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Propagation of small perturbations in synchronized oscillator networks

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Abstract. – We study the propagation of a harmonic perturbation of small amplitude on a network of coupled identical phase oscillators prepared in a state of full synchronization. The perturbation is externally applied to a single oscillator, and is transmitted to the other oscillators through coupling. Numerical results and an approximate analytical treatment, valid for random and ordered networks, show that the response of each oscillator is a rather well-defined function of its distance from the oscillator where the external perturbation is applied. For small distances, the system behaves as a dissipative linear medium: the perturbation amplitude decreases exponentially with the distance, while propagating at constant speed. We suggest that the pattern of interactions may be deduced from measurements of the response of individual oscillators to perturbations applied at different nodes of the network.

Synchronization phenomena have recently attracted a great deal of attention [1]. The emergence of synchronous behaviour in large ensembles of interacting dynamical elements is a paradigmatic form of collective self-organization, typical of a vast class of natural systems ranging from complex chemical reactions to large-scale biological processes [2, 3]. Many of these phenomena are successfully reproduced by relatively simple mathematical models. The most basic form of synchronized dynamics, which can be achieved in an ensemble of identical periodic oscillators subject to attractive coupling, is full synchronization [4]. In such state, the individual motions of all oscillators coincide.

A network of fully synchronized coupled oscillators may be viewed as an active medium in a highly coherent collective state. A relevant question regarding the dynamics of this system is how the medium responds to an external perturbation which affects its coherence. Due to the coupling between oscillators, the perturbation should spread through the medium. Moreover, if dissipative mechanisms are acting and the state of full synchronization is stable, the perturbation should die out as the distance from the point where it is applied becomes larger. The properties of this propagation phenomenon provide a dynamical characterization of the synchronized medium, much like the propagation of a defect in an extended system. This is the problem that we address in this letter.

Kuramoto's model of coupled phase oscillators [5] provides the basis for our model. The state of each oscillator is described by a phase variable $\phi_i \in [0, 2\pi)$ which, in the absence of coupling, rotates at constant frequency, $\dot{\phi}_i = \omega_i$. A network of coupled oscillators is governed by the equations

$$\dot{\phi}_i = \omega_i + k \sum_{j=1}^{N} J_{ij} \sin\left(\phi_j - \phi_i\right), \tag{1}$$

where k is the coupling strength. Here, $J_{ij} = 1$ if oscillator i is coupled to oscillator j, and $J_{ij} = 0$ otherwise. The connection (or adjacency) matrix $\mathcal{J} = \{J_{ij}\}$ is not necessarily symmetric. The underlying connection network is therefore a directed graph, where the links pointing to the node occupied by a given oscillator i start at those oscillators which enter the equation of motion for ϕ_i.

If the oscillators are identical, $\omega_i = \omega_j$ for all i, j. Without generality loss, their natural frequencies are chosen to be $\omega_i = 0$, and the coupling strength is fixed at $k = 1$. To represent the external perturbation, one of the oscillators —say, $i = 1$— is also coupled to an oscillating force of strength ϵ and frequency Ω. Starting from eq. (1), the equation of motion for ϕ_i can be written as

$$\dot{\phi}_i = \sum_{j=1}^{N} J_{ij} \sin\left(\phi_j - \phi_i\right) + \epsilon\delta_{i1} \sin(\Omega t - \phi_1), \tag{2}$$

where δ_{ij} is the Kronecker symbol. In the absence of external forcing ($\epsilon = 0$), the ensemble admits a fully synchronized state where $\phi_i = \phi^*$ for all i, with constant ϕ^*. The stability of this state can be analytically proven for some specific forms of the connection matrix $\mathcal{J}$. In particular, full synchronization is stable in the case where the number of nonvanishing elements in all rows of $\mathcal{J}$ is the same, *i.e.* when the number $z_i = \sum_j J_{ij}$ of connections ending at each oscillator is the same, $z_i = z$, for all i [6]. In this case, the network is a regular directed graph [7]. The following study is mainly focused on this kind of network.

