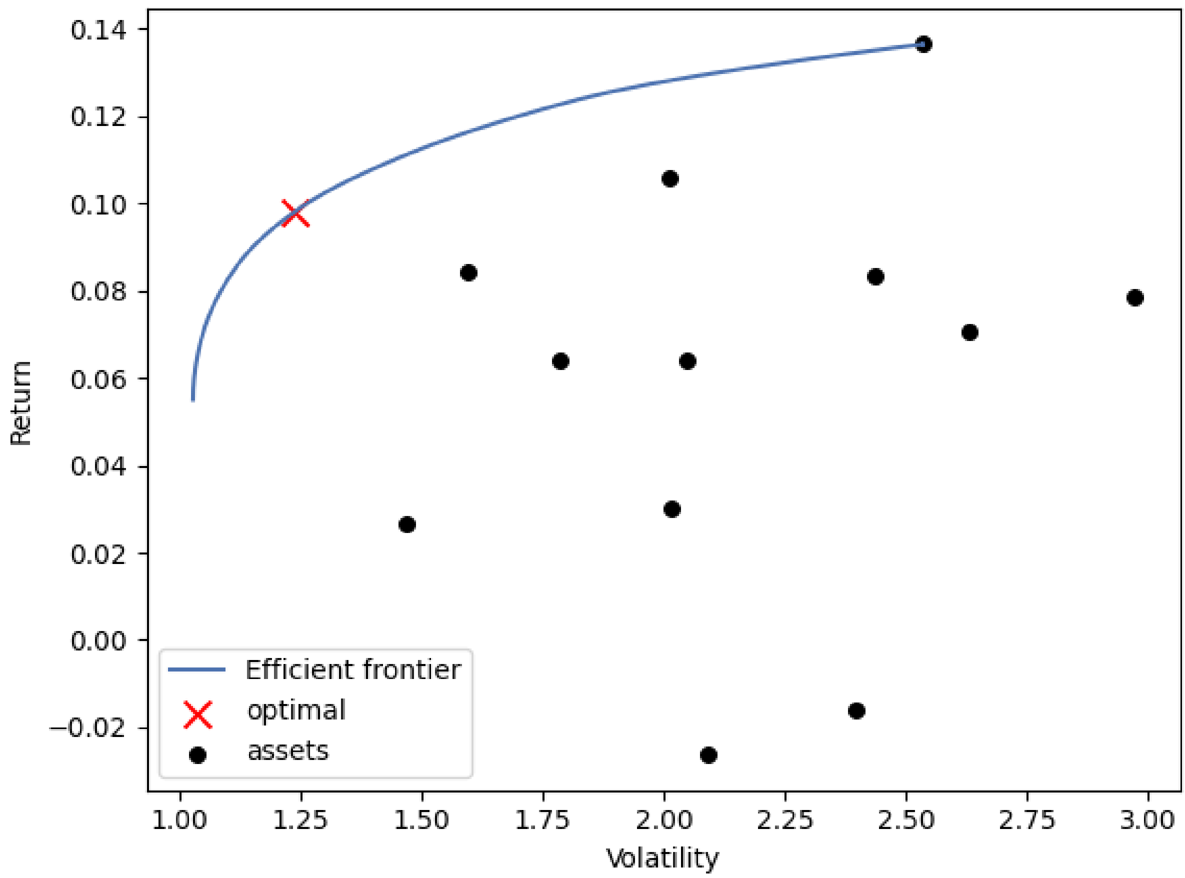


Figure 10: Efficient Frontier and max Sharpe Ratio portfolio (daily).

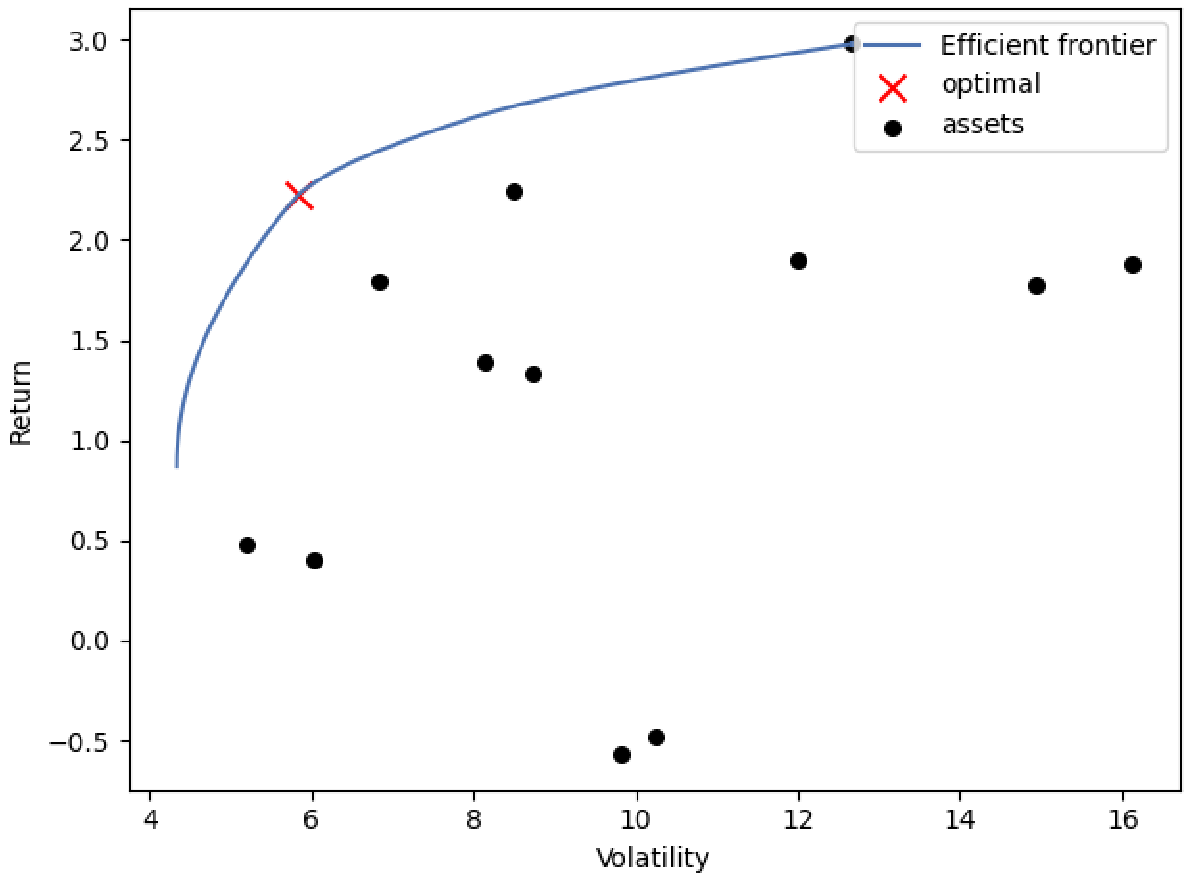


Figure 11: Efficient Frontier and max Sharpe Ratio portfolio (monthly).

It is easy to see that, in both cases, individual assets are almost always below the frontier (they are rarely on the frontier). This is because the best returns for certain risk levels can only be achieved by combining several assets and thus taking advantage of the lower volatility resulting from diversification. The returns and volatilities of individual assets are shown below.

	Returns	Volatility
ALERION CLEAN POWER	0.136426	2.535002
FINECOBANK SPA	0.063798	2.047249
STELLANTIS	0.083343	2.436674
AMPLIFON	0.106012	2.013177
EXPRIVIA	0.078338	2.970048
CEMBRE	0.063941	1.787161
SNAM	0.026390	1.466430
DAVIDE CAMPARI MILANO	0.084170	1.596858
CAIRO COMMUNICATION	-0.026319	2.089487
TELECOM ITALIA RSP	-0.016270	2.397783
RATTI	0.030152	2.015187
JUVENTUS FOOTBALL CLUB	0.070779	2.632614

	Returns	Volatility
ALERION CLEAN POWER	2.978817	12.657981
FINECOBANK SPA	1.330502	8.732252
STELLANTIS	1.900335	11.997645
AMPLIFON	2.241812	8.496024
EXPRIVIA	1.881972	16.125422
CEMBRE	1.393285	8.127573
SNAM	0.480906	5.188157
DAVIDE CAMPARI MILANO	1.794153	6.832159
CAIRO COMMUNICATION	-0.569407	9.819158
TELECOM ITALIA RSP	-0.479865	10.250530
RATTI	0.397481	6.019705
JUVENTUS FOOTBALL CLUB	1.776662	14.947052

Figure 12: Returns and volatilities of individual stocks (daily and monthly)

Given these data, the portfolio that maximises the Sharpe Ratio has the following characteristics:

	Returns	**Volatility**
Max Sharpe Porfolio (daily)	0,098	1,238
Max Sharpe Porfolio (monthly)	2,227	5,843

From now on, we are going to implement the analyses done so far, no longer only studying the stock market with the various stocks, but also taking into account an equity benchmark. The latter is a benchmark, in our case represented by the FTSE ITALIA ALL SHARE, which is used as an objective reference index to compare the portfolio's performance with market trends. The purpose of a benchmark is in fact to assess the typical risks of the market in which the portfolio invests and to provide a convenient tool to help investors evaluate the results they will get from managing a given portfolio of securities.

In particular, our benchmark is composed of all the components of the FTSE MIB index[9], FTSE Italia Mid Cap[10] and FTSE Italia Small Cap[11].

On this data, we calculated the percentage returns for both daily and monthly. Then we calculated all the relative statistics on them; so we computed mean, standard deviation, variance, kurtosis and skewness. The summary table below shows them. Please note that the first column of values refers to the daily data where, instead, the second column refers to the monthly values.

	FTSE ITALIA ALL SHARE - TOT RETURN IND	FTSE ITALIA ALL SHARE - TOT RETURN IND
mean	0.000418	0.008754
std	0.013779	0.058111
var	0.000190	0.003377
skew	-1.331955	-0.559048
kurtosis	15.221102	2.992421

Figure 13: FTSE ITALIA ALL SHARE statistics (daily and monthly)

[9] The main equity market benchmark index that measures the performance of 40 Italian stocks weighing around 80 per cent of domestic market capitalisation
[10] An index related to the 60 most liquid and well-capitalised publicly traded stocks not included in the FTSE MIB index
[11] An index that brings together dozens of small-cap companies. Overall, the companies included in the index represent about a quarter of the market capitalisation of the Italian stock market and occupy the top position in daily trading

As already anticipated, this data allows us to understand what the "Italian stock market performance" was (since we are taking companies with more than 80% market capitalisation). Again, these data are useful to be considered as benchmarks and therefore need to be compared with other data, in our case those relating to our portfolio. Below we summarise the data for the same statistics as in the previous table for the average of our 12 selected stocks. Please note that the first column refers to daily data and the second column to monthly data.

mean	0.000584	0.012606
std	0.021656	0.099328
var	0.000487	0.010945
skew	0.255564	0.431625
kurtosis	9.475686	2.823587

Figure 14: Selected securities statistics (daily and monthly)

As can easily be seen, the average returns for our portfolio provide returns that are almost 40% higher than the returns of our benchmark. This is obviously done at the cost of a higher variance and standard deviation (implying volatility).

This means that our portfolio will be more volatile and therefore riskier than the benchmark, but at the same time will allow for much higher returns.

While as far as skewness and kurtosis are concerned, it is due to take a step back. Kurtosis measures the greater or lesser skewness of a data distribution. In particular:

- Kurtosis index $K = 0$: normal form;
- Kurtosis index $K > 0$: leptokurtic form;
- Kurtosis index $K < 0$: platykurtic form.

As can be noted, both the benchmark and our portfolio have a leptokurtic shape, with more data in the middle around the mean (as is to be expected from the time series), but our portfolio has a lower peak. This means that it will have slightly more weight on the tails of the distribution and therefore slightly more risk (as the standard deviation also tells us).

The skewness instead is a measurement of the distortion of symmetrical distribution or asymmetry in a data set. A distribution is positively skewed when its tail is more pronounced on the right side than it is on the left and negatively skewed viceversa.

Here one can see that the skewness of the fund is more pronounced on the left while that of the portfolio on the right. This means that for the benchmark most values are on the right side of the average when negative skewness is present. As such, the most extreme values are on the left side (when negative skewness is present) while for the portfolio most values end up being on the left side of the average and the most extreme values are on the right side (it can therefore happen that there are large gains).

Let’s now move to the beta part.

Beta measures the systematic risk of a stock as part of the capital asset pricing model. It is a measure of the systematic risk of a security.

It measures the expected change in a security's return for each percentage point change in the market return. The expected rate of return of a security varies linearly with the beta value of the security. Specifically:

- A beta of 1 indicates that the security or mutual fund has an average volatility comparable to that of the entire market;
- A beta of less than 1 indicates that the investment is less volatile than the market;
- A beta greater than 1 indicates that the investment is more volatile than the market.

Beta therefore might be a parameter that investors take into account according to their risk appetite; in particular: investors with a lower risk appetite might prefer investments with a lower beta, while more risk-prone investors might look for opportunities with a higher beta.

Its formula is thus represented by

$$\beta_p = \frac{Cov(r_p, r_b)}{Var(r_b)}$$

where r_p indicates the returns of the portfolio, while r_b indicates that we are referring to the returns of the benchmark.

The betas of all securities in the portfolio can be freely consulted in the table below.

Daily	
ALERION CLEAN POWER	0.43
FINECOBANK SPA	1.05
STELLANTIS	1.30
AMPLIFON	0.64
EXPRIVIA	0.69
CEMBRE	0.41
SNAM	0.69
DAVIDE CAMPARI MILANO	0.63
CAIRO COMMUNICATION	0.73
TELECOM ITALIA RSP	1.00
RATTI	0.31
JUVENTUS FOOTBALL CLUB	0.74

Monthly	
ALERION CLEAN POWER	0.53
FINECOBANK SPA	0.94
STELLANTIS	1.56
AMPLIFON	0.59
EXPRIVIA	1.08
CEMBRE	0.88
SNAM	0.46
DAVIDE CAMPARI MILANO	0.59
CAIRO COMMUNICATION	1.00
TELECOM ITALIA RSP	1.11
RATTI	0.48
JUVENTUS FOOTBALL CLUB	0.73

Figure 15: Betas (daily and monthly)

It may be interesting to comment on the portfolio beta for both daily and monthly. They were calculated as a matrix multiplication between the betas calculated above and the respective weights of the optimal portfolio found in step 5. The result is a beta for the portfolio with a daily timeframe of 0.55. More specifically, a beta of 0.55 means that, on average, the security or portfolio moves only 0,55% relative to a change of 1% in the market index.

