Neutron Stars

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December 2021

1 Introduction

This small document serves as a guide into the theoretical tools implemented in the program Neutron Stars, which solves the Newtonian equations of equilibrium for a neutron star and their general relativistic corrections, the Tolman-Oppenheimer-Volkoff (TOV) equations. In the first section, a short detour over neutron stars and the possible equations of state for their interior are given; the numerical methods used in the program are shown in the second section, while in the third one the results for each equation of state at a fixed central density are plotted.

2 Neutron stars

2.1 A general overview

Neutron stars are a possible final stage of the evolution of stars. This stage is reached when the helium in the star's core is burned up to heavier elements like carbon and oxygen: then, the nuclear processes stop, the temperature decreases, the star shrinks and the pressure in the core increases. If the mass of the star is above a certain value, new fusion processes can still be started; however, a big contraction of the star leads to a bigger pressure in the core and this pressure can blow away the outer layers of the star. What is left is a compact core and if the initial mass was higher then eight solar masses we can have a neutron star: in fact, in the core we have enough mass to go on with the fusion of carbon and oxygen while in the immediate external region of the core, hydrogen and helium fusions take place, keeping the whole system in equilibrium with gravity. Once iron is produced in the inner regions, the reactions here stop and there is not enough pressure resisting to gravity. As a result, we have a collapse in which beta reactions occur between nuclei and electrons to form neutron-richer elements, and the density increases until we get a core which is neutron-dominated. The remnant of the outer layers is ejected in a shock wave, that is, a supernova. If the left core mass is higher than some solar masses, the collapse ends in a black hole formation, otherwise, as we have said, a neutron star is born. To start modelizing the interior of the neutron star, we will start by considering a neutron star with only degenerate neutrons. In this case, notice that the hydrostatic equilibrium is kept by the Pauli principle, which tells that two fermions cannot occupy the same slot in the phase space: therefore, the neutrons fill the phase space with increasing momentum when the density increases and this turns in a degenerate pressure which compensates the self gravity of the neutron star. In what follows, we consider both the non relativistic and the full relativistic case for a so called pure neutron star: the latter was the model used in 1939 by Oppenheimer for solving the TOV. Of course, this is a very rough approximation, since we should take into account the presence of small fractions of other particles as electrons, protons, muons, hyperons and quarks: then more sophisticated models can be built taking into account nuclear interactions. As an output, the mass and the radius of the star is given, since these are the two main astrophysical observables and could be used as constraints on the models for the interior of the star.

2.2 Equilibrium equations

In the next sections, we will set $c = 1$ but we keep the value of $\hbar$. As we have said, there are two kind of forces which keep the equilibrium for the star: the particles pressure and the gravitational pull. Combining them using the Newtonian gravitational law gives:

$$\frac{dp(r)}{dr} = -\frac{G\rho(r)m(r)}{r^2} \tag{1}$$

and

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 \tag{2}$$

which is a system of coupled ordinary differential equations yelding to the pressure and the mass as functions of the radius. These equations can be solved only if we are given an equation of state which expresses the pressure as a function of the density, and appropriate boundary conditions, in our case $m(r = 0) = 0$ and $p(r = 0) = 0$.

A correction for this system is given by general relativity: considering an isotropic, static and ideal fluid in equilibrium and its energy-momentum tensor, the field equations replace the expression for the derivative of the pressure:

$$\frac{dp(r)}{dr} = -\frac{G\rho(r)m(r)}{r^2}[1 + \frac{p(r)}{\rho(r)}][1 + \frac{4\pi r^3 p(r)}{m(r)}][1 - \frac{2Gm(r)}{r}]^{-1} \tag{3}$$

Together with the definition of the mass function, we get the so called TOV equations. Since the three terms in this correction are bigger then one, the new equation strengths the gravitational attraction, i.e. we expect a bigger mass in a smaller radius with respect to Newton.

