Chapter 7

Light and Color

7.1 Introduction

Visual images are created by the patterns of light falling on the retina of the human eye. The goal of an imaging system is to use an image sensor to capture the light from a scene, to convert the sensor response to electronic form, and to store and/or transmit this data. At the end of the chain, a display device converts the electronic data back to an optical image that should appear as similar as possible to the original scene when viewed by a human viewer, i.e., patterns and colors should be accurately reproduced. In the previous chapters, image signals have simply been represented as a scalar value between 0 and 1, where this scalar value represents a level of gray ranging from black to white. In the chapter, the exact nature of the image signal is revealed for both gray-scale and color images. In the case of color images, it is shown to be a three-dimensional vector quantity. This chapter is based on the more detailed book by the author [Dubo 10]. This book can be consulted for more details and references to the literature.

7.2 Light

Light is electromagnetic radiation with wavelengths roughly in the band from 350 to 780 nm. This is the range of wavelengths to which the human eye is sensitive and has a response (although response below 400 nm and above 700 nm is very small). The study of light as radiant energy, independently of its effect on the human visual system, is *radiometry.* This radiant energy is denoted Q_e, measured in Joules (J).

In imaging, we are generally more interested in the power of light passing through a given surface (say a sensor element) at a given place and time than in total energy. There are a number of standard units that measure various forms of radiant power. *Radiant flux* (also called radiant power), denoted P_e, is the radiant energy emitted, transferred or received by a given surface per unit of time. We can write $P_e = \frac{dQ_e}{dt}$, measured in Watts (W).

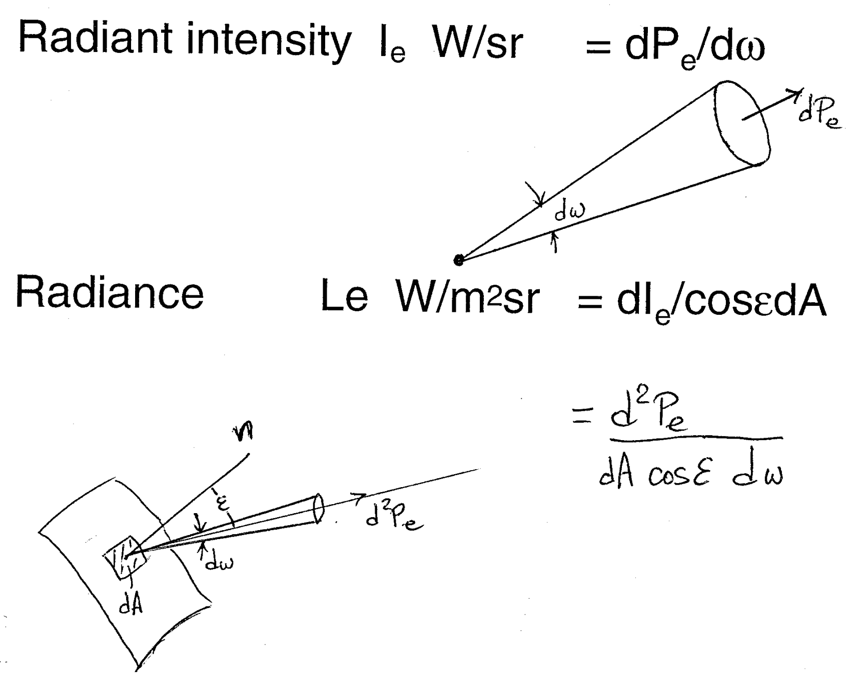

Figure 7.1: Illustration of concepts of radiant intensity and radiance.

The spectral distribution of light as a function of optical wavelength is described by the *power density spectrum* $P_e(\lambda)$, where $P_e(\lambda)\,d\lambda$ is the incremental power contained in the range of wavelength $[\lambda, \lambda + d\lambda)$. Of course power is non-negative, so $P_e(\lambda) \geq 0$. Total power is given by $P_e = \int_0^\infty P_e(\lambda)\,d\lambda$, but we often write $P_e = \int_{\lambda_{\min}}^{\lambda_{\max}} P_e(\lambda)\,d\lambda$ to limit our consideration to wavelengths in a given range of interest, e.g., 350 to 780 nm.

Radiant flux refers to an entire surface, say a whole image sensor or CRT display. We are usually interested in more localized measures. For example *irradiance* E_e measures the radiant flux per unit area falling on a surface at a particular location, $E_e = \frac{dP_e}{dA}$, measured in W/m^2. This would be relevant in measuring the relative intensity of light falling at a particular point on an image sensor. A closely related concept is *radiant emittance*, denoted $M_e = \frac{dP_e}{dA}$ in W/m^2, measuring the radiant flux per unit area emitted by a surface at a particular location. This would be relevant in measuring the relative intensity of light emitted by a CRT display at a particular point on the screen.

Two final quantities of interest are *radiant intensity* and *radiance*. Radiant intensity, denoted I_e, is the radiant flux emitted by a point source of light in a given direction, per unit of solid angle, $I_e = \frac{dP_e}{d\omega}$, measured in W/sr, where sr stands for steradian, the unit of solid angle. Radiance is the radiant flux emitted by a surface in a given direction per unit of area and solid angle. These concepts are illustrated in Fig. 7.1. All of these quantities can be represented as power density spectra as a function of λ, for example $E_e(\lambda)$ for irradiance.

Fig.7.2 illustrates some typical power density spectra of light. As we shall see, the relative power at different wavelengths determines the perceived color. Note that the power density spectrum of light from a red helium-neon laser is concentrated at a single wavelength λ_0, about 635 nm. This can be approximated by a Dirac delta function $\delta(\lambda - \lambda_0)$.

Fig. 7.3 shows the transmission as a function of wavelength, $t(\lambda)$, of a plastic filter described as 'dark lemon'. This would also be the power density spectrum of light coming out of the filter if white light with a flat spectrum were shone through

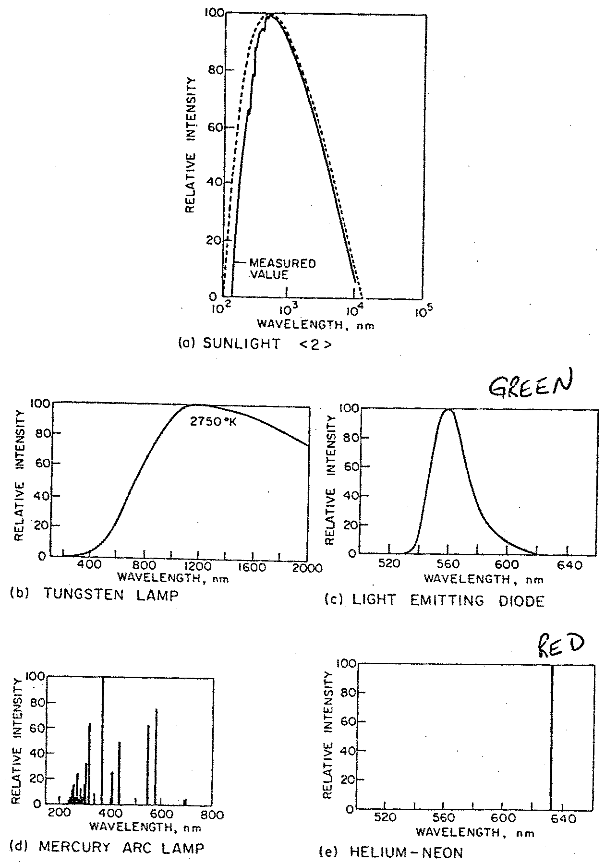

Figure 7.2: Illustration of typical power density spectra of various lights.

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7.3 Colorimetry

The color sensation perceived by a viewer is determined by the power spectral density of the light incident on the retina, $f(\lambda)$. The visual system is only sensitive to electromagnetic radiation in a certain range $\mathcal{V} = (\lambda_{\min}, \lambda_{\max})$, where $\lambda_{\min} \approx 370$nm and $\lambda_{\max} \approx 730$nm. We denote the set of all non-negative finite-energy spectral

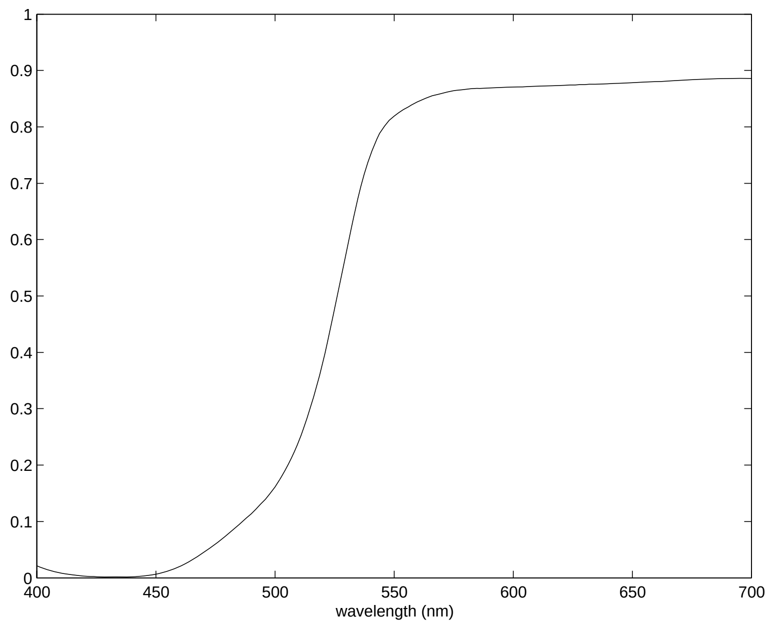

Figure 7.3: Spectral transmission of dark lemon plastic filter.

densities on this range by

$$\mathcal{P} = \{f(\lambda), \lambda_{\min} \leq \lambda \leq \lambda_{\max} \mid f(\lambda) \geq 0, \int_{\lambda_{\min}}^{\lambda_{\max}} |f(\lambda)| \, d\lambda < \infty\}. \tag{7.1}$$

This is a subset of the vector space $L^1(\mathcal{V})$ of absolutely-integrable functions on $\mathcal{V}$:

$$L^1(\mathcal{V}) = \{f(\lambda), \lambda_{\min} \leq \lambda \leq \lambda_{\max} \mid \int_{\lambda_{\min}}^{\lambda_{\max}} |f(\lambda)| \, d\lambda < \infty\}. \tag{7.2}$$

It is a *psychophysical* observation that different spectral densities can give rise to identical color sensations in a human viewer. There are physiological explanations for this, but the theory that follows is based strictly on psychophysical observations. If two spectral densities $C_1(\lambda)$ and $C_2(\lambda)$ produce an identical color sensation in a human viewer under the same viewing conditions, they are said to form a *metameric pair*. We denote this

$$C_1(\lambda) \triangleq C_2(\lambda). \tag{7.3}$$

Empirical psychophysical observations about such metameric equivalence allows us to establish a strong mathematical structure for color measurement. We will present these observations as needed and develop the consequences. These properties are sometimes called Grassmann's laws.

7.3.1 Properties of metamerism

Metamerism, as expressed by equation (7.3), is a *relation* on the set $\mathcal{P}$. The first empirical observation is transitivity:

Transitivity (G1). If $C_1(\lambda) \triangleq C_2(\lambda)$ and $C_2(\lambda) \triangleq C_3(\lambda)$, then $C_1(\lambda) \triangleq C_3(\lambda)$.

If this is coupled with the obvious facts that $C(\lambda) \triangleq C(\lambda)$ and that $C_1(\lambda) \triangleq C_2(\lambda)$ implies $C_2(\lambda) \triangleq C_1(\lambda)$, we conclude that metamerism is an equivalence relation on $\mathcal{P}$ (see Appendix 7.A).

For any $C(\lambda) \in \mathcal{P}$, we denote

$$[C] = \{C_1(\lambda) \in \mathcal{P} \mid C_1(\lambda) \triangleq C(\lambda)\}, \tag{7.4}$$

the set of all spectral densities metamerically equivalent to $C(\lambda)$; this is an equivalence class. Metameric classes are either identical or disjoint, and form a partition of $\mathcal{P}$ (Appendix 7.A). We refer to $[C]$ as a *color*. Sometimes we write $[C(\lambda)]$ to indicate the metameric equivalence class containing the spectral density $C(\lambda)$. Equality of colors is denoted with an ordinary = sign, $[C_1] = [C_2]$, meaning that the equivalence classes are identical.

The following two empirical facts show that we can scale and add colors.

Scaling (G2). If $C_1(\lambda) \triangleq C_2(\lambda)$ then $\alpha C_1(\lambda) \triangleq \alpha C_2(\lambda)$ for any real, nonnegative α.

Property G2 means that if $[C] = \{C_1(\lambda) \in \mathcal{P} \mid C_1(\lambda) \triangleq C(\lambda)\}$ then $[\alpha C] = \{\alpha C_1(\lambda) \in \mathcal{P} \mid C_1(\lambda) \triangleq C(\lambda)\}$. We denote this color $\alpha[C]$ without ambiguity.

Addition (G3). $C(\lambda) + C_1(\lambda) \triangleq C(\lambda) + C_2(\lambda)$ for arbitrary $C(\lambda)$ if and only if $C_1(\lambda) \triangleq C_2(\lambda)$.

Corollary to G3. *If* $C_1(\lambda) \triangleq C_2(\lambda)$ *then* $C_1(\lambda) + C_3(\lambda) \triangleq C_2(\lambda) + C_4(\lambda)$ *if and* *only if* $C_3(\lambda) \triangleq C_4(\lambda)$.

