The Stability of Lagrangian Points in the Circular Restricted Three Body Problem

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1 Introduction

At 12.20 UTC on the 25^{th} December, 2021, the James Webb Space Telescope (JWST) launched from Guiana Space Centre on a month–long journey to its destination: a halo orbit around the L_2 Lagrange point of the Earth–Sun system. Since launch it has conducted infrared observations within our galaxy, such as the Carina Nebula [2], and of other galaxies such as Stephan's Quintet [3].

Of course, it is an immense challenge to place a telescope in space, and among these challenges is deciding where to put it. The orbit of JWST has been carefully chosen to let it carry out its mission with relatively little rocket thrust keeping it in place. More thorough rationale for the orbit dimensions can be found in the Observatory Characteristics section of the JWST user documentation [1].

In this essay, we shall investigate the decision to send JWST to L_2 by calculating the locations of Lagrangian points and analysing their stability. To do this, we'll need to setup the Circular Restricted Three Body Problem (CR3BP) mathematically and define what it means for a Lagrangian point to be stable. We begin by studying some rudimentary Hamiltonian mechanics that will make our analysis easier.

2 Rudiments of Hamiltonian Mechanics

2.1 Lagrangians

To start with, I ought to recommend Hamill's book *A Student's guide to Lagrangians and Hamiltonians*. I found his concise but rigorous treatment of the subject particularly illuminating, and it is from here that I will take several important results, largely omitting thorough justification, to use in subsequent discussions.

Definition 2.1 (Lagrangian)**.** For a system of N particles in d dimensions, let $n = Nd$ be the total number of coordinates.

We give the Lagrangian, $L\colon \mathbb{R}^{2n+1} \to \mathbb{R}$, in terms of *generalised coordinates*[1], $q_1, \ldots, q_n$, their corresponding velocities $\dot{q}_1, \ldots \dot{q}_n$, and time t.

$$L = L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, \dot{q}_i, t),$$

where $T\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ is kinetic energy, and $V\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ is the interaction, or potential energy.

Theorem 2.2 (Lagrange's equation)**.** *For any single arbitrary coordinate q,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{1}$$

[1] for a detailed discussion of generalised coordinates see [7, pp. 5–8]

We use this without proof, though a derivation can be pieced together from [7, § 2.2] and *Hamilton's principle* [7, § 3.2].

We will now define the concept of momentum in generalised coordinates. You may be familiar with $p_x = m\dot{x}$ for a single component in Cartesian, and generalised momentum is similar.

Definition 2.3. A particle is free if it has no interaction, that is $V = 0$.

Observe that the Lagrangian of any such particle in Cartesian coordinates is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

Therefore we find

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x,$$

and similarly for y and z. So we define generalised momentum to have the same property.

Definition 2.4 (Generalised momentum)**.** For any arbitrary coordinate, q_i, the corresponding component of generalised momentum is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

This completes the groundwork for formal consideration of the Hamiltonian.

2.2 Hamiltonians

Definition 2.5 (Hamiltonian)**.** For a system of N particles in d dimensions, let $n = Nd$ be the total number of coordinates.

We give the Hamiltonian, $H\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ in terms of generalised coordinates, momenta, and time as follows:

$$H = H(q_i, p_i, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(q_i, \dot{q}_i, t).$$

This is, in general, equal to the total energy, $T + V$, but not always.

One vital ingredient in formulating equations of motion in any Hamiltonian problem is *Hamilton's equations.* We could use them without proof, but I'll present a short justification omitting the discussion on Legendre transformations [7, § 4.1].

Theorem 2.6 (Hamilton's Equations)**.** *For each coordinate q_i and its momentum p_i,*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{$\star$}$$

Sketch proof. The Hamiltonian is essentially the *Legendre transformation* of the Lagrangian with *active variables*[2] $\dot{q}_1, \ldots, \dot{q}_n$. The important feature of applying this transformation is that it immediately gives the first equality of ($\star$).

For the second, we have that

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}.$$

Now using Lagrange's equation (1) and the definition of p_i we find

$$\dot{p}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}. \qquad \square$$

2.3 Canonical Transformations

The second ingredient for obtaining equations of motion in Hamiltonian systems is canonical transformations. Suppose that we wish to transform coordinates from q_i and their corresponding momenta p_i into Q_i and their corresponding momenta P_i in a way that conserves Hamilton's equations. That is, we have ($\star$) and wish to obtain

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{2}$$

for a new Hamiltonian $K(Q_i, P_i, t)$. It can be shown (as in [7, pp. 111–112]) that the following is a sufficient condition to guarantee equations (2) hold:

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{\mathrm{d}F}{\mathrm{d}t}. \tag{3}$$

The F in (3) can be any arbitrary function of the coordinates (new and old), momenta and time. Such a function is called a *generating function.*

Definition 2.7 (Generating function)**.** This is any $F = F(q_i, p_i, Q_i, P_i, t)$ that satisfies (3).

Of course, there are many possible generating functions for a canonical transformation, as laid out by Hamill [7, p. 114], but for our purposes we will study only one such form.

Proposition 2.8. *Given a generating function $F = F(p_i, Q_i, t)$ with*

$$q_i = -\frac{\partial F}{\partial p_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i}, \tag{4}$$

then $K = H + \frac{\partial F}{\partial t}$ is a Hamiltonian satisfying (2).