2.3 Pure neutron stars

Next, provide an equation of state by modelling the interior of the star. Begin with the simple case of a pure neutron gas: the distribution of the particles follow the Fermi-Dirac statistic

$$f(E) = \frac{1}{e^{\frac{E-\mu}{k_b T}} + 1} \tag{4}$$

and therefore the quantities

$$n = \int \frac{g f(k)}{(2\pi\hbar)^3} d^3k \tag{5}$$

$$\epsilon = \int \frac{g E(k) f(k)}{(2\pi\hbar)^3} d^3k \tag{6}$$

$$p = \int \frac{g k^2 f(k)}{E(k)(2\pi\hbar)^3} d^3k \tag{7}$$

define the number, the energy and the pressure of the neutron gasses in a unit phase space box, with g the degeneracy and $\frac{1}{(2\pi\hbar)^3}$ the phase space volume element. For these free fermions hold the following condition

$$(\mu - m_e) >> k_b T \tag{8}$$

which allows us to consider the system at zero temperature and approximate the Fermi-Dirac function with a Heaviside function centered at the Fermi level (strictly speaking, this is a Sommerfeld expansion at zero order). Hence we get:

$$n = \int_0^{k_F} \frac{g f(k) 8\pi k^2}{(2\pi\hbar)^3} dk = \frac{k_F^3}{3\pi^2\hbar^3} \tag{9}$$

or

$$k_F = \hbar (\frac{3\pi^2 \rho}{m_N})^{1/3} \tag{10}$$

with ρ the rest mass density,

$$\epsilon = \int_0^{k_F} \frac{g E(k) f(k) 8\pi k^2}{(2\pi\hbar)^3} dk = \frac{\epsilon_0}{8}[(2x^3 + x)(1+x^2)^{1/2} - \sinh^{-1}(x)] \tag{11}$$

$$p = \int_0^{k_F} \frac{g k^2 f(k) 8\pi k^2}{E(k)(2\pi\hbar)^3} dk = \frac{\epsilon_0}{24}[(2x^3 - 3x)(1+x^2)^{1/2} + 3\sinh^{-1}(x)] \tag{12}$$

where $\epsilon_0 = \frac{m_N^4}{\pi^2\hbar^3}$ and $x = \frac{k_F}{m_N}$.
As a first step we can consider the non-relativistic regime, where $k_F << m_N$ or $x << 1$. The above quantities become:

$$\epsilon = \rho \tag{13}$$

$$p = \frac{\hbar^2}{15\pi^2 m_N} (\frac{3\pi^2\rho}{m_N})^{5/3} = K_{nr} \epsilon^{5/3} \tag{14}$$

This particular form for the equation of state is called polytropical: it can be thrown in the Newton structure equations system or in the TOV system to get

the mass and the radius of a neutron star. Notice that while $m(r = 0) = 0$, $\epsilon(r = 0) \neq 0$ and so will be the pressure in the center of the star.
The full relativistic equation of state cannot be given explicitly, due to the complicated analytic structure of the p, ϵ expressions. We can take two routes to solve this problem. We can give reasonable values for the pressure or the energy density, i.e. we specify the LHS of these expressions, let's say for the pressure, we bring the value to the RHS and solve numerically the equation. This will give a set of Fermi momenta which can be casted in the equation for the energy to get two arrays of solutions ready to be plugged in the structure equations. Otherwise, we can give a range of Fermi momenta to both the equations, we can extract two sets of points, one for the ϵ and one for the p, and then interpolate them. The result of this interpolation will be plugged in the equilibrium equations. In the presented results, the second route was taken, as it seemed more computationally convenient.

2.4 Including the interactions

Interactions could be introduced in very different ways, so instead of focusing on a single method we can incorporate more of them if we follow the article of Read et al. The main idea is to split the neutron star into two sections, the crust and the core. For the crust, we adopt a piecewise polytropic equation of state which is based on a SLy model. This equation is "kept fixed" if we vary the core model, i.e. we assume with a little loss of generality that all neutron stars have the same crust. For the core, we use a piecewise polytropic type equation of state again, but this time we vary the composition of the core and the method we used to get the parameters of the equation of state. All the parameters and the details of each model are presented in Appendix B, C of Read.

2.4.1 Building the piecewise polytropic

We start from a pressure polytropic in the rest mass density:

$$p(\rho) = K\rho^{\Gamma} \tag{15}$$

The energy density can be found using the first law of thermodynamics and reads

$$\epsilon = (1 + a)\rho + \frac{1}{\Gamma - 1} K\rho^{\Gamma} \tag{16}$$

with a constant of integration which has to be fixed. Next, consider a sequence of rest mass density intervals: each interval has its own polytropic equation for the pressure and its own energy density equation. The first constant a_0 is fixed by requiring

$$\lim_{\rho \to 0} \frac{\epsilon}{\rho} = 1 \tag{17}$$

i.e. no energy at all when we reach the star surface; this gives $a_0 = 0$. The other constants can be fixed by requiring the continuity of the energy density on the endpoints of each interval and yelds to:

$$a_i = \frac{\epsilon(\rho_{i-1})}{\rho_{i-1}} - 1 - \frac{K_i}{\Gamma_i - 1}\rho_{i-1}^{\Gamma_i - 1}$$