For a monthly timeframe, on the other hand, the beta turns out to be 0.58, so very similar to the previous one and the same kind of reasoning applies.

Finally, we can derive the Security Market Line.

First, we must make a very brief introduction to the Capital Asset Pricing Model. The CAPM was proposed by William Sharpe as a model of risk and return in his 1964 article. The same model was also proposed in other related articles by Jack Treynor (1962) and John Lintner (1965).

The CAPM uses the optimal choices investors make to identify the efficient portfolio as the market portfolio of all stocks and securities in the market. To obtain this remarkable result, we make three assumptions regarding the behavior of investors:

- Investors can buy and sell all securities at competitive market prices;
- Investors hold only efficient portfolios of traded securities;
- Investors have homogeneus expectations regarding the volatilities, correlations, and expected returns of securities.

If these assumptions hold, then we know that all investors will demand the same efficient portfolio of risky securities by only tailoring the investment in risk-free securities to their particular risk appetite.

The general formula formula to represent the CAPM equation is

$$R_i = R_f + \beta_i * (R_m - R_f)$$

where R_i is the expected and thus required return on a stock i, R_f risk-free rate, β_i is the beta of stock i and R_m is the market expected return. In particular, the last term represents the risk premium.

Using the CAPM, therefore, it is possible to graphically represent the SML. The SML shows the expected return for each security as a function of its beta with the market taking into account only systematic risk. It is therefore represented by a straight line on the graph, with the expected return on the y-axis (vertical) and the systemic risk (beta) on the x-axis (horizontal). The slope of the line represents the market risk premium, i.e. the additional rate of return required by investors to take on systemic risk.

We have therefore now graphically represented the security market line for two random stocks and our portfolio. For the latter we have used the beta explained above and the expected return derived from the Mean Variance optimal portfolio allocation. It is important to note that we previously converted our annual risk-free rate of 3% to a daily risk-free rate of 0.042% in order to have consistency between time periods.

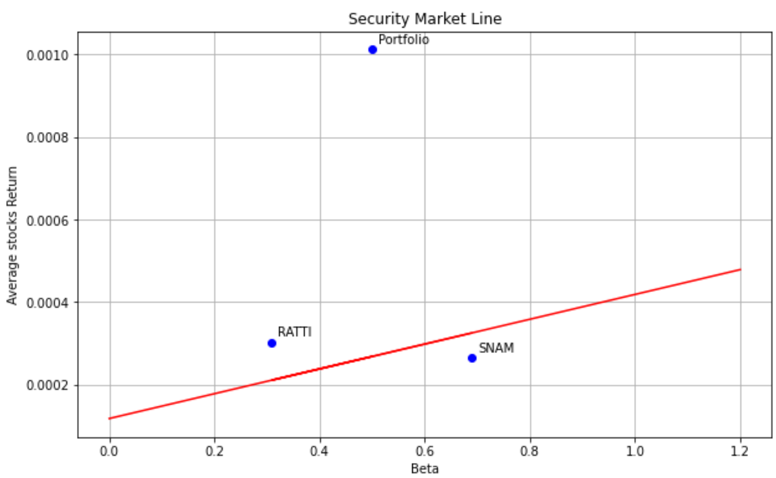


Figure 16: SML of portfolio (daily) and 2 random stocks.

In this case, the securities are very close to the SML, which implies that they are correctly valued by the market.

In general, if the expected return of a security or portfolio is above the SML, it is said to be undervalued and may represent an investment opportunity. Conversely, if it is below the SML, it is said to be overvalued and may not be a profitable investment. As expected, one can see that most of our stocks are underpriced and are therefore an opportunity. The greater the distance from the straight line, the greater the investment opportunity. Our portfolio therefore represents a great financial opportunity according to SML.

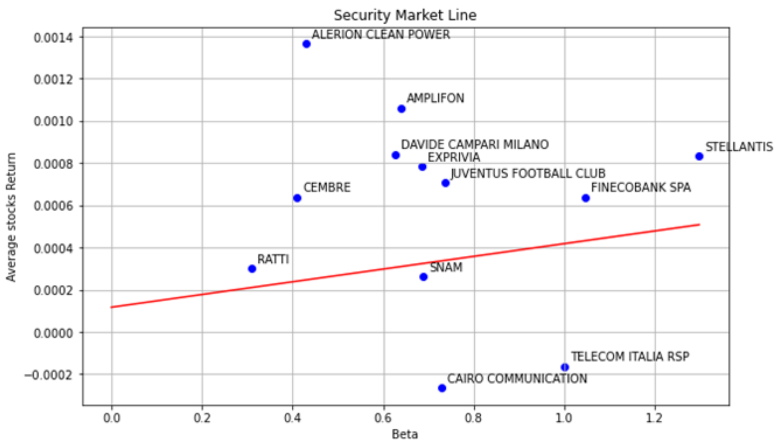


Figure 17: SML of all the stocks of the portfolio (daily)

We did the same work and reasoning for the monthly periods. Note the difference in the interest rate, which this time becomes a monthly risk-free rate of 0.25%. We represent the graphs in the same order as before.

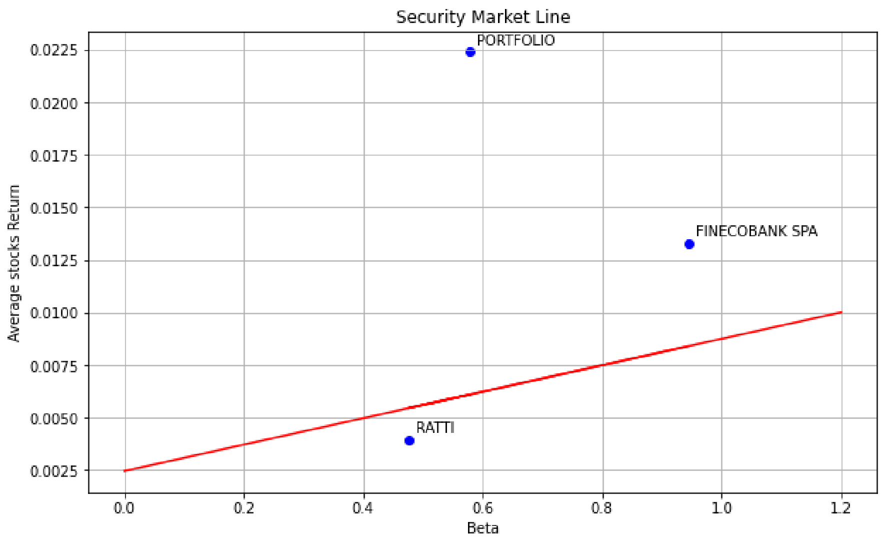


Figure 18: SML of portfolio (monthly) and 2 random stocks.

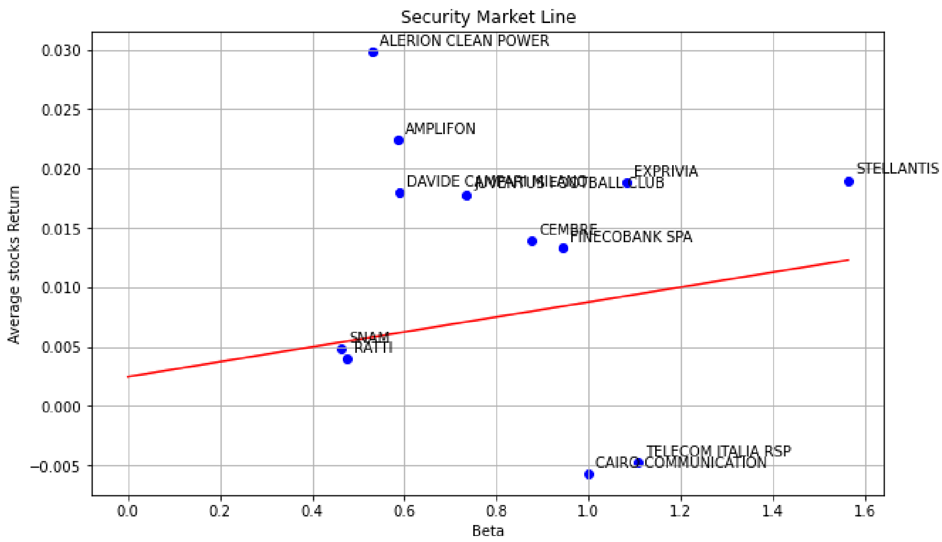


Figure 19: SML of all the stocks of the portfolio (monthly)

The same reasoning applies even with a different timeframe.

1.4 Black-Littermann approach

In this section we will focus on the Black-Litterman approach a further mathematical model for portfolio allocation developed in 1991 at Goldman Sachs by Fischer Black and Robert Litterman. The peculiarity of this approach is that it takes investors' expectations (and thus their experience) regarding the possible future performance of securities (views) and combines them with market expectations (prior) to produce an expected return (to be estimated) that takes both factors into account (posterior).

A very strong assumption of the Black-Litterman world is that market returns are normally distributed with a true hidden mean μ and covariance Σ. Infact, according to this approach,

$E_{investor}(R) \sim Q + \varepsilon$, where $\varepsilon \sim N(0, \Omega)$. The meaning is that the error in views ε has an average of 0 indicating the fact that there is no bias in the investor's assumptions.

First, let us define the priors. Black and Litterman pointed out that a natural choice for this prior is the market's estimate of return, which is incorporated in the market capitalisation[12] of the asset, the risk aversion parameter δ, and the variance-covariance matrix of our securities Σ. We specify that for the calculation of the δ, we used the closing market prices of the index FTSE ITALIA ALL SHARE, resulting in a risk aversion parameter of 1.78 for the daily data and 2,57

[12] Market capitalisation data were taken from the website of Borsa Italiana.

for monthly data. Once we have obtained the priors, we have represented them in the following figure.

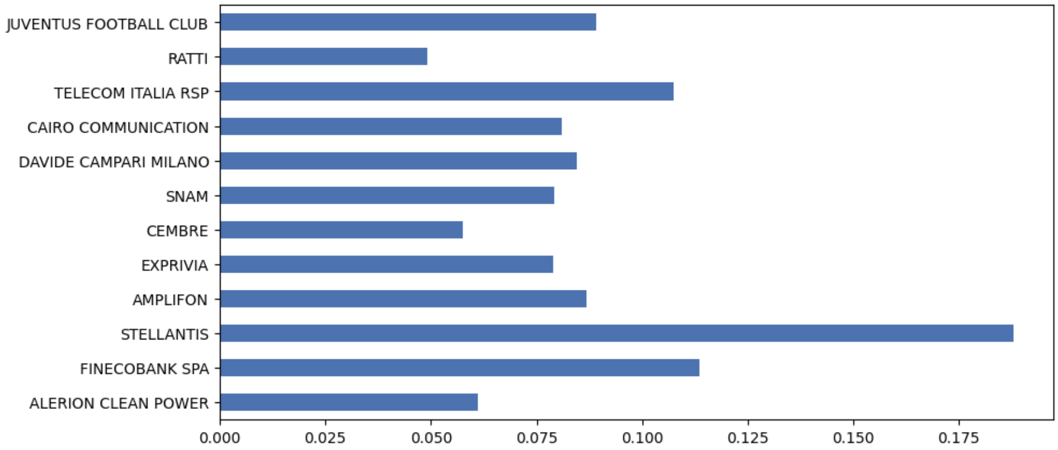


Figure 20: Priors daily

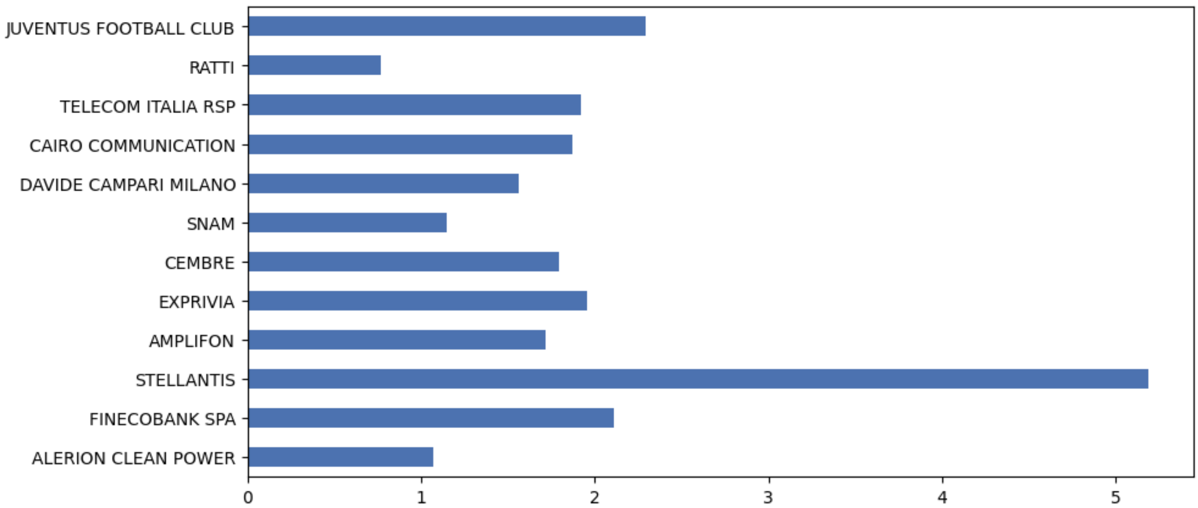


Figure 21: Priors monthly

We then calculated the posteriors. We define the vector Q representing the views (the following are, for example, for the monthly part), two absolute and two relative:

1. STELLANTIS will drop 5% (absolute)
2. DAVIDE CAMPARI MILANO will rise by 5% (absolute)
3. ALERION CLEAN POWER will outperforms STELLANTIS by 7% (relative)
4. CEMBRE will underperform DAVIDE CAMPARI MILANO by 50% (relative)

In addition to Q, it is necessary to define the picking matrix P, which will allow us to indicate which security the view refers to.

Next, we defined the confidences that allow us to define the confidence we have in our prediction. This part allows to take into account the error in judgement, since nobody is 100% certain of what will happen. Then through the variance-covariance matrix we defined Ω, the uncertainty matrix.

Finally, we calculated the posteriors using the same method as the priors, but this time as a function of risk aversion, views, priors, market cap, variance-covariance matrix and Ω.