[2] The q_i and t are called *passive variables* as they do not partake in the transformation

Proof. (Let's agree to do away with sums and subscripts for convenience) If we set $\tilde{F} = F + pq$, then this, too, is a generating function and

$$\begin{aligned}\frac{\mathrm{d}\tilde{F}}{\mathrm{d}t} &= \frac{\partial \tilde{F}}{\partial p}\dot{p} + \frac{\partial \tilde{F}}{\partial q}\dot{q} + \frac{\partial \tilde{F}}{\partial Q}\dot{Q} + \frac{\partial \tilde{F}}{\partial t} \\ &= \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial Q}\dot{Q} + p\dot{q} + q\dot{p} + \frac{\partial F}{\partial t}.\end{aligned}$$

Substituting this into (3) we get

$$K = H + \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial p} + q\right)\dot{p} + \left(\frac{\partial F}{\partial Q} + p\right)\dot{Q}$$

which, using (4) gives the desired result of $K = H + \frac{\partial F}{\partial t}$. □

3 The Circular Restricted Three Body Problem

3.1 Simplification

The Circular Restricted Three Body Problem is a simplification of the General Three Body Problem, though there are difficulties in solving this directly as outlined by Grutzelius [6]. The problem concerns three bodies, bound by their gravity, two of which (Sun and Earth) are considerably larger than the third (Satellite), and so the effect of the third body on the Sun and Earth can be neglected. We shall further suppose the Satellite moves in the same plane as the Sun and Earth, thus restricting the problem to 2 dimensions[3].

The goal is to find the motion of the Satellite given its initial condition (start position and velocity); we already know the motion of the Sun and Earth by the two problem as presented in appendix A, and we shall assume this motion is circular. We follow Valtonen and Karttunen [12, §§ 5.1–5.5], although alternative approaches can be found in [8, § 2.3][4] and [5, §§ 5.3–5.4].

We first nondimensionalize, making the units are as nice as possible. We choose:

- the origin to be the centre of mass (CoM)
- the total mass of the Sun and Earth to be 1, and the mass of the Earth to be μ. ($0 < \mu < \frac{1}{2}$ and the mass of the Sun is $1 - \mu$)
- the distance between the Sun and Earth to be 1, so that, by the definition of CoM, the distance between the Sun and CoM is μ while the distance between the Earth and CoM is $1 - \mu$.

[3]Though the assumptions made here are broad, they turn out to closely match reality in many contexts, such as the Sun–Earth–JWST system we're concerned with.

[4]This text uses a Newtonian approach which avoids prior work on Hamiltonian mechanics at the cost of obscuring some conceptual features of the problem.

- units of time so that the angular velocity of the Sun and Earth is 1.

This has the effect of removing all constants in the equations except for μ; from the first two changes the gravitational constant, G, equals 1.

Finally we will change coordinate frames from *Inertial* (Fixed) into *Synodic* (Rotating). If the position of the Satellite is $(\underline{\xi}, \underline{\eta})$ in the fixed frame, then its position in the rotating frame will be denoted (ξ, η), related by the following:

$$\begin{aligned} \underline{\xi} &= \xi \cos t - \eta \sin t, \\ \underline{\eta} &= \xi \sin t + \eta \cos t. \end{aligned} \tag{5}$$

In the Synodic frame, the Sun and Earth remain fixed at positions $(-\mu, 0)$ and $(1-\mu, 0)$ respectively. Let ρ_1 and ρ_2 be the distances Sun to Satellite and Earth to Satellite respectively. Figure 1 shows the complete setup in the two coordinate frames.

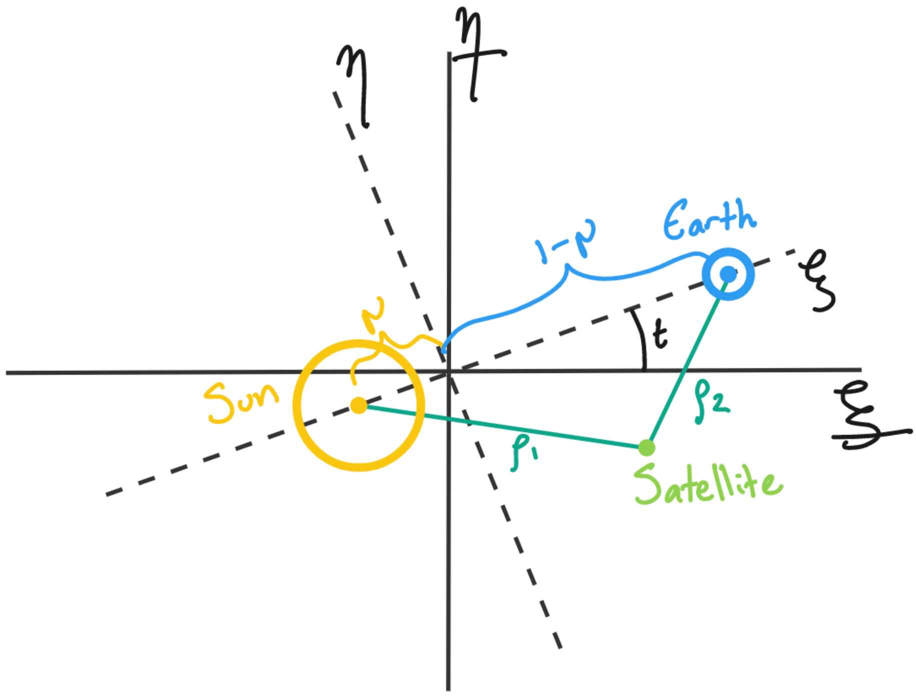

Figure 1: Illustration of Inertial and Synodic coordinate frames. (Not to scale)

3.2 Mathematical Formulation

We start in the Inertial frame as the Hamiltonian of the satellite is straightforward to formulate here. We will then use a canonical transformation to obtain equations in the Synodic frame.

