Discontinuous Galerkin approximation on polyhedric mesh for the Elastoplastic problem

Project for the course of Numerical Analysis for Partial Differential Equations

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Pure elasticity:

$\sigma = \mathcal{D}^{el}\varepsilon$ (Stress-Strain relation)

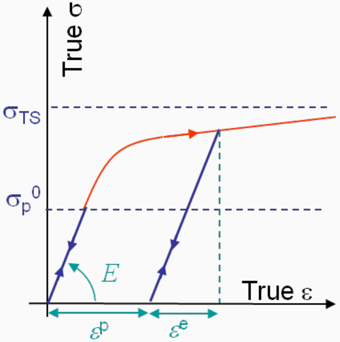
$\mathcal{D}^{el}\tau = 2\mu\tau + \lambda Tr\left(\tau\right)\mathbb{I}_3$ (Hooke's law)

$\mu,\ \lambda\ \rightarrow$ Lamè coefficients

Plasticity:

$\sigma = \mathcal{D}^{ep}\varepsilon$ (Stress-Strain relation)

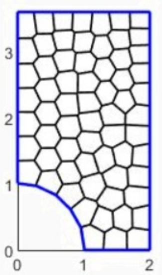
$\mathcal{D}^{ep} = \mathcal{D}^{ep}\left(\varepsilon\right)$ (Elastoplastic law)



Strain decomposition
image taken from "Fracture Mechanics Online Class" ([8])

Discontinuous Galerkin method

- It allows to capture discontinuities in the solution
- High order approximation
- Flexible description of heterogeneous domains
- High level of parallelism
- It robustly support polyhedral meshes



Example of polyhedral mesh

Discontinuous Galerkin for the Elastic problem

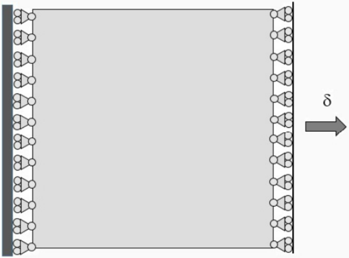
$$
\begin{cases}
-\nabla \cdot \sigma\left(u\left(x,y\right)\right) = f\left(x,y\right) & \text{in } \Omega \subset \mathbb{R}^2 \\
\\
\text{boundary conditions}
\end{cases}
$$

- $u =$ displacement (unknown)
- $f \in L^2\left(\Omega\right)$
- $\sigma : \Omega \to \mathbb{S} = \{3 \times 3 \text{ symmetric tensors}\}$
- $\sigma\left(u\right) = \mathcal{D}^{el}\varepsilon\left(u\right)$ (Hooke's law)
- $\varepsilon\left(u\right) = \frac{1}{2}\left(\nabla u + \left(\nabla u\right)^T\right)$

Boundary conditions for 2D problems

$$\begin{cases} -\nabla \cdot \sigma(u(x,y)) = f(x,y) & \text{in } \Omega \subset \mathbb{R}^2 \\ u \cdot n = u_{D,n} & \text{on } \Gamma_{D,n} \\ u \cdot t = u_{D,t} & \text{on } \Gamma_{D,t} \\ n^T \sigma(u) n = h_n & \text{on } \Gamma_{N,n} \\ t^T \sigma(u) n = h_t & \text{on } \Gamma_{N,t} \end{cases}$$

- $u = [u_x, u_y]$
- $\Gamma_{D,n} \cup \Gamma_{N,n} = \partial\Omega$
- $\Gamma_{D,t} \cup \Gamma_{N,t} = \partial\Omega$
- $n =$ outer normal direction
- $||t|| = 1, \quad t \cdot n = 0$