We then obtained the following results:

	Prior	Posterior
ALERION CLEAN POWER	0.061021	0.081879
FINECOBANK SPA	0.113632	0.060259
STELLANTIS	0.187920	0.079331
AMPLIFON	0.086886	0.047521
EXPRIVIA	0.079003	0.037635
CEMBRE	0.057479	0.035508
SNAM	0.079148	0.042226
DAVIDE CAMPARI MILANO	0.084426	0.034842
CAIRO COMMUNICATION	0.080951	0.038094
TELECOM ITALIA RSP	0.107356	0.054448
RATTI	0.049020	0.019798
JUVENTUS FOOTBALL CLUB	0.089062	0.042795

	Prior	Posterior
ALERION CLEAN POWER	1.067435	0.444395
FINECOBANK SPA	2.106158	0.638786
STELLANTIS	5.189550	0.374607
AMPLIFON	1.718226	0.563371
EXPRIVIA	1.953563	0.521449
CEMBRE	1.793770	0.622097
SNAM	1.148708	0.524942
DAVIDE CAMPARI MILANO	1.562388	0.202276
CAIRO COMMUNICATION	1.871617	0.491654
TELECOM ITALIA RSP	1.922176	0.518649
RATTI	0.764432	0.283988
JUVENTUS FOOTBALL CLUB	2.291591	0.458057

Figure 22: Prior and Posterior for (daily and monthly)

Finally, we calculated weights and statistics, which are summarised in the following figures.

Portfolio (daily) Black Litterman	
Optimal weights	
ALERION CLEAN POWER	0,2137
FINECOBANK SPA	0,1243
STELLANTIS	0,1922
AMPLIFON	0,0827
EXPRIVIA	0,0379
CEMBRE	0,0450
SNAM	0,0676
DAVIDE CAMPARI MILANO	0,0364
CAIRO COMMUNICATION	0,0439
TELECOM ITALIA RSP	0,0996
RATTI	0,0000
JUVENTUS FOOTBALL CLUB	0,0568
Portfolio statistics	
Expected return	0,0610
Volatility	0,2150
Variance	0,0462
Skewness	-0,9509
Kurtosis	11,1100
Sharpe ratio	0,1442

Portfolio (monthly) Black Litterman	
Optimal weights	
ALERION CLEAN POWER	0,0801
FINECOBANK SPA	0,1637
STELLANTIS	0,0000
AMPLIFON	0,1439
EXPRIVIA	0,0045
CEMBRE	0,1654
SNAM	0,2473
DAVIDE CAMPARI MILANO	0,0000
CAIRO COMMUNICATION	0,0358
TELECOM ITALIA RSP	0,0953
RATTI	0,0356
JUVENTUS FOOTBALL CLUB	0,0285
Portfolio statistics	
Expected return	0,5460
Volatility	0,7860
Variance	0,6178
Skewness	-0,5647
Kurtosis	1,3955
Sharpe ratio	0,8352

Figure 23: Black-Litterman portfolios (daily and monthly)

Comparing this portfolio with the previous ones, we notice slightly lower returns. The reason lies in the fact that in the Mean Variance Model we used a risk aversion coefficient reflecting our expectations, whereas in this case, we extrapolated risk aversion directly from the market. In any case, it is not difficult to see that, in this last portfolio, we rather preferred lower returns, with the aim of obtaining a much better standard deviation.

1.5 Bayesian Asset Allocation

In this section we will perform a Bayesian Asset Allocation. First, let us recall the Bayes Theorem as we have studied it:

$$f_{po}(\theta | Y_T) = \frac{\mathcal{L}(\theta | Y_T)\, \pi(\theta)}{f(Y_T)}$$

where

- θ is a given parameter;
- Y_T is the matrix of past information T x N (since each component of (y_1, ... , Y_T)is composed by N variables;

- f_{po} (θ| Y_T) is the pdf of the random variable generated by the estimate, given the past informations;
- L (θ| Y_T) = $\prod_{t=1}^{T}$ f(Yt|θ), that is the loglikelihood function;
- π (θ) = f_{pr} (θ);
- f(Y_T) is the marginal density of past information: it is a normalization factor.

The function to maximize when we look to the asset allocation process is the following:

Max Rt U(W_{T+1})

with respect to w, where W_{T+1} is the wealth at time T and it is a function of the previous wealth at time T, weights (to maximize), return of the risky asset at time T+1 and return of the risk-free rate at time T.

For the standard Bayesian model, we get first of all a new mean and standard deviation. We use the assumptions according to which the mean is equals to mean of our vector of returns + 1*standard deviation, while for the standard deviation we take the square root of the diagonal of the new variance-covariance matrix, computed from the previous one multiplying by 2. In this part the same theoretical part regarding the true mean and the true variance-covariance matrix of the Black-Litterman model holds.

(daily)	prior_mean	prior_std
ALERION CLEAN POWER	2,6714	3,5850
FINECOBANK SPA	2,1110	2,8952
STELLANTIS	2,5200	3,4460
AMPLIFON	2,1192	2,8471
EXPRIVIA	3,0484	4,2003
CEMBRE	1,8511	2,5274
SNAM	1,4928	2,0738
DAVIDE CAMPARI MILANO	1,6810	2,2583
CAIRO COMMUNICATION	2,0632	2,9550
TELECOM ITALIA RSP	2,3815	3,3910
RATTI	2,0453	2,8499
JUVENTUS FOOTBALL CLUB	2,7034	3,7231

(monthly)	prior_mean	prior_std
ALERION CLEAN POWER	15,6368	17,9011
FINECOBANK SPA	10,0628	12,3493
STELLANTIS	13,8980	16,9672
AMPLIFON	10,7378	12,0152
EXPRIVIA	18,0074	22,8048
CEMBRE	9,5209	11,4941
SNAM	5,6691	7,3372
DAVIDE CAMPARI MILANO	8,6263	9,6621
CAIRO COMMUNICATION	9,2498	13,8864
TELECOM ITALIA RSP	9,7707	14,4964
RATTI	6,4172	8,5131
JUVENTUS FOOTBALL CLUB	16,7237	21,1383

Figure 24: Prior mean and Prior standard deviation (daily and monthly)

Finally, our results on bayes allocation led us to the following results:

Portfolio (daily) Bayesian	
Optimal weights	
ALERION CLEAN POWER	0,0766
FINECOBANK SPA	0,0962
STELLANTIS	0,1194
AMPLIFON	0,0755
EXPRIVIA	0,1183
CEMBRE	0,0539
SNAM	0,0573
DAVIDE CAMPARI MILANO	0,0609
CAIRO COMMUNICATION	0,0809
TELECOM ITALIA RSP	0,1037
RATTI	0,0509
JUVENTUS FOOTBALL CLUB	0,1065
Portfolio statistics	
Expected return	0,0591
Volatility	1,2608
Variance	1,5896
Skewness	0,0233
Kurtosis	0,7247
Sharpe ratio	0,0231

Portfolio (monthly) Bayesian	
Optimal weights	
ALERION CLEAN POWER	0,0736
FINECOBANK SPA	0,0818
STELLANTIS	0,1165
AMPLIFON	0,0672
EXPRIVIA	0,1631
CEMBRE	0,0769
SNAM	0,0380
DAVIDE CAMPARI MILANO	0,0579
CAIRO COMMUNICATION	0,0869
TELECOM ITALIA RSP	0,0759
RATTI	0,0408
JUVENTUS FOOTBALL CLUB	0,1214
Portfolio statistics	
Expected return	1,3824
Volatility	6,5983
Variance	43,5376
Skewness	0,0489
Kurtosis	0,2954
Sharpe ratio	0,0311

Figure 25: Bayesian portfolios (daily and monthly)

The results obtained are definitely in line with the previous ones. The small differences are mainly due to the fact that we assumed different priors than in the Black-Litterman Approach. Further observations are given in the concluding section, where the main statistics are summarised, and a single portfolio given by the average of the 4 final allocations is considered.

1.6 Global Minimum Variance Portfolio Allocation

As far as the Global Minimum Variance Portfolio Allocation is concerned, we can say that all the arguments made for the paragraph on the Efficient Frontier apply. In addition to the above, we can define the GMVP as the portfolio with the lowest variance (and hence standatd deviation) on the efficient frontier. We have therefore used the same techniques as in the Sharpe Ratio portfolio definition, but in this case, of course, our objective function will not be the maximisation of the Sharpe Ratio, but the minimisation of the variance.

For anything not specified in this section, what is said in the Efficient Frontier section applies.

Based on the previous data, we then calculated the GMVP, which is depicted in the following figures (shown as optimal), for both daily and monthly data.

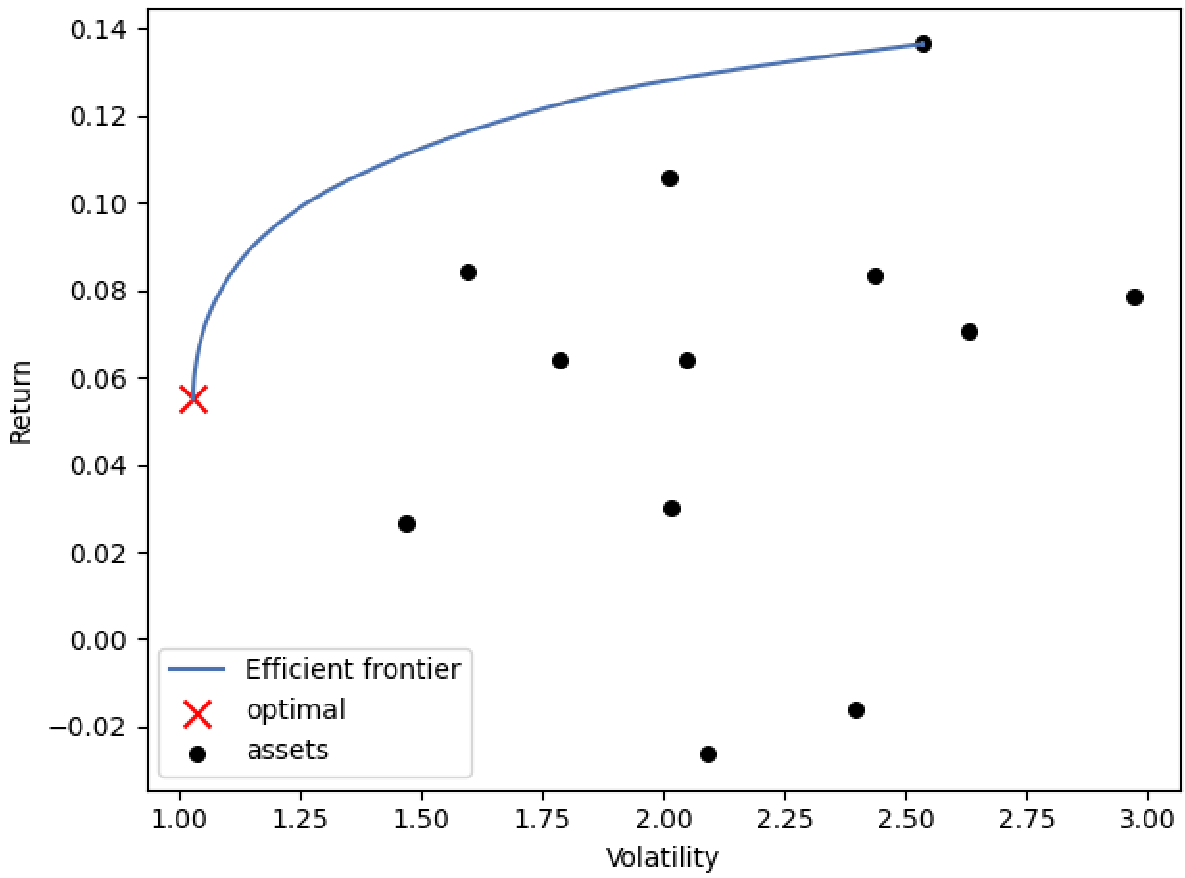


Figure 26: Global Minimum Variance Portfolio (daily)

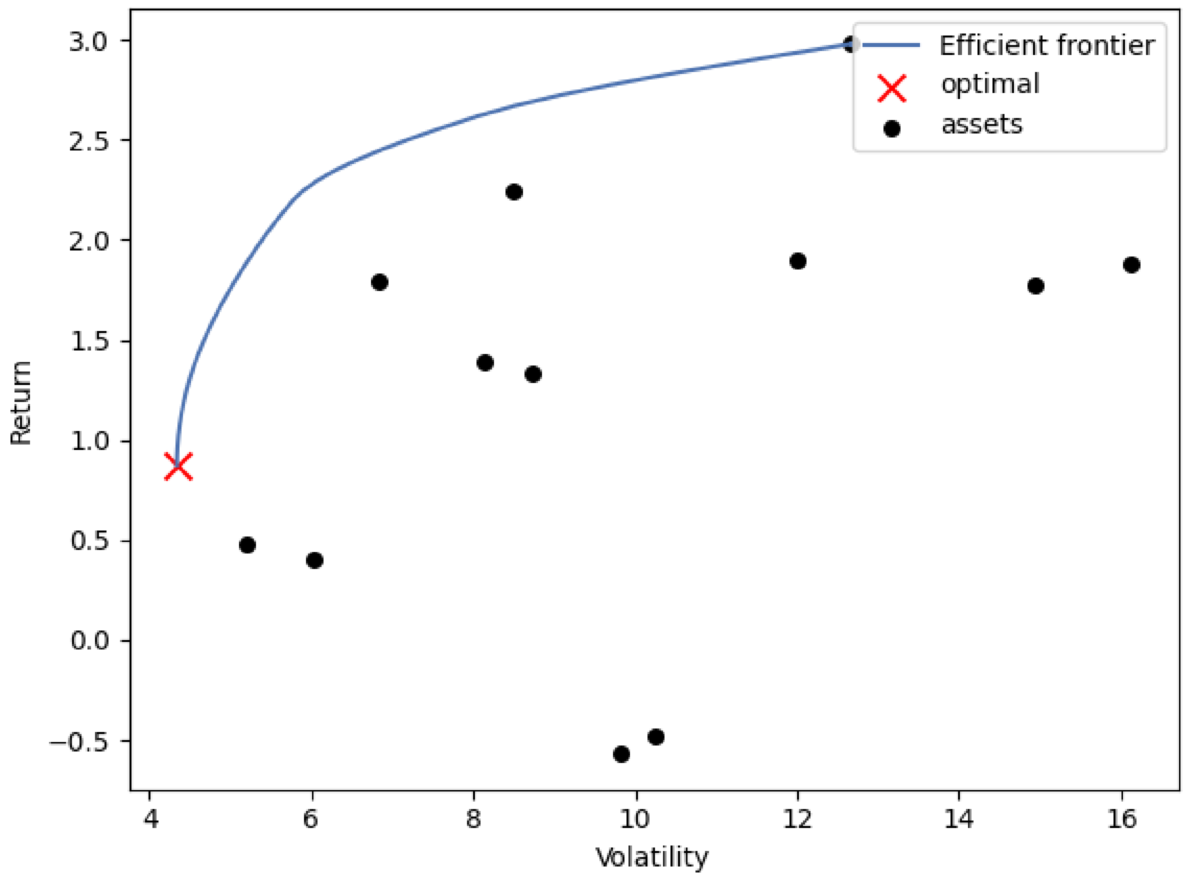


Figure 27: Global Minimum Variance Portfolio (monthly)

Below are the main statistics of our portfolios, as we have done for all the others.