A small perturbation of the fully synchronized state produces a deviation of the same order ϵ as the perturbation. For $\epsilon \to 0$, the solution to eq. (2) can be found by writing $\phi_i = \phi^* + \epsilon\psi_i$, and expanding up to the first order in ϵ, which yields

$$\dot{\psi}_i = \sum_{j=1}^{N} J_{ij}\left(\psi_j - \psi_i\right) + \delta_{i1} \exp[\mathrm{i}\Omega t]. \tag{3}$$

In the last term, we have replaced $\sin(\Omega t)$ by $\exp[\mathrm{i}\Omega t]$, for convenience in the mathematical treatment. After transients have elapsed, the solution to eq. (3) has the form $\psi_i(t) = A_i \exp[\mathrm{i}\Omega t]$. The complex amplitude A_i is obtained from the linear system

$$\mathcal{M}\boldsymbol{A} = \boldsymbol{e}_1 \,, \tag{4}$$

where $\boldsymbol{A} = (A_1, A_2, \ldots, A_N)$, $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, and $\mathcal{M} = (z + \mathrm{i}\Omega)\mathcal{I} - \mathcal{J}$, with $\mathcal{I}$ the $N \times N$ identity matrix. The amplitudes are, thus, $\boldsymbol{A} = \mathcal{M}^{-1}\boldsymbol{e}_1$. For a given connection matrix $\mathcal{J}$, they can be found by numerically inverting $\mathcal{M}$.

Figure 1 shows typical results for the moduli $|A_i|$ and the phases φ_i of the amplitudes, $A_i \equiv |A_i| \exp[\mathrm{i}\varphi_i]$, in a 10^3-oscillator random network. Each oscillator is coupled to $z = 2$ neighbours, which are chosen at random from the whole ensemble (avoiding self and multiple connections). The three data sets of the figure show results for the same random network and different perturbation frequencies Ω. Each dot represents the modulus or phase of a single oscillator i as a function of its distance d_i to oscillator 1, where the external perturbation is

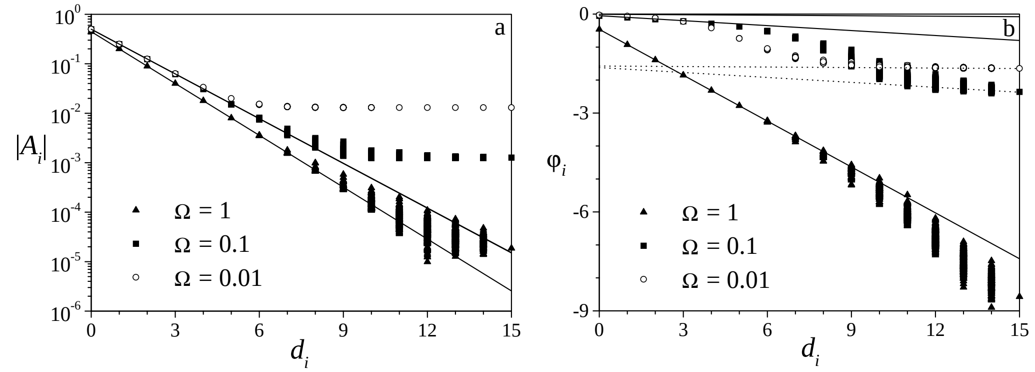

Fig. 1 – Moduli (a) and phases (b) of the individual amplitudes A_i in a random network ($z = 2$) of 10^3 fully synchronized oscillators, as functions of the distance to the perturbed oscillator. Each dot corresponds to the amplitude of a single oscillator. The three data sets have been obtained for the same realization of the network but with different perturbation frequencies Ω. Full and dotted lines are analytical predictions for small and large distances, respectively.

applied. This distance is defined as the number of connections along the shortest (directed) path starting at oscillator 1 and ending at i. For this particular network, d_i varies between 0 (for $i = 1$) and 15.

The numerical results of fig. 1 have been obtained by inversion of the matrix $\mathcal{M}$ in eq. (4). We have verified that, as expected, these results are reproduced by numerical integration of the equations of motion (2), for sufficiently small values of the perturbation amplitude ($\epsilon \lesssim 10^{-4}$).

It turns out that the modulus $|A_i|$ has a rather well-defined dependence on the distance d_i. For small distances, it decays exponentially, and its value is essentially the same for all the oscillators at a given distance. As d_i grows, however, the values of $|A_i|$ become more dispersed as a function of d_i, especially for large perturbation frequencies. Moreover, the exponential decay breaks down, and the variation of $|A_i|$ with the distance becomes much slower. The values of $|A_i|$ for large distances depend on the frequency Ω. Numerical results for different realizations of the random network show that these values can strongly differ between realizations, and seem to be determined by the maximum value of d_i in a given network.