Proof. Apply (G3) twice and use transitivity. By (G3), $C_1(\lambda) + C_3(\lambda) \triangleq C_2(\lambda) + C_3(\lambda)$, and again by (G3), $C_2(\lambda) + C_3(\lambda) \triangleq C_2(\lambda) + C_4(\lambda)$ if and only if $C_3(\lambda) \triangleq C_4(\lambda)$. Transitivity (G1) establishes the result. □

Property (G3) allows us to define addition of colors. Let $[C_1]$ and $[C_2]$ be two colors. Define the sum of $[C_1]$ and $C_2]$ to be

$$[C_1] + [C_2] = [C_1(\lambda) + C_2(\lambda)] \tag{7.5}$$

where $C_1(\lambda)$ is any element of the class $[C_1]$ and $C_2(\lambda)$ is any element of the class $[C_2]$. This assignment is well-defined if it is independent of the particular choices $C_1(\lambda)$ and $C_2(\lambda)$ within $[C_1]$ and $[C_2]$ respectively. Indeed, let $C_{11}(\lambda)$ and $C_{12}(\lambda)$ be two arbitrary elements of $[C_1]$ and $C_{21}(\lambda)$ and $C_{22}(\lambda)$ be two arbitrary elements of $[C_2]$. Then

$$\begin{aligned} C_{11}(\lambda) + C_{21}(\lambda) &\triangleq C_{11}(\lambda) + C_{22}(\lambda) \\ &\triangleq C_{12}(\lambda) + C_{22}(\lambda) \end{aligned}$$

by applying (G3) twice. Thus addition of colors as defined by equation (7.5) is well defined.

7.3.2 Extension of metameric properties to $L^1(\mathcal{V})$

Since we have both multiplication by a real scalar and addition of colors, it is tempting to identify the color space as a vector space. However, we are missing a crucial property of vector spaces, namely that every element must have a negative in the space, since $-C(\lambda)$ is not an admissible spectral density. Thus, we now extend the definition of equivalence to the entire vector space $L^1(\mathcal{V})$. Any element $C(\lambda)$ of $L^1(\mathcal{V})$ can be written as $C_a(\lambda) - C_b(\lambda)$ where $C_a(\lambda), C_b(\lambda) \in \mathcal{P}$; this representation is not unique. Intuitively, based on how color matches are carried out in practice, we would like to say that two arbitrary elements $C_1(\lambda)$ and $C_2(\lambda)$ of $L^1(\mathcal{V})$ are equivalent if and only if

$$C_{1a}(\lambda) + C_{2b}(\lambda) \triangleq C_{2a}(\lambda) + C_{1b}(\lambda), \tag{7.6}$$

where since both $C_{1a}(\lambda) + C_{2b}(\lambda)$ and $C_{2a}(\lambda) + C_{1b}(\lambda)$ are elements of $\mathcal{P}$, such metameric equivalence is meaningful. However, this would have to hold no matter how $C_1(\lambda)$ and $C_2(\lambda)$ are decomposed as the difference of elements of $\mathcal{P}$. We introduce a specific decomposition and show that it extends metameric equivalence to $L^1(\mathcal{V})$ and then show that the property of equation (7.6) holds for any decomposition.

For an arbitrary $C(\lambda) \in L^1(\mathcal{V})$, define the following elements of $\mathcal{P}$:

$$C^+(\lambda) = \begin{cases} C(\lambda) & \text{if } C(\lambda) \geq 0, \\ 0 & \text{if } C(\lambda) < 0. \end{cases}$$

$$C^-(\lambda) = \begin{cases} 0 & \text{if } C(\lambda) \geq 0, \\ -C(\lambda) & \text{if } C(\lambda) < 0. \end{cases}$$

Both $C^+(\lambda) \in \mathcal{P}$ and $C^-(\lambda) \in \mathcal{P}$, and $C(\lambda) = C^+(\lambda) - C^-(\lambda)$. We now define the following relation on $L^1(\mathcal{V})$:

$$C_1(\lambda) \boxed{=} C_2(\lambda) \quad \text{if and only if} \quad C_1^+(\lambda) + C_2^-(\lambda) \triangleq C_2^+(\lambda) + C_1^-(\lambda). \tag{7.7}$$

Since both $C_1^+(\lambda) + C_2^-(\lambda)$ and $C_2^+(\lambda) + C_1^-(\lambda)$ are elements of $\mathcal{P}$, we can test for metameric equivalence. Note that if $C_1(\lambda) \in \mathcal{P}$ and $C_2(\lambda) \in \mathcal{P}$, then $C_1(\lambda) \boxed{=} C_2(\lambda)$ if and only if $C_1(\lambda) \triangleq C_2(\lambda)$, so that this relation is an extension of metameric equivalence on $\mathcal{P}$ to a relation on $L^1(\mathcal{V})$. In fact, we will show that $\boxed{=}$ is also an equivalence relation.

Proposition 7.1. The relation $\boxed{=}$ on $L^1(\mathcal{V})$ is an equivalence relation.

Proof. We must show that reflexivity, symmetry and transitivity of $\triangleq$ imply reflexivity, symmetry and transitivity of $\boxed{=}$. See Appendix 7.B for the proof. □

To turn the set of equivalence classes into a vector space, we must also verify that properties G2 and G3 continue to hold for the relation $\boxed{=}$.

Theorem 7.1. If $C_1(\lambda) \boxed{=} C_2(\lambda)$, then $\alpha C_1(\lambda) \boxed{=} \alpha C_2(\lambda), \forall \alpha \in \mathbb{R}$ (G2′)

Proof. See Appendix 7.B. $\square$

Theorem 7.2. $C(\lambda)+C_1(\lambda) \mathrel{\boxed{=}} C(\lambda)+C_2(\lambda)$ for arbitrary $C(\lambda) \in L^2(\mathcal{V})$ if and only if $C_1(\lambda) \mathrel{\boxed{=}} C_2(\lambda)$ (G3′).

This is not an obvious result since the regions where $C(\lambda)+C_1(\lambda)$ is positive and negative do not coincide with those of $C(\lambda)$ or $C_1(\lambda)$, and similarly for $C(\lambda)+C_2(\lambda)$. We first establish the following simpler results.

Proposition 7.2. Let $C_1(\lambda), C_2(\lambda) \in \mathcal{P}$. Then $C_1(\lambda) - C_2(\lambda) \mathrel{\boxed{=}} 0$ if and only if $C_1(\lambda) \triangleq C_2(\lambda)$.

Proof. See Appendix 7.B. $\square$

Proposition 7.3. Let $C_1(\lambda), C_2(\lambda) \in L^2(\mathcal{V})$. Then

$$C_1(\lambda) \mathrel{\boxed{=}} C_2(\lambda) \quad \text{if and only if} \quad C_1(\lambda) - C_2(\lambda) \mathrel{\boxed{=}} 0$$

Proof. By Proposition 7.2, $C_1^+(\lambda)+C_2^-(\lambda) \triangleq C_2^+(\lambda)+C_1^-(\lambda)$ if and only if $(C_1^+(\lambda)+C_2^-(\lambda)) - (C_2^+(\lambda)+C_1^-(\lambda)) \mathrel{\boxed{=}} 0$, i.e., $C_1(\lambda) - C_2(\lambda) \mathrel{\boxed{=}} 0$. $\square$

Corollary to Proposition 3: $C_1(\lambda) \mathrel{\boxed{=}} C_2(\lambda)$ *if and only if* $C_1(\lambda) = C_2(\lambda) + C_0(\lambda)$ *where* $C_0(\lambda) \mathrel{\boxed{=}} 0$.

Proof. $C_1(\lambda) - C_2(\lambda) \mathrel{\boxed{=}} 0$ means $C_1(\lambda) - C_2(\lambda) \in [0]$. Thus, denoting $C_0(\lambda) = C_1(\lambda) - C_2(\lambda)$, the result follows. $\square$

With these propositions, the proof of Theorem 7.2 is now straightforward.
Proof of Theorem 7.2: By proposition 7.3, $C(\lambda)+C_1(\lambda) \mathrel{\boxed{=}} C(\lambda)+C_2(\lambda)$ if and only if $(C(\lambda)+C_1(\lambda)) - (C(\lambda)+C_2(\lambda)) \mathrel{\boxed{=}} 0$, i.e., $C_1(\lambda) - C_2(\lambda) \mathrel{\boxed{=}} 0$, which again by Proposition 7.3 occurs if and only if $C_1(\lambda) \mathrel{\boxed{=}} C_2(\lambda)$. $\blacksquare$

We can also use Proposition 7.3 to show that the extended definition of equivalence satisfies the intuitive property of equation (7.6).

Theorem 7.3. Suppose that $C_1(\lambda) = C_{1a}(\lambda) - C_{1b}(\lambda)$ and $C_2(\lambda) = C_{2a}(\lambda) - C_{2b}(\lambda)$ where $C_{1a}(\lambda), C_{1b}(\lambda), C_{2a}(\lambda), C_{2b}(\lambda) \in \mathcal{P}$. Then $C_1(\lambda) \mathrel{\boxed{=}} C_2(\lambda)$ if and only if $C_{1a}(\lambda) + C_{2b}(\lambda) \triangleq C_{2a}(\lambda) + C_{1b}(\lambda)$.

Proof. By Proposition 7.3 $C_1(\lambda) \boxminus C_2(\lambda)$ if and only if $(C_{1a}(\lambda) - C_{1b}(\lambda)) - (C_{2a}(\lambda) - C_{2b}(\lambda)) \boxminus 0$, or equivalently, if and only if $(C_{1a}(\lambda) + C_{2b}(\lambda)) - (C_{2a}(\lambda) + C_{1b}(\lambda)) \boxminus 0$. Applying Proposition 7.3 again, this hold if and only if $C_{1a}(\lambda) + C_{2b}(\lambda) \boxminus C_{2a}(\lambda) + C_{1b}(\lambda)$. But since both $C_{1a}(\lambda) + C_{2b}(\lambda)$ and $C_{2a}(\lambda) + C_{1b}(\lambda)$ are elements of $\mathcal{P}$, this is true if and only if $C_{1a}(\lambda) + C_{2b}(\lambda) \triangleq C_{2a}(\lambda) + C_{1b}(\lambda)$. □

Another way to view Proposition 7.3 is

$$C_1(\lambda) \boxminus C_2(\lambda) \qquad \text{if and only if} \qquad C_1(\lambda) = C_2(\lambda) + C_0(\lambda)$$

where $C_0(\lambda) \boxminus 0$, i.e., $C_0(\lambda) \in [0]$, and so

$$[C(\lambda)] = \{C(\lambda) + C_0(\lambda) \mid C_0(\lambda) \in [0]\}. \tag{7.8}$$

Now $[0]$ is a subspace of the vector space $L^2(\mathcal{V})$, since if $C(\lambda) \in [0]$, i.e., $C(\lambda) \boxminus 0$, then $\alpha C(\lambda) \boxminus [0], \quad \forall \alpha \in \mathbb{R}$ by Theorem 7.1, and if $C_1(\lambda), C_2(\lambda) \in [0]$ then $C_1(\lambda) + C_2(\lambda) \in [0]$ (apply Theorem 7.2 twice). Thus $[C(\lambda)]$ is a coset of the subspace $[0]$ in $L^2(\mathcal{V})$. It follows that our color space, the set of equivalence classes of $\boxminus$ in $L^2(\mathcal{V})$ is a *vector space*, denoted $\mathcal{C}$. By Theorem 7.2, addition of equivalence classes is a well-defined operation.

We define a *physical color* $[C]$ to be an element of the color space such that the corresponding equivalence class $\{C_1(\lambda) \mid C_1(\lambda) \boxminus C(\lambda)\}$ contains at least one element of $\mathcal{P}$, i.e., $[C] \cap \mathcal{P} \neq \emptyset$. The set of physical colors forms a *subset* of $\mathcal{C}$ (not a subspace) denoted $\mathcal{C}_P$.

Theorem 7.4. The set of physical colors $\mathcal{C}_P$ is a convex subset of $\mathcal{C}$.

Proof. Let $[C_1], [C_2] \in \mathcal{C}_P$. We need to show that $\alpha[C_1] + (1-\alpha)[C_2] \in \mathcal{C}_P$ for all $0 \leq \alpha \leq 1$. Let $C_1(\lambda)$ and $C_2(\lambda)$ be elements of the equivalence classes $[C_1]$ and $[C_2]$ that are members of $\mathcal{P}$. It follows that $\alpha C_1(\lambda) + (1-\alpha) C_2(\lambda) \in \mathcal{P}$ for any $\alpha \in [0,1]$, and since it is a member of the equivalence class $\alpha[C_1] + (1-\alpha)[C_2]$, it follows that $\alpha[C_1] + (1-\alpha)[C_2] \in \mathcal{C}_P$. □

7.3.3 Bases for the vector space $\mathcal{C}$

The next empirical observation allows us to establish the dimension of the vector space $\mathcal{C}$.

Dimension (G4). Four colors are always linearly dependent. For $C(\lambda), C_1(\lambda), C_2(\lambda), C_3(\lambda)$ arbitrary elements of $\mathcal{P}$, there always exist $\alpha \geq 0$ and $\alpha_1, \alpha_2, \alpha_3$, not all zero, such that

$$\alpha C(\lambda) + \sum_{i \in A} \alpha_i C_i(\lambda) \triangleq \sum_{i \in \overline{A}} \alpha_i C_i(\lambda)$$

where $A \subset \{1, 2, 3\}$ and $\overline{A} = \{1, 2, 3\} \backslash A$. This is *not* the case for three colors.

In other words, either $C(\lambda)$ matches a weighted combination of the three $C_i(\lambda)$, or a weighted combination of $C(\lambda)$ and one of the $C_i(\lambda)$ matches a weighted combination of the other two, or a weighted combination of $C(\lambda)$ and two of the $C_i(\lambda)$ matches the third. This result allows us to determine the dimension of the color space $\mathcal{C}$.

Theorem 7.5. The dimension of $\mathcal{C}$ is 3.