Definition 3.1. In the Inertial frame, the Hamiltonian of the Satellite is

$$H_I = \frac{1}{2}\left(p_{\underline{\xi}}^2 + p_{\underline{\eta}}^2\right) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}$$

where $p_{\underline{\xi}}$ and $p_{\underline{\eta}}$ are the $\underline{\xi}$ and $\underline{\eta}$ components of momentum.

Lemma 3.2. *The Hamiltonian of the Satellite in the Synodic frame is*

$$H_S = \frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + \eta p_\xi - \xi p_\eta - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}$$

where p_ξ and p_η are the ξ and η components of momentum.

Proof. Recall equations (5) for the transformation of coordinates. Observe that the function

$$F(p_{\underline{\xi}}, p_{\underline{\eta}}, \xi, \eta) := -(\xi\cos t - \eta\sin t)p_{\underline{\xi}} - (\xi\sin t + \eta\cos t)p_{\underline{\eta}}$$

is of the form discussed in proposition 2.8 and satisfies

$$\underline{\xi} = -\frac{\partial F}{\partial p_{\underline{\xi}}}, \qquad \underline{\eta} = -\frac{\partial F}{\partial p_{\underline{\eta}}}.$$

This is a generating function for this transformation so we can find that the new momenta are

$$\begin{aligned} p_\xi &= -\frac{\partial F}{\partial \xi} = p_{\underline{\xi}}\cos t + p_{\underline{\eta}}\sin t \\ p_\eta &= -\frac{\partial F}{\partial \eta} = p_{\underline{\eta}}\cos t - p_{\underline{\xi}}\sin t. \end{aligned}$$

Note that $p_\xi^2 + p_\eta^2 = p_{\underline{\xi}}^2 + p_{\underline{\eta}}^2$, so, by proposition 2.8, the new Hamiltonian is

$$\begin{aligned} H_S &= H_I + \frac{\partial F}{\partial t} \\ &= \frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) - \frac{\mu}{\rho_2} - \frac{1-\mu}{\rho_1} + \eta p_\xi - \xi p_\eta. \end{aligned}$$ □

We can now use Hamilton's equations ($\star$) to obtain the equations of motion in the following proposition.

Proposition 3.3 (Equations of motion)**.** *The position of the Satellite (ξ, η) satisfies*

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \frac{\partial \Omega}{\partial \xi} \\ \ddot{\eta} + 2\dot{\xi} &= \frac{\partial \Omega}{\partial \eta} \end{aligned} \tag{$\star\star$}$$

where $\Omega\colon \mathbb{R}^2 \to \mathbb{R}$ *is the so called* Effective potential function*, whose properties will be of great interest in what follows:*

$$\Omega(\xi, \eta) := \frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}.$$

Proof. Directly from $(\star)$ we have

$$\begin{aligned}
\dot{\xi} &= \frac{\partial H_S}{\partial p_\xi} = p_\xi + \eta \\
\dot{\eta} &= \frac{\partial H_S}{\partial p_\eta} = p_\eta - \xi \\
\dot{p_\xi} &= -\frac{\partial H_S}{\partial \xi} = p_\eta + \frac{\partial}{\partial \xi}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) \\
\dot{p_\eta} &= -\frac{\partial H_S}{\partial \eta} = -p_\eta + \frac{\partial}{\partial \eta}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)
\end{aligned}$$

and note that

$$\ddot{\xi} - \dot{\eta} = \dot{p_\xi}, \quad \ddot{\eta} + \dot{\xi} = \dot{p_\eta}.$$

Hence we have, by substituting for p_ξ and p_η as well as the above, that

$$\begin{aligned}
\left(\ddot{\xi} - \dot{\eta}\right) &= (\dot{\eta} + \xi) + \frac{\partial}{\partial \xi}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) \\
\left(\ddot{\eta} + \dot{\xi}\right) &= \left(\eta - \dot{\xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right).
\end{aligned}$$

Setting Ω to be as defined above and rearranging, this gives equations $(\star\star)$. □

3.3 The Potential Function

The potential function Ω is central to studying the CR3BP thanks to a few crucial properties. First is the consideration of *conserved quantities.* Notice that due to the assumptions we've made, namely that the Satellite has no effect on the Sun and Earth, Newton's third law is violated and consequently both Energy and Momentum conservation cannot be used. Instead, we must be clever in finding a conserved quantity known as the *Jacobi constant.*