The polytropical constants are not required as parameters since they can be evaluated by asking for the continuity of the pressure on the endpoints of each interval:

$$K_{i+1} = \frac{p(\rho_i)}{\rho_i^{\Gamma_{i+1}}}$$

To sum up, the best fit of the interior of a neutron star made of interacting particles, is given by a patch of two piecewise polytropic equations of state, one for the crust and one for the core, with fixed tropes and parameters depending by the method of evaluating the nuclear interactions and the particles taken into account.

3 Numerical algorithms

To get the mass and the pressure as functions of the radius, i.e. solve the equilibrium equations, we need to go numerical.

3.1 Runge-Kutta method

From all the algorithms, an adaptive-step Runge-Kutta was chosen. The basic idea behind a Runge-Kutta is to make an intermadiate step for the evaluation of the function and we illustrate it by looking at the so called RK2 method. We can rewrite $\frac{dy}{dt} = f(t, y)$ as

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y)dt \tag{18}$$

and then expand $f(t, y)$ in Taylor series around the center of the $[t_i, t_{i+1}]$ interval. Then the integral can be rewritten as

$$\int_{t_i}^{t_{i+1}} f(t, y)dt = hf(t_i + \frac{h}{2}, y(t_i + \frac{h}{2})) \tag{19}$$

but since we don't know y at $t_i + \frac{h}{2}$, we evaluate it through Euler method, i.e.

$$y(t_i + \frac{h}{2}) = y_i + \frac{h}{2}f(t_i, y_i) \tag{20}$$

so

$$y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)) \tag{21}$$

We see that we have used an intermediate step to get a better value for the slope, and in turn a better value for the y_{i+1} (order $O(h^3)$).

The method implemented to solve the structure equations in the program Neutron Stars, is the RK45: it makes use of six intermediate evaluations and of an adaptive step. This means that, once given a certain tolerance parameter ξ and a maximum step size h_{max}, at every cycle the algorithm compares the evaluated solution with another solution calculated with an higher order RK. Then the difference ϵ between this two solutions is matched with the tolerance and we ask:

- if $\epsilon > \xi$ decrease the step
- if $\epsilon < \xi$ increase the step

This philosophy grants a large step and a saving of computational effort when the function is smooth, and grants a small step, i.e. a more careful evaluation of the solution, when the function varies sharply or oscillates. Finally, a particular attention was put on the outcomes of each intermediate evaluation, since it was noticed that few of them could become negative, generate an overflow and then a NAN result. To fix this, given the fact that the functions which have to be integrated are relatively well-behaved, and so the overflow occurred only for a couple of times over a huge amount of evaluating cycles, in the final algorithm we impose to break the calculation when one step becomes negative and go on to the next cycle.

3.2 Cubic spline

In this subsection the interpolation method used for the full relativistic pure neutron star is presented.
Given $n+1$ data points (x_i, y_i) with $a < x_1 < ... < b$ we call cubic spline a function such that:

- it is $C^2[a, b]$
- on each interval of the data x_i it is a degree 3 polynomial
- it fits the datas y_i

therefore we can write it like this

$$S(x) = C_i(x) \qquad on \quad x_{i-1} < x < x_i \tag{22}$$

with the conditions like

$$C_i(x_{i-1}) = y_{i-1} \qquad i = 1, ..., n \tag{23}$$

$$C_i(x_i) = y_i \qquad i = 1, ..., n \tag{24}$$

$$C_i'(x_i) = C_{i+1}'(x_i) \qquad i = 1, ..., n-1 \tag{25}$$

$$C_i''(x_i) = C_{i+1}''(x_i) \qquad i = 1, ..., n-1 \tag{26}$$

As we can understand by counting, we miss two equations to get a $4n \times 4n$ system. Therefore we impose two boundary conditions on the endpoints:

$$C_1''(x_0) = C_n''(x_n) = 0 \tag{27}$$

Next, we need to implement an effective method to solve this system. Calling $S''(x_i) = M_i$ we have the following identities:

$$S''(x_i) = C_i''(x_i) = C_{i+1}''(x_i) = M_i \qquad i = 1, ..., n-1 \tag{28}$$

$$S''(x_0) = C_1''(x_0) = M_0 \qquad S''(x_n) = C_n''(x_n) = M_n \tag{29}$$

Working with second derivatives is nice, because they are all linear functions since the $S(x)$ is cubic. Then, calling $h_i = x_i - x_{i-1}$ we use Lagrangian interpolation for each C_i'' of the subintervals, i.e.