Proof. Using G4 and previous results, we must show that we can find three linearly independent colors that span $\mathcal{C}$. The proof is in Appendix 7.B. □

Three linearly independent colors $[P_1]$, $[P_2]$ and $[P_3]$ form a basis for $\mathcal{C}$ and are referred to as *primaries*. By convention, the intensities of primaries are chosen so that the sum of one unit of each matches a selected reference white, e.g., equal energy white. Thus, if $[P_1']$, $[P_2']$ and $[P_3']$ are linearly independent elements of $\mathcal{C}$ and $[W]$ is the reference white, we would get the unique representation of $[W]$

$$[W] = W_1[P_1'] + W_2[P_2'] + W_3[P_3'].$$

We then choose as primaries $[P_i] = W_i[P_i']$, $i = 1, 2, 3$. An example of a set of primaries would be the equivalence classes containing the spectral densities of the light emitted by the red, green and blue phosphors used in television displays. There is no requirement for the $[P_i]$ to be physical colors, and in fact they are not for several widely used color systems.

The coefficients in the expansion of a color with respect to a set of primaries are called *tristimulus values*. For any $[C] \in \mathcal{C}$ we have

$$[C] = C_1[P_1] + C_2[P_2] + C_3[P_3] \tag{7.9}$$

where the C_i are tristimulus values. We generally use uppercase letters with an appropriate subscript related to the primaries to denote tristimulus values. Any color can be represented by a triple of tristimulus values with respect to a set of primaries and thus could be plotted on a three-dimensional set of coordinate axes. It should be emphasized that in such a diagram, usual Cartesian concepts such as distance and orthogonality are not meaningful.

7.3.4 Computation of tristimulus values

Given an arbitrary spectral density $C(\lambda) \in \mathcal{P}$, we would like to compute the tristimulus values of its metameric class with respect to a given set of primaries $[P_1]$, $[P_2]$ and $[P_3]$. As in linear system theory, we can write

$$C(\lambda) = \int C(\mu)\delta(\lambda - \mu)\, d\mu \tag{7.10}$$

where $\delta(\cdot)$ is the Dirac delta function. Let $[\delta(\lambda - \mu)]$ be the color corresponding to a narrowband monochromatic light at wavelength μ; it is referred to as a *pure spectrum color*. Of course in practice we would use an approximation to the ideal delta function. For each μ, assume that we can determine the tristimulus values for this spectrum color:

$$[\delta(\lambda - \mu)] = \bar{p}_1(\mu)[P_1] + \bar{p}_2(\mu)[P_2] + \bar{p}_3(\mu)[P_3], \tag{7.11}$$

or equivalently

$$\delta(\lambda - \mu) \boxminus \bar{p}_1(\mu)P_1(\lambda) + \bar{p}_2(\mu)P_2(\lambda) + \bar{p}_3(\mu)P_3(\lambda), \tag{7.12}$$

where $P_i(\lambda)$ is an arbitrary element of $[P_i]$ for $i = 1, 2, 3$. Applying (G2′) and (G3′), we get

$$\begin{aligned} C(\lambda) &\boxed{=} \int_{\lambda_{\min}}^{\lambda_{\max}} C(\mu)\,(\bar{p}_1(\mu)P_1(\lambda) + \bar{p}_2(\mu)P_2(\lambda) + \bar{p}_3(\mu)P_3(\lambda))\; d\mu \\ &\boxed{=} \left(\int_{\lambda_{\min}}^{\lambda_{\max}} C(\mu)\bar{p}_1(\mu)\, d\mu\right) P_1(\lambda) + \left(\int_{\lambda_{\min}}^{\lambda_{\max}} C(\mu)\bar{p}_2(\mu)\, d\mu\right) P_2(\lambda) \\ &\quad + \left(\int_{\lambda_{\min}}^{\lambda_{\max}} C(\mu)\bar{p}_3(\mu)\, d\mu\right) P_3(\lambda). \end{aligned}$$

It follows that

$$C_i = \int_{\lambda_{\min}}^{\lambda_{\max}} C(\lambda)\bar{p}_i(\lambda)\, d\lambda. \tag{7.13}$$

The $\bar{p}_i(\lambda)$, as functions of λ, are referred to as color matching functions. Note that

$$\int P_j(\lambda)\bar{p}_i(\lambda)\, d\lambda = \delta_{ij}. \tag{7.14}$$

Color matching functions were determined by the CIE (Commission International d'Eclairage) in 1931 for the primaries given by

$$\begin{aligned} B(\lambda) &= \delta(\lambda - 435.8) \\ G(\lambda) &= \delta(\lambda - 546.1) \\ R(\lambda) &= \delta(\lambda - 700.0) \end{aligned}$$

with the reference white being equal energy white. The resulting color matching functions are referred to as $\bar{b}(\lambda)$, $\bar{g}(\lambda)$ and $\bar{r}(\lambda)$, see Fig. 7.4. Fortunately, this experiment only needs to be done for one set of primaries. Color matching functions for any other set of primaries can be determined by a change of basis operation, as shown in the next section.

7.3.5 Transformation of primaries

Since primaries form a basis for the color vector space, the change of representation from one set of primaries to another is a change of basis operation. Let $[P_1]$, $[P_2]$,

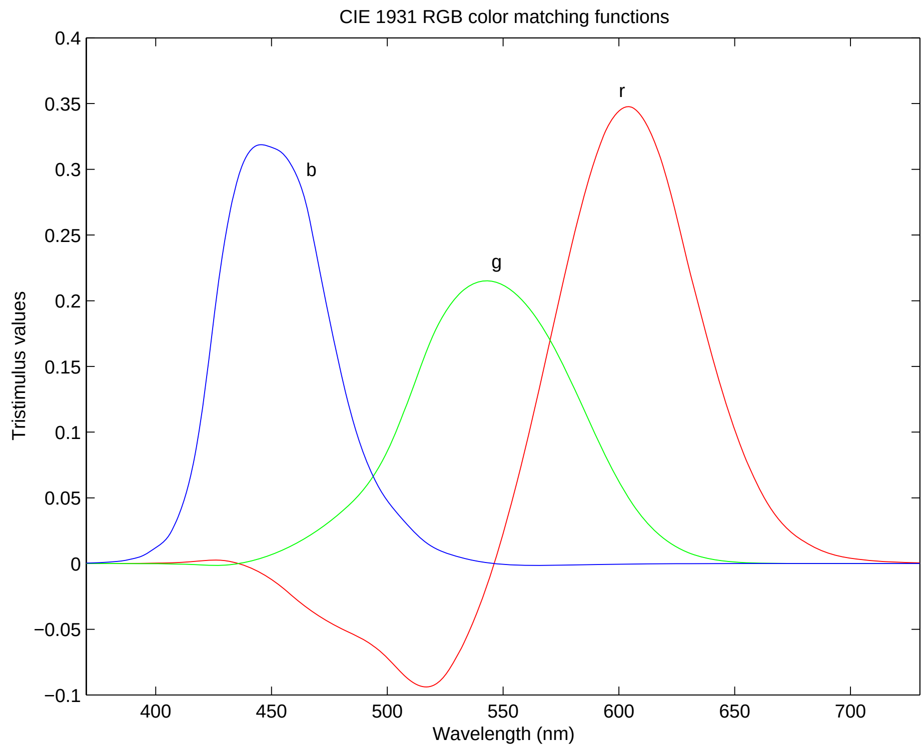

Figure 7.4: Color matching functions of the CIE 1931 RGB primaries, monochromatic lights at wavelengths 700.0 nm, 546.1 nm and 435.8 nm respectively.

$[P_3]$ be a set of primaries, and $[\tilde{P}_1]$, $[\tilde{P}_2]$, $[\tilde{P}_3]$ a different set. We can express the primaries $[P_j]$ in terms of the $[\tilde{P}_k]$ as

$$[P_j] = \sum_{k=1}^{3} a_{kj}[\tilde{P}_k], \quad j = 1, 2, 3. \tag{7.15}$$

This allows us to transform the representations of arbitrary colors. Specifically

$$\begin{aligned}
[C] &= \sum_{j=1}^{3} C_j [P_j] \\
&= \sum_{j=1}^{3} C_j \sum_{k=1}^{3} a_{kj} [\tilde{P}_k] \\
&= \sum_{k=1}^{3} \left(\sum_{j=1}^{3} C_j a_{kj} \right) [\tilde{P}_k],
\end{aligned}$$

from which we identify

$$\tilde{C}_k = \sum_{j=1}^{3} C_j a_{kj}, \qquad k = 1, 2, 3. \tag{7.16}$$

This can be written in matrix form as

$$\begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \tag{7.17}$$

or $\tilde{\mathbf{C}} = \mathbf{AC}$. Furthermore, it is clear that $\mathbf{C} = \mathbf{A}^{-1}\tilde{\mathbf{C}}$.

This matrix operation can also be used to transform the color matching functions for the primaries $[P_j]$ to color matching functions for the primaries $[\tilde{P}_k]$, recognizing that color matching functions specify tristimulus values for each λ:

$$\begin{bmatrix} \bar{\tilde{p}}_1(\lambda) \\ \bar{\tilde{p}}_2(\lambda) \\ \bar{\tilde{p}}_3(\lambda) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \bar{p}_1(\lambda) \\ \bar{p}_2(\lambda) \\ \bar{p}_3(\lambda) \end{bmatrix}. \tag{7.18}$$

The relationship between primaries can also be written in matrix form as

$$\begin{bmatrix} [P_1] \\ [P_2] \\ [P_3] \end{bmatrix} = \mathbf{A}^T \begin{bmatrix} [\tilde{P}_1] \\ [\tilde{P}_2] \\ [\tilde{P}_3] \end{bmatrix}. \tag{7.19}$$

There are many sets of primaries used in practice in various contexts. The most important is the CIE 1931 XYZ primaries, since this is the basic set of primaries in terms of which all other sets of primaries are usually defined. These primaries were initially defined such that all the color-matching functions were positive everywhere and so that the Y tristimulus value was equal to the luminance (see Section 7.4). As a result the Y color-matching function is equal to the relative luminous efficiency function $V(\lambda)$ of photometry. Another consequence is that the resulting primaries are not physical colors. If we denote the RGB primaries of Fig. 7.4 as $[R_{31}]$, $[G_{31}]$ and $[B_{31}]$ (the subscript referring to 1931), the XYZ primaries are defined by the matrix equation

$$\begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix} = \begin{bmatrix} .4184 & -.0912 & .0009 \\ -.1587 & .2524 & -.0026 \\ -.0828 & .0157 & .1786 \end{bmatrix} \begin{bmatrix} [R_{31}] \\ [G_{31}] \\ [B_{31}] \end{bmatrix}. \tag{7.20}$$

If $[C]$ is an arbitrary color which can be represented by

$$[C] = C_R[R_{31}] + C_G[G_{31}] + C_B[B_{31}] = C_X[X] + C_Y[Y] + C_Z[Z] \tag{7.21}$$

then we convert between the two sets of primaries by

$$\begin{bmatrix} C_R \\ C_G \\ C_B \end{bmatrix} = \begin{bmatrix} .4184 & -.1587 & -.0828 \\ -.0912 & .2524 & .0157 \\ .0009 & -.0026 & .1786 \end{bmatrix} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \tag{7.22}$$

and inversely,

$$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} = \begin{bmatrix} 2.769 & 1.752 & 1.130 \\ 1.000 & 4.591 & 0.060 \\ 0.000 & 0.057 & 5.593 \end{bmatrix} \begin{bmatrix} C_R \\ C_G \\ C_B \end{bmatrix}. \tag{7.23}$$

We can apply this at each λ to obtain the XYZ color matching functions from the RGB color-matching functions of Fig. 7.5. The result is illustrated in Fig. 7.5.

Based on the above discussions, we can immediately identify four methods that are frequently used to identify a set of primaries $[P_i]$, $i = 1, 2, 3$.

1. The representation of each of the primaries as a linear combination of the XYZ primaries is specified directly, as in equation (7.19), where $[X]$, $[Y]$ and

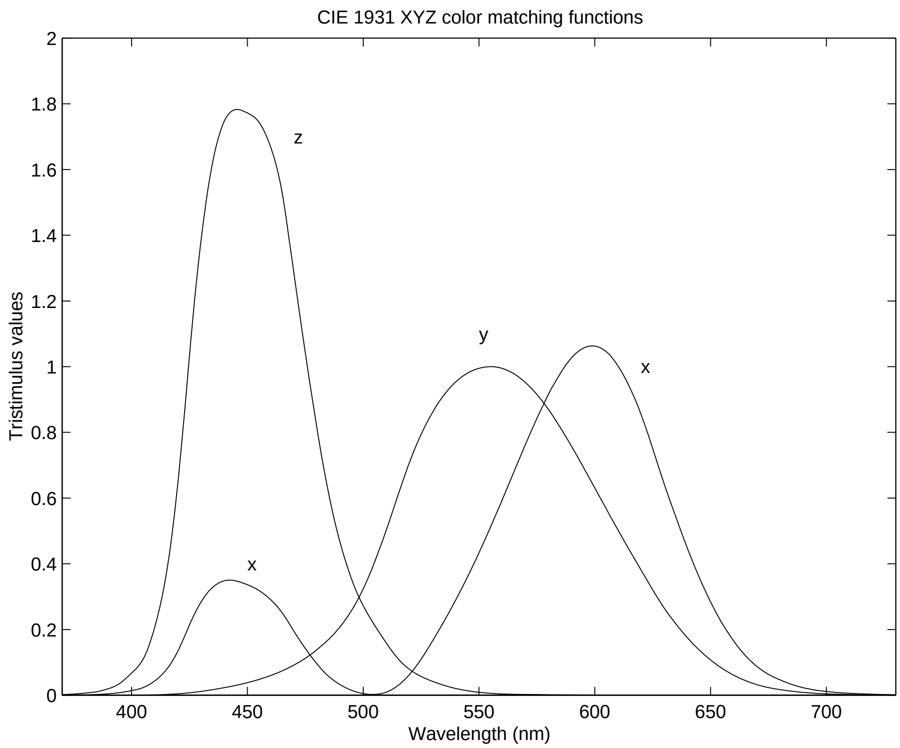

Figure 7.5: Color matching functions of the CIE 1931 XYZ primaries.