The trick is to multiply the equations of motion $(\star\star)$ by $\dot{\xi}$ and $\dot{\eta}$ respectively:

$$\begin{aligned}
\dot{\xi}\ddot{\xi} - 2\dot{\xi}\dot{\eta} &= \dot{\xi}\frac{\partial \Omega}{\partial \xi} \\
\dot{\eta}\ddot{\eta} + 2\dot{\xi}\dot{\eta} &= \dot{\eta}\frac{\partial \Omega}{\partial \eta}.
\end{aligned}$$

We then add the two equations to obtain

$$\dot{\xi}\ddot{\xi} + \dot{\eta}\ddot{\eta} = \dot{\xi}\frac{\partial\Omega}{\partial\xi} + \dot{\eta}\frac{\partial\Omega}{\partial\eta},$$

and by the chain rule this is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}^2 + \dot{\eta}^2\right) = \frac{\mathrm{d}\Omega}{\mathrm{d}t}.$$

Integrating, and noticing $\dot{\xi}^2 + \dot{\eta}^2$ is just the square of velocity, v,

$$v^2 = 2\Omega - C$$

where C is the *Jacobi constant*, which is conserved in time.

What makes the Jacobi constant so useful is that Ω is a function only of position. So once we know the value of C given initial conditions, we can confine the trajectory of the Satellite to a region which ensures its conservation is respected. As $v^2 \geqslant 0$ for all time, the Satellite moves in a region where

$$\Omega \geqslant \frac{1}{2}C.$$

Figure 2 shows[5] some *forbidden regions* (the shaded areas) for the Satellite trajectory given a value of C when $\mu = 0.11$.

Example 3.4. Take the $C = 3.56$ diagram in figure 2: trajectories starting close to the Sun and Earth cannot cross into the shaded region. However, there is the possibility of an orbit transiting from the area close to Earth to the area around the Sun.

According to an article from Astronomy [4], an unusual discovery was made on September 3 2002. An amateur astronomer discovered a strange object orbiting the Earth, which turned out to be the third stage of the Saturn V rocket from Apollo 12. This small, man-made object went on an incredible journey orbiting the Sun after leaving Earth in 1971, and then returning 30 years later. If the Jacobi constant for this orbit was calculated, I speculate[6] it would produce a forbidden region similar to that in the $C = 3.56$ diagram of figure 2.

Second, the potential function allows us to easily identify the locations of *Lagrangian points*.

[5]This figure has been created in the style of [12, p. 121]. However, I believe the source uses incorrect values for C; the numbers given are for $J = \frac{1}{2}C$, a different definition of the Jacobi constant.

[6]Unfortunately this is a rather poor estimate as the moon also has a huge effect on the rocket's orbit. We would need to solve a restricted four body problem to obtain anything accurate in this case.