$$C_i''(x) = M_{i-1}\frac{x_i - x}{h_i} + M_i\frac{x - x_{i-1}}{h_i} \qquad for \quad x \in [x_{i-1}, x_i] \tag{30}$$

This equation can be integrated twice and the additive constants are fixed using the conditions $C_i(x_{i-1}) = M_{i-1}$ and $C_i(x_i) = M_i$. We obtain

$$C_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + (y_{i-1} - \frac{M_{i-1}h_i^2}{6})\frac{x_i - x}{h_i} + (y_i - \frac{M_i h_i^2}{6})\frac{x - x_{i-1}}{h_i} \tag{31}$$

$$for \quad x \in [x_{i-1}, x_i]$$

However, even if this form for the C_i can be rewritten as a degree-three polynomial by reshuffling the terms, the M_i functions are still unknown, so we need to find them using the other equations and boundary conditions; this means we need to re-derive what we have found, evaluating it at the endpoints of each interval and impose the continuity of the derivatives, i.e. $C_{i+1}'(x_i) = C_i'(x_i)$

$$C_{i+1}'(x_i) = -M_i\frac{h_{i+1}}{2} + \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{M_{i+1} - M_i}{6}h_{i+1} \tag{32}$$

$$C_i'(x_i) = M_i\frac{h_i}{2} + \frac{y_i - y_{i-1}}{h_i} - \frac{M_i - M_{i-1}}{6}h_i \tag{33}$$

This yelds to

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = d_i \tag{34}$$

where

$$a_i = \frac{h_i}{h_i + h_{i+1}}$$

$$b_i = 2$$

$$c_i = \frac{h_{i+1}}{h_i + h_{i+1}}$$

$$d_i = 6f[x_{i-1}, x_i, x_{i+1}]$$

and

$$f[x_{i-1}, x_i, x_{i+1}] = \frac{\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}}{h_i + h_{i+1}} \qquad i = 1, ..., n-2$$

$$f = 0 \qquad otherwise$$

The system of equations is now reduced into a tridiagonal-matrix-form

$$\begin{pmatrix} 2 & 0 & 0 & ... & 0 & \\ a_1 & 2 & c_1 & ... & 0 & \\ 0 & 0 & ... & 0 & 0 & \\ 0 & ... & a_{n-1} & 2 & c_{n-1} & \\ 0 & ... & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ . \\ . \\ . \\ M_n \end{pmatrix} = \begin{pmatrix} d_0 \\ . \\ . \\ . \\ d_n \end{pmatrix} \tag{35}$$

which can be solved using Thomas algorithm, i.e. by mapping the old coefficients into new ones that take the form

$$c'_1 = \frac{c_1}{b_1} \tag{36}$$

$$c'_i = \frac{c_i}{b_i - a_i c'_{i-1}} \tag{37}$$

and

$$d'_1 = \frac{d_1}{b_1} \tag{38}$$

$$d'_i = \frac{d_i - a_i d'_{i-1}}{b_i - a_i c'_{i-1}} \tag{39}$$

Now the system is ready to be inverted and solved by applying a simple substitution technique.

4 Results

4.1 Non Relativistic Pure Neutron Star

Newton mass: 0.37 solar masses
Newton radius: 20.71 km
TOV mass: 0.38 solar masses
TOV radius: 20.04 km