$[Z]$ take the role of the $[\tilde{P}_i]$. In other words, the matrix $\mathbf{A}^T$ is specified. For example, the 1976 CIE Uniform Chromaticity Scale (UCS) primaries are given by

$$\begin{aligned} [U'] &= 2.25[X] + 2.25[Z] \\ [V'] &= [Y] - 2[Z] \\ [W'] &= 3[Z] \end{aligned} \tag{7.24}$$

and so we have

$$\mathbf{A} = \begin{bmatrix} 2.25 & 0 & 0 \\ 0 & 1 & 0 \\ 2.25 & -2 & 3 \end{bmatrix}.$$

2. The matrix equation to calculate the tristimulus values of an arbitrary color

with respect to the primaries $[P_i]$ as a function of the tristimulus values for the XYZ primaries is specified directly, i.e., the matrix $\mathbf{A}^{-1}$ is specified. For the same example of the 1976 UCS primaries with $\mathbf{A}$ as above, we have

$$\begin{aligned}\begin{bmatrix} C_{U'} \\ C_{V'} \\ C_{W'} \end{bmatrix} &= \mathbf{A}^{-1} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}.\end{aligned} \tag{7.25}$$

3. The spectral densities $P_i(\lambda)$ of one member of each of the equivalence classes $[P_i]$ is provided. For example, this could be the spectral density of the light emitted from the red, green and blue phosphors of a CRT display. In this case, the XYZ tristimulus values of each of the primaries can be computed using equation (7.13) with the known XYZ color matching functions of Fig. 7.5. Fig. 7.6 shows the power spectral densities of the phosphors in a typical Sony trinitron CRT. These have been normalized so that their sum matches the reference white D_{65} and the integral of equation (7.13) can be approximated by a sum of products spaced every 2 nm. Applying this, we find

$$\begin{bmatrix} [R_C] \\ [G_C] \\ [B_C] \end{bmatrix} = \begin{bmatrix} 0.4641 & 0.2597 & 0.0357 \\ 0.3055 & 0.6592 & 0.1421 \\ 0.1808 & 0.0811 & 0.9109 \end{bmatrix} \begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix}. \tag{7.26}$$

4. A set of three color-matching functions $\bar{p}_i(\lambda)$, assumed to be a linear combination of $\bar{x}(\lambda)$, $\bar{y}(\lambda)$ and $\bar{z}(\lambda)$, is provided. An example would be the spectral sensitivities of the L, M and S type cone photoreceptors in the human retina. Given these, we must find the matrix $\mathbf{A}^{-1}$ relating the two sets of color matching functions, which is unique and easily determined using standard methods of linear algebra.

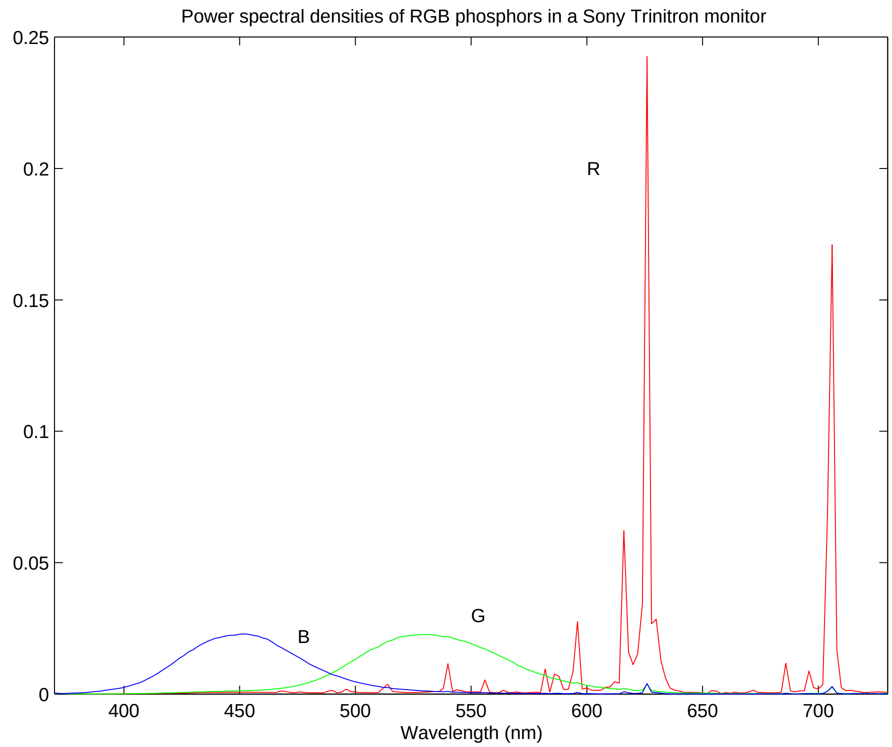

Figure 7.6: Spectral densities of typical SONY CRT phosphors.

7.4 Luminance and Chromaticity

Luminance is a measure of relative brightness. If two lights have equal luminance, they appear to be equally bright to a viewer, independently of their chromatic attributes. Although it may be difficult to judge if, say, a red light and a blue light have equal brightness when viewing them side by side, this judgement is easier if they are viewed in alternation one after the other. At a low frequency of switching, we can see the display flipping red, blue, red, blue, ... However, as the switching frequency increases and passes a certain limit, the two colors merge into one, which flickers if they have different brightness. The intensity of one of the lights can be adjusted until the flickering disappears. At this point, the two lights have equal perceptual brightness. This brightness depends on the power density spectrum of the light. A light with a spectrum concentrated near 550 nm appears brighter than a light of equal total power with a spectrum concentrated near 700 nm. This property is captured by the *relative luminous efficiency* curve $V(\lambda)$, shown in Fig.7.7. The curve tells us that a monochromatic light at wavelength λ_0 with power density spectrum $\delta(\lambda - \lambda_0)$ appears equally bright as a monochromatic light with power density spectrum $V(\lambda_0)\delta(\lambda - \lambda_{\max})$, where $\lambda_{\max} \approx 555$ nm.

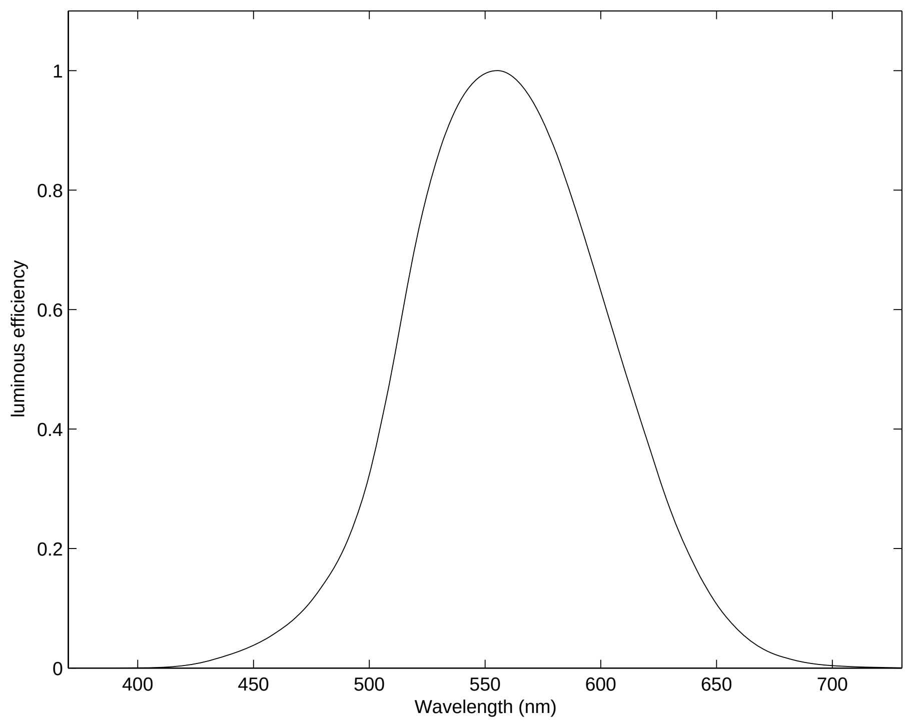

Figure 7.7: Relative luminous efficiency $V(\lambda)$.

For an arbitrary colored light with power density spectrum $C(\lambda)$, the luminance is defined as

$$C_L = K_m \int C(\lambda)V(\lambda)\,d\lambda \tag{7.27}$$

where K_m is a constant that determines the units of luminance. Thus, two different lights with power density spectra $C_1(\lambda)$ and $C_2(\lambda)$ will appear equally bright if $C_{1L} =$

C_{2L}, although they may look different in their chromatic attributes. According to the definition of luminance, if the power density spectrum is multiplied by a constant α, say $C_1(\lambda) = \alpha C(\lambda)$, then the luminance is also multiplied by α, $C_{1L} = \alpha C_L$; the light appears brighter but its chromatic aspect remains unchanged. Luminance is also additive: if $C(\lambda) = C_1(\lambda) + C_2(\lambda)$, then $C_L = C_{1L} + C_{2L}$. From this, we conclude that if

$$C(\lambda) \boxed{=} C_1 P_1(\lambda) + C_2 P_2(\lambda) + C_3 P_3(\lambda)$$

then $C_L = C_1 P_{1L} + C_2 P_{2L} + C_3 P_{3L}$, where $P_{iL} = K_m \int P_i(\lambda) V(\lambda)\, d\lambda$. The P_{iL} are called *luminosity coefficients* of the primaries $[P_i], i = 1, 2, 3$.

In terms of the tristimulus values, if $[C] = C_1[P_1] + C_2[P_2] + C_3[P_3]$, then the tristimulus values of $\alpha[C]$ are αC_1, αC_2 and αC_3. This corresponds to moving along a line passing through the origin and through the point (C_1, C_2, C_3) in a 3D color space, as illustrated in Fig. 7.8 for RGB primaries. (Note: Although the RGB axes are drawn as orthogonal in Fig. 7.8, we have not established any meaningful concept of orthogonality in the color vector space. The axes are drawn as orthogonal only for convenience. Since lines and planes remain lines and planes under any change of basis, the arguments made in this section hold independently of the existence of orthogonality.) The chromatic properties of the light are determined by a specification of this line, while luminance is proportional to the distance from the origin. One way of specifying the line is by giving the coordinates of its intersection with the plane $C_1 + C_2 + C_3 = 1$, i.e., find γ such that the tristimulus values of $\gamma[C] = \gamma C_1[P_1] + \gamma C_2[P_2] + \gamma C_3[P_3]$ lie on the said plane. It follows that $\gamma C_1 + \gamma C_2 + \gamma C_3 = 1$, and so $\gamma = 1/(C_1 + C_2 + C_3)$. The corresponding values, termed *chromaticity coordinates* are

$$c_i = \frac{C_i}{C_1 + C_2 + C_3}, \quad i = 1, 2, 3. \tag{7.28}$$

They are dependent since $c_3 = 1 - c_1 - c_2$, so only two need be specified. Note that we systematically denote tristimulus values with uppercase letters and chromaticity coordinates with (the same) lowercase letters.

The set of colors plotted in the $c_1 c_2$ plane is termed a chromaticity diagram, as shown in Fig. 7.9 for the CIE 1931 RGB primaries. The chromaticities of the

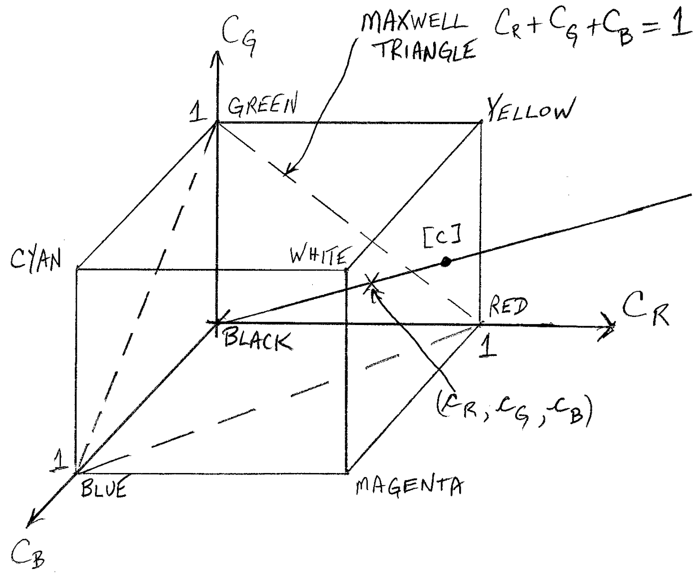

Figure 7.8: Illustration of the concept of chromaticity in RGB space. The triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is known as the Maxwell triangle.

spectral colors $[\delta(\lambda - \lambda_0)]$ lie along the the sharkfin-shaped curve. The straight line joining the extreme spectral colors is called the line of purples. All physical colors have chromaticities that lie in the region bounded by the spectrum locus and the line of purples. Note that for a normalized set of primaries, such that $[P_1] + [P_2] + [P_3] = [W]$ where $[W]$ is a reference white, then $W_1 = W_2 = W_3 = 1$ and thus $w_1 = w_2 = w_3 = \frac{1}{3}$, i.e., the chromaticity coordinates of reference white are all $\frac{1}{3}$.

7.4.1 Determination of tristimulus values from luminance and chromaticities

Suppose that for a given color, we are given the luminance C_L and the chromaticities c_1 and c_2 with respect to some set of primaries $[P_1]$, $[P_2]$ and $[P_3]$ (of course $c_3 = 1 - c_1 - c_2$). We know the primaries, so the luminosity coefficients P_{1L}, P_{2L}, P_{3L} are known. We want to find the tristimulus values C_1, C_2, C_3.