Portfolio (daily) GMVP	
Optimal weights	
ALERION CLEAN POWER	0,0801
FINECOBANK SPA	0,0004
STELLANTIS	0,0000
AMPLIFON	0,0483
EXPRIVIA	0,0314
CEMBRE	0,1823
SNAM	0,2305
DAVIDE CAMPARI MILANO	0,1646
CAIRO COMMUNICATION	0,0733
TELECOM ITALIA RSP	0,0000
RATTI	0,1621
JUVENTUS FOOTBALL CLUB	0,0269
Portfolio statistics	
Expected return	0,0550
Volatility	1,0280
Variance	1,0568
Skewness	-1,4831
Kurtosis	19,4659
Sharpe ratio	0,0243

Portfolio (monthly) GMVP	
Optimal weights	
ALERION CLEAN POWER	0,0884
FINECOBANK SPA	0,0187
STELLANTIS	0,0000
AMPLIFON	0,0186
EXPRIVIA	0,0000
CEMBRE	0,0043
SNAM	0,4007
DAVIDE CAMPARI MILANO	0,1374
CAIRO COMMUNICATION	0,0000
TELECOM ITALIA RSP	0,0408
RATTI	0,2910
JUVENTUS FOOTBALL CLUB	0,0000
Portfolio statistics	
Expected return	0,8710
Volatility	4,3337
Variance	18,7810
Skewness	-0,3698
Kurtosis	1,0349
Sharpe ratio	0,0448

Figure 28: Statistcs of GMVP (daily and monthly)

1.7 Conclusion

In this section, we will focus on the conclusions of our exercise, compare the various portfolios, and provide appropriate explanations for divergences and similarities. Next, we will define average statistics for a portfolio that takes into account, albeit minimally, all the assumptions and the various results of the individual approaches.

Below are the various results for better visualisation for comparison purposes.

Portfolio (daily) without constraint	
Portfolio statistics	
Expected return	0,1012
Volatility	1,1795
Variance	1,3913
Skewness	-0,6194
Kurtosis	9,6695
Sharpe ratio	0,0604

Portfolio (daily) Black Litterman	
Portfolio statistics	
Expected return	0,0610
Volatility	0,2150
Variance	0,0462
Skewness	-0,9509
Kurtosis	11,1100
Sharpe ratio	0,1442

Portfolio (daily) Bayesian	
Portfolio statistics	
Expected return	0,0591
Volatility	1,2608
Variance	1,5896
Skewness	0,0233
Kurtosis	0,7247
Sharpe ratio	0,0231

Portfolio (daily) GMVP	
Portfolio statistics	
Expected return	0,0550
Volatility	1,0280
Variance	1,0568
Skewness	-1,4831
Kurtosis	19,4659
Sharpe ratio	0,0243

Portfolio (monthly) without constraint	
Portfolio statistics	
Expected return	2,1417
Volatility	5,2194
Variance	27,2424
Skewness	-0,2291
Kurtosis	1,1482
Sharpe ratio	0,4046

Portfolio (monthly) Black Litterman	
Portfolio statistics	
Expected return	0,5460
Volatility	0,7860
Variance	0,6178
Skewness	-0,5647
Kurtosis	1,3955
Sharpe ratio	0,8352

Portfolio (monthly) Bayesian	
Portfolio statistics	
Expected return	1,3824
Volatility	6,5983
Variance	43,5376
Skewness	0,0489
Kurtosis	0,2954
Sharpe ratio	0,0311

Portfolio (monthly) GMVP	
Portfolio statistics	
Expected return	0,8710
Volatility	4,3337
Variance	18,7810
Skewness	-0,3698
Kurtosis	1,0349
Sharpe ratio	0,0448

Figure 29: Statistics of all portfolios (daily and monthly)

The first observation is that, as was logical to expect, the statistics of the portfolios turn out to be more or less consistent with each other. In fact, having taken the same 12 assets into consideration, we find it reasonable (even if with differences justified by different assumptions, such as risk aversion or different priors) that they produce very similar results. Indeed, beyond the individual assumptions and objective functions, the portfolios are all based on the same input data, i.e. asset prices (daily and monthly) and many of the assumptions are shared (rational investor, public information).

Spostandoci invece sulle differenze esse sono in primis giustificate dalle diverse objectives function dei singoli modelli e dalle diverse assumptions. Ad esempio, per definizione, Black-Litterman e Bayes approach tengono conto di views e probabilità soggettive del singolo investitore (e quindi dell'esperienza) che ponderano con quelle del mercato, mentre il Mean-Variance approach tiene conto solo di queste ultime. Nel Mean-Variance model abbiamo assunto il nostro personale coefficiente di avversione al rischio, mentre nella parte del Black-Litterman approach abbiamo considerato il coefficiente di avversione al rischio del mercato. Inoltre, Ancora, una delle ragioni alla base delle differenze tra Bayesian Allocation e Black-

Moving instead to the differences, these are primarily justified by the different objective functions of the individual models and the different assumptions.

For example, by definition, the Black-Litterman and Bayes approaches consider views and subjective probabilities of the individual investor (and thus experience) which weigh up against those of the market, whereas the Mean-Variance approach only takes the latter into account. In the Mean-Variance model, we assumed our own risk aversion coefficient, whereas in the Black-Litterman approach, we considered the market's risk aversion coefficient. Moreover, again, one of the reasons for the differences between Bayesian Allocation and Black-Litterman Allocation is that in the former we took into account prior returns shifted by an amount equal to the standard deviation and a variance-covariance matrix that was twice as large as the initial one. One result that might be perplexing is that the GMVP volatility is higher than in the Black-Litterman case. In reality, this result is totally justified when we look at the (more negative) views and consider the different risk aversion parameters taken into account. So it is true that the GMVP represents the portfolio with the lowest volatility, but this may not be true when entering the views: to give a concrete example, if an investor is expecting a significant market crash and much higher volatility than in previous years while the market is not of the same opinion, the classical approach will not take this potential drop in returns and increase in volatility into account, while the Black-Litterman will.

Overall, the portfolio differences arise mainly from variations in expected returns, volatilities, correlations, investor views, prior beliefs, and optimization objectives. The different assumptions, models, and subjective inputs utilized in each approach lead to divergent portfolio compositions and statistics.

To conclude, we decided to define the statistics of an average portfolio, created from a linear combination of all 4 portfolios (considering equal weights). We decided to assign equal weights of 25% each, as we believe that the 4 approaches are characterised by offsetting shortcomings and advantages. The final statistics are shown in the following figure.

Portfolio (daily) Linear Combination	
Portfolio statistics	
Expected return	0,0691
Volatility	0,9208
Variance	1,0210
Skewness	-0,7575
Kurtosis	10,2425
Sharpe ratio	0,0630

Portfolio (monthly) Linear Combination	
Portfolio statistics	
Expected return	1,2353
Volatility	4,2344
Variance	22,5447
Skewness	-0,2787
Kurtosis	0,9685
Sharpe ratio	0,3289

Figure 30: Statistics of a portfolio given by a linear combination of the four portfolios

The construction of these statistics is aimed at trying to minimise the possible errors given by the individual assumptions and methodologies (robustness). In doing so, we can see that the individual portfolios have very similar characteristics to this average portfolio.

2: Climate Change Risk

The introduction discusses the importance of understanding the economic cost of climate change and its impact on the macroeconomy. It highlights the challenges in measuring this cost due to the long-term nature of the consequences of global warming.

The paper proposes using forward-looking equity prices as a way to assess the economic cost of climate change. The authors present a climate change model that incorporates the interaction between global warming and economic growth, as well as the risks associated with temperature changes.
So basically, temperature is a source of long-term risk in economic growth and this risk is imputed in current asset valuations and risk premiums.

The model predicts that temperature risks are reflected in asset valuations and risk premiums, and the authors use empirical evidence to support these predictions.

They find that low-frequency temperature variations have a significant negative effect on asset valuations and carry a positive risk premium. The paper also explores the cross-sectional variation in temperature risks and their relationship with long-run growth risks in asset cash flows. They conclude that climate-change risk is already incorporated into asset prices and use the temperature elasticities of equity valuations to estimate the social cost of carbon emissions.

What differentiates this paper from the others is that in Dell, Jones, and Olken (2012), Bansal and Ochoa (2012), and Colacito, Hoffmann, and Phan (2019) the effect of temperature changes on economic growth is examined, here instead the paper focuses on forward-looking equity valuations and asset returns that incorporate the impact of global warming on risk premia, which cannot be identified from past growth rate data. The other literature in this field shows how the paper is innovative and in particular Jagannathan, Ravikumar, and Sammon (2018) show that incorporating environmental concerns in investment portfolio decisions helps reduce exposure to systematic risks, which corroborates our key argument that climate change is a source of macroeconomic risks.

2.1 Climate Change and Asset Prices

Various studies show us that there has been a dramatic increase in temperatures since 1970 and that there is a much more significant trend of subsequent dramatic increases.

The paper presents a simplified model that considers the interaction between economic development and climate change and the dangers of global warming damage. The model seeks to understand how asset prices and risk premiums are affected by global warming. The condensed model provides analytical answers and a clear understanding of the welfare and cost implications of temperature-related risks.

$$\Delta c_{t+1} = \mu + \sigma_\eta \eta_{t+1} + D_{t+1} \,,$$

Where ΔC_{t+1} is the log of the aggregate consumption growth, η_{t+1} is consumption growth innovation and D_{t+1} are damages caused by climate changes that can be rewritten as below

$$D_{t+1} = N_{t+1} d \,,$$

where $d < 0$ is the decrease in consumption growth caused by temperature, and Nt+1 is a Poisson process whose intensity varies over time and rises with the temperature: (paper suppose a constant increase of temperature)

$$\pi_t = \ell_0 + \ell_1 T_t \,,$$

where T_t is global temperature relative to its pre-industrial level with ℓ_0, $\ell_1 > 0$. This is because if temperature does not affect the probability of future economic damage (in the case where $\ell 1 = 0$), then temperature changes have no effect on long-run economic growth and thus, would have no effect on marginal utility (meaning that they would not be priced in)

The global temperature anomaly is driven by

$$\begin{aligned} T_{t+1} &= \chi \mathcal{E}_{t+1} \,, \\ \mathcal{E}_{t+1} &= \nu \mathcal{E}_t + \Theta(\mu + \sigma_\eta \eta_{t+1}) + \sigma_\zeta \zeta_{t+1} \,, \end{aligned}$$

where ε_{t+1} are carbon emissions, $\nu \in (0, 1)$ determines the persistence of carbon emissions, temperature, Θ measures carbon intensity of consumption, and $\chi > 0$ is climate sensitivity to emissions and $\zeta_{t+1} \sim$ i.i.d. $N(0, 1)$.

Anthropogenic carbon emissions, driven by endogenous industrial emissions and exogenous innovation, contribute to temperature changes and the likelihood of climate-related economic damages. This feedback loop between economic growth and temperature has implications for asset prices. Temperature fluctuations are considered a source of economic risk, as higher temperatures increase the probability of damages and lead to a decline in economic growth. Moreover, emission shocks have persistent effects on temperature, amplifying long-run risks.

The specific disaster model used is not crucial; what matters is that temperature fluctuations influence the distribution of future consumption growth, representing long-run risks.

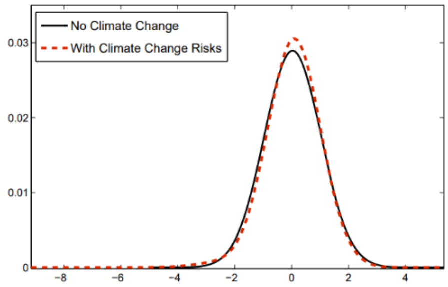


Figure 31: Global warming with respect to consumption growth

This figure shows the implications of global warming on the distribution of (normalized) consumption growth. Assuming a temperature increase of 2%, we can see how this induces long-term negative tail risks in consumption growth.
Let us now instead consider a representative agent with recursive preferences to assess the impact of climate change on consumption decisions and financial activities.

The utility function of this representative agent at time t is given by:

$$U_t = \left\{(1-\delta)C_t^{1-\frac{1}{\psi}} + \delta\left(\mathbb{E}_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

where U_{t+1} is the continuation value of a consumption plan starting in t + 1, δ is the rate of time preference, γ is the coefficient of relative risk aversion, and ψ is the intertemporal elasticity of substitution (IES).

Recursive preferences provide a distinction between willingness to substitute consumption across time and across various states of nature, in contrast to the power-utility formulation that is frequently used in integrated assessment models of climate change. Usually, this version is preferred since they can solve the well-known risk-free rate, equity premium and volatility riddles by accounting for the combined dynamics of total cash flows and equity prices.

Through Euler condition we can derive the log of the intertemporal marginal rate of substitution (IMRS), given by

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1)r_{c,t+1}$$

Where $r_{c,t+1}$ is the endogenous return on wealth, and

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$$

Of course, if γ = (1 / ψ) then the IMRS is equal to the constant relative risk aversion (CRRA) specification.

If we maximize the life-time utility function, then we will see that in this setting it is proportional to the wealth-consumption ratio and it is given by

$$\frac{U_t}{C_t} = \left[(1-\delta)Z_t\right]^{\frac{\psi}{\psi-1}}$$

Where Zt ≡ (Wt/Ct) is the aggregate wealth-consumption ratio. Be aware that aggregate valuation Zt represents agents' expectations for future expected growth, uncertainty, and tail risks that are caused by climate change as it is the (normalized) present value of current and future consumption.

