(2) Lipschitz bound on f:

$$f(\xi) = \frac{1}{\sqrt{1+\varepsilon^2\xi^2}}.$$

$$f'(\xi) = \frac{-\varepsilon^2\xi}{(1+\varepsilon^2\xi^2)^{\frac{3}{2}}}.$$

$f' \to 0$ as $\xi \to \pm\infty$. So $\max|f'|$ at a turning point.

$$f''(\xi) = \frac{2\varepsilon^4\xi^2 - \varepsilon}{(1+\varepsilon^2\xi^2)^{\frac{5}{2}}}.$$

$$f''(\xi^*) = 0 \quad \text{for } \xi^* = \frac{\pm 1}{\varepsilon\sqrt{2}}.$$

$$|f'(\xi^*)| = \frac{\frac{1}{\sqrt{2}}\varepsilon}{(1+\frac{1}{2})^{\frac{3}{2}}} = \frac{2\varepsilon}{\sqrt{3}^3}.$$

By MVT, Lipschitz bound on f:

$$\underline{|f(\xi_1) - f(\xi_2)| \leq \frac{2\varepsilon}{\sqrt{3}^3}|\xi_1 - \xi_2|.}$$

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(2) $f^{\varepsilon}(\xi) = \dfrac{1}{\sqrt{1+\varepsilon^2 \xi^2}}$.

$$\int_{\mathbb{T}} \partial_t (u_2 - u_1)(u_2 - u_1)\, dx = \kappa \int_{\mathbb{T}} \partial_x \big(f(\partial_x w_2) \partial_x u_2 - f(\partial_x w_1) \partial_x u_1 \big)(u_2 - u_1)\, dx$$

$$\frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} (u_2 - u_1)^2 dx = -\kappa \int_{\mathbb{T}} (\partial_x u_2 - \partial_x u_1) \underbrace{\big(f(\partial_x w_2) \partial_x u_2 - f(\partial_x w_1) \partial_x u_1 \big)} dx.$$

$$= f(\partial_x w_2) \partial_x u_2 - f(\partial_x w_2) \partial_x u_1 + f(\partial_x w_2) \partial_x u_1 - f(\partial_x w_1) \partial_x u_1$$

$$= f(\partial_x w_2)(\partial_x u_2 - \partial_x u_1) + (f(\partial_x w_2) - f(\partial_x w_1)) \partial_x u_1.$$

$$\frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} |u_2 - u_1|^2 dx = -\kappa \int_{\mathbb{T}} f(\partial_x w_2)(\partial_x u_2 - \partial_x u_1)^2 + (f(\partial_x w_2) - f(\partial_x w_1))(\partial_x u_2 - \partial_x u_1) \partial_x u_1\, dx.$$

Same

Symmetry.

$$\frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} |u_2 - u_1|^2 dx = -\kappa \int_{\mathbb{T}} f(\partial_x w_1)(\partial_x u_2 - \partial_x u_1)^2 + (f(\partial_x w_1) - f(\partial_x w_2))(\partial_x u_1 - \partial_x u_2) \partial_x u_2\, dx$$

$$\frac{d}{dt} \int_{\mathbb{T}} |u_2 - u_1|^2 dx = -\kappa \int_{\mathbb{T}} (f(\partial_x w_2) + f(\partial_x w_1))(\partial_x u_2 - \partial_x u_1)^2 + (f(\partial_x w_2) - f(\partial_x w_1)) \underline{(\partial_x u_2 - \partial_x u_1)(\partial_x u_1 + \partial_x u_2)}\, dx.$$

$$= -\kappa \int_{\mathbb{T}} (f(\partial_x w_2) + f(\partial_x w_1)) \underline{(\partial_x u_2 - \partial_x u_1)}^2 + (f(\partial_x w_2) - f(\partial_x w_1))((\partial_x u_2)^2 - (\partial_x u_1)^2)\, dx.$$

~~Reverse triangle inequality~~ $\cancel{(\partial_x u_2)^2 - (\partial_x u_1)^2 \le (\partial_x u_2 - \partial_x u_1)^2}$

$$\frac{d}{dt}\int_{\mathbb{T}} |u_2-u_1|^2\,dx + \kappa\int_{\mathbb{T}} \big(f(\partial_x w_2)+f(\partial_x w_1)\big)(\partial_x u_2 - \partial_x u_1)^2\,dx$$

$$= -\kappa \int_{\mathbb{T}} \underbrace{\big(f(\partial_x w_2) - f(\partial_x w_1)\big)}\big((\partial_x u_2)^2 - (\partial_x u_1)^2\big)\,dx.$$

You want ^{2} to get Lipschitz bound.
Maybe use a Cauchy ~~w~~ ε-inequality?

$$\le \kappa\int_{\mathbb{T}} \underbrace{\frac{1}{2\cancel{\varepsilon}}\big((\partial_x u_2)^2-(\partial_x u_1)^2\big)^2}_{\text{Looks bad for contractivity.}} + \frac{1}{2}\cancel{\varepsilon}\big(f(\partial_x w_2)-f(\partial_x w_1)\big)^2\,dx.$$

Regardless,

$$\le \kappa \int_{\mathbb{T}} \frac{2\varepsilon}{\sqrt{3}^3}\, |\partial_x w_2 - \partial_x w_1|\, \underbrace{\big((\partial_x u_2)^2 - (\partial_x u_1)^2\big)}_{\text{Looks like } \|\partial_x u_2\|_{L^2}^2 - \|\partial_x u_1\|_{L^2}^2.}\,dx.$$

Maybe it is Cauchy?

$$\le \frac{2\kappa\varepsilon}{\sqrt{3}^3}\int_{\mathbb{T}} \frac{1}{2}|\partial_x w_2-\partial_x w_1|^2 + \frac{1}{2}\big((\partial_x u_2)^2-(\partial_x u_1)^2\big)^2\,dx$$

$$= \frac{\cancel{2}\kappa\varepsilon}{\sqrt{3}^3}\left(\cancel{\tfrac{1}{2}}\,\|\partial_x w_2 - \partial_x w_1\|_{L^2}^2 + \cancel{\tfrac{1}{2}}\,\|(\partial_x u_2)^2 - (\partial_x u_1)^2\|_{L^2}^2\right)$$

Can introduce ε if needed

Bounded by $\|\partial_x u_2 - \partial_x u_1\|_{L^2}^2$?

Generalisation:

$$\partial_t u - \mathcal{K}\,\partial_x\left(f^\varepsilon(\partial_x u)\,\partial_x u\right) = 0, \quad u(x,0) = u_0(x).$$

f^ε uniformly Lipschitz with some small $\varepsilon \to 0$:

$$\left|f^\varepsilon(\xi_1) - f^\varepsilon(\xi_2)\right| \leq L_\varepsilon\,|\xi_1 - \xi_2|.$$

where $L_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Aim is to prove that

$A : H^1 \to H^1$ is a contractive map.

Start with $A: C^2 \to C^2$ and conclude by density.

requires continuity?

AIM:

$$\|u_1 - u_2\|_{L^2([0,T];H^1)} \leq \frac{1}{\theta}\,\|w_1 - w_2\|_{L^2([0,T];H^1)}.$$

for some $\theta > 1$. (provided ε is small enough).

u_i are solutions to $\begin{cases} \partial_t u_i - \mathcal{K}\,\partial_x\left(f^\varepsilon(\partial_x w_i)\,\partial_x u_i\right) = 0. \\ u_i(x,0) = u_0(x). \end{cases}$