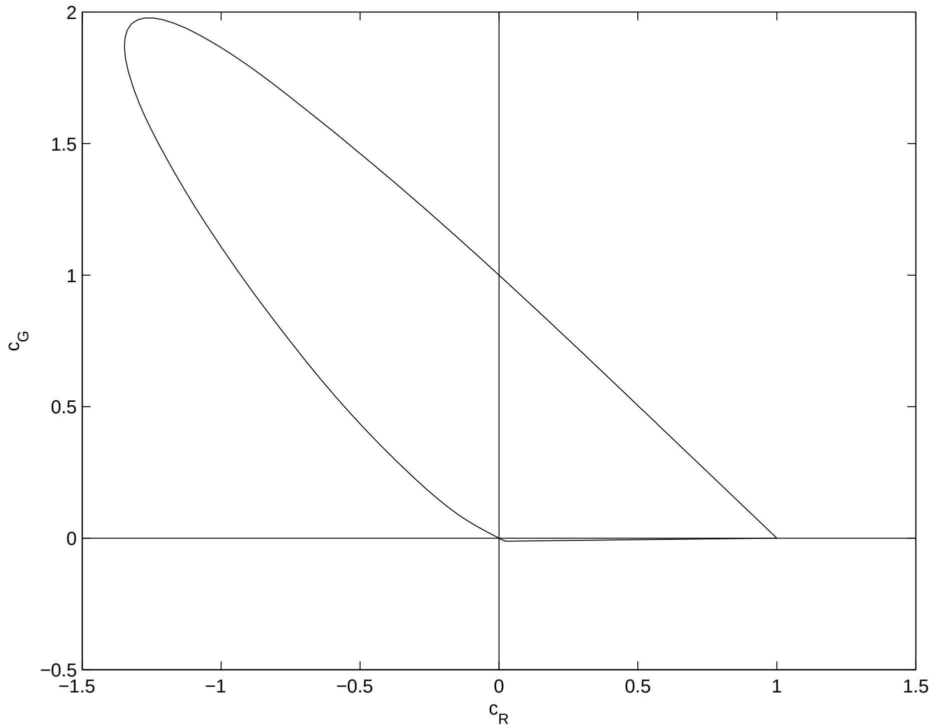

Figure 7.9: Chromaticity diagram for CIE 1931 RGB primaries.

Solution: We know that $C_L = C_1P_{1L} + C_2P_{2L} + C_3P_{3L}$. Dividing by $C_1 + C_2 + C_3$, we obtain

$$\frac{C_L}{C_1 + C_2 + C_3} = c_1P_{1L} + c_2P_{2L} + c_3P_{3L}.$$

Now multiplying by C_i for $i = 1, 2, 3$ yields

$$C_Lc_i = C_i(c_1P_{1L} + c_2P_{2L} + c_3P_{3L})$$

and thus

$$C_i = \frac{C_Lc_i}{c_1P_{1L} + c_2P_{2L} + c_3P_{3L}}, \quad i = 1, 2, 3. \tag{7.29}$$

7.4.2 Additive reproduction of colors

As we have mentioned, the tristimulus values allow us to *numerically* specify colors in terms of a set of primaries, which may or may not be physical colors. However, to actually *synthesize* colors, say on a CRT, we add (superpose) lights corresponding to three physical primaries, weighted by the tristimulus values, which must be positive. Not all colors can be synthesized in this way, and the judicious choice of primaries determines what fraction of the set of all physical colors can in fact be synthesized with the given primaries. In terms of the chromatic attribute, this is best visualized with a chromaticity diagram.

Proposition 7.4. On a chromaticity diagram, the chromaticities of all colors that are realized by the sum of a positive quantity of three physical colors lie within a triangle whose vertices are the chromaticities of the three colors.

Proof. We first show that the chromaticities of all colors that are realized by the sum of a positive quantity of *two* physical colors lie on the straight line joining the chromaticities of the two colors. Then, adding this sum to a positive quantity of the third color will yield chromaticities that lie within the stated triangle. To understand why the chromaticities of a sum of two colors lie on the straight line joining their chromaticities, we observe that in the three-dimensional color space, the sum of two colors lies on the plane that contains the lines through the origin and each of these two colors. The chromaticities of the sum lie on the intersection of this plane and the Maxwell triangle, which is a straight line. The projection of this straight line on the chromaticity diagram is also a straight line.

More formally, let the primaries with respect to which we are computing chromaticities be $[P_1]$, $[P_2]$ and $[P_3]$, and the three physical colors under consideration be $[A]$, $[B]$ and $[C]$. Thus, $[A] = A_1[P_1] + A_2[P_2] + A_3[P_3]$ and $a_i = A_i/(A_1 + A_2 + A_3)$, with similar expressions for $[B]$ and $[C]$. Now let $[Q] = \alpha_1[A] + \alpha_2[B]$ where $\alpha_1, \alpha_2 > 0$, so that $Q_i = \alpha_1 A_i + \alpha_2 B_i$ for $i = 1, 2, 3$. Then the chromaticities of this mixture are given by

$$q_i = \frac{\alpha_1 A_i + \alpha_2 B_i}{\alpha_1(A_1 + A_2 + A_3) + \alpha_2(B_1 + B_2 + B_3)}.$$

Cross-multiplying this equation, and substituting $A_i = a_i(A_1 + A_2 + A_3)$ and $B_i = b_i(B_1 + B_2 + B_3)$, gives

$$\alpha_1(A_1+A_2+A_3)a_i+\alpha_2(B_1+B_2+B_3)b_i = (\alpha_1(A_1+A_2+A_3)+\alpha_2(B_1+B_2+B_3))q_i$$

or rearranging

$$\alpha_1(A_1 + A_2 + A_3)(a_i - q_i) + \alpha_2(B_1 + B_2 + B_3)(b_i - q_i) = 0, \quad i = 1, 2, 3.$$

Now, using similar triangles, we observe that the segment of the straight line joining (a_1, a_2) and (b_1, b_2) is described by

$$\frac{q_1 - a_1}{b_1 - q_1} = \frac{q_2 - a_2}{b_2 - q_2} = \gamma > 0$$

or equivalently

$$(a_i - q_i) + \gamma(b_i - q_i) = 0, \quad i = 1, 2.$$

The conclusion follows. □

This is illustrated in the xy chromaticity diagram of Fig. 7.10. The chromaticities of red, green and blue primaries typical of the phosphors of a modern CRT are shown. The subset of all possible colors that can be reproduced on this CRT have chromaticities that lie within the indicated triangle. This explains why red, green and blue are used as primaries in additive color reproduction systems.

The following sections give an overview of a selected set of important color representations that are used in practice, several of which have already been mentioned. There are in fact many others.

7.5 Linear Color Representations

This section presents several important representations of color space based on the tristimulus values with respect to different sets of primaries. Since they are all related to each other by a linear transformation, we refer to them here as linear color representations.

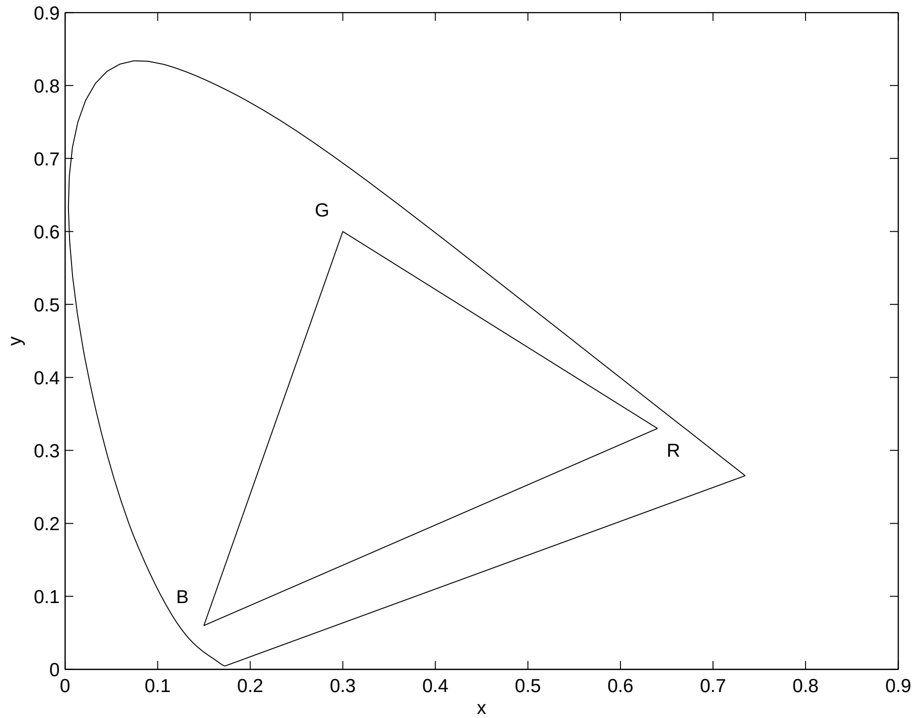

Figure 7.10: Chromaticities of colors that can be reproduced by addition of positive quantities of three primaries.

7.5.1 CIE Monochromatic RGB (1931)

This representation, already mentioned in Section 7.3.4, is based on monochromatic red, green and blue lights, with equal-energy white used as the reference white.

$$
\begin{aligned}
[R_{31}] &= k_R[\delta(\lambda - 700.0)] \\
[G_{31}] &= k_G[\delta(\lambda - 546.1)] \\
[B_{31}] &= k_B[\delta(\lambda - 435.8)] \\
[E] &= k_E[1]
\end{aligned}
$$

The luminosity coefficients of the three primaries are given by equation(7.27) as

$$R_L = K_m k_R V(700.0) \quad G_L = K_m k_G V(546.1) \quad B_L = K_m k_B V(435.8).$$

In order to satisfy $[R_{31}] + [G_{31}] + [B_{31}] = [E]$, it was found that $R_L : G_L : B_L = 1 : 4.5907 : 0.0601$. Letting $K_m k_R = k$, this gives

$$K_m k_G = 4.5907 \frac{V(700.0)}{V(546.1)} k \approx 0.0191k$$
$$K_m k_B = 0.0601 \frac{V(700.0)}{V(435.8)} k \approx 0.0137k$$

The measured color-matching functions were given in Fig. 7.4.

7.5.2 CIE 1931 XYZ

The XYZ primaries were introduced in Section 7.3.5 in terms of the 1931 RGB primaries in equation (7.20) which is reproduced here for convenience

$$\begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix} = \begin{bmatrix} .4184 & -.0912 & .0009 \\ -.1587 & .2524 & -.0026 \\ -.0828 & .0157 & .1786 \end{bmatrix} \begin{bmatrix} [R_{31}] \\ [G_{31}] \\ [B_{31}] \end{bmatrix}.$$

It follows that

$$\begin{aligned} [X] + [Y] + [Z] &= 0.1769[R_{31}] + 0.1769[G_{31}] + 0.1769[B_{31}] \\ &= 0.1769[E] \end{aligned}$$

Then, computing the luminosity coefficients of these primaries,

$$\begin{aligned} \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} &= \begin{bmatrix} .4184 & -.0912 & .0009 \\ -.1587 & .2524 & -.0026 \\ -.0828 & .0157 & .1786 \end{bmatrix} \begin{bmatrix} R_L \\ G_L \\ B_L \end{bmatrix} \\ &= \begin{bmatrix} .4184 & -.0912 & .0009 \\ -.1587 & .2524 & -.0026 \\ -.0828 & .0157 & .1786 \end{bmatrix} \begin{bmatrix} R_L \\ 4.5907R_L \\ 0.0601R_L \end{bmatrix} = \begin{bmatrix} 0 \\ R_L \\ 0 \end{bmatrix} \end{aligned}$$

Thus, $[X]$ and $[Z]$ have zero luminance, while $[Y]$ has the same luminance as R_{31}. The change in the luminance of the reference white $[E]$ was made so that the peak value of $\bar{y}(\lambda)$, which is the same as $V(\lambda)$, is 1.0. XYZ is the standard color system from which other color systems are described.

	Red	Green	Blue	White, D_{65}
x	0.640	0.300	0.150	0.3127
y	0.330	0.600	0.060	0.3290
z	0.030	0.100	0.790	0.3582

Table 7.1: XYZ chromaticities of ITU-R Rec. 709 red, green and blue primaries

For a given color $[C]$, if we know its luminance C_L and chromaticities c_X, c_Y and c_Z, the tristimulus values are easily found since $X_L = Z_L = 0$ and we use $Y_L = 1$. Thus

$$C_X = \frac{C_L c_X}{c_Y} \quad C_Y = C_L \quad C_Z = \frac{C_L c_Z}{c_Y}$$

A chromaticity diagram is obtained by plotting c_Y versus c_X.

7.5.3 ITU-R Rec. 709 Primaries

These primaries are representative of the phosphors used in modern CRTs. They are defined in terms of their XYZ chromaticities and the given reference white. This section will serve to illustrate many of the calculations done in colorimetry. These primaries could be referred to as $[R_{709}]$, $[G_{709}]$ and $[B_{709}]$, but will refer to them here simply as $[R]$, $[G]$ and $[B]$ for simplicity. The primaries and reference white are specified in Table 7.1. Of course the Z-chromaticities do not *need* to be provided in the table, since the sum of the values in each column is 1.

