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Phase Transitions in Active Rotator Systems

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In order to study the statistical dynamics of a large population of limit cycle oscillators or excitable elements, an active rotator model is introduced. This is defined dynamically as a stochastic version of a relaxational planar model (with external field or anisotropy) modified by an additional constant driving force. Its numerical study based on a mean field treatment revealed the existence of a peculiar ordered phase in which individual motions are organized into a macroscopic rhythm. Two possible types of transition to this ordered phase are also found.

A new class of nonequilibrium cooperative systems is obtained by a simple modification of an equilibrium planar model with Hamiltonian

$$H=-\sum_{i>j}K_{ij}\cos(\phi_i-\phi_j)-B\sum_i\cos\nu\phi_i\,, \tag{1}$$

where ϕ_i $(i=1, 2, \cdots, N)$ denotes the phase of the i-th spin and the last term represents the effect of an external magnetic field if $\nu=1$ or uniaxial anisotropy if $\nu=2$. It is known that purely relaxational dynamics in the form

$$\frac{d\phi_i}{dt}=-\frac{\partial H}{\partial\phi_i}+\eta_i(t) \tag{2}$$

gives correct equilibrium distribution $\exp(-H/T)$ provided the random forces $\eta_i(t)$ are chosen to be Gaussian with statistical averages

$$\langle\eta_i(t)\rangle=0\,,$$
$$\langle\eta_i(t)\eta_j(t')\rangle=2T\delta_{ij}\delta(t-t')\,. \tag{3}$$

Let Eq. (2) be modified by adding a constant driving force ω (assumed positive) as

$$\frac{d\phi_i}{dt}=\omega-\frac{\partial H}{\partial\phi_i}+\eta_i(t) \tag{4}$$

by which our rotators become "activated". Without interaction and random forces, the motion of each rotator is given by

$$\frac{d\phi_i}{dt}=\omega-b\sin\nu\phi_i\,, \tag{5}$$

where $b=\nu B$. If $\nu=1$, Eq. (5) gives a phase model representing a limit cycle oscillator or an excitable element such as a nerve cell, and this is conveniently called a ring model.[1)] In fact,[1),2)] $\phi_i(t)$ increases monotonically when $|\omega/b|>1$, and then we have an oscillator with frequency

$$\tilde{\omega}=2\pi\left[\int_{-\pi}^{\pi}\frac{d\phi}{\omega-b\sin\phi}\right]^{-1}=\omega\sqrt{1-(b/\omega)^2}\,. \tag{6}$$

When $|\omega/b|\lesssim 1$, we have a stable-unstable pair of equilibria relatively close to each other. As a result, the system behaves very differently in its course back to the resting state (i.e., stable equilibrium) depending on whether or not the initial disturbance from the resting state is strong enough to force the system beyond the "potential barrier" peaked near the unstable equilibrium. This feature is known to be most basic to excitable systems. Dynamical units as described by Eq. (5) also appear in some other physical contexts, e.g., some electric circuits with periodic forcing,[3] the Josephson junction as described by an overdamped phase motion,[4] and a certain phenomenological model for explaining the non-Ohmic charge-density wave conductivity.[5] In some of these physical problems, the rotator models are generalized to include many components,[6] although the phase slippage between interacting pair never occurs in those models while it is the key process to the phenomena of the present concern.

In this communication, we will be interested in some phase transitions shown by our active rotator system (4) with (3). The case $\nu=1$ corresponds to a community either of oscillators (smooth or relaxational) or of excitable elements. We also included the case $\nu=2$ in our study for the sake of completeness. Its previous studies[7] in connection with phase transition are restricted to vanishing b-term, which corresponds to perfectly smooth oscillators. Throughout the present work, we will restrict ourselves to a numerical analysis carried out for a rather special form of interaction in which each rotator couples to all the others with equal strength of the order of N^{-1} or $K_{ij}=N^{-1}K$. A dynamical version of the mean field treatment then holds exactly as discussed previously.[7] Although not directly related to the present work, it should be mentioned that there exists another interesting mean field approach to populations of smooth oscillators by Yamaguchi et al.[8] in which the orbital softness is so important that the phase description does not work.