As shown in fig. 1b, the phase φ_i is negative for all distances and its absolute value increases with d_i. This implies that the response of oscillators to the perturbation is increasingly retarded as the distance grows. For small distances we find a linear regime, with a constant phase shift $\Delta\varphi$ between oscillators whose distances differ by one. Consequently, the perturbation propagates at constant speed $v = |\Omega/\Delta\varphi|$. For larger distances, before the regime of exponential decay in $|A_i|$ breaks down (cf. figs. 1a and b), this linear dependence is lost. After an intermediate zone of faster variation, where the dispersion of the phase as a function of the distance is larger, the dependence of φ_i on d_i becomes much less pronounced. The linear regime is particularly well defined for large perturbation frequencies, while the large-distance behaviour is clearly seen for small Ω.

The dependence of A_i on d_i can be understood by relating the connection matrix $\mathcal{J}$ with the distribution of distances in the network. The starting point is the solution to eq. (4), $\boldsymbol{A} = \mathcal{M}^{-1}\boldsymbol{e}_1$. Since the eigenvalues of the connection matrix $\mathcal{J}$ are all less than or equal to z in modulus [6], the matrix $\mathcal{M}^{-1} = [(z + \mathrm{i}\Omega)\mathcal{I} - \mathcal{J}]^{-1}$ can be expanded as a series of powers of $\mathcal{J}$ (for $\Omega \neq 0$). The amplitude of the deviation from the fully synchronized state for the i-th oscillator turns out to be

$$A_i = \sum_{m=0}^{\infty} (z + \mathrm{i}\Omega)^{-m-1} J_{i1}^{(m)}, \tag{5}$$

where $J_{ij}^{(m)}$ is an element of the m-th power of the connection matrix: $\mathcal{J}^m = \{J_{ij}^{(m)}\}$. The matrix $\mathcal{J}^m$ bears explicit information about the metric structure of the network. Specifically, the element $J_{ij}^{(m)}$ equals the number of directed paths of length m starting at node j and ending at node i [7]. Taking $j = 1$, *i.e.* the node at which the external perturbation is applied, we have in particular that $J_{i1}^{(m)} = 0$ for $m < d_i$ and $J_{i1}^{(m)} \neq 0$ for $m = d_i$. If $m > d_i$, $J_{i1}^{(m)}$ is zero if no path of length m joins oscillators 1 and i, and positive otherwise.

For oscillator i, thus, the first contribution to the sum in eq. (5) is the term with $m = d_i$. If $z/N \ll 1$ and d_i is small, there is essentially only one path of length d_i from 1 to i. This is realized by noticing that the probability of having more than one path of small length between two given oscillators is, at most, of order z/N. Consequently, for the majority of nodes at small distances from the perturbed oscillator, we have $J_{i1}^{(d_i)} = 1$. The total fraction $n(d_i)$ of nodes with small d_i is also small, $n(d_i) \approx z^{d_i}/N$ and, therefore, the possibility that a node at distance d_i is also connected by a path of length slightly larger than d_i can be neglected. This implies that, for oscillators at small distances d_i, the sum in eq. (5) is dominated by the first nonzero term:

$$A_i \approx (z + \mathrm{i}\Omega)^{-d_i-1} = \left(z^2 + \Omega^2\right)^{-\frac{d_i+1}{2}} \exp\left[-\mathrm{i}(d_i+1)\tan^{-1}\frac{\Omega}{z}\right]. \tag{6}$$

This dominance is enhanced as $|z+\mathrm{i}\Omega|$ becomes large, since higher-order terms are weighted by increasing powers of this number. In particular, the contribution of the first nonzero term becomes increasingly important as the frequency Ω grows. In the right-hand side of eq. (6), the exponential dependence of $|A_i|$ and the linear dependence of φ_i on d_i, with a phase shift $\Delta\varphi = \tan^{-1}(\Omega/z)$, are apparent. Full straight lines in fig. 1 show the quantitative agreement of this prediction with numerical results for small distances. Note that, as expected, the exponential approximation improves for larger Ω.