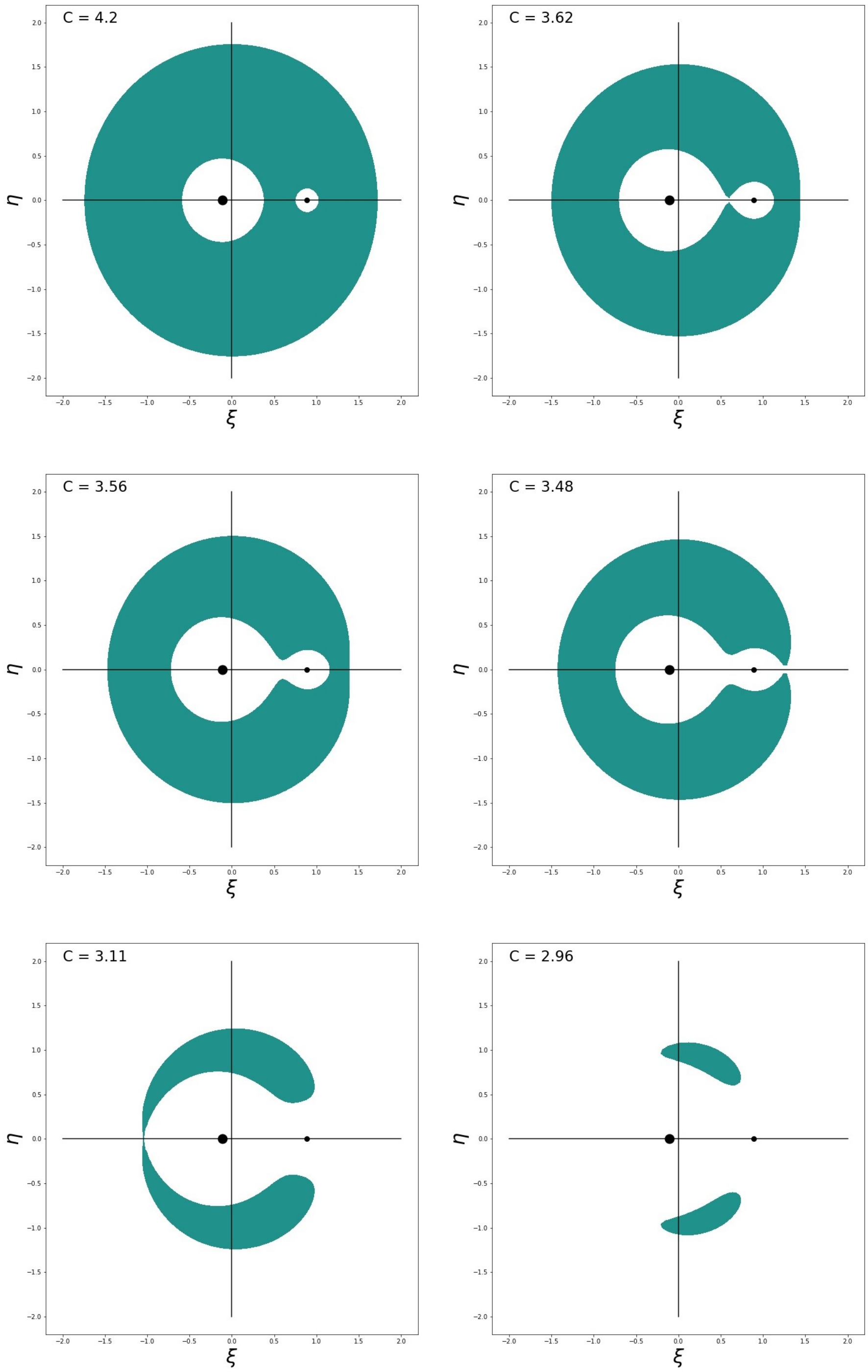

Figure 2: Some forbidden regions for various values of the Jacobi constant when $\mu = 0.11$.

Definition 3.5 (Lagrangian points)**.** These are the fixed points of the CR3BP. That is, where $\dot{\xi} = 0$ and $\dot{\eta} = 0$ for all time.

Subsequently considering equations ($\star\star$), this is true only if $\frac{\partial \Omega}{\partial \xi} = 0$ and $\frac{\partial \Omega}{\partial \eta} = 0$, thus the Lagrangian points are the critical points of Ω.

We therefore need only find critical points of Ω to obtain the Lagrangian points. An interactive plot of this function can be found at `https://www.geogebra.org/calculator/dzvrkvne`, while a plot for $\mu = 0.11$ can be seen in figure 3. Notice that L_1, L_2 and L_3 are saddles, while L_4 and L_5 are local minima. This gives a clue as to the stability of such points, which will be discussed in section 4.

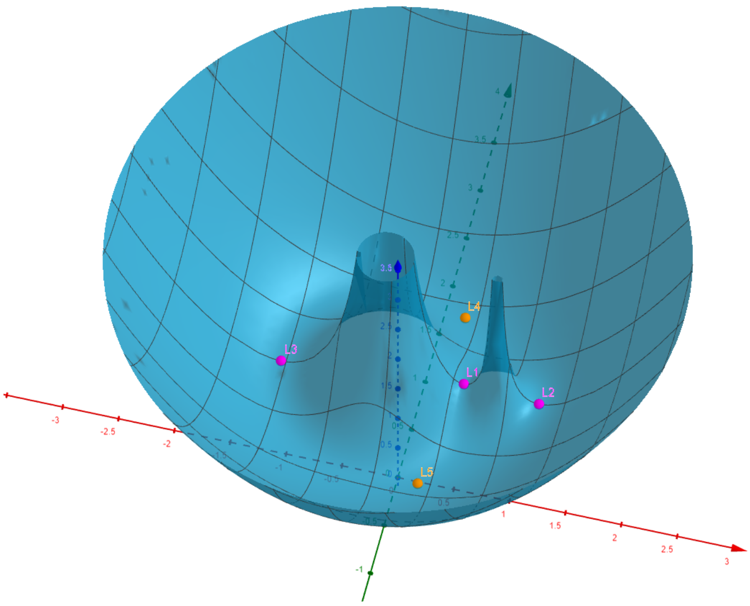

Figure 3: A plot of the potential function for $\mu = 0.11$, taken from GeoGebra.

3.4 Lagrangian Points

We will now find the Lagrangian points mathematically. As discussed earlier, we do this by finding the critical points of the potential function. Recall its definition:

$$\Omega(\xi, \eta) := \frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}$$

where ρ_1 and ρ_2 are given by

$$\rho_1 = \left((\xi + \mu)^2 + \eta^2\right)^{\frac{1}{2}}$$
$$\rho_2 = \left((\xi + \mu - 1)^2 + \eta^2\right)^{\frac{1}{2}}.$$

Computing the derivatives (which is a little bit of work) gives

$$0 = \frac{\partial \Omega}{\partial \xi} = \xi - (1-\mu)\frac{\xi+\mu}{\rho_1^3} - \mu\frac{\xi+\mu-1}{\rho_2^3}, \tag{6}$$

$$0 = \frac{\partial \Omega}{\partial \eta} = \eta - (1-\mu)\frac{\eta}{\rho_1^3} - \mu\frac{\eta}{\rho_2^3}. \tag{7}$$

Note first that (7) has a trivial solution $\eta = 0$ (this will produce L_1, L_2, L_3). Substituting $\rho_1 = |\xi + \mu|$ and $\rho_2 = |\xi + \mu - 1|$ into (6) we have

$$\xi - (1-\mu)\frac{\xi+\mu}{|\xi+\mu|^3} - \mu\frac{\xi+\mu-1}{|\xi+\mu-1|^3} = 0.$$

To solve this requires some case analysis for the signs on absolute values. We can ignore $\xi = -\mu$ and $\xi = 1-\mu$ as these are the positions of the Sun and Earth.