Example of mixed boundary condition

Weak formulation for the elastic problem

- $W_h^{DG} = \{w \in \left[L^2(\Omega)\right]^2 : w|_K \in [\mathbb{P}^{p_k}(K)]^2 \, \forall K \in \mathcal{T}_h\}$
- $\mathcal{A}_\theta : W_h^{DG} \times W_h^{DG} \to \mathbb{R}$:
  $\mathcal{A}_\theta(u,v) = \sum_{i=1}^{N} \int_{K_i} \sigma(u) : \varepsilon(v)\, ds \; - \; \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \{\{\sigma(u)\}\} : [[\tilde{v}]]\, dl$
  $- \; {\color{blue}\theta} \sum_{F \in \mathcal{F}_h^{i,D}} \int_F [[u]] : \{\{\widetilde{\sigma(v)}\}\} dl \; + \; \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \gamma \, [[u]] : [[\tilde{v}]]\, dl$
- $F : W_h^{DG} \to \mathbb{R}$:
  $Fv = \int_{\Omega_h} fv ds + \sum_{F \in \left(\mathcal{F}_{h,n}^{D} \cup \mathcal{F}_{h,t}^{D}\right)} \int_F g \{\{\widetilde{\sigma(v)}\}\} dl + \sum_{F \in \left(\mathcal{F}_{h,n}^{N} \cup \mathcal{F}_{h,t}^{N}\right)} \int_F h \tilde{v} dl$

Where $g = u_{D,n}\mathbf{n} + u_{D,t}\mathbf{t}$, $\; h = h_n\mathbf{n} + h_t\mathbf{t}$ and:

$$\begin{cases} \tilde{v} = (v \cdot \mathbf{n})\, \mathbf{n} & \text{on } \Gamma_{D,n} \setminus \Gamma_{D,t} \\ \tilde{v} = (v \cdot \mathbf{t})\, \mathbf{t} & \text{on } \Gamma_{D,t} \setminus \Gamma_{D,n} \\ \tilde{v} = v & \text{otherwise} \end{cases} \qquad \begin{cases} \widetilde{\sigma} = \left(\mathbf{n}^T \sigma \mathbf{n}\right) \mathbf{n} & \text{on } \Gamma_{N,n} \setminus \Gamma_{N,t} \\ \widetilde{\sigma} = \left(\mathbf{t}^T \sigma \mathbf{n}\right) \mathbf{t} & \text{on } \Gamma_{N,t} \setminus \Gamma_{N,n} \\ \widetilde{\sigma} = \sigma & \text{otherwise} \end{cases}$$

Find $\mathbf{u_h} \in \mathbf{W_h^{DG}}$ s.t. :

$$\mathcal{A}_\theta(\mathbf{u_h}, \mathbf{v_h}) = F\mathbf{v_h} \quad \forall \mathbf{v_h} \in \mathbf{W_h^{DG}}$$

Linear system

- $\mathbf{V}_{ij} = \sum_{k=1}^{N} \int_{K_k} \sigma(\phi_j) : \varepsilon(\phi_i)\, ds$
- $\mathbf{IT}_{\mathbf{ij}} = \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \{\{\sigma(\phi_j)\}\} : [[\tilde{\phi}_i]] dl$
- $\mathbf{S}_{\mathbf{ij}} = \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \gamma\, [[\phi_j]] : [[\tilde{\phi}_i]] dl$
- $\mathbf{f}_{\mathbf{i}} = F\phi_i$
- $\mathbf{u} = \left[u_1^x, u_2^x, \ldots, u_{N_h}^x, u_1^y, u_2^y, \ldots, u_{N_h}^y\right]^T$
- $\mathbf{K} = \mathbf{V} - \mathbf{IT} - \theta \cdot \mathbf{IT}^T + \mathbf{S}$

$$\text{find } \mathbf{u} \in \mathbb{R}^{2N_h} \text{ s.t. } \mathbf{Ku} = \mathbf{f}$$

Theoretical results

Let $u_h \in W_h^{DG}$ *be the DGFEM solution*

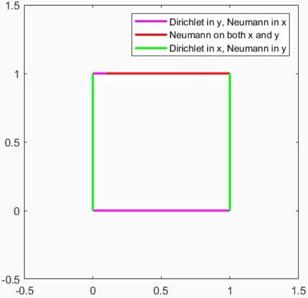
- $||u - u_h||_{DG}^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2(s_K - 1)}}{p_K^{2(r_K - 3/2)}} \left(||\mathcal{E}u||_{H^{r_K}(T_K)}^2 + ||\mathcal{E}\sigma(u)||_{\mathcal{H}^{r_K}(T_K)}^2 \right)$
- $||u - u_h||_{L^2}^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2s_K}}{p_K^{2(r_K - 1/2)}} \left(||\mathcal{E}u||_{H^{r_K}(T_K)}^2 + ||\mathcal{E}\sigma(u)||_{\mathcal{H}^{r_K}(T_K)}^2 \right) \quad \text{if } \theta = 1$
- $||u - u_h||_{L^2}^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2(s_K - 1)}}{p_K^{2(r_K - 3/2)}} \left(||\mathcal{E}u||_{H^{r_K}(T_K)}^2 + ||\mathcal{E}\sigma(u)||_{\mathcal{H}^{r_K}(T_K)}^2 \right) \quad \text{if } \theta \in \{-1, 0\}$