D_{65} is a reference white specified by the CIE and is meant to be typical of daylight. Normalization is provided by setting the luminance of reference white to unity, $D_L = 1$. We will simply denote reference white by $[D]$ in this section. It follows that

$$\begin{aligned} D_X &= \frac{d_X}{d_Y} = 0.9505 \\ D_Y &= 1 \\ D_Z &= \frac{d_Z}{d_Y} = 1.0888 \end{aligned}$$

To find the XYZ tristimulus values of the red, green and blue primaries, we need to find their luminances R_L, G_L and B_L. These are determined through the constraint $[R] + [G] + [B] = [D]$. This equation implies the matrix equation

$$\begin{bmatrix} R_X \\ R_Y \\ R_Z \end{bmatrix} + \begin{bmatrix} G_X \\ G_Y \\ G_Z \end{bmatrix} + \begin{bmatrix} B_X \\ B_Y \\ B_Z \end{bmatrix} = \begin{bmatrix} D_X \\ D_Y \\ D_Z \end{bmatrix}$$

Using $R_X = R_L \frac{r_X}{r_Y}$ and so on in the above equation with the chromaticities from Table 7.1 gives

$$\begin{bmatrix} 1.\overline{93} & 0.5 & 2.5 \\ 1.0 & 1.0 & 1.0 \\ 0.\overline{09} & 0.1\overline{6} & 13.1\overline{6} \end{bmatrix} \begin{bmatrix} R_L \\ G_L \\ B_L \end{bmatrix} = \begin{bmatrix} 0.9505 \\ 1.0 \\ 1.0888 \end{bmatrix}$$

Solving this matrix equation, we find

$$\begin{bmatrix} R_L \\ G_L \\ B_L \end{bmatrix} = \begin{bmatrix} 0.2127 \\ 0.7152 \\ 0.0722 \end{bmatrix}$$

and substituting into the expressions $R_X = R_L \frac{r_X}{r_Y}$ etc., we find

$$\begin{bmatrix} [R] \\ [G] \\ [B] \end{bmatrix} = \underbrace{\begin{bmatrix} 0.4125 & 0.2127 & 0.0193 \\ 0.3576 & 0.7152 & 0.1192 \\ 0.1804 & 0.0722 & 0.9502 \end{bmatrix}}_{\mathbf{A}^T} \begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix}.$$

To convert tristimulus values between the two sets of primaries, suppose that

$$\begin{aligned} [C] &= C_R[R] + C_G[G] + C_B[B] \\ &= C_X[X] + C_Y[Y] + C_Z[Z] \end{aligned}$$

Then

$$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} = \underbrace{\begin{bmatrix} 0.4125 & 0.3576 & 0.1804 \\ 0.2127 & 0.7152 & 0.0722 \\ 0.0193 & 0.1192 & 0.9502 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} C_R \\ C_G \\ C_B \end{bmatrix}$$

and

$$\begin{bmatrix} C_R \\ C_G \\ C_B \end{bmatrix} = \underbrace{\begin{bmatrix} 3.2405 & -1.5372 & -0.4985 \\ -0.9693 & 1.8760 & 0.0416 \\ 0.0556 & -0.2040 & 1.0573 \end{bmatrix}}_{\mathbf{A}^{-1}} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}.$$

The chromaticities of colors that can be physically reproduced with these primaries are shown in Fig. 7.10, and the color matching functions, obtained by transforming the XYZ color matching functions using $\mathbf{A}^{-1}$ are shown in Fig. 7.11.

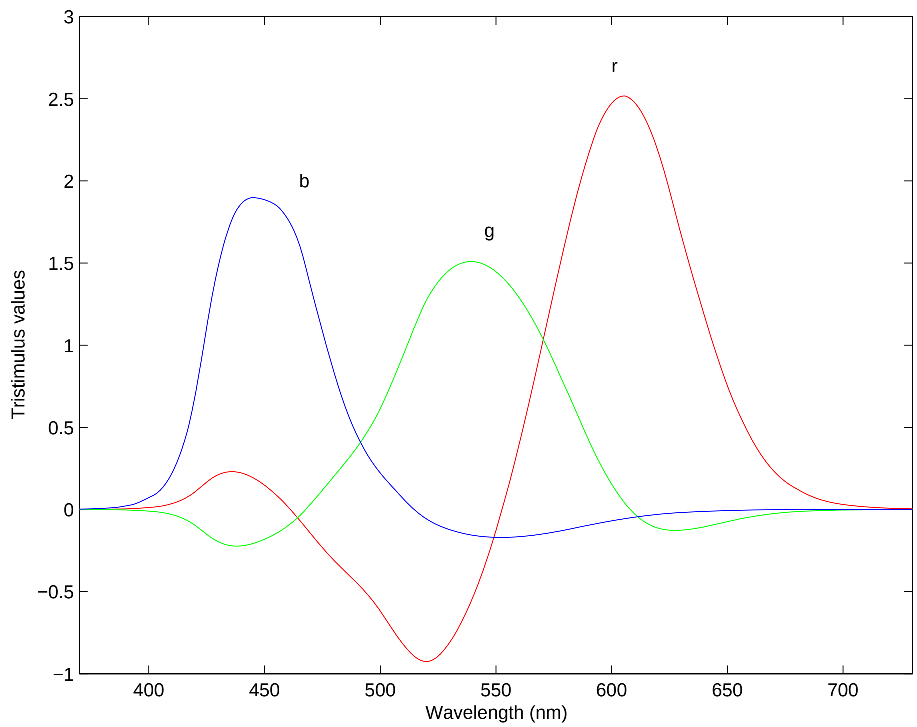

Figure 7.11: Color matching functions of Rec. 709 RGB primaries.

7.6 Perceptually Uniform Color Representations

The color representations that we have seen so far have the disadvantage that they are not perceptually uniform. By perceptually uniform, we mean that equal distances in the color space correspond to equal perceptual differences. There have been many attempts to determine color representations that are more perceptually uniform than the representations seen until now. There are two aspects to this: perceptual uniformity with respect to the chromatic aspect, and perceptual uniformity with respect to the brightness.

To illustrate the concept of perceptual nonuniformity in the chromatic aspect, consider Fig. 7.12. The ellipses in this diagram give the loci of chromaticities that are just noticeably different from each of 25 representative colors, for a constant luminance, based on experiments performed by MacAdam. The ellipses are enlarged by a factor of 10 to improve visibility on the figure. From this, it is clear that a much larger change in the chromaticity of a green light is required for it to be noticeable than for a red light, and in turn for a blue light. Thus other spaces were sought that would be more perceptually uniform in terms of chromaticity.

7.6.1 CIE Uniform Chromaticity Scale

In 1960, the CIE introduced the Uniform Chromaticity Scale (UCS), with new primaries $[U]$, $[V]$ and $[W]$ defined by

$$\begin{aligned}
[U] &= 1.5[X] + 1.5[Z] \\
[V] &= [Y] - 3[Z] \\
[W] &= 2[Z].
\end{aligned}$$

This was replaced in 1976 by the new UCS given by

$$\begin{aligned}
[U'] &= 2.25[X] + 2.25[Z] \\
[V'] &= [Y] - 2[Z] \\
[W'] &= 3[Z].
\end{aligned}$$

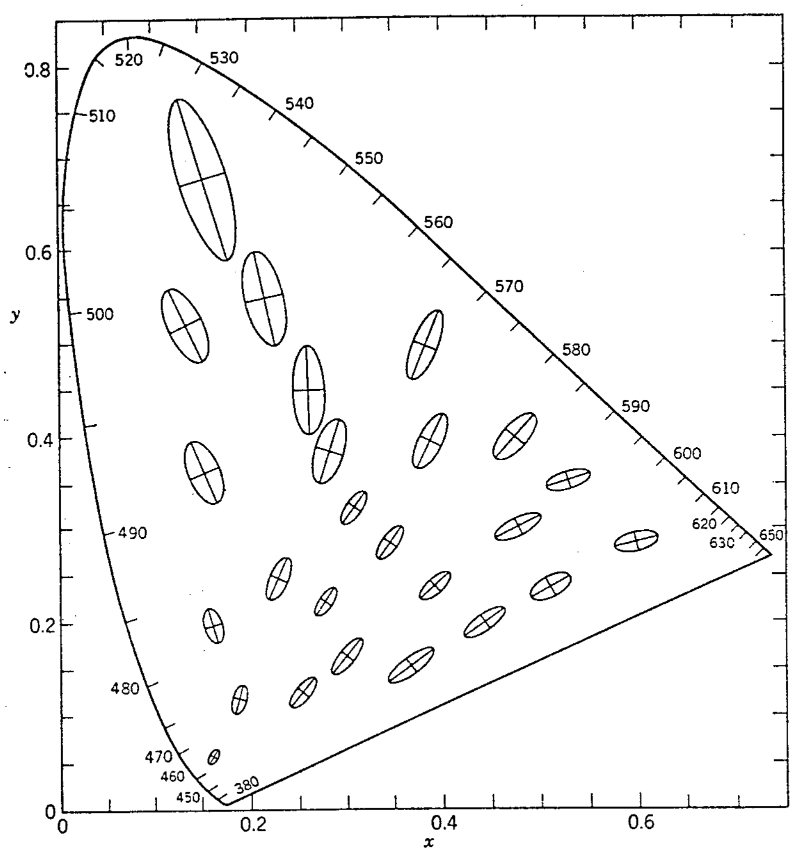

Figure 7.12: MacAdam ellipses in an XYZ chromaticity diagram.

The latter space is the one used now. The equations for transformation of tristimulus values for a color $[Q]$ are easily found to be

$$\begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} = \begin{bmatrix} 2.25 & 0 & 0 \\ 0 & 1 & 0 \\ 2.25 & -2 & 3 \end{bmatrix} \begin{bmatrix} Q_{U'} \\ Q_{V'} \\ Q_{W'} \end{bmatrix}$$

and to go the other way

$$\begin{bmatrix} Q_{U'} \\ Q_{V'} \\ Q_{W'} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix}$$

It is interesting that the tristimulus value for the new primary $[V']$ is again equal to the luminance, even though this primary is different from the primary $[Y]$ in the XYZ space. This highlights that there is no single primary whose tristimulus value is the luminance. Primaries and tristimulus values cannot be considered in isolation. All three must be considered simultaneously.

The chromaticities in the 1976 UCS space are given by

$$q_{U'} = \frac{4Q_X}{Q_X + 15Q_Y + 3Q_Z}$$
$$q_{V'} = \frac{9Q_Y}{Q_X + 15Q_Y + 3Q_Z}$$

The MacAdam ellipses in 1960 UCS chromaticity diagram are shown in Fig. 7.13, and are clearly much more uniform than in the XYZ space, but they are still not perfectly uniform, where they would all become equal-sized circles.

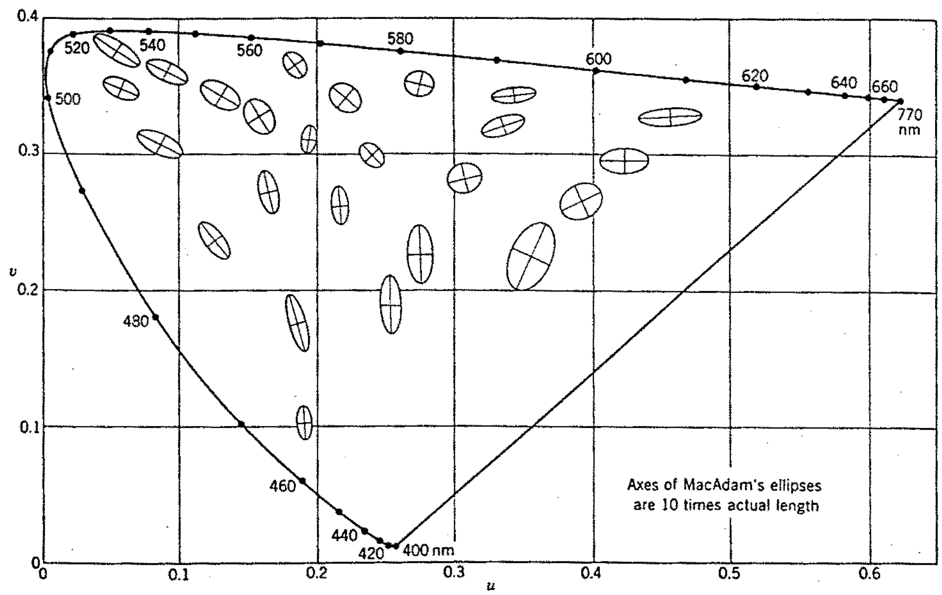

Figure 7.13: MacAdam ellipses in an 1960 UCS chromaticity diagram.

7.7 Nonlinear Color Representations

7.7.1 CIELUV and CIELAB

Two nonlinear representations are now widely used. They have reasonable perceptual uniformity. They are called $L^*u^*v^*$ or CIELUV and $L^*a^*b^*$ or CIELAB. These spaces attempt to be perceptually uniform in both brightness and chromatic aspect. As with other color coordinate systems, the actual values obtained depend on the choice of a reference white, denoted $[W]$ (e.g., D_{65}). Assume that a color $[Q]$ is specified by its tristimulus values in XYZ space: $[Q] = Q_X[X] + Q_Y[Y] + Q_Z[Z]$. Similarly, the reference white is specified as $[W] = W_X[X] + W_Y[Y] + W_Z[Z]$. Of course, the luminances of $[Q]$ and $[W]$ are given by $Q_L = Q_Y$ and $W_L = W_Y$.

The brightness component Q_{L^*} is the same in both CIELUV and CIELAB spaces, and is defined by

$$Q_{L^*} = \begin{cases} 903.3\frac{Q_Y}{W_Y} & \frac{Q_Y}{W_Y} \leq 0.008856 \\ 116\left(\frac{Q_Y}{W_Y}\right)^{\frac{1}{3}} - 16 & \frac{Q_Y}{W_Y} > 0.008856 \end{cases}$$

Since we assume that $0 \leq Q_Y \leq W_Y$, it follows that $0 \leq Q_{L^*} \leq 100$. A difference of about 1 in Q_{L^*} is near the threshold of discrimination.