Case 1 (L_1). When $-\mu < \xi < 1-\mu$ we have

$$\xi - \frac{1-\mu}{(\xi+\mu)^2} + \frac{\mu}{(\xi+\mu-1)^2} = 0,$$

Case 2 (L_2). When $\xi > 1-\mu$ we have

$$\xi - \frac{1-\mu}{(\xi+\mu)^2} - \frac{\mu}{(\xi+\mu-1)^2} = 0,$$

Case 3 (L_3). And when $\xi < -\mu$ we have

$$\xi + \frac{1-\mu}{(\xi+\mu)^2} + \frac{\mu}{(\xi+\mu-1)^2} = 0.$$

In all three cases, we are essentially solving a fifth order polynomial (not exactly straightforward algebraically, but relatively simple using a computer). Figure 4 shows the results. We have now found the first three Lagrangian points for all values of μ.

Next we consider the case where $\eta \neq 0$. Dividing equation (7) through by η, then with a small dose of trickery, multiplying by $\xi + \mu$ gives

$$(\xi+\mu) - (1-\mu)\frac{\xi+\mu}{\rho_1^3} - \mu\frac{\xi+\mu}{\rho_2^3} = 0.$$

We can now subtract equation (6) to obtain

$$\mu - \frac{\mu}{\rho_2^3} = 0,$$

meaning $\rho_2 = 1$. We can also multiply though by $\xi + \mu - 1$ instead of $\xi + \mu$ and similarly obtain $\rho_1 = 1$. Now solving the simultaneous equations

$$1 = \rho_1 = \left((\xi+\mu)^2 + \eta^2\right)^{\frac{1}{2}},$$

$$1 = \rho_2 = \left((\xi+\mu-1)^2 + \eta^2\right)^{\frac{1}{2}}.$$

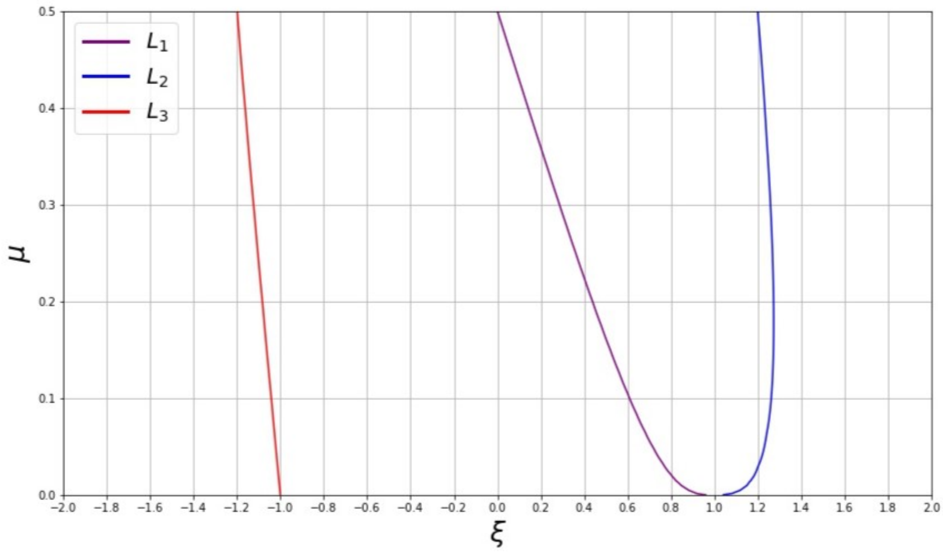

Figure 4: Graph of Lagrange point locations varying with μ.

We conclude that the positions of L_4 and L_5 are $(\frac{1}{2}-\mu, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}-\mu, -\frac{\sqrt{3}}{2})$ respectively. This completes the derivation for all 5 Lagrangian points.

Remark. An important observation is the link between the Lagrangian points and *periodic solutions* of the General Three Body Problem, discussed in [8, pp. 23–25]. L_1, L_2, L_3 all lie co-linear with the Sun and Earth and are thus the Euler periodic solutions, while L_4 and L_5 form an equilateral triangle with the Sun and Earth so are the Lagrange periodic solutions. These are the two earliest known exact solutions to the G3BP, so it is elegant yet unsurprising they turn out to be important in the CR3BP.

4 Stability Analysis

4.1 Linearization

The final aim of this essay to determine the stability of Lagrangian points by linearising the equations of motion in their vicinity. An article by Szebehely gives a thorough case by case analysis [10], though we shall follow a different method. To begin with we'll give a precise notion of what is meant by *linearization* and *stability*.

Definition 4.1 (Linearization)**.** Given the ODE $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and f continuously differentiable, then close to the fixed point x^* we have

$$f(x) = f(x^*) + Df(x^*)(x - x^*) + o(|x - x^*|).$$

Setting $y = x - x^*$ and ignoring the $o(|x - x^*|)$ terms, we can deduce the linearised equation

$$\dot{y} = Df(x - x^*)y.$$

Remark. A lot can be deduced about the behaviour of solutions near the fixed points by finding eigenvalues of $Df(x^*)$. Importantly, if an eigenvalue has positive real part, then x^* will be unstable in the linear system and consequently unstable in the non-linear system.