In section 3 of chapter 1, we want to arrive at the analytical solution for the log of wealth $z_t = A_0 + A_1 T_t$. consumption:

To resolve this formula and understand what are the "black box" A0 and A1 we need to start from two known formulas:

1) Formula for the log of the intertemporal marginal rate of substitution (IMRS), given by:

   $$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1}$$

2) Formula for the log-linear approximation of wealth return

   $$r_{c,t+1} = \kappa_0 + \Delta c_{t+1} + z_{t+1} - \kappa_1 z_t \,,$$

We then, can exploit the Euler condition:

$$\mathbb{E}_t\left[\exp(m_{t+1} + r_{c,t+1})\right] = 1$$

We can now substitute the two previous formulas and the calculations are shown below. Please note that $\mathbb{E} = \mathbb{E}_t$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta - \frac{\theta}{\psi}\Delta_{C_{t+1}} + (\theta - 1)\, r_{c,t+1} + r_{c,t+1}\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta - \frac{\theta}{\psi}\Delta_{C_{t+1}} + \theta\, r_{c,t+1} - r_{c,t+1} + r_{c,t+1}\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; - \frac{\theta}{\psi}\Delta_{C_{t+1}} + \theta\, r_{c,t+1}\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; - \frac{\theta}{\psi}\Delta_{C_{t+1}} + \theta\,(k_0 + \Delta c_{t+1} + z_{t+1} - k_1 z_t)\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; - \frac{\theta}{\psi}\Delta_{C_{t+1}} + \theta\, k_0 + \theta\Delta c_{t+1} + \theta z_{t+1} - \theta k_1 z_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + \Delta_{C_{t+1}}\left(\theta - \frac{\theta}{\upsilon}\right) + \theta\, k_0 + \theta\Delta c_{t+1} + \theta\,(z_{t+1} - \; k_1 z_t)\right)\right] = 1$$

Since $\psi = \frac{1-\gamma}{1-\frac{1}{\psi}}$

If we assume that $\gamma = \frac{1}{\psi}$ then $\theta = 1$ and we get

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + \Delta_{C_{t+1}}\left(1 - \frac{1}{\upsilon}\right) + \theta\, k_0 + \theta\Delta c_{t+1} + \theta\,(z_{t+1} - \; k_1 z_t)\right)\right] = 1$$

We then arrive at the conclusion

$$\mathbb{E}_t\left[\exp\left(\theta \ln \delta + (1-\gamma)\Delta c_{t+1} + \theta\kappa_0 + \theta(z_{t+1} - \kappa_1 z_t)\right)\right] = 1$$

Where in particular

$$\kappa_1 = \frac{e^{\bar{z}}}{e^{\bar{z}}-1}, \quad \kappa_0 = \kappa_1 \bar{z} - \ln(e^{\bar{z}} - 1),$$

with Z_t as the unconditional mean of the log wealth-consumption ratio. Plugging in the dynamics of consumption growth and the conjecture for Z_t yields:

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + (1-\gamma)\left(\mu \; + \sigma_\eta\, \eta_{t+1} \; + D_{t+1}\right) \; + \theta\, k_0 \; + \theta\left(A_0 \; + A_1 T_{t+1} \; - k_1 A_0 \; - k_1 A_1 T_t\right)\right)\right]$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + (1-\gamma)\mu \; + (1-\gamma)\sigma_\eta \eta_{t+1} \; + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + \theta A_0 \; + \theta A_1 T_{t+1} \; - \theta k_1 A_0 \; - \theta k_1 A_1 T_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + (1-\gamma)\mu \; + (1-\gamma)\sigma_\eta \eta_{t+1} \; + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + A_0\, \theta\,(1-k_1) \; + \theta A_1 T_{t+1} \; - \theta k_1 A_1 T_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + (1-\gamma)\mu \; + (1-\gamma)\sigma_\eta \eta_{t+1} \; + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + A_0\, \theta\,(1-k_1) \; + \theta A_1 T_{t+1} \; - \theta k A_1 T_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + (1-\gamma)\mu \; + (1-\gamma)\sigma_\eta \eta_{t+1} \; + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + A_0\, \theta\,(1-k_1) \; + \theta A_1\left(\chi\, \nu\, \varepsilon_t + \Theta\, \chi\, \mu \; + \Theta\, \chi\, \sigma_\eta\, \eta_{t+1} \; + \chi\, \sigma_\varsigma\, \varsigma_{t+1}\right) \; - \theta k_1 A_1 T_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + (1-\gamma)\mu \; + (1-\gamma)\sigma_\eta \eta_{t+1} \; + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + A_0\, \theta\,(1-k_1) \; + \theta A_1 \chi\, \nu\, \varepsilon_t + \theta A_1 \Theta\, \chi\, \mu \; + \theta A_1 \Theta\, \chi\, \sigma_\eta\, \eta_{t+1} \; + \theta A_1 \chi\, \sigma_\varsigma\, \varsigma_{t+1} \; - \theta k_1 A_1 T_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + \mu\; (1-\gamma+\theta A_1 \Theta\, \chi) + \sigma_\eta \eta_{t+1}\; (1-\gamma+\theta A_1 \Theta\, \chi) + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + A_0\, \theta\,(1-k_1) \; + \theta A_1 \nu\, T_t + \theta A_1 \chi \sigma_\varsigma \varsigma_{t+1} - \theta k_1 A_1 T_t\right)\right] = 1$$

$$\mathbb{E}\left[\exp\left(\theta \ln \delta \; + \mu\; (1-\gamma+\theta A_1 \Theta\, \chi) + \sigma_\eta \eta_{t+1}\; (1-\gamma+\theta A_1 \Theta\, \chi) + (1-\gamma)D_{t+1} \; + \theta\, k_0 \; + A_0\, \theta\,(1-k_1) \; + \theta A_1 T_t\,(\nu - k_1) - \theta k_1 A_1 T_t\right)\right] = 1$$

And so, just by interchanging the terms and eliminating the exponential since we are dealing with the log of wealth consumption, we finally obtain

$$\mathbb{E}_t\Bigg[\theta\ln\delta + (1-\gamma+\chi\Theta\theta A_1)\mu + \theta\kappa_0 + (1-\kappa_1)\theta A_0 + \theta(\nu-\kappa_1)A_1T_t + (1-\gamma+\chi\Theta\theta A_1)\sigma_\eta\eta_{t+1} + \chi\theta A_1\sigma_\zeta\zeta_{t+1} + (1-\gamma)D_{t+1}\Bigg] = 1\,.$$

We can now solve for this expectation

$$\theta\ln\delta \ + \mu\ (1-\gamma+\theta A_1\Theta\,\chi) + \theta k_0 + (1-k_1)\theta A_0 + \theta\,(\nu-k_1)A_1T_t + \mathbb{E}\left[(1-\gamma+\theta A_1\Theta\,\chi)\sigma_\eta\eta_{t+1} + \chi\theta A_1\sigma_\zeta\zeta_{t+1} + (1-\gamma)D_{t+1}\right] = 1$$

Since η_{t+1} is distributed like a normal distribution, we know that its expectation is 0 and the formula becomes

$$\theta\ln\delta \ + \mu\ (1-\gamma+\theta A_1\Theta\,\chi) + \theta k_0 + (1-k_1)\theta A_0 + \theta\,(\nu-k_1)A_1T_t + (1-\gamma)\,(\ell_0+\ell_1T_t)\ d + \mathbb{E}\left[\chi\theta A_1\,(\epsilon_{t+1}-\nu\epsilon_t-\theta\,(\mu+\sigma_\eta+\mu_{t+1}))\right] = 1$$

$$\theta\ln\delta \ + \mu\ (1-\gamma+\theta A_1\Theta\,\chi) + \theta k_0 + (1-k_1)\theta A_0 + \theta\,(\nu-k_1)A_1T_t + (1-\gamma)\pi_t\phi d + \mathbb{E}\left[\chi\theta A_1\sigma_\zeta\frac{(\epsilon_{t+1}-\nu\epsilon_t-\Theta\,(\mu+\sigma_\eta+\mu_{t+1}))}{\sigma_\zeta}\right] = 1$$

$$\theta\ln\delta \ + \mu\ (1-\gamma+\theta A_1\Theta\,\chi) + \theta k_0 + (1-k_1)\theta A_0 + \theta\,(\nu-k_1)A_1T_t + (1-\gamma)\ell_0\phi\ d + \ell_1T_t\phi d\,(1-\gamma) +$$

$$+\,\mathbb{E}\left[\chi\theta A_1\epsilon_{t+1} - \nu\epsilon_t\chi\theta A_1 - \theta\mu\chi\Theta A_1 - \chi\theta A_1\Theta\sigma_\eta + \chi\theta A_1\Theta\eta_{t+1}\right] = 1$$

$$\theta\ln\delta \ + \mu\ (1-\gamma+\theta A_1\Theta\,\chi) + \theta k_0 + (1-k_1)\theta A_0 + (\theta\,(\nu-k_1)A_1 + \ell_1\phi d\,(1-\gamma))T_t +$$

$$+\,\ell_0\,(1-\gamma)\phi\ d - \upsilon T_t\theta A_1 - \chi\theta A_1\Theta\mu - \chi\theta A_1\Theta\sigma_\eta\mathbb{E}\left[\chi\theta A_1\epsilon_{t+1}\right] = 1$$

and finally we get

$$\begin{aligned} 0 = \ & \theta\ln\delta + (1-\gamma+\chi\Theta\theta A_1)\mu + \theta\kappa_0 + (1-\kappa_1)\theta A_0 + 0.5\,(1-\gamma+\chi\Theta\theta A_1)^2\sigma_\eta^2 \\ & + 0.5\,(\chi\theta A_1\sigma_\zeta)^2 + \ell_0\phi\big\{(1-\gamma)\ d\big\} + \Big(\theta(\nu-\kappa_1)A_1 + \ell_1\phi\big\{(1-\gamma)\ d\big\}\Big)T_t\,, \end{aligned}$$

Using the MGF of the Poisson distribution we obtain our unknowns, in particular

$$\begin{aligned} A_0(\kappa_1-1) = \ & \ln\delta + \kappa_0 + \Big(1-\frac{1}{\psi}+\chi\Theta A_1\Big)\mu + \frac{\ell_0}{\theta}\phi\big\{(1-\gamma)\ d\big\} \\ & + 0.5\,\theta\bigg[\Big(1-\frac{1}{\psi}+\chi\Theta A_1\Big)^2\sigma_\eta^2 + \Big(\chi A_1\sigma_\zeta\Big)^2\bigg]\,. \end{aligned}$$

$$A_1 = \frac{\ell_1}{\theta}\frac{\phi\big\{(1-\gamma)d\big\}}{\kappa_1-\nu}\,,$$

With $k_1 > 1$ determined endogenously by the mean of the wealth-consumption ratio.

Here we can distinguish 2 cases, in particular when the representative agent prefers the early resolution of uncertainty, and formally we can say that

$$\gamma > \frac{1}{\psi}$$

then the wealth-consumption ratio declines with temperature. In contrast, if the standard power-utility specification is equal to the risk aversion, then the wealth-consumption ratio responds positively to temperature fluctuations (This is definitely stranger than the previous case)

Now we again find the formula for the IMRS but with an innovation, namely we consider the IMRS conditional on the information we have up to time t.

$$m_{t+1} - \mathbb{E}_t\big[m_{t+1}\big] = -\lambda_\eta \sigma_\eta \eta_{t+1} - \lambda_\zeta \sigma_\zeta \zeta_{t+1} - \lambda_D (N_{t+1} - \pi_t)$$

where

$$\begin{aligned} \lambda_\eta &= \gamma + (1-\theta)\chi\Theta A_1 \\ \lambda_\zeta &= (1-\theta)\chi A_1 \\ \lambda_D &= \gamma d \end{aligned}$$

Then, we can solve for m_{t+1} and using the solution for z_t defined above, we can derive the dynamics of the intertemporal marginal rate of substitution:

$$m_{t+1} = m_0 - (\theta-1)(\kappa_1 - \nu)A_1 T_t - (\gamma - (\theta-1)\chi\Theta A_1)\sigma_\eta \eta_{t+1} - (1-\theta)\chi A_1 \sigma_\zeta \zeta_{t+1} - \gamma D_{t+1}$$

Taking in mind that m_0 is

$$m_0 = \theta \ln\delta + (\theta-1)\kappa_0 - \Big(\gamma - (\theta-1)\chi\Theta A_1\Big)\mu + (\theta-1)(\kappa_1 - 1)A_0$$

Now using the Euler condition defined above we can finally derive the risk-free rate:

$$r_{f,t} = r_f + \ell_1 \left(\frac{\theta-1}{\theta}\phi\{(1-\gamma)d\} - \phi\{-\gamma d\}\right) T_t$$

Again, taking into account that r_f is

$$\begin{aligned} r_f = &-\theta\ln\delta + \gamma\mu - (\theta-1)\left(\kappa_0 + (1-\kappa_1)A_0 + \chi\Theta A_1 \mu\right) \\ &- \Big[\big(\gamma - (\theta-1)\chi\Theta A_1\big)^2 \sigma_\eta^2 + \big((1-\theta)A_1\chi\sigma_\zeta\big)^2\Big] - \ell_0\phi\{-\gamma d\} \end{aligned}$$

Thanks to these algebraic steps, exploiting the dynamics of the IMRS and the Euler condition we are able to have a new interest rate calculated on the IMRS that take these innovations into account.

We can now say that since (as we will see more clearly later) temperature risks entail risk premiums and affect asset valuations only if they affect the distribution of future economic growth, a new risk premium needs to be calculated that takes these factors into account, and in particular a revised CAPM formula.
This is because the beneficial impact of economic expansion confounds the negative impact of endogenous temperature changes. Therefore, it is essential to take into account the mitigating effect of growth fluctuations to accurately determine the only negative influence of temperature. Similarly, excluding growth controls could cause estimation errors for the temperature risk premium.