Let us now take a suitable time unit in which K is set to unity. Thus we are left with three independent parameters ω, b and T. The normalized number density of the rotators having phase ϕ at time t is defined by

$$n(\phi,\,t)=N^{-1}\sum_{j=1}^{N}\delta(\phi_j(t)-\phi)\,. \tag{7}$$

Because we always assume N to be infinitely large, $n(\phi,\,t)$ represents a macrovariable. Then its fluctuation of the order of $1/\sqrt{N}$ may be neglected. In what follows, we will therefore not distinguish between $n(\phi,\,t)$ and its statistical average. Let Eq.(4) be expressed in terms of $n(\phi,\,t)$ as

$$\frac{d\phi_i}{dt}=\omega-b\,\sin\nu\phi_i-\int_{-\pi}^{\pi}d\phi'\sin(\phi_i-\phi')\,n(\phi',\,t)+\eta_i(t)\,. \tag{8}$$

This equation is equivalent to the Fokker-Planck equation for the probability distribution $P(\phi,\,t)$ of ϕ_i,

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\phi}\{\omega-b\,\sin\nu\phi-\int_{-\pi}^{\pi}d\phi'\sin(\phi-\phi')\,n(\phi',\,t)\}P+T\frac{\partial^2 P}{\partial\phi^2}\,. \tag{9}$$

Since different ϕ_i's are statistically independent for the uniform interaction model we are working with, $P(\phi, t)$ may be identified with $n(\phi, t)$. In this way, Eq. (9) takes the form of a nonlinear diffusion equation for $n(\phi, t)$,

$$\frac{\partial n(\phi, t)}{\partial t} = -\frac{\partial}{\partial \phi}\{\omega - b\sin\nu\phi - \int_{-\pi}^{\pi} d\phi' \sin(\phi-\phi') n(\phi', t)\} n(\phi, t) + T\frac{\partial^2 n(\phi, t)}{\partial \phi^2}, \tag{10}$$

which is subject to the periodic boundary condition, $n(\phi+2\pi, t) = n(\phi, t)$, and normalization condition, $\int_{-\pi}^{\pi} d\phi\ n(\phi, t) = 1$. Our particular concern is the behavior of $n(\phi, t)$ as $t \to \infty$. In solving Eq. (10) numerically, it was first transformed to a set of ordinary differential equations for the spatial Fourier amplitudes of $n(\phi, t)$, and then approximated this set by retaining typically the first 30 modes except near the phase-transition points where a sufficient accuracy required 50 modes. A typical T-b phase diagram obtained for each value of ν is shown in Fig. 1. Some phase diagrams for different ω values are depicted on a common T-b plane in Fig. 2.

We first look into the case $\nu=1$. Figures 1(a) and 2(a) show that there are two distinct regimes in the parameter space. They are a time-periodic regime and a stationary regime, which are indicated by P and S, respectively. The time-evolution of the distribution $n(\phi, t)$ obtained for some representative sets of parameter values (i.e., points (A) to (C) in Fig. 1(a)) is presented in Fig. 3(a). In the time-periodic regime, extensive physical quantities are generally expected to show oscillatory behavior, while they remain constant in the stationary regime. It is interesting to observe that the phase boundary between P and S consists of two segments with an angular point of connection. In order to see how these parts of the transition line are different, we investigated the variation in some system characteristics as we move along the loop indicated by a square in Fig. 1(a). The quantities actually calculated are the frequency $\tilde{\omega}$ of n (set identically to zero in the stationary regime) and an "order parameter" σ defined by

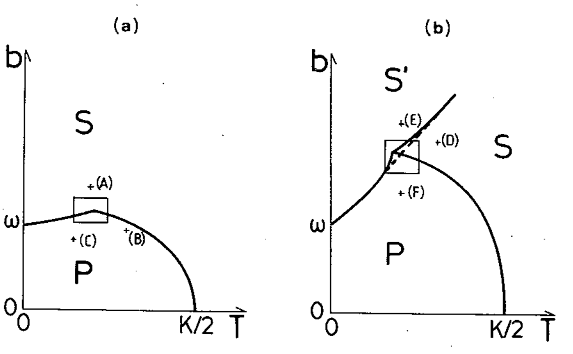

Fig. 1. Typical phase diagram obtained from the long-time behavior of the solution of Eq. (10); (a): $\nu=1$, $\omega=0.5$; (b): $\nu=2$, $\omega=0.5$. P represents a time-periodic regime, and S and S' stationary regimes. Some physical quantities are calculated at points (A) to (F) and also along the loops indicated by squares (also see the text and Figs. 3 and 4).