Definition 4.2 (Lyapunov stability)**.** A fixed point x^* of ODE $\dot{x} = f(x)$ is Lyapunov stable if for every neighbourhood U of x^* there exists a neighbourhood $V \subset U$ such that any $x_0 \in V$ has $x(t) \in U$ for all $t \geq 0$, where $x(t)$ denotes the solution to the ODE with initial condition x_0.

Any fixed point that is not Lyapunov Stable is unstable.

For the ODE system given by $(\star\star)$, we can set $\gamma = \dot{\xi}$ and $\delta = \dot{\eta}$ to obtain the 4 dimensional system

$$\begin{aligned}
\dot{\xi} &= \gamma \\
\dot{\eta} &= \delta \\
\dot{\gamma} &= 2\delta + \frac{\partial \Omega}{\partial \xi} \\
\dot{\delta} &= -2\gamma + \frac{\partial \Omega}{\partial \eta}.
\end{aligned}$$

So we have the ODE $\dot{x} = f(x)$ where $x = (\xi, \eta, \gamma, \delta)$ and $f(x) = \left(\gamma, \delta, 2\delta + \frac{\partial \Omega}{\partial \xi}, -2\gamma + \frac{\partial \Omega}{\partial \eta}\right)$.

Hence, linearising around each Lagrangian point given by $(\xi^*, \eta^*, 0, 0)$, we obtain the matrix

$$Df(\xi^*, \eta^*, 0, 0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial^2 \Omega}{\partial \xi^2} & \frac{\partial^2 \Omega}{\partial \xi \partial \eta} & 0 & 2 \\ \frac{\partial^2 \Omega}{\partial \xi \partial \eta} & \frac{\partial^2 \Omega}{\partial \eta^2} & -2 & 0 \end{pmatrix}.$$

4.2 Eigenvalue Analysis

For this subsection, we follow the method outlined by Rubinsztejn in their article [9]. It was mentioned earlier that stability can be determined by finding eigenvalues for the linearized system. The matrix given above has eigenvalues that solve

$$\lambda^4 + \left(4 - \frac{\partial^2 \Omega}{\partial \xi^2} - \frac{\partial^2 \Omega}{\partial \eta^2}\right)\lambda^2 + \left(\frac{\partial^2 \Omega}{\partial \xi^2}\frac{\partial^2 \Omega}{\partial \eta^2} - \left(\frac{\partial^2 \Omega}{\partial \xi \partial \eta}\right)^2\right) = 0. \qquad (\star\star\star)$$

Proposition 4.3. *Lagrangian points L_1, L_2, L_3 cannot have purely imaginary solutions to $(\star\star\star)$ and consequently are unstable.*

Proof. We have at L_1, L_2, L_3 that $\frac{\partial^2\Omega}{\partial\xi^2} = 1 + 2\alpha$, $\frac{\partial^2\Omega}{\partial\eta^2} = 1 - \alpha$ and $\frac{\partial^2\Omega}{\partial\xi\partial\eta} = 0$ where $\alpha = \frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}$.

Observe that at each point $\alpha > 1$. This is shown clearly in [12, p. 127]. Equation $(\star\star\star)$ becomes

$$\lambda^4 + (2-\alpha)\lambda^2 + (1+2\alpha)(1-\alpha) = 0$$

and solving for $\Lambda = \lambda^2$ gives solutions

$$\Lambda_\pm = \frac{\alpha - 2 \pm \sqrt{9\alpha^2 - 8\alpha}}{2}.$$

But as $\alpha > 1$, we can see $\Lambda_+ > 0$ while $\Lambda_- < 0$. This means of the four eigenvalues, only two of them can be purely imaginary, while two are real. In particular, one eigenvalue is positive real, which causes these Lagrangian points to be unstable. □

Proposition 4.4. *Lagrangian points L_4, L_5 can have purely imaginary solutions to $(\star\star\star)$ provided $\mu < \frac{1}{2} - \sqrt{\frac{23}{108}}$.*

Proof. We'll deal with L_4 as L_5 is similar. Recall its position is $(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2})$, and that $\rho_1 = \rho_2 = 1$. Using these we find $\frac{\partial^2\Omega}{\partial\xi^2} = \frac{3}{4}$, $\frac{\partial^2\Omega}{\partial\eta^2} = \frac{9}{4}$ and $\frac{\partial^2\Omega}{\partial\xi\partial\eta} = \frac{3\sqrt{3}}{4}(1-2\mu)$.

Equation $(\star\star\star)$ becomes

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

and again solving for $\Lambda = \lambda^2$ gives solutions

$$\Lambda_\pm = \frac{-1 \pm \sqrt{1 - 27\mu(1-\mu)}}{2}.$$

Now, when $\mu < \frac{1}{2} - \sqrt{\frac{23}{108}}$, it can be shown that $1 - 27\mu(1-\mu) > 0$. Also, $1 - 27\mu(1-\mu) < 1$ since $\mu > 0$. This means that both solutions for Λ are negative real, and so all four eigenvalues are purely imaginary. □

As stated by Rubinsztejn, this means that the motion of solutions close to L_4 and L_5 is bounded, so these are Lyapunov stable fixed points.