Using the solution for Z_t, and the dynamics of consumption growth (already defined) to substitute these two elements in formula of the log-linear return approximation we obtain

$$r_{c,t+1} = r_c - (\kappa_1 - \nu)A_1 T_t + (1 + \chi\Theta A_1)\sigma_\eta \eta_{t+1} + A_1 \chi \sigma_\zeta \zeta_{t+1} + D_{t+1}$$

Considering that

$$r_c = A_0(1 - \kappa_1) + \kappa_0 + (1 + \chi\Theta A_1)\mu$$

Now, using the solution of the IMRS and the return dynamics we get

$$\ln \mathbb{E}_t[R_{c,t+1}] - r_{f,t} = (\gamma + (1-\theta)\chi\Theta A_1)(1 + \chi\Theta A_1)\sigma_\eta^2 + (1-\theta)(A_1\chi\sigma_\zeta)^2 + \gamma d^2(\ell_0 + \ell_1 T_t)$$

thanks to these algebraic passages we are able to rewrite the CAPM formula, not as we are used to seeing it, but the CAPM formula implemented taking the 'environment' factor into account.

$$\ln \mathbb{E}_t\big[R_{c,t+1}\big] - r_{f,t} = \underbrace{(1 + \chi\Theta A_1)\big(\gamma + (1-\theta)\chi\Theta A_1\big)\sigma_\eta^2}_{Growth\ \ Premium} + \underbrace{(1-\theta)(A_1\chi\sigma_\zeta)^2}_{Temp-Premium} + \underbrace{\gamma d^2(\ell_0 + \ell_1 T_t)}_{Damage-Premium} .$$

Breaking the formula down into 3, we see how it differs from the original CAPM: The first term is the risk premium for variations in consumption growth and incorporates the impact of endogenous temperature risks. The second and third terms represent the premiums for exogenous variations in temperature and for temperature-induced damage risks.
To summarize, let us try to understand why an implementation of the CAPM formula is necessary. The paper sets out to analyze and argue how consumption growth increases temperature, which in turn affects the economy by propagating damages. So, intuitively, let us look at the 3 main reasons:

1. A positive consumption growth shock has contradictory effects; while it is positive in the short term, it is negative in the long term due to the rise in industrial carbon emissions and temperature, which increases the risk of future damages.
2. Temperature hazards have a negative cost because a positive exogenous innovation in temperature increases marginal utility
3. Changes in temperature increase the volatility of marginal utility and expose the economy to the danger of destruction.

Since, as explained above, climate change influences consumption dynamics (and vice versa), it is intuitive to think that assets that are highly exposed to consumption growth 'risks' are consequently highly influenced by climate change risks.
After indexed by i, a cross-section of equity securities that feature heterogenous exposure to consumption risks

$$\begin{aligned}\Delta d_{i,t+1} &= \varphi_i \Delta c_{t+1} + \sigma_i u_{i,t+1} \\ &= \varphi_i \big(\mu + \sigma_\eta \eta_{t+1} + D_{t+1}\big) + \sigma_i u_{i,t+1}\,,\end{aligned}$$

Where φ_i is the measure of long-run risk in dividends (our dividend beta), and μ distributed as a normal identically and independent distributed, then we have that from Euler equation the risk premium of the asset i is:

$$\ln \mathbb{E}_t\big[R_{i,t+1}\big] - r_{f,t} \;=\; \beta_{i,\eta}\lambda_\eta\sigma_\eta^2 \;+\; \beta_{i,\zeta}\lambda_\zeta\sigma_\zeta^2 \;+\; \beta_{i,D}\lambda_D(\ell_0 + \ell_1 T_t)\,,$$

As a result, we have shown that cross-sectional differences of consumption risks in the dividends of assets (in particular, as formula shows we are dealing with 3 different sources of risks) result in cross-sectional differences in the temperature risks of asset returns.
The three different sources of risk are

$$\begin{aligned}\beta_{i,\eta} &= \varphi_i + \kappa_{i,1}\chi\Theta B_{i,1}\,, \\ \beta_{i,\zeta} &= \kappa_{i,1}\chi B_{i,1}\,, \\ \beta_{i,D} &= \varphi_i d\,,\end{aligned}$$

We can now substitute these 3 sources of risk in the formula.

Considering that $B_{i,1}$ is the asset-specific counterpart to A1, which measures temperature (semi) elasticity of aggregate wealth. Then $K_{1,i}$ <1 is the constant of log-linearization and his formula is given by

$$B_{i,1} = \ell_1 \frac{\frac{1-\theta}{\theta}\phi\{(1-\gamma)\,d\} + \phi\{(\varphi_i - \gamma)\,d\}}{1 - \kappa_{i,1}\nu}$$

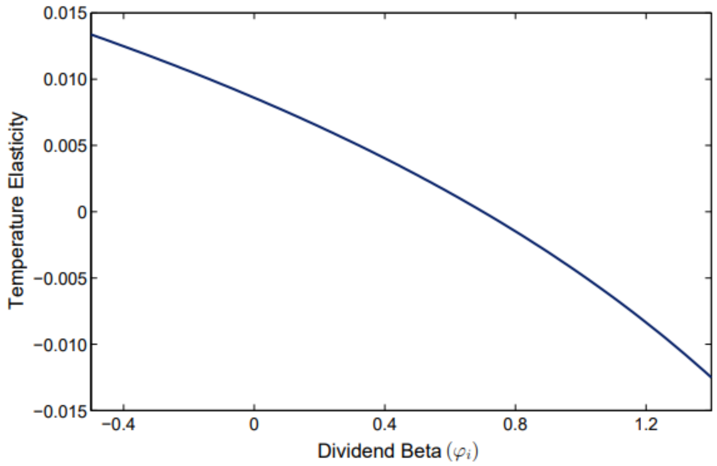
with $\kappa_{i,1} = \frac{e^{\bar{z}_i}}{e^{\bar{z}_i}+1}$

And z bar defined as the mean of the log price-dividend ratio.

As this first equation shows we can see that, exactly as in the consumption growth risk model, the dividend beta is determined but also a part is added that considers exposure to exogenous temperature variations.

The second equation then, shows the temperature beta of asset i and this also shows the exogenous temperature variations.

Let's now analyze this figure



It is designed to roughly match the dynamics of annual consumption and temperature. As the figure shows, while the relationship between dividend betas and temperature elasticities is non-linear it is strongly negative. Similarly, temperature betas are inversely related to dividend betas, so the correlation between the two of them is negative. This means that assets that have a negative or relatively low dividend beta have positive temperature elasticity and temperature beta.

After explaining the model, Chapter 2 only deals with Empirical Evidence, so we skip it for obvious reasons.

2.2 The Social Cost of Carbon Based on Asset Prices

The paper demonstrates that climate risk is embedded in asset prices (since investor are afraid about the long run rise in temperature and other reasons explained above), due to that we can find important information about the cost of climate change directly in the prices of the asset.

Now instead, the paper will summarize not anymore, the costs embedded in asset prices but the cost embedded in the welfare.

What's differ this paper from the others is that on one hand here the authors compute the SCC directly from the capital market data and so the SCC estimate incorporates forward-looking asset prices, reflecting market expectations for emissions, economic losses, abatement efforts and technological progress in addressing climate change (effectively let capital markets reveal the cost of carbon emissions); Other studies, on the other hand, have measured using integrated evaluation models, thus basically relying on several assumptions, some of which are highly uncertain.

In particular, we can formally define a function called marginal utility of emissions where the social cost of carbon emissions (SCC) (that summarize all the welfare costs) is given by

$$SCC_t = -\frac{\partial U_t}{\partial \mathcal{E}_t} \Big/ \frac{\partial U_t}{\partial C_t}$$

where U_t is the life-time utility of the agent.

This formula tells us, for example, that a slight increase in current emissions has a ripple effect on future temperatures, leading to amplified damages and risks in the economy down the line.

So, at the end, the SCC is a good proxy to estimate the extent to which current consumption needs to increase in order to offset the anticipated future losses resulting from climate change.

If then, in addition to the marginal utility function of emissions, we consider the maximised lifetime utility, which takes into account multiple factors that influence an individual's overall welfare, it is possible to examine how emissions influence an individual's overall utility and well-being (or rather malaise). The maximized lifetime utility is given by:

$$\frac{U_t}{C_t} = \omega Z_t^{\frac{\psi}{\psi-1}}$$

If we then calculate from this formula the derivatives of utility with respect to consumption and emissions, we obtain the marginal utility function of consumption and the marginal utility function of emissions respectively. These derivatives represent the change in the individual's utility for a marginal increase in consumption and emissions.

The derivative of utility with respect to consumption (often called the marginal utility function of consumption) indicates how much the individual's utility changes with changes in consumption while the derivative of utility with respect to emissions (the marginal utility function of emissions) represents the change in the individual's utility for a marginal increase in emissions. The two results are

$$\frac{\partial U_t}{\partial C_t} = \omega Z_t^{\frac{1}{1-1/\psi}},$$
$$\frac{\partial U_t}{\partial \mathcal{E}_t} = \frac{\partial U_t}{\partial Z_t}\frac{\partial Z_t}{\partial \mathcal{E}_t} = \frac{\omega}{1-1/\psi} Z_t^{\frac{1/\psi}{1-1/\psi}} \frac{\partial Z_t}{\partial \mathcal{E}_t}$$

So only now, we can calculate the SCC as a measure of the elasticity of Zt to emissions. Taking then the derivative of Z_t with respect to the current emissions we can finally show that

$$\begin{aligned}\frac{\partial Z_t}{\partial \mathcal{E}_t} &= \sum_{j=0}^{\infty} E_t\left[\frac{\partial C_{t+j}}{\partial \mathcal{E}_t}\frac{1}{C_t}M_{t\to t+j} + \frac{\partial M_{t\to t+j}}{\partial \mathcal{E}_t}\frac{C_{t+j}}{C_t}\right] \\ &= (1-1/\psi)\sum_{j=0}^{\infty} E_t\left[\frac{\partial C_{t+j}}{\partial \mathcal{E}_t}\frac{1}{C_t}M_{t\to t+j}\right] + \sum_{j=0}^{\infty} E_t\left[\delta^j\left(\frac{C_{t+j}}{C_t}\right)^{1-1/\psi}\frac{\partial S_{t\to t+j}}{\partial \mathcal{E}_t}\right]\end{aligned}$$

And so we can represent the SCC as

$$SCC_t = \frac{\sum_{j=0}^{\infty} E_t\left[-\frac{\partial C_{t+j}}{\partial \mathcal{E}_t}M_{t\to t+j}\right]}{W_t} C_t + Q,$$

where (C_{t+j} / $\partial \varepsilon_t$) is the loss in time (t+j) consumption induced by a marginal increase in current emissions and $M_{t\to t+j}$ is the stochastic discount factor.

All the first term represents the discounted value of future consumption stream damages resulting from an increase in current emissions, relative to current wealth (Wt).

The second term, represented below

$$Q = -\frac{1}{1-1/\psi}\sum_{j=0}^{\infty} E_t\left[\delta^j\left(\frac{C_{t+j}}{C_t}\right)^{1-1/\psi}\frac{\partial S_{t\to t+j}}{\partial \mathcal{E}_t}\right]\frac{C_t}{Z_t}$$

captures the incremental effect of emissions on future utility.

We can now also compute the SCC by directly estimating the elasticity of the wealth-consumption ratio using capital market data. In particular, the social cost of carbon emissions can be measured in the data by the (semi) elasticity of asset prices to temperature, so this means that taking the correspondent derivatives the formula became:

$$SCC_t \;=\; \frac{\psi}{\psi-1}\,\frac{-\partial \log Z_t}{\partial T_t}\,\frac{\partial T_t}{\partial \mathcal{E}_t}\,C_t\,.$$

We can also see this formula in order to rewrite it as a term in the wealth-consumption ratio. The model solution for the SCC is given by

$$SCC_t \;=\; \frac{\psi}{\psi-1}\,(-A_1)\,\chi\,C_t \;=\; \frac{\Phi}{\gamma-1}\,\chi\,C_t\,,$$

where A_1 is (as we have already defined) the temperature (semi) elasticity of the wealth-consumption ratio

2.3 Conclusion

In conclusion, this study demonstrates that the economic cost of rising temperatures due to climate change can be measured by examining forward-looking equity prices. The analysis reveals that low-frequency temperature variations have a significant negative impact on asset valuations and carry a positive risk premium, indicating that climate risk is already incorporated into asset prices, providing valuable information about the costs associated with climate change.

By utilizing the information embedded in asset valuations, the study estimates the social cost of carbon emissions (SCC) using a semi-parametric capital-market approach. The results indicate a substantial SCC (as one would expect), reflecting the market's concern about the impact of climate change on long-term economic growth and risk. The SCC estimates suggest that society would be willing to sacrifice a significant portion of world gross domestic product to eliminate global industrial emissions.

Overall, this research underscores the importance of considering forward-looking information in asset prices when assessing the economic impact and costs associated with rising temperatures. By incorporating climate risk into economic models, valuable insights can be gained into the potential consequences of climate change and the need for appropriate policy responses.