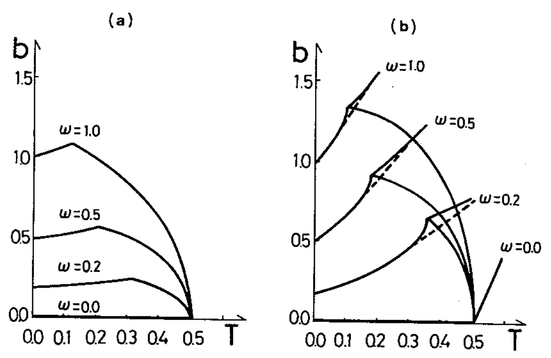

Fig. 2. Phase diagrams for some different ω values; (a): $\nu=1$, (b): $\nu=2$.

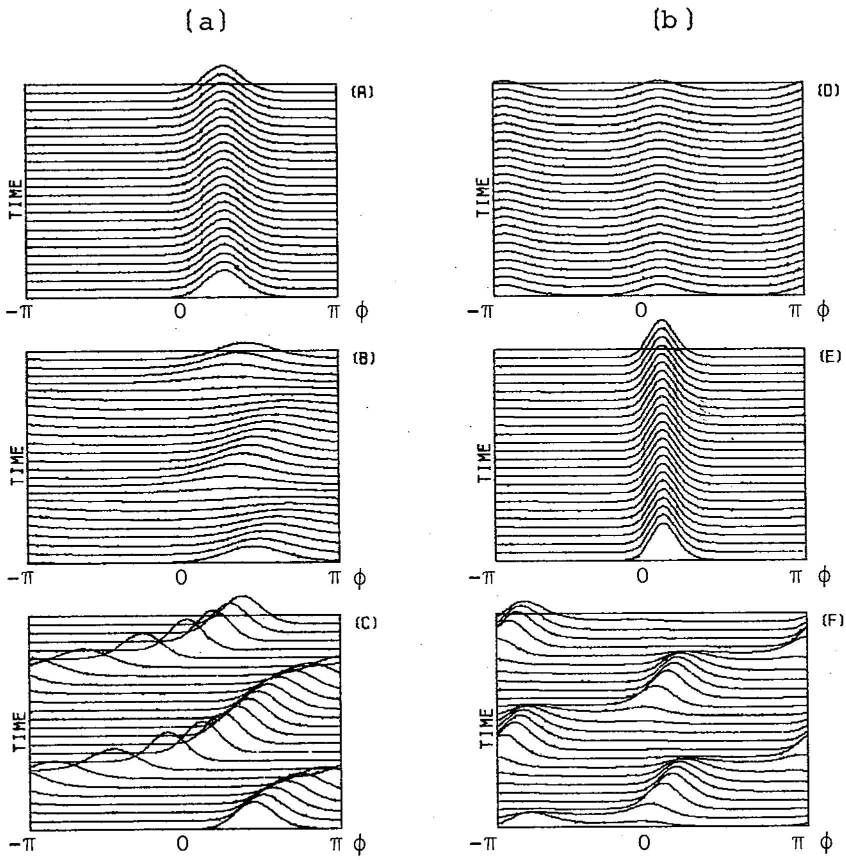

Fig. 3. Time-evolutions of the number density as $t\to\infty$ obtained for points (A) to (F) in Fig. 1.

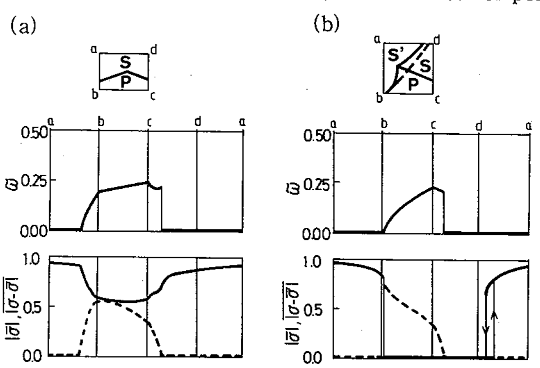

Fig. 4. Variation of some physical quantities along the square loops indicated in Fig. 1. The solid and dashed curves in the lowest figures represent $|\bar{\sigma}|$ and $\overline{|\sigma-\bar{\sigma}|}$ respectively. For further details, see the text.