The chromatic components for the CIELUV system are based on the chromaticities in the 1976 CIE UCS system:

$$q_{U'} = \frac{4Q_X}{Q_X + 15Q_Y + 3Q_Z}$$
$$q_{V'} = \frac{9Q_Y}{Q_X + 15Q_Y + 3Q_Z}$$

and similarly, for the reference white,

$$w_{U'} = \frac{4W_X}{W_X + 15W_Y + 3W_Z}$$
$$w_{V'} = \frac{9W_Y}{W_X + 15W_Y + 3W_Z}$$

The CIELUV u^* and v^* components are then defined as

$$Q_{u^*} = 13Q_{L^*}(q_{U'} - w_{U'})$$
$$Q_{v^*} = 13Q_{L^*}(q_{V'} - w_{V'})$$

The chromatic components for the CIELAB system are described directly from the XYZ tristimulus values. If Q_X/W_X, Q_Y/W_Y and Q_Z/W_Z are all greater than 0.008856, then

$$Q_{a^*} = 500\left(\left(\frac{Q_X}{W_X}\right)^{\frac{1}{3}} - \left(\frac{Q_Y}{W_Y}\right)^{\frac{1}{3}}\right)$$
$$Q_{b^*} = 200\left(\left(\frac{Q_Y}{W_Y}\right)^{\frac{1}{3}} - \left(\frac{Q_Z}{W_Z}\right)^{\frac{1}{3}}\right)$$

If any of the values are less than 0.008856, a linear segment is used as for L^*, although such small values are not normally encountered.

7.7.2 Device Representation

The light output of a cathode ray tube (CRT) is related to the voltage applied approximately by a power law

$$\text{intensity} = \text{voltage}^{\gamma}$$

or better

$$\text{intensity} = (\text{voltage} + \epsilon)^{\gamma}.$$

To compensate for this, the RGB values are usually gamma corrected. In this

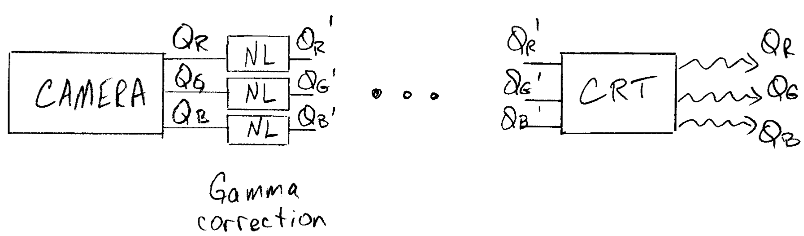

Figure 7.14: Gamma correction for CRT display.

figure, Q_R, Q_G and Q_B are the tristimulus values with respect to a set of red, green and blue primaries that correspond to the phosphors of the CRT. $Q'_R = NL(Q_R)$, $Q'_G = NL(Q_G)$ and $Q'_B = NL(Q_B)$ are the gamma-corrected values, where NL is a nonlinear function that should invert the nonlinearity of the CRT.

ITU-R Recommendation 709 also gives a standard form for gamma correction:

$$Q'_R = \begin{cases} 4.5Q_R & Q_R \leq 0.018 \\ 1.099Q_R^{0.45} - 0.099 & 0.018 < Q_R \leq 1.0 \end{cases}$$

with similar expressions for Q'_G and Q'_B. The inverse law is

$$Q_R = \begin{cases} 0.2\dot{2}Q'_R & Q'_R \leq 0.081 \\ \left(\frac{Q'_R+0.099}{1.099}\right)^{2.2\dot{2}} & 0.081 < Q'_R \leq 1.0 \end{cases}$$

again with similar expressions for Q_G and Q_B. Note that the gamma-corrected RGB space consisting of values Q'_R, Q'_G, Q'_B is much more perceptually uniform than the linear RGB space of tristimulus values Q_R, Q_G, Q_B and may be a suitable and much lower complexity replacement for the more uniform CIELUV and CIELAB spaces. Remember that because of the nonlinear transformation, Q'_R, Q'_G, Q'_B are *not* tristimulus values. The Rec. 709 gamma-correction law is shown in Fig. 7.15

7.7.3 Luma-Color-Difference Representation

The luma-color-difference representation (and its variants) is one of the most widely used spaces in image compression and transmission. It is used in standard television, as well as in digital compression schemes like JPEG and MPEG. It is based on gamma-corrected RGB values seen in the previous section. In this color system, the luma carries the brightness information and the color differences carry the chromatic information. It is a nonlinear space, so again these coordinates are *not* tristimulus values, although they are related to tristimulus values by a reversible nonlinear transformation. One of the advantages of this color space is that the color difference components can be subsampled with respect to the luma component, say by a factor of two in the horizontal and vertical dimension, with little impact on the perceived

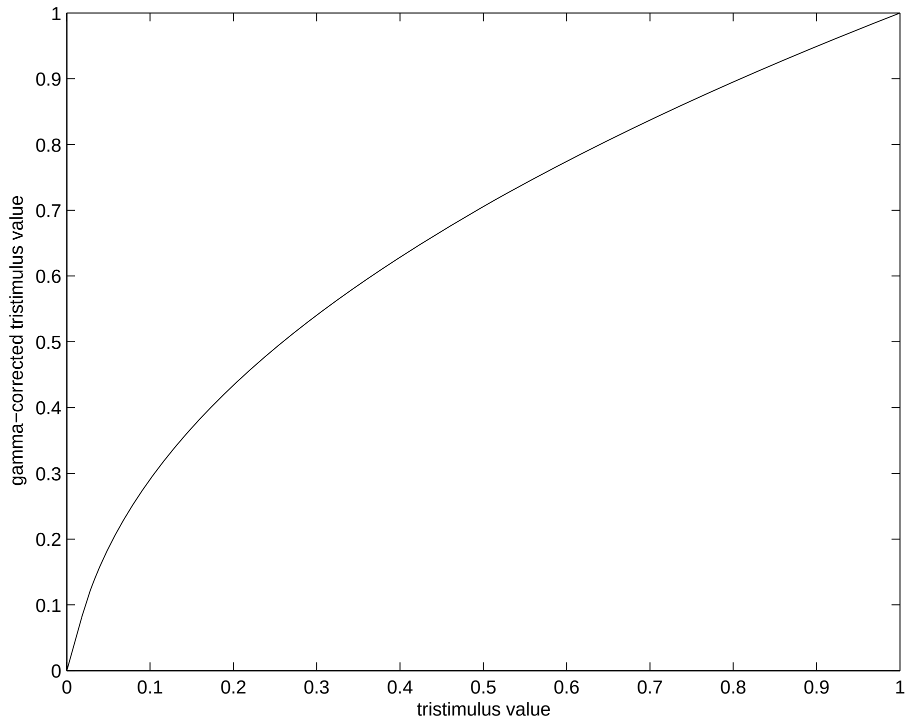

Figure 7.15: ITU-R Rec. 709 gamma-correction law.

quality of the color image. The transformation to this space is defined by ITU-R Rec. 601

$$\begin{aligned} Q'_Y &= 0.299Q'_R + 0.587Q'_G + 0.114Q'_B \\ Q_{P_B} &= 0.564(Q'_B - Q'_Y) \\ Q_{P_R} &= 0.713(Q'_R - Q'_Y) \end{aligned}$$

where Q'_R, Q'_G, Q'_B are gamma-corrected RGB values, assumed to lie between 0 and 1. It follows that $0 \leq Q'_Y \leq 1$ and $-0.5 \leq Q_{P_B}, Q_{P_R} \leq 0.5$. This can be expressed in matrix form as

$$\begin{bmatrix} Q'_Y \\ Q_{P_B} \\ Q_{P_R} \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ -0.169 & -0.331 & 0.5 \\ 0.5 & -0.419 & -0.081 \end{bmatrix} \begin{bmatrix} Q'_R \\ Q'_G \\ Q'_B \end{bmatrix}$$

For 8-bit integer values between 0 and 255, we have

$$
\begin{aligned}
Q'_{Y_{219}} &= 16 + 219 Q'_Y \\
Q_{C_B} &= 128 + 224 Q_{P_B} \\
Q_{C_R} &= 128 + 224 Q_{P_R}
\end{aligned}
$$

Appendices

A. Equivalence relations

Let $\mathcal{A}$ be a set. A relation R is a subset of $\mathcal{A} \times \mathcal{A}$. If $(x_1, x_2) \in R$, we write $x_1 \; R \; x_2$. Examples of relations on $\mathbb{R}$ are $x_1 = x_2$ and $x_1 < x_2$.

A relation is said to be an *equivalence relation* if it satisfies the three following properties:

$$\begin{array}{ll} \text{reflexivity:} & x \; R \; x \\ \text{symmetry:} & x_1 \; R \; x_2 \Rightarrow x_2 \; R \; x_1 \\ \text{transitivity:} & x_1 \; R \; x_2 \text{ and } x_2 \; R \; x_3 \Rightarrow x_1 \; R \; x_3 \end{array}$$

Of the above examples, $x_1 = x_2$ is an equivalence relation and $x_1 < x_2$ is not.

Let $\sim$ be an equivalence relation on a set $\mathcal{A}$, and define $[z] = \{x \in \mathcal{A} \mid x \sim z\}$ for any $z \in \mathcal{A}$. This set is called the equivalence class containing z. Let $[z]$ and $[y]$ be two equivalence classes. Then either $[z] = [y]$ or $[z] \cap [y] = \emptyset$. A consequence of this is that if y is any element of $[z]$, then $[y] = [z]$, i.e., any element of the equivalence class can be used as the class representative. A second consequence is that every element $x \in \mathcal{A}$ belongs to one and only one equivalence class. Thus, the set of equivalence classes forms a partition of $\mathcal{A}$.

B. Proofs

Proposition 7.1: *The relation $\boxminus$ on $L^2(\mathcal{V})$ is an equivalence relation.*

Proof. reflexivity: $C^+(\lambda) + C^-(\lambda) \triangleq C^+(\lambda) + C^-(\lambda)$ by reflexivity of $\triangleq$, and thus $C(\lambda) \boxminus C(\lambda)$.
symmetry: If $C_1(\lambda) \boxminus C_2(\lambda)$ then $C_1^+(\lambda) + C_2^-(\lambda) \triangleq C_2^+(\lambda) + C_1^-(\lambda)$. By symmetry of $\triangleq$, $C_2^+(\lambda) + C_1^-(\lambda) \triangleq C_1^+(\lambda) + C_2^-(\lambda)$ and thus $C_2(\lambda) \boxminus C_1(\lambda)$.
transitivity: Suppose that $C_1(\lambda) \boxminus C_2(\lambda)$ and $C_2(\lambda) \boxminus C_3(\lambda)$. Then

$$\begin{aligned} &C_1^+(\lambda) + C_2^-(\lambda) \triangleq C_2^+(\lambda) + C_1^-(\lambda) \quad \text{and} \\ &C_2^+(\lambda) + C_3^-(\lambda) \triangleq C_3^+(\lambda) + C_2^-(\lambda). \end{aligned}$$

Applying (G3),

$$C_1^+(\lambda) + C_2^-(\lambda) + C_3^-(\lambda) \triangleq C_2^+(\lambda) + C_1^-(\lambda) + C_3^-(\lambda) \quad \text{and}$$
$$C_2^+(\lambda) + C_3^-(\lambda) + C_1^-(\lambda) \triangleq C_3^+(\lambda) + C_2^-(\lambda) + C_1^-(\lambda).$$

Applying transitivity of $\triangleq$, $C_1^+(\lambda)+C_2^-(\lambda)+C_3^-(\lambda) \triangleq C_3^+(\lambda)+C_2^-(\lambda)+C_1^-(\lambda)$, and applying G3 again, $C_1^+(\lambda) + C_3^-(\lambda) \triangleq C_3^+(\lambda) + C_1^-(\lambda)$, i.e., $C_1(\lambda) \boxminus C_3(\lambda)$. □

Theorem 7.1 *If* $C_1(\lambda) \boxminus C_2(\lambda)$, *then* $\alpha C_1(\lambda) \boxminus \alpha C_2(\lambda), \forall \alpha \in \mathbb{R}$ (G2′).