Appendices

1A. Statistics of Return (daily) for each stock

mean	std	var	skew	kurtosis
0,000406	0,023372	0,000546	-0,33418	11,76002
0,000651	0,031899	0,001018	2,543398	18,73465
0,000224	0,030732	0,000944	1,037096	11,64504
-9,95E-06	0,022295	0,000497	-0,04167	6,072629
0,000833	0,024367	0,000594	-0,4513	5,447453
0,000271	0,035381	0,001252	-0,55022	95,22676
0,000589	0,019402	0,000376	0,12863	3,770014
0,000227	0,021178	0,000449	-0,76273	12,12068
6,63E-05	0,019296	0,000372	0,042383	6,822686
0,000295	0,027254	0,000743	-0,03407	7,081486
0,000334	0,019525	0,000381	0,023583	8,983823
0,000281	0,028823	0,000831	0,16743	7,527692
0,000638	0,020472	0,000419	-0,12316	3,54211
0,000842	0,015969	0,000255	-0,25051	8,497894
-0,0003	0,024346	0,000593	0,363006	6,996303
0,000478	0,017434	0,000304	0,129019	4,642196
-0,00041	0,036265	0,001315	0,298588	28,05753
0,000369	0,015757	0,000248	-1,12254	14,04545
0,001364	0,02535	0,000643	1,667672	12,30779
0,000454	0,016433	0,00027	-0,87812	10,62039
0,000425	0,013914	0,000194	-0,65107	8,27335
0,000258	0,015819	0,00025	-0,25663	6,837787
0,000112	0,016355	0,000267	1,726548	51,81701
0,000392	0,019772	0,000391	-0,43601	5,724191
-0,00051	0,04596	0,002112	1,15722	18,79387
0,000695	0,016222	0,000263	0,229758	10,74182
0,000447	0,020925	0,000438	-0,86005	12,01426
0,000277	0,016102	0,000259	0,909269	11,69815
0,000205	0,023874	0,00057	0,115671	5,091296
-0,00016	0,023978	0,000575	0,136715	16,89247
-0,00025	0,024316	0,000591	0,549174	16,36421
0,000145	0,018494	0,000342	0,829051	15,31522
-5,53E-05	0,017638	0,000311	0,692252	6,943695
0,000197	0,019352	0,000375	2,785542	27,30906
0,000277	0,015678	0,000246	-0,51385	12,47392
0,000417	0,016271	0,000265	-0,51338	6,121101
0,000295	0,015194	0,000231	-0,52981	6,267642
0,001136	0,023665	0,00056	0,265946	3,643639
0,00106	0,020132	0,000405	-0,39284	6,042009
0,000347	0,020832	0,000434	0,205784	3,936181
0,000682	0,026508	0,000703	5,097964	164,3416
0,000625	0,022514	0,000507	-0,38762	4,645737
0,000149	0,031716	0,001006	2,659886	22,38893
0,00084	0,019313	0,000373	-0,12307	2,629586
0,000766	0,02209	0,000488	1,072804	10,50961

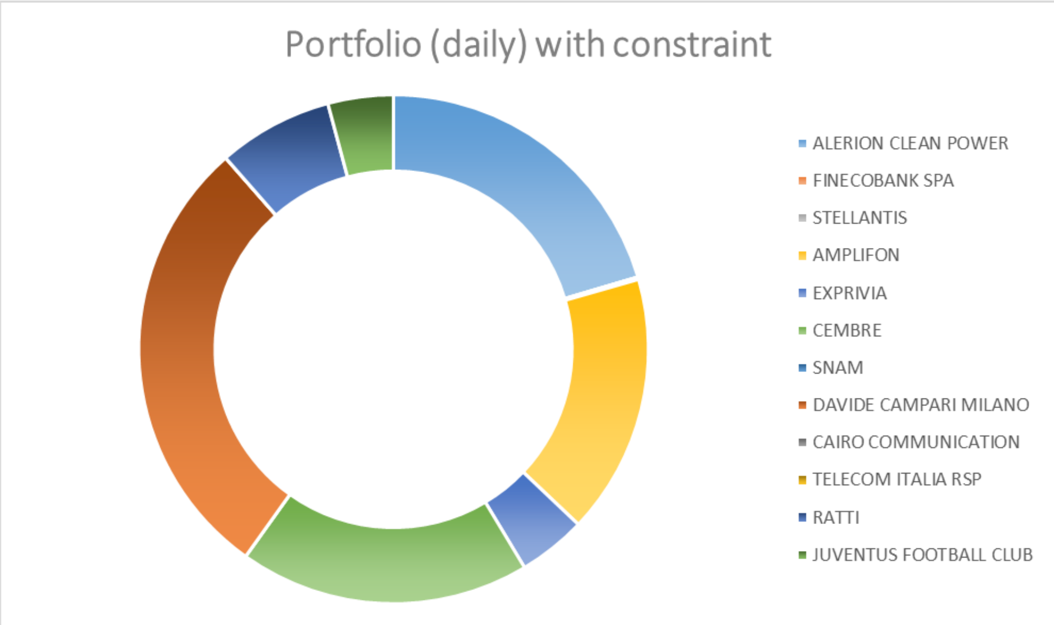
0,000155	0,015416	0,000238	1,144579	17,857
0,000353	0,017686	0,000313	-1,25548	16,60338
0,000276	0,020478	0,000419	3,326866	63,37883
0,000205	0,025356	0,000643	1,058778	11,71146
-0,00026	0,020895	0,000437	0,460281	5,103383
-0,00041	0,025273	0,000639	2,346586	22,72959
1,88E-05	0,033016	0,00109	3,060998	24,97467
0,000309	0,021331	0,000455	-0,2802	8,991718
0,000178	0,015705	0,000247	-0,73719	12,08021
0,000264	0,014664	0,000215	-1,18956	16,77614
0,000132	0,01807	0,000327	-0,96875	17,03315
4,38E-06	0,022712	0,000516	0,824777	13,36916
0,000701	0,017313	0,0003	0,045202	12,55556
0,000559	0,034195	0,001169	1,42594	9,653295
0,000168	0,023384	0,000547	0,559861	7,20517
9,34E-05	0,032786	0,001075	2,306217	18,72592
0,000783	0,0297	0,000882	1,858959	14,58101
0,000342	0,022684	0,000515	1,328083	32,1541
0,000708	0,026326	0,000693	0,257385	8,056556
0,000651	0,024904	0,00062	0,276887	9,66522
-0,00068	0,030338	0,00092	1,528531	14,39402
-0,00035	0,023067	0,000532	1,139049	10,68162
0,000385	0,02082	0,000433	0,177573	2,661309
0,000134	0,015832	0,000251	-0,17244	5,006214
0,00054	0,019822	0,000393	-0,03472	4,400473
0,000234	0,018277	0,000334	-0,0358	2,903422
0,000236	0,020711	0,000429	0,587699	9,153348
0,000589	0,017343	0,000301	2,213981	33,35556
0,000301	0,036058	0,0013	2,938235	48,01215
6,14E-05	0,024567	0,000604	-0,19183	11,0501
0,000208	0,017994	0,000324	0,382652	3,795574
0,000322	0,014251	0,000203	-0,35435	13,48908
0,000302	0,020152	0,000406	0,943686	12,83185
0,000443	0,028443	0,000809	1,228547	7,867968
-0,00015	0,023925	0,000572	1,164558	21,90285
0,000639	0,0173	0,000299	-0,14956	12,54592
0,000639	0,017872	0,000319	0,149451	3,629554
0,000278	0,018609	0,000346	0,44172	5,074727
7,12E-05	0,023431	0,000549	1,974864	14,32842
0,000773	0,017558	0,000308	0,380876	1,789594
0,000179	0,024192	0,000585	0,423539	7,429094
0,000479	0,02735	0,000748	-0,15099	5,811315
-0,00032	0,028695	0,000823	0,432306	10,20825

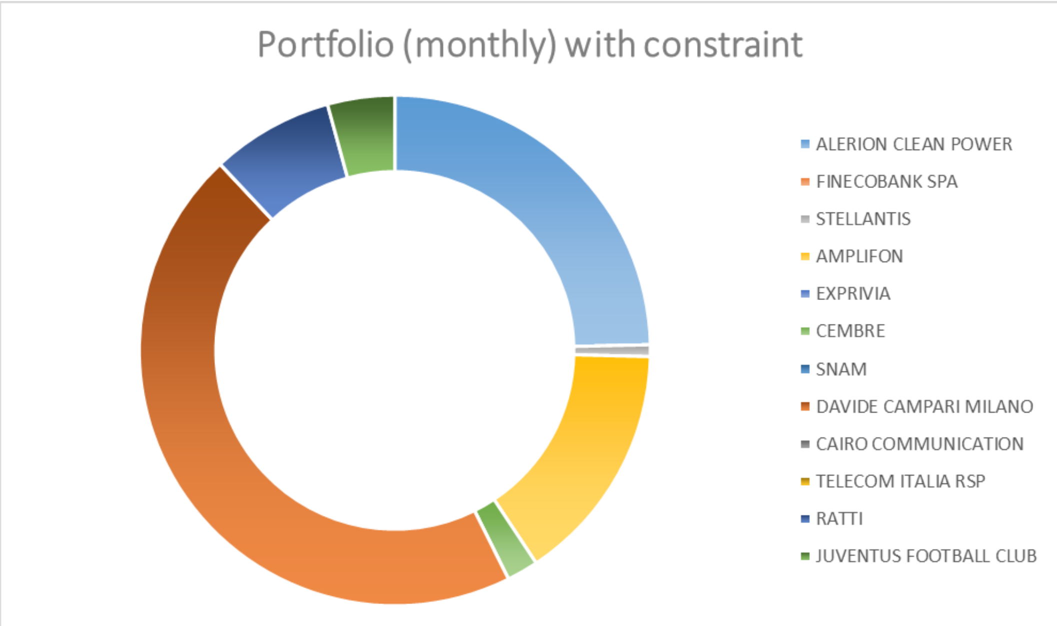
2A. Statistics of Return (monthly) for each stock

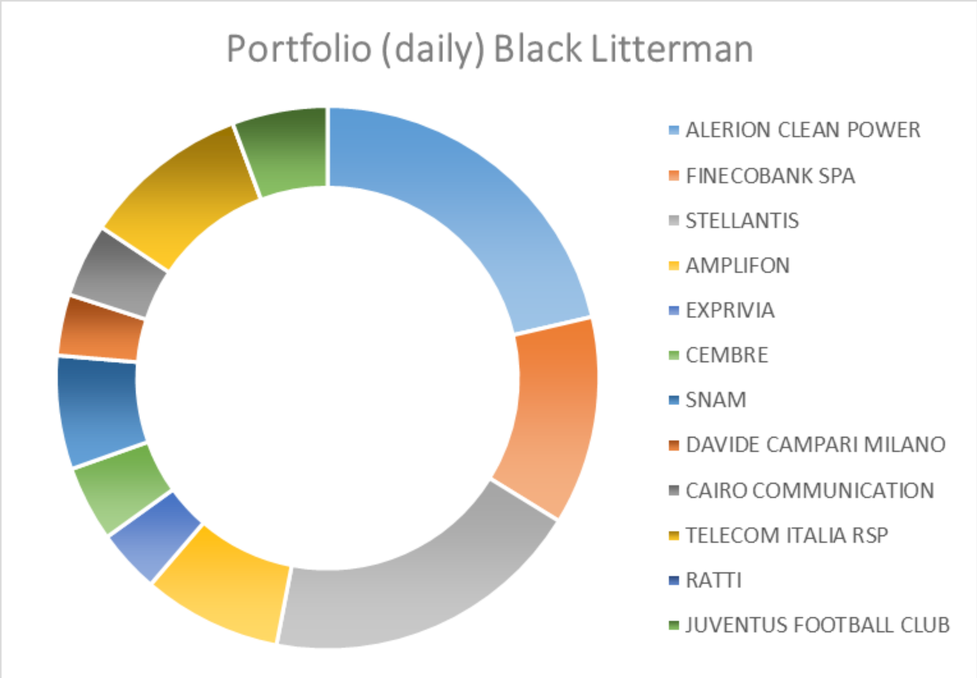
mean	std	var	skew	kurtosis
0,009081	0,1109	0,012299	0,199375	2,783641
0,025371	0,298663	0,089199	7,564373	68,29475
0,00606	0,169458	0,028716	3,573282	22,04243
-0,00164	0,094643	0,008957	-0,43078	0,917258
0,019003	0,119976	0,014394	-0,20891	0,918015
-0,00238	0,119726	0,014334	1,101458	4,288343
0,013266	0,095181	0,00906	-0,01289	-0,24261
0,004823	0,096087	0,009233	-0,39814	2,061415
-0,00182	0,094501	0,00893	-0,99607	2,691812
0,006076	0,121319	0,014718	-0,20417	1,73632
0,007513	0,090725	0,008231	-0,64957	0,928548
0,005356	0,12853	0,01652	0,400285	0,476941
0,013305	0,087323	0,007625	-0,08072	-0,02148
0,017942	0,068322	0,004668	-0,37307	0,535194
-0,00846	0,098983	0,009798	0,057502	1,350172
0,009895	0,074975	0,005621	-0,15292	0,84969
-0,00994	0,148801	0,022142	-0,41797	6,254002
0,007173	0,063889	0,004082	-0,05887	0,812603
0,029788	0,12658	0,016022	1,866077	4,568616
0,009654	0,072351	0,005235	-0,95041	2,019844
0,008326	0,047759	0,002281	-0,09213	-0,48007
0,005887	0,076749	0,00589	-0,28604	0,124453
0,002867	0,082805	0,006857	0,770655	3,264089
0,008585	0,089881	0,008079	-0,45584	2,371492
-0,01985	0,165638	0,027436	0,870033	3,336793
0,014444	0,064984	0,004223	-0,10385	-0,0629
0,009443	0,093299	0,008705	-0,71012	2,245517
0,005097	0,066716	0,004451	0,048577	2,163393
0,004633	0,110065	0,012114	0,069176	0,905163
-0,0048	0,102505	0,010507	0,699612	1,493415
-0,00643	0,108017	0,011668	1,227989	4,178957
0,001347	0,06268	0,003929	0,038799	0,941785
-0,00166	0,076645	0,005874	0,98356	2,9234
0,002494	0,070685	0,004996	1,927756	6,26645
0,005497	0,064675	0,004183	-0,57469	0,942722
0,009252	0,076865	0,005908	-0,50913	0,611437
0,00805	0,060763	0,003692	-0,20575	0,426364
0,027151	0,132083	0,017446	0,173483	0,209869
0,022418	0,08496	0,007218	-0,69364	1,41831
0,007116	0,093886	0,008815	-0,06939	0,101221
0,018038	0,168753	0,028477	4,247417	27,85373
0,012943	0,097506	0,009507	-0,20474	2,125835
0,005942	0,20138	0,040554	5,032969	35,71909
0,018713	0,09284	0,008619	-0,54751	0,111475
0,016045	0,098721	0,009746	0,939728	2,115911