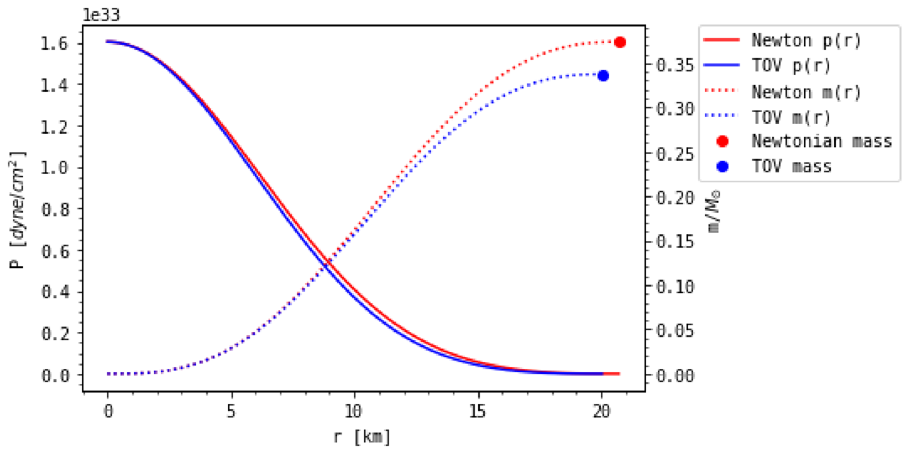

Figura 1: Non relativistic equation of state for a pure neutron star.

4.2 Relativistic Pure Neutron Star

Newton mass: 1.29 solar masses
Newton radius: 10.98 km
TOV mass: 0.71 solar masses
TOV radius: 9.22 km

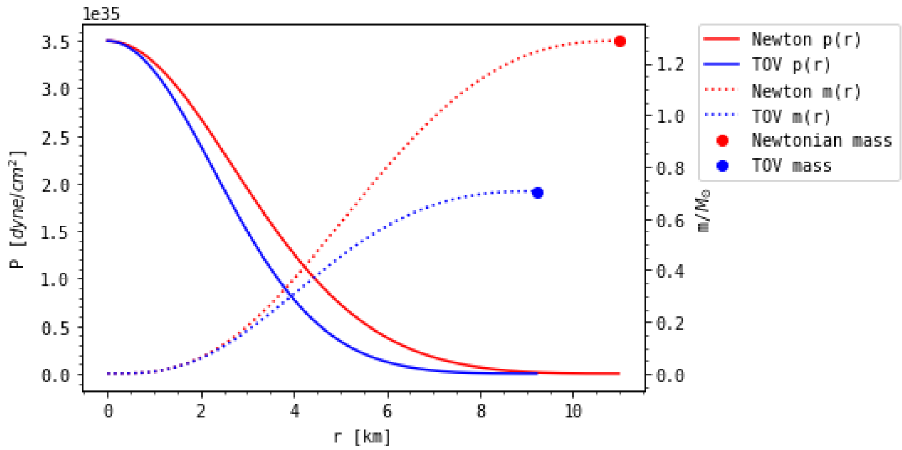

Figura 2: Relativistic equation of state for a pure neutron star.

4.3 Example of Read Neutron Star

Newton mass: 21.25 solar masses
Newton radius: 20.38 km
TOV mass: 2.02 solar masses
TOV radius: 11.35 km
Notice the completely nosense of Newton results: this is due to the fact that the parameters choice was made in a purely GR regime.

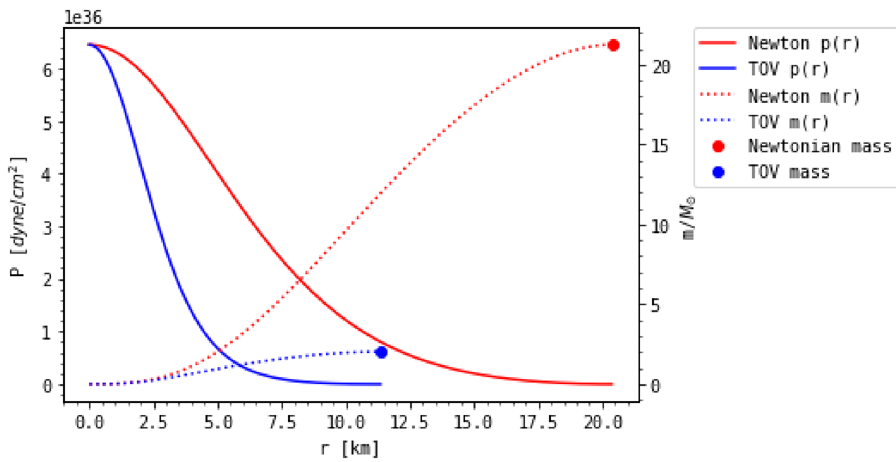

Figura 3: PAL6-type Read equation of state.

5 Bibliography

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