$$\sigma(t)=\int_{-\pi}^{\pi}d\phi e^{i\phi}n(\phi,t) \tag{11}$$

or related quantities $\bar{\sigma}$ and $\overline{|\sigma-\bar{\sigma}|}$, the bar meaning a long-time average. From the behavior of $\tilde{\omega}$ as shown in Fig. 4(a), it is implied that $\tilde{\omega}$ changes continuously across the low T segment of the transition line, whereas it shows a discontinuity across the high T segment. Intuitively, the $S\to P$ transition at lower T may be described as follows. As a result of the decreasing "pinning force", the stationary n becomes increasingly difficult to maintain itself, and below some threshold value of b, n comes to show a sliding motion; its mean velocity $\tilde{\omega}$ is at first very small due to virtual pinning. Thus the transition is easily understood as basically reflecting the element behavior governed by Eq. (5). In contrast, the transition in the large-T region is essentially cooperative in nature, and should rather be understood as an extrapolation from the transition occurring at vanishing b. In fact, in this limit, our original system (4) with ϕ_i replaced by $\phi_i+\omega t$ represents an ordinary plane rotator system, and hence the existence of transition is clear in the mean field framework. Although we will not go into bifurcation-theoretical argument in the present communication, it is only mentioned that the onset of the sliding motion at lower T is characterized by a saddle-node type bifurcation; the bifurcation at higher T is in contrast a Hopf type for which the oscillation starts with a finite frequency and an infinitesimal amplitude. Such an interpretation is consistent with the order parameter variation presented in Fig. 4(a). Note that at the onset of the sliding motion, the quantity $\overline{|\sigma-\bar{\sigma}|}$ remains infinitesimal. This is simply because for the most period the quantities $\sigma(t)$ and $\bar{\sigma}$ are essentially identical. Finally, it is remarked that no hysteresis was observed in our $S\leftrightarrow P$ transition.

We now move to the case $\nu=2$. Figures 1(b) and 2(b) show that other than a periodic regime P, there are two stationary regimes S and S', which makes contrast to the previous case. This comes from system's symmetry, i.e., the property that Eq. (10) is invariant under the translation $\phi\to\phi+\pi$. The regime S where $n(\phi+\pi)=n(\phi)$ in fact reflects such a symmetry which is however broken in S'. This is clear from Fig. 3(b) where the profiles of n for some sets of parameter values are illustrated. Another qualitative difference from the previous case is that the transitions $S'\leftrightarrow S$ and $S'\leftrightarrow P$ generally show a hysteresis. Thus, both the upper and lower stability limits are indicated in each of Figs. 1(b) and 2(b). Analogously to the previous case, the $S'\to P$ transition is characterized by the onset of a sliding motion of n, while the $S\to P$ transition is described by a Hopf bifurcation. The order parameter behavior given in Fig. 4(b), which shows a remarkable contrast to Fig. 4(a), is easy to interpret in terms of the system symmetry mentioned. To be specific, the distribution $n(\phi,t)$ as $t\to\infty$ is invariant under $\phi\to\phi+\pi$ in P as well as in S, if a suitable time-displacement is undertaken simultaneously, which means that $\bar{\sigma}$ vanishes identically in these regimes.

Our phase diagrams include some known limiting cases:

(1) $T=0$

This corresponds to the absence of random forces. The rotators are then perfectly synchronous, i.e., $\phi_i(t)$ are identical and obey Eq. (5). Clearly, the $S\leftrightarrow P$ (or $S'\leftrightarrow P$) transition occurs at $b=\omega$.

(2) $b=0$

The system then reduces to a usual planar model (without external field and anisotropy) by the simultaneous transformation $\phi_i\to\phi_i+\omega t$ $(i=1,2,\cdots,N)$ in Eq. (4). The mean field treatment such as presented above immediately gives the critical value of T as $T_c=K/2$ for the onset of a spontaneous polarization.

(3) $\omega=0$

This gives an equilibrium planar model with external field or anisotropy. The equilibrium solution of Eq. (10) satisfies the transcendental equation

$$n(\phi)=C\exp\left\{\frac{1}{T}[B\cos\nu\phi+K\int_{-\pi}^{\pi}d\phi'\cos(\phi-\phi')n(\phi')]\right\}, \tag{12}$$

where C is the normalization constant.

The above three special cases may be extended perturbatively to some nontrivial asymptotic regimes in the parameter space. However, this problem together with bifurcation-theoretical interpretation of the whole phase diagram will be the subject of a separate work.

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