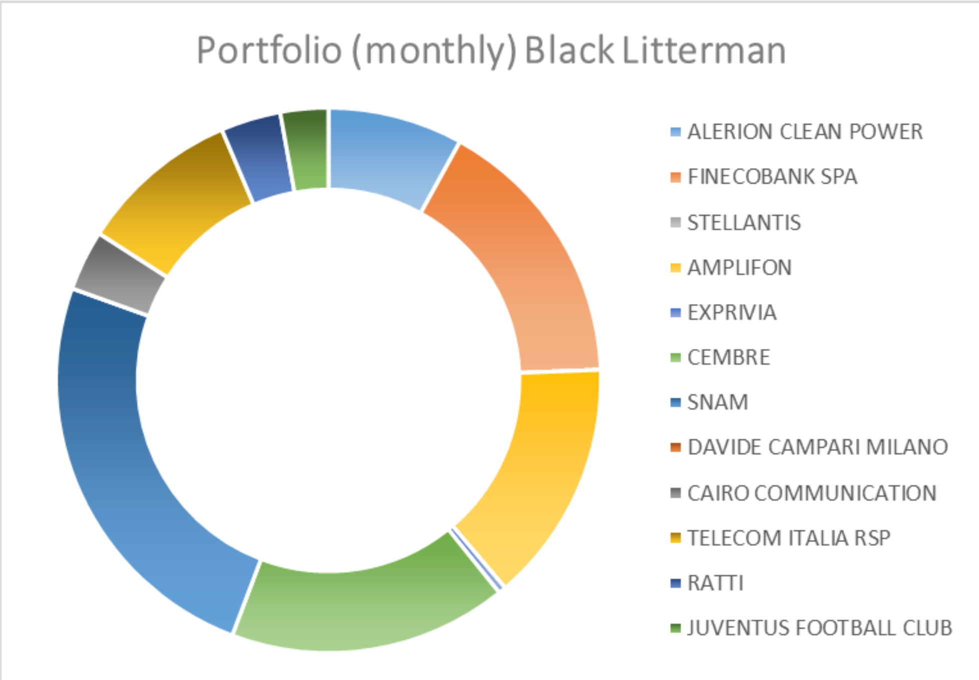
0,003043	0,066134	0,004374	-0,17023	1,828403
0,00682	0,069351	0,00481	-0,32175	0,975164
0,007661	0,113081	0,012787	1,01354	4,967156
0,004407	0,120889	0,014614	1,052472	2,637892
-0,00569	0,098192	0,009642	0,347199	0,53835
-0,01137	0,094244	0,008882	1,838077	9,004813
0,004041	0,221997	0,049283	5,491516	41,95842
0,006626	0,096729	0,009357	-0,52425	1,837072
0,003916	0,073167	0,005353	-0,19882	2,008081
0,004809	0,051882	0,002692	-0,21579	-0,43875
0,002029	0,075984	0,005774	0,638265	2,882873
-0,00034	0,110389	0,012186	1,086287	3,286347
0,014377	0,069641	0,00485	0,048517	0,866182
0,011859	0,161487	0,026078	1,436088	3,986474
0,001946	0,090795	0,008244	-0,09311	1,225609
0,002841	0,189936	0,036076	5,091598	32,3966
0,01882	0,161254	0,026003	2,376395	13,78794
0,00748	0,110513	0,012213	1,477028	10,63492
0,017767	0,149471	0,022341	1,329077	5,840026
0,015675	0,13161	0,017321	0,9431	4,599985
-0,01715	0,115394	0,013316	0,749973	2,391781
-0,01146	0,058369	0,003407	1,178698	4,901279
0,008509	0,097631	0,009532	0,67666	0,83741
0,002798	0,071705	0,005142	-0,10304	0,8109
0,01063	0,077544	0,006013	-0,12135	-0,30288
0,00464	0,080257	0,006441	0,505178	3,319207
0,004353	0,085564	0,007321	-0,11698	0,732861
0,012114	0,07338	0,005385	2,054093	9,317987
-0,0033	0,098243	0,009652	1,631835	5,719456
0,000357	0,108326	0,011734	0,762903	1,4229
0,002514	0,055275	0,003055	0,52958	1,44371
0,006412	0,05673	0,003218	0,229023	2,59577
0,003975	0,060197	0,003624	0,217624	1,62343
0,014283	0,178887	0,032001	2,1468	7,152488
-0,00376	0,120066	0,014416	3,240227	20,45101
0,012894	0,069013	0,004763	0,143738	1,593918
0,013933	0,081276	0,006606	-0,08436	3,619977
0,007045	0,096234	0,009261	0,200366	0,245683
-0,0004	0,093271	0,008699	1,415495	4,32955
0,015557	0,063613	0,004047	0,299077	-0,46098
0,003569	0,111282	0,012384	0,273132	-0,18782
0,010685	0,128054	0,016398	-0,19062	0,225876
-0,00626	0,137955	0,019032	1,0062	3,357589

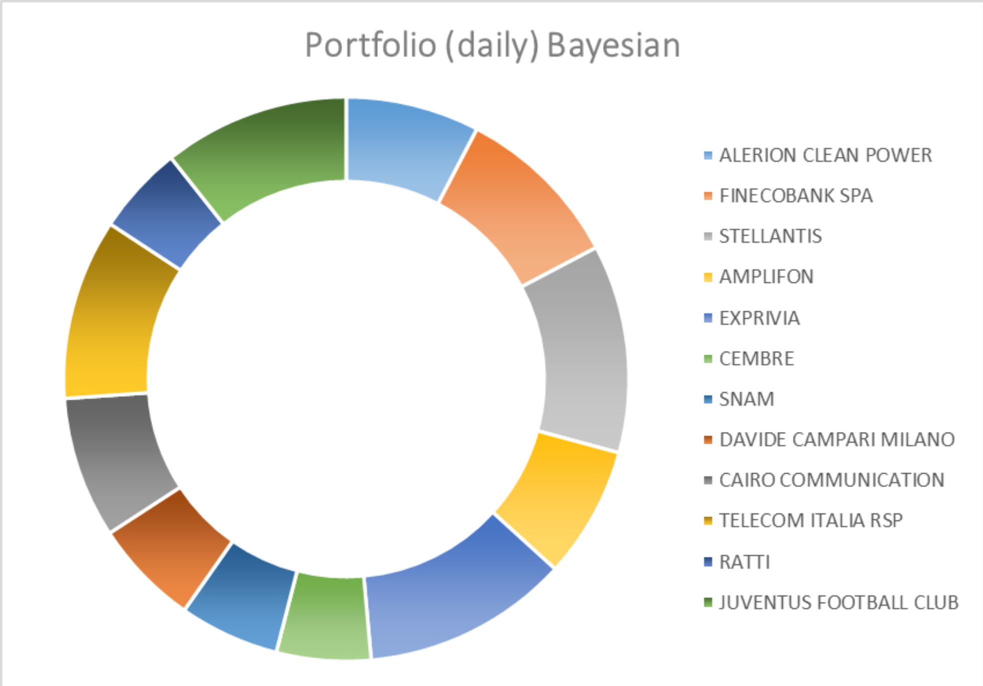
3A. Optimal weights of our portfolios (daily and monthly)

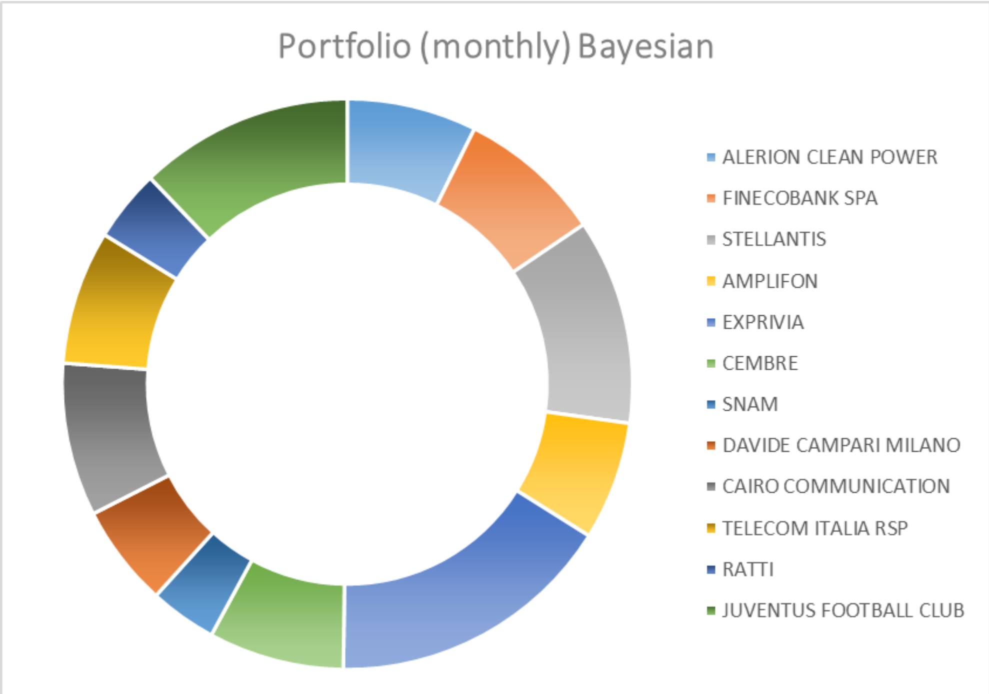


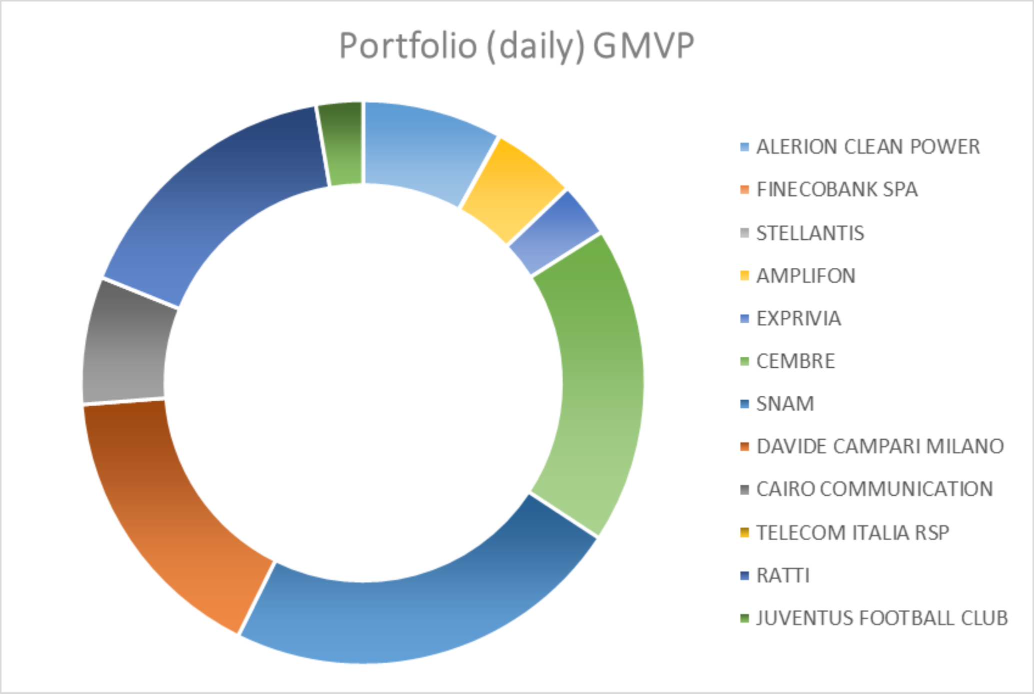



Portfolio (daily) Black Litterman
ALERION CLEAN POWER
FINECOBANK SPA
STELLANTIS
AMPLIFON
EXPRIVIA
CEMBRE
SNAM
DAVIDE CAMPARI MILANO
CAIRO COMMUNICATION
TELECOM ITALIA RSP
RATTI
JUVENTUS FOOTBALL CLUB


Portfolio (monthly) Black Litterman
ALERION CLEAN POWER
FINECOBANK SPA
STELLANTIS
AMPLIFON
EXPRIVIA
CEMBRE
SNAM
DAVIDE CAMPARI MILANO
CAIRO COMMUNICATION
TELECOM ITALIA RSP
RATTI
JUVENTUS FOOTBALL CLUB


Portfolio (daily) Bayesian
ALERION CLEAN POWER
FINECOBANK SPA
STELLANTIS
AMPLIFON
EXPRIVIA
CEMBRE
SNAM
DAVIDE CAMPARI MILANO
CAIRO COMMUNICATION
TELECOM ITALIA RSP
RATTI
JUVENTUS FOOTBALL CLUB


Portfolio (monthly) Bayesian
ALERION CLEAN POWER
FINECOBANK SPA
STELLANTIS
AMPLIFON
EXPRIVIA
CEMBRE
SNAM
DAVIDE CAMPARI MILANO
CAIRO COMMUNICATION
TELECOM ITALIA RSP
RATTI
JUVENTUS FOOTBALL CLUB


Portfolio (daily) GMVP
ALERION CLEAN POWER
FINECOBANK SPA
STELLANTIS
AMPLIFON
EXPRIVIA
CEMBRE
SNAM
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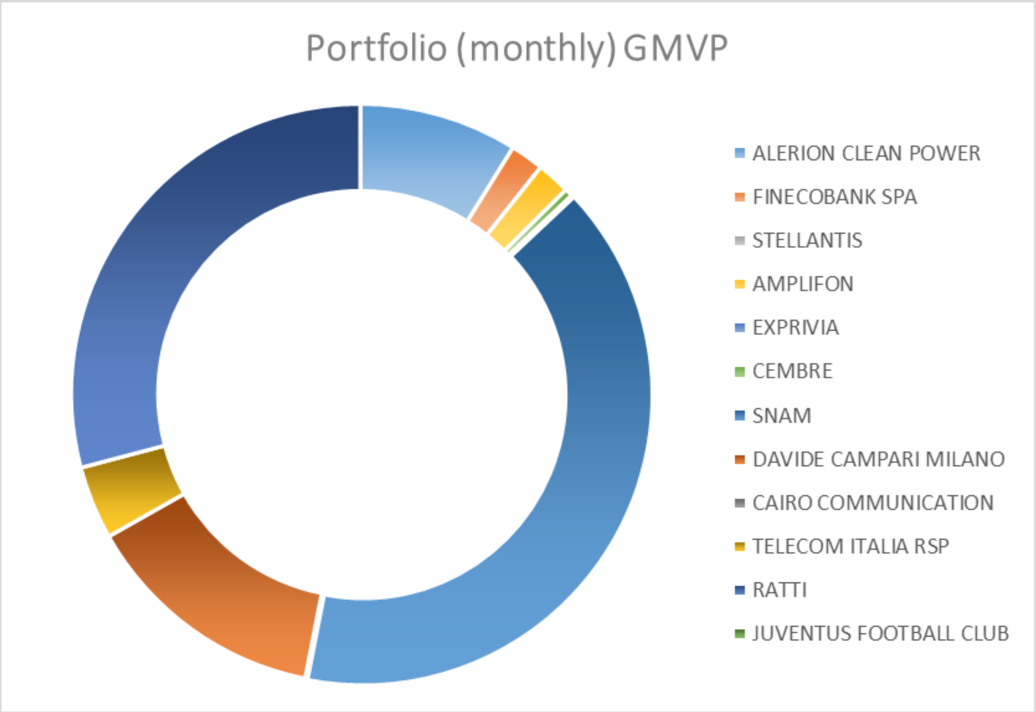
CAIRO COMMUNICATION
TELECOM ITALIA RSP
RATTI
JUVENTUS FOOTBALL CLUB


Portfolio (monthly) GMVP
ALERION CLEAN POWER
FINECOBANK SPA
STELLANTIS
AMPLIFON
EXPRIVIA
CEMBRE
SNAM
DAVIDE CAMPARI MILANO
CAIRO COMMUNICATION
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