Proof. Consider first the case $\alpha \geq 0$. By hypothesis, $C_1^+(\lambda) + C_2^-(\lambda) \triangleq C_2^+(\lambda) + C_1^-(\lambda)$. Applying G2, $\alpha(C_1^+(\lambda) + C_2^-(\lambda)) \triangleq \alpha(C_2^+(\lambda) + C_1^-(\lambda))$, i.e. $\alpha C_1^+(\lambda) + \alpha C_2^-(\lambda) \triangleq \alpha C_2^+(\lambda) + \alpha C_1^-(\lambda)$. Now, for $\alpha \geq 0$, it follows that $(\alpha C)^+ = \alpha C^+$ and $(\alpha C)^- = \alpha C^-$, and so

$$(\alpha C_1)^+(\lambda) + (\alpha C_2)^-(\lambda) \triangleq (\alpha C_2)^+(\lambda) + (\alpha C_1)^-(\lambda), \text{ i.e.,}$$

$$\alpha C_1(\lambda) \boxminus \alpha C_2(\lambda).$$

Now consider the case $\alpha < 0$. Then,

$$(\alpha C)^+ = |\alpha| C^- = -\alpha C^- \quad \text{and}$$
$$(\alpha C)^- = |\alpha| C^+ = -\alpha C^+.$$

Now applying G2 with $-\alpha$,

$$-\alpha(C_1^+(\lambda) + C_2^-(\lambda)) \triangleq -\alpha(C_2^+(\lambda) + C_1^-(\lambda))$$

$$(\alpha C_1)^-(\lambda) + (\alpha C_2)^+(\lambda) \triangleq (\alpha C_2)^-(\lambda) + (\alpha C_1)^+(\lambda)$$

Rearranging,

$$(\alpha C_1)^+(\lambda) + (\alpha C_2)^-(\lambda) \triangleq (\alpha C_2)^+(\lambda) + (\alpha C_1)^-(\lambda), \text{ i.e.,}$$

$$\alpha C_1(\lambda) \boxminus \alpha C_2(\lambda)$$

□

Proposition 7.2: *Let $C_1(\lambda), C_2(\lambda) \in \mathcal{P}$. Then $C_1(\lambda) - C_2(\lambda) \boxminus 0$ if and only if $C_1(\lambda) \triangleq C_2(\lambda)$.*

Proof. Let $\mathcal{I}_1 = \{\lambda \mid C_1(\lambda) \geq C_2(\lambda)\}$ and $\mathcal{I}_2 = \{\lambda \mid C_1(\lambda) < C_2(\lambda)\}$. Then $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ and $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{V}$. Thus

$$(C_1(\lambda) - C_2(\lambda))^+ = \begin{cases} C_1(\lambda) - C_2(\lambda) & \lambda \in \mathcal{I}_1 \\ 0 & \lambda \in \mathcal{I}_2 \end{cases}$$

and

$$(C_1(\lambda) - C_2(\lambda))^- = \begin{cases} 0 & \lambda \in \mathcal{I}_1 \\ C_2(\lambda) - C_1(\lambda) & \lambda \in \mathcal{I}_2 \end{cases}$$

Note that

$$(C_1(\lambda) - C_2(\lambda))^+ + C_2(\lambda) = \begin{cases} C_1(\lambda) & \lambda \in \mathcal{I}_1 \\ C_2(\lambda) & \lambda \in \mathcal{I}_2 \end{cases}$$

and

$$(C_1(\lambda) - C_2(\lambda))^- + C_1(\lambda) = \begin{cases} C_1(\lambda) & \lambda \in \mathcal{I}_1 \\ C_2(\lambda) & \lambda \in \mathcal{I}_2 \end{cases}$$

Thus trivially

$$(C_1(\lambda) - C_2(\lambda))^+ + C_2(\lambda) \triangleq (C_1(\lambda) - C_2(\lambda))^- + C_1(\lambda).$$

It follows from the corollary to G3 that if $(C_1(\lambda) - C_2(\lambda))^+ \triangleq (C_1(\lambda) - C_2(\lambda))^-$, i.e., $C_1(\lambda) - C_2(\lambda) \boxminus 0$, then $C_1(\lambda) \triangleq C_2(\lambda)$ and conversely, if $C_1(\lambda) \triangleq C_2(\lambda)$ then $(C_1(\lambda) - C_2(\lambda))^+ \triangleq (C_1(\lambda) - C_2(\lambda))^-$. □

Theorem 7.5: *The dimension of $\mathcal{C}$ is 3.*

Proof. Property (G4) implies that we can find three elements of $\mathcal{P}$, $P_1(\lambda)$, $P_2(\lambda)$, $P_3(\lambda)$ such that no one of them can match a linear combination of the other two. It follows that $[P_1(\lambda)]$, $[P_2(\lambda)]$, and $[P_3(\lambda)]$ are linearly independent in $\mathcal{C}$. Otherwise there would exist $\alpha_1, \alpha_2, \alpha_3$, not all zero, such that $\alpha_1[P_1] + \alpha_2[P_2] + \alpha_3[P_3] \boxminus [0]$, which contradicts the above statement. We need only prove that $[P_1]$, $[P_2]$ and $[P_3]$

span $\mathcal{C}$ to establish the proof. Let $C(\lambda)$ be an element of $L^2(\mathcal{V})$. If $C(\lambda) \in \mathcal{P}$ or $-C(\lambda) \in \mathcal{P}$, it is clear that we can find $\alpha_1, \alpha_2, \alpha_3$ such that $C(\lambda) \boxed{=} \alpha_1 P_1(\lambda) + \alpha_2 P_2(\lambda) + \alpha_3 P_3(\lambda)$. Now suppose that neither of the above two conditions hold and that $C(\lambda)$ is an arbitrary element of $\mathcal{C}$. Thus $C(\lambda) = C^+(\lambda) - C^-(\lambda)$ where both $C^+(\lambda)$ and $C^-(\lambda)$ are nonzero elements of $\mathcal{P}$. From (G4) we can find $\alpha_1^+, \alpha_2^+, \alpha_3^+$ such that

$$C^+(\lambda) \boxed{=} \alpha_1^+ P_1(\lambda) + \alpha_2^+ P_2(\lambda) + \alpha_3^+ P_3(\lambda),$$

and similarly, we can find $\alpha_1^-, \alpha_2^-, \alpha_3^-$ such that

$$C^-(\lambda) \boxed{=} \alpha_1^- P_1(\lambda) + \alpha_2^- P_2(\lambda) + \alpha_3^- P_3(\lambda).$$

It follows that

$$C^+(\lambda) - C^-(\lambda) \boxed{=} (\alpha_1^+ - \alpha_1^-)P_1(\lambda) + (\alpha_2^+ - \alpha_2^-)P_2(\lambda) + (\alpha_3^+ - \alpha_3^-)P_3(\lambda)$$

or in other words

$$[C] = (\alpha_1^+ - \alpha_1^-)[P_1] + (\alpha_2^+ - \alpha_2^-)[P_2] + (\alpha_3^+ - \alpha_3^-)[P_3]$$

and so $[P_1]$, $[P_2]$ and $[P_3]$ do indeed span $\mathcal{C}$, and so $\mathcal{C}$ must be of dimension 3. $\square$

Problems

1. The web color *goldenrod* that we will denote $[Q]$ is specified by the RGB values 218, 165, 32, on a scale from 0 to 255. Thus they can be assumed to be $Q'_R = 0.8549$, $Q'_G = 0.6471$, $Q'_B = 0.1255$ on a scale from 0 to 1. We assume that these are gamma-corrected values, according to the Rec. 709 gamma law (Section 7.7.2), and that the primaries are the Rec. 709 RGB primaries, normalized with respect to reference white D_{65} (Section 7.5.3). The goal of this problem is to determine representations of this color in other color coordinate representations. Determine the following (show your work):

(a) the tristimulus values Q_R, Q_G, Q_B in the Rec. 709 RGB color representation (Section 7.5.3);

(b) the luminance Q_L and the chromaticities q_R, q_G, q_B in the Rec. 709 representation;

(c) the XYZ tristimulus values Q_X, Q_Y, Q_Z and the corresponding chromaticities q_X, q_Y, q_Z (Section 7.6);

(d) the 1976 $U'V'W'$ tristimulus values $Q_{U'}$, $Q_{V'}$, $Q_{W'}$ and the corresponding chromaticities $q_{U'}$, $q_{V'}$, $q_{W'}$ (Section 7.6.1);

(e) the CIELAB coordinates Q_{L^*}, Q_{a^*}, Q_{b^*} (Section 7.7.1);

(f) the Luma and color differences $Q_{Y'}$, Q_{P_B}, Q_{P_R} (Section 7.7.3).

* You can visualize this color in any Windows program that lets you specify the RGB values of a color. For example, in Microsoft Word, draw a shape like a rectangle and set the fill color using "More Colors – Custom" and enter the red, green and blue values in the boxes.

2. As stated at the end of Section 7.3.5 in point 4, the spectral absorption curves of the three types of cone photoreceptors in the human retina should be a linear combination of any set of three color-matching functions. The outputs

of these receptors can be considered to be tristimulus values with respect to some set of primaries that we will call $[L]$, $[M]$, $[S]$ (which stands for long, medium and short). It has been found that the tristimulus values with respect to these primaries for a color $[C]$, denoted C_l, C_m, C_s, can be obtained from the XYZ tristimulus values by

$$\begin{bmatrix} C_l \\ C_m \\ C_s \end{bmatrix} = \begin{bmatrix} 0.4002 & 0.7076 & -0.0808 \\ -0.2263 & 1.1653 & 0.0457 \\ 0.0 & 0.0 & 0.9182 \end{bmatrix} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}$$

(a) Determine and plot the color matching functions for the LMS primaries, denoted $\bar{l}(\lambda)$, $\bar{m}(\lambda)$, $\bar{s}(\lambda)$. The data for the xyz color-matching functions are given on the course web page.

(b) Express the primaries $[L]$, $[M]$, $[S]$ in terms of the primaries $[X]$, $[Y]$, $[Z]$. What color is $[L] + [M] + [S]$?

(c) Why are these primaries called $[L]$, $[M]$ and $[S]$?

(d) Determine the LMS tristimulus values of the color *goldenrod* of problem 1. Can you give a physical interpretation (in terms of your eye) of these tristimulus values?

3. The Bayer color sampling strategy induces a new set of color signals from the original RGB values (assume Rec. 709) as follows:

$$\begin{bmatrix} f_L \\ f_{C1} \\ f_{C2} \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ -0.25 & 0.5 & -0.25 \\ -0.25 & 0.0 & 0.25 \end{bmatrix} \begin{bmatrix} f_R \\ f_G \\ f_B \end{bmatrix}$$

These can be considered to be tristimulus values with respect to a new set of primaries denoted $[L]$, $[C1]$, $[C2]$.

(a) Determine and plot the color matching functions for the LC1C2 primaries, denoted $\bar{l}(\lambda)$, $\bar{c1}(\lambda)$, $\bar{c2}(\lambda)$. The data for the XYZ color-matching functions are given on the course web page.

(b) Express the primaries $[L]$, $[C1]$, $[C2]$ in terms of the primaries $[R]$, $[G]$, $[B]$ and in terms of the primaries $[X]$, $[Y]$, $[Z]$.

(c) Determine the LC1C2 tristimulus values of the color *goldenrod* of problem 1 and of the reference white D_{65}.

4. The recommendation 709 RGB primaries can be expressed in terms of the CIE XYZ primaries by

$$\begin{bmatrix} [R] \\ [G] \\ [B] \end{bmatrix} = \begin{bmatrix} 0.4125 & 0.2127 & 0.0193 \\ 0.3576 & 0.7152 & 0.1192 \\ 0.1804 & 0.0722 & 0.9502 \end{bmatrix} \begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix}.$$

Consider the cyan, magenta and yellow (CMY) primaries used in printing. These are given by $[C] = [B] + [G]$, $[M] = [R] + [B]$ and $[YE] = [R] + [G]$.

(a) Determine the *tristimulus values* of $[C]$, $[M]$ and $[YE]$ with respect to the XYZ primaries. Compute the XYZ *chromaticities* of $[C]$, $[M]$ and $[YE]$ and plot them on an xy chromaticity diagram. Comment on the suitability of cyan, magenta and yellow as primaries for an additive color display device like a cathode ray tube (CRT).

(b) Suppose that $[C]$, $[M]$ and $[YE]$ as in part (a) are taken as primaries in a color system. Determine the tristimulus values of a monochromatic light $\delta(\lambda - 510nm)$ with respect to these primaries. You will need to use the XYZ color matching functions. Carefully explain all steps. Can the given light be *physically* synthesized as a sum of a positive quantity of the $[C]$, $[M]$ and $[YE]$ primaries?

(c) Determine the tristimulus values of the color *goldenrod* of question 1 with respect to the $[C]$, $[M]$ and $[YE]$ primaries. Can this color be *physically* synthesized as a sum of a positive quantity of the $[C]$, $[M]$ and $[YE]$ primaries?

5. The EBU (European Broadcasting Union) primaries, have the following specification

	Red	Green	Blue	White D_{65}
x	0.640	0.290	0.150	0.3127
y	0.330	0.600	0.060	0.3290
z	0.030	0.110	0.790	0.3582

Assume that the reference white has unit luminance $D_L = 1.0$ and that $[R] + [G] + [B] = [D_{65}]$.

(a) Find the XYZ tristimulus values of the reference white $[D_{65}]$, i.e., D_X, D_Y and D_Z.

(b) Using $[R]+[G]+[B] = [D_{65}]$, determine the luminances of the three primaries, R_L, G_L and B_L.

(c) Now find the XYZ tristimulus values of the three primaries, i.e. if $[R] = R_X[X] + R_Y[Y] + R_Z[Z]$, find R_X, R_Y, R_Z, and similarly for $[G]$ and $[B]$.

(d) If an arbitrary color $[Q]$ is written

$$[Q] = Q_X[X] + Q_Y[Y] + Q_Z[Z] = Q_R[R] + Q_G[G] + Q_B[B]$$

determine the matrix relations to find Q_X, Q_Y, Q_Z from Q_R, Q_G, Q_B and vice-versa.

(e) Plot an xy chromaticity diagram showing the triangles of chromaticities reproducible with the EBU RGB primaries.

(f) Compute and plot the color matching functions for the EBU RGB primaries by transforming the XYZ color matching functions using the results of (d).

(g) For the three spectral densities $Q_1(\lambda)$, $Q_2(\lambda)$ and $Q_3(\lambda)$ in the attached table, compute their chromaticities and plot them on an xy chromaticity diagram. Would they make good primaries for a physical color image synthesis system such as a CRT? Explain.

Q_1	Q_2	Q_3	λ
.19	.00	.60	400 nm
.20	.00	.63	
.20	.00	.64	
.20	.00	.63	
.20	.00	.62	
.20	.02	.59	450 nm
.20	.06	.53	
.19	.19	.43	
.18	.31	.31	
.16	.43	.20	
.13	.52	.10	500 nm
.08	.61	.05	
.06	.67	.02	
.04	.69	.01	
.03	.69	.00	
.04	.67	.00	550 nm
.08	.64	.00	
.14	.60	.00	
.22	.55	.00	
.32	.49	.00	
.41	.43	.00	600 nm
.50	.38	.00	
.56	.33	.00	
.63	.28	.00	
.67	.25	.00	
.71	.23	.00	650 nm
.75	.21	.00	
.77	.20	.00	
.79	.19	.00	
.80	.19	.00	
.81	.18	.00	700 nm