Example 4.5. A fantastic example of this stability are the so called *Trojan asteroids* in the Sun–Jupiter system. Many small bodies have stable orbits around the L_4 and L_5 Lagrangian points in this system, leading to two large clusters of asteroids called the "Greeks" and "Trojans". The discovery of these asteroids was made in 1906 by German astrophotographer Max Wolf [11], but only recently in 2021 has a space mission set out to explore them.

5 Concluding Remarks

Our discoveries have shown that the orbit of JWST around L_2 is unstable. Though this may be surprising at first, given that the spacecraft will need to make adjustments to its trajectory approximately every 21 days [1], it should be clear that sending the telescope to a stable Lagrangian point is risky from studying Trojan asteroids. These stable fixed points may accumulate many small celestial objects posing the threat of colliding with the craft — not an issue faced by the unstable Lagrangian points L_1, L_2, L_3. In addition, L_2's proximity to Earth and distance from the sun makes it a perfect choice for the new infrared telescope's location.

Furthermore, the discovery of Trojan asteroids may change our understanding of the solar system. Having orbits that are stable for all time means they have existed since the beginning; they're left over from the solar system's formation. The space mission to explore these asteroids will consequently provide valuable information on how the solar system formed, a prominent question in modern astrophysics.

References

[1] JWST User Documentation (JDox). JWST User Documentation Website, 2016 - 2023. URL https://jwst-docs.stsci.edu. Provided by the SAO/NASA Astrophysics Data System.

[2] NASA's Webb Reveals Cosmic Cliffs, Glittering Landscape of Star Birth, 2022. URL https://www.nasa.gov/image-feature/goddard/2022/nasa-s-webb-reveals-cosmic-cliffs-glittering-landscape-of-star-birth. Accessed on 28th February 2023.

[3] NASA's Webb Sheds Light on Galaxy Evolution, Black Holes, 2022. URL https://www.nasa.gov/image-feature/goddard/2022/nasa-s-webb-reveals-cosmic-cliffs-glittering-landscape-of-star-birth. Accessed on 28th February 2023.

[4] Doug Adler. How a long-gone Apollo rocket returned to Earth, 2020. URL https://astronomy.com/news/2020/05/how-a-long-gone-apollo-rocket-returned-to-earth. Accessed on 24th March 2023.

[5] Urs Frauenfelder and Otto Van Koert. *The restricted three-body problem and holomorphic curves*. Springer, 2018.

[6] JH Grutzelius. The Three Body Problem, 2004.

[7] Patrick Hamill. *A Student's Guide to Lagrangians and Hamiltonians*. Cambridge University Press.

[8] Zdzislaw Musielak and Billy Quarles. *Three body dynamics and its applications to exoplanets*. Springer, 2017.

[9] Ari Rubinsztejn. Stability of Lagrange points - Three Body Problem, 2018. URL https://gereshes.com/2018/12/17/stability-of-the-lagrange-points-three-body-problem/. Accessed on 13th April 2023.

[10] Victor Szebehely. Stability of the points of equilibrium in the restricted problem. *The Astronomical Journal*, 72:7, 1967.

[11] David Dezell Turner. How Were the Trojan Asteroids Discovered and Named?, 2021. URL https://www.nasa.gov/feature/goddard/2021/how-were-the-trojan-asteroids-discovered-and-named. Accessed on 13th April 2023.

[12] Mauri Valtonen and Hannu Karttunen. *The three-body problem*. Cambridge University Press, 2006.

A The Two Body Problem

We are now concerned with two objects orbiting each other by their own gravity. To solve this problem mathematically, it first makes sense to understand Kepler's laws of planetary motion for the One Body Problem. They are as follows:

1. A planet orbits the sun in an ellipse, with the sun at one focus of the ellipse.
2. A line connecting the planet to the sun sweeps out equal areas in equal time intervals.
3. If the orbital period is P, and the radius r, then $P^2 = kr^3$ for some constant k.

Kepler found these laws purely by observation, but were later derived by Newton from his law of gravitation.

To solve the two body problem, we first place the centre of mass at the origin of our inertial coordinate system. Let the two bodies have masses m_1, m_2, and position vectors $\underline{r}_1$, $\underline{r}_2$. By the definition of centre of mass,

$$\frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2} = \underline{0}.$$

Letting $\underline{r} = \underline{r}_2 - \underline{r}_1$ be the vector of mass 2 from mass 1,

$$\underline{r}_1 = -\frac{m_2}{m_1 + m_2}\underline{r} \qquad \text{and} \qquad \underline{r}_2 = \frac{m_1}{m_1 + m_2}\underline{r}.$$

This amounts to saying that both masses are in the same direction as $\underline{r}$ but with different scalars; the motion of both objects is the same shape with different amplitudes.

We can also find that

$$\begin{aligned}
\underline{r} &= \underline{r}_2 - \underline{r}_1 \\
\frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2\underline{r}_2}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\underline{r}_1}{\mathrm{d}t^2} \\
&= -\frac{Gm_1}{|\underline{r}|^3}\underline{r} - -\frac{Gm_2}{|\underline{r}|^3}\underline{r} \\
&= -\frac{G(m_1 + m_2)}{|\underline{r}|^3}\underline{r},
\end{aligned}$$

which is mathematically identical to the One Body Problem.