For large distances, such that $z^{d_i} \sim N$, it is not any longer possible to insure that the first contribution to A_i is given by only one path of length d_i, nor that higher-order contributions are relatively negligible. In fact, one can argue that the number of paths of length m ending at a given node i scales as $J_{i1}^{(m)} \sim z^m$ for large m. This can be proven inductively, noting that this number is z times the number of paths of length $m-1$ ending at the oscillators to which i is coupled. This result is confirmed by the calculation of the average value of $J_{ij}^{(m)}$ as an element of $\mathcal{J}^m$, under the hypothesis that the elements J_{ij} of $\mathcal{J}$ are independent random variables. Assuming, thus, $J_{i1}^{(m)} = J_0 z^m$, where the constant J_0 may sensibly depend on the specific realization of the random network, we find

$$A_i \approx J_0 \sum_{m=d_i}^{\infty} (z + \mathrm{i}\Omega)^{-m-1} z^m = \frac{J_0}{\Omega}\left(1 + \frac{\Omega^2}{z^2}\right)^{-d_i/2} \exp\left[-\mathrm{i}\left(\frac{\pi}{2} + d_i \tan^{-1}\frac{\Omega}{z}\right)\right]. \tag{7}$$

Comparing with eq. (6), we realize that the dependence of $|A_i|$ on d_i is now much slower, especially, for small Ω. On the other hand, the phase shift between oscillators whose distances differ by one, $\Delta\varphi = \tan^{-1}(\Omega/z)$, is the same as for small distances. When Ω is small, the only difference is an additional shift of $-\pi/2$. This is clearly confirmed by numerical results: the dotted lines in fig. 1b are displaced by $-\pi/2$ with respect to the corresponding full lines, which stand for the small-d_i analytical prediction. An independent confirmation of eq. (7) is

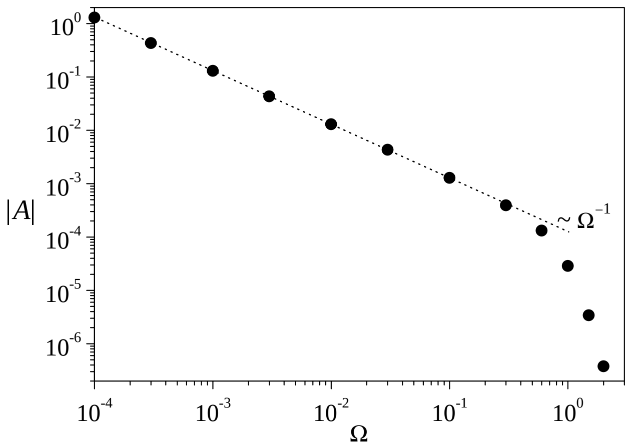

Fig. 2 – Amplitude modulus at the maximal distance, $d_i = 15$, for the 10^3-oscillator random network considered in fig. 1, as a function of the perturbation frequency Ω. The dotted line has slope -1.

given by the fact that for small Ω, large d_i, and a given connection network, the amplitude modulus should behave as $|A_i| \sim \Omega^{-1}$. This is shown in fig. 2 for the same network as in fig. 1, at the maximal distance, $d_i = 15$. Note that this behaviour can be interpreted as a resonance-like phenomenon in the response of the oscillators —whose natural frequency is $\omega_i = 0$— to the perturbation. The external harmonic action induces a larger effect when its frequency is closer to the individual frequency of the dynamical components of the system.

Though the results discussed so far apply to random oscillator networks, our analytical approach can be used for arbitrary connection patterns. In the special case of regular networks we can even obtain exact results. Consider, for instance, a directed ring, *i.e.* a linear array with periodic boundary conditions, where each oscillator is coupled to its nearest neighbour to the left ($z = 1$). In this situation, $J_{ij} = 1$ if $i = (j + 1) \mod N$ and 0 otherwise, which implies that $J_{i1}^{(m)}$ vanishes unless $m = d_i + kN$ $(k = 0, 1, 2, \ldots)$. Explicitly calculating the amplitude from eq. (5), we find

$$A_i = \left[1 - (1 + \mathrm{i}\Omega)^{-N}\right]^{-1} (1 + \mathrm{i}\Omega)^{-d_i - 1}. \tag{8}$$

Here, the amplitude modulus decays exponentially for all d_i and the phase shift between oscillators whose distances differ by one is constant, $\Delta\varphi = \tan^{-1}\Omega$.

Another situation where the decay of the amplitude modulus is purely exponential is the case of a tree-like connection network. Trees are graphs which contain no cycles. In directed trees, consequently, there is at most one path joining any two nodes, and therefore $z = 1$. Under such conditions, only one term in eq. (5) contributes to the amplitude, and

$$A_i = (1 + \mathrm{i}\Omega)^{-d_i - 1}. \tag{9}$$

Those nodes which cannot be reached from the perturbed oscillator have $A_i = 0$.

If the number z_i of connections ending at each node is not uniform over the network, eq. (3) can still be reduced to (4). Now, however, $\mathcal{M} = \mathcal{Z} + \mathrm{i}\Omega\mathcal{I} - \mathcal{J}$, where $\mathcal{Z}$ is the diagonal matrix with elements $Z_{ij} = z_i\delta_{ij}$. In this situation, we cannot insure that the eigenvalues of the connection matrix $\mathcal{J}$ are bounded in modulus, and therefore we are in general not able to expand $\mathcal{M}^{-1}$ in order to obtain an expression similar to eq. (5). Whereas we may numerically

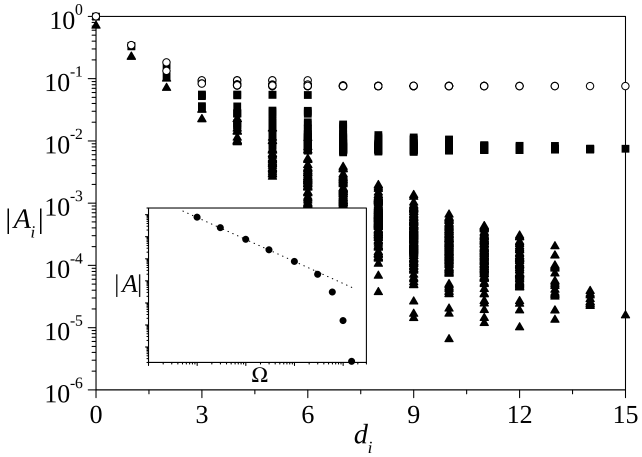

Fig. 3 – Amplitude modulus as a function of the distance from the perturbed oscillator in a 10^3-oscillator random network where the number of neighbours of each node is $z_i = 1$, 2 or 3 with equal probability. Symbols are as in fig. 1. The inset shows the amplitude modulus at the maximal distance, $d_i = 15$, as a function of the perturbation frequency. Axis scales in the inset are the same as in fig. 2. The dotted line has slope -1.

solve eq. (4) by inverting $\mathcal{M}$, our analytical approach does not apply anymore. Moreover, for such an arbitrary connection pattern the stability of the fully synchronized state is not guaranteed, and has to be separately analyzed for each realization of the network.

Figure 3 shows numerical results for the amplitude modulus as a function of the distance from the perturbed oscillator, for a 10^3-oscillator random network with a random distribution of z_i. Specifically, z_i is chosen to be 1, 2, or 3 with equal probability, in such a way that the average value of z_i equals the value of z for the network considered in figs. 1 and 2. Once z_i has been determined for each node, its z_i neighbours are selected at random. For the network realization of fig. 3 the maximal distance is, again, $d_i = 15$. We find that, qualitatively, $|A_i|$ shows the same dependence on d_i and Ω as for the regular graph. As may have been expected, however, there is a larger dispersion in the amplitude modulus for elements at a given distance from the perturbed oscillator, especially, at large frequencies. The value of $|A|$ at the maximal distance, shown in the inset of fig. 3, exhibits the same dependence on the perturbation frequency as in the former case.

In summary, we have shown that the response of a network of synchronized phase oscillators to a local harmonic perturbation of external origin exhibits two well-defined regimes, depending on the distance to the point at which the perturbation is applied. At small distances, the system behaves as a dissipative medium. For a variety of connection patterns, the perturbation propagates through the network at constant velocity, while its amplitude decreases exponentially with the distance. In random networks, this behaviour holds up to certain threshold distance, beyond which the variation of the amplitude with the distance is much slower. The smaller the perturbation frequency, the larger the amplitude at long distances and the smaller the threshold. Our analytical approach, which applies to regular graphs, suggests that the existence of these two regimes is a consequence of the geometrical properties of the network, associated with the number of paths connecting the perturbed oscillator with any other node at a given distance. The analysis is linear, and the solutions for harmonic perturbations studied here can be combined to describe any other perturbation of sufficiently small amplitude. Nonlinear effects in the propagation of perturbations, which are expected to play a role for stronger perturbations, will be the subject of a future contribution.

Let us finally point out that the present results establish a direct link between the dynamics of an ensemble of coupled oscillators and the geometry of the connection network. This suggests a method to explore the structure of connections if the individual activity of oscillators is accessible through measurements. Determining the response of each oscillator to external perturbations applied to different elements of the ensemble could make it possible to reconstruct the underlying interaction pattern.

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