with $s_K = \min(p_K + 1, r_K)$ *and* r_K *is the regularity degree of* u.

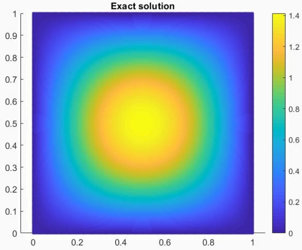
$u \in H^r(\Omega)\ \forall r \in \mathbb{N} \ \rightarrow$ *Exponential decay of the errors w.r.t. the polynomials degree p.*

Numerical results for the elastic problem

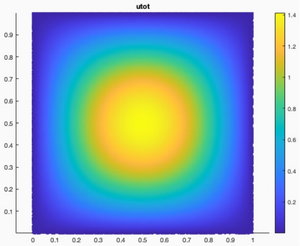
$$u_{ex}(x,y) = [\sin(\pi x)\sin(\pi y)\,,\ \sin(\pi x)\sin(\pi y)]$$



Representation of mixed boundary conditions

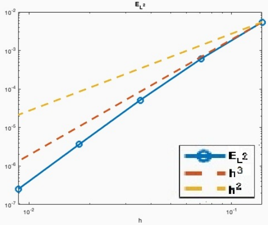


Total displacement of the exact solution

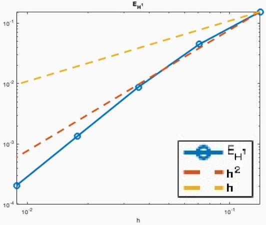


Total displacement of the numerical solution

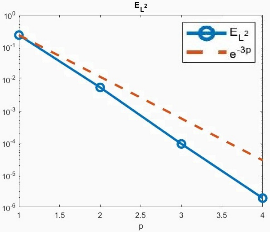
Numerical Results for the elastic problem



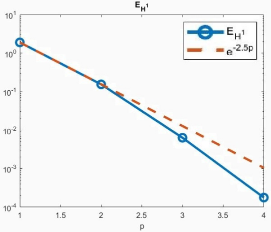
Convergence in h of $||u_{ex} - u_h||_{L^2}$ *with* $p = 2$



Convergence in h of $||u_{ex} - u_h||_{H^1}$ *with* $p = 2$

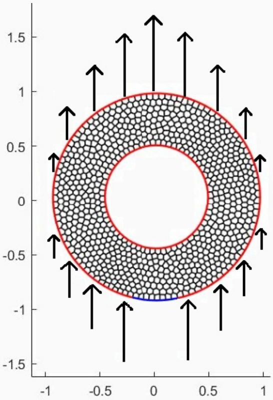


Convergence in p of $||u_{ex} - u_h||_{L^2}$ *with* $N = 100$

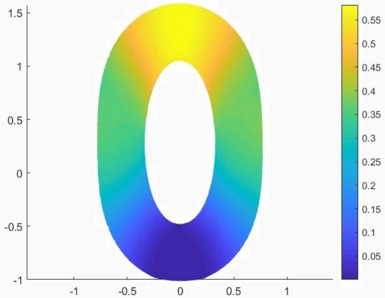


Convergence in p of $||u_{ex} - u_h||_{H^1}$ *with* $N = 100$

Numerical Results for the elastic problem



Computational mesh and boundary conditions



Displacement applied to the original domain

The Elastoplastic problem

$$\sigma(u) = \mathcal{D}^{ep}(u)\,\varepsilon(u)$$

To be specified:

- Elastic behaviour $\rightarrow \mathcal{D}^{ep} = \mathcal{D}^{el}$ (Hooke's law)
- Yield criterion $\Psi(\sigma)$: $\begin{cases} \Psi(\sigma) < 0 & \rightarrow \quad \text{Elastic response} \\ \Psi(\sigma) = 0 & \rightarrow \quad \text{Elastoplastic response} \end{cases}$
- Elastoplastic behaviour $\rightarrow \mathcal{D}^{ep} = ???$

$$\mathcal{D}^{ep} = \mathcal{D}^{ep}(u) \quad \Longrightarrow \quad \text{Newton algorithm}$$

Quasi-static loading and Newton algorithm

Result: u_N *(final displacement)*
Input: N, F, MaxIt;
1. set: $i = 0$;
while $i < N$ *(loading phase)* **do**

Quasi-static loading and Newton algorithm

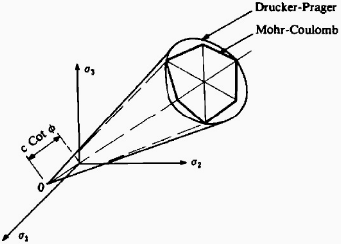
Result: u_N *(final displacement)*
Input: N, F, MaxIt;
1. set: $i = 0$;
while $i < N$ *(loading phase)* **do**
  2. define $F_i = \frac{i}{N} F$;
  3. compute an initial guess u_i^0 ;
  4. set: $k = 0$
  while *not "super-converging" and k* $<$ *MaxIt (Newton algorithm)* **do**
    5. compute the constitutive law $\mathcal{D}^{ep}\left(u_i^k\right)$;
    6. compute u_i^{k+1} using the Newton update rule;
    7. $k = k + 1$;
  end
  8. update *i* conveniently;
end

Loading procedure with Newton algorithm

The Drucker-Prager yield criterion

$$\Psi\left(\sigma\right) = \sqrt{\frac{1}{2}}|\mathcal{I}_D\sigma| + \frac{\eta}{3}\mathbb{I}_3 : \sigma - c$$

where $\mathcal{I}_D\sigma$ is the deviatoric part of σ



Mohr-Coulomb and Drucker-Prager yield surfaces

image taken from [2]

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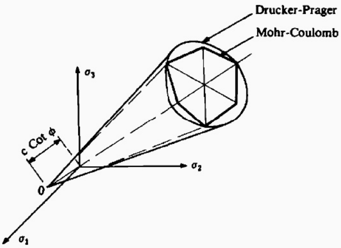

Mohr-Coulomb and Drucker-Prager yield surfaces

image taken from [2]

Plane-strain approach:

- $\varepsilon^{el}_{xz} = \varepsilon^{el}_{yz} = 0$
- $\varepsilon^{pl}_{xz} = \varepsilon^{pl}_{yz} = 0$
- $\varepsilon_{zz} = \varepsilon^{el}_{zz} + \varepsilon^{pl}_{zz} = 0$

Solution in closed form for Drucker-Prager criterion

At every iteration we need to assembly:

- $T_k(u) = \sigma(u) = \mathcal{D}^{ep}(u)\,\varepsilon(u)$
- The linearized tensor T_k^0 s.t. $\sigma(u) = T_k^0 \varepsilon(u) + o(|\varepsilon(u)|)$

We need the **trials**:

- $\sigma_k^{tr} = \mathcal{D}^{el}\left(\varepsilon_k(u) - \varepsilon_{k-1}^{pl}(u)\right)$
- $p_k^{tr} = \mathbb{I}_3 : \sigma_k^{tr} = Tr\left(\sigma_k^{tr}\right)$
- $s_k^{tr} = \mathcal{I}_D \varepsilon_k(u)$
- $\rho_k^{tr} = 2\mu||\varepsilon_k(u)||_{L^2}$
- $n_k^{tr} = \frac{s_k^{tr}}{||\varepsilon_k(u)||_{L^2}}$

Solution in closed form for Drucker-Prager criterion

1. If $\Psi\left(\sigma_k^{tr}\right) \leq 0$:
   - $T_k = \sigma_k^{tr}$
   - $T_k^0 = \mathcal{D}^{el}$
   - $\varepsilon_k^{pl}(u) = \varepsilon_{k-1}^{pl}(u)$

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   - $T_k^0 = \mathcal{D}^{el}$
   - $\varepsilon_k^{pl}(u) = \varepsilon_{k-1}^{pl}(u)$
2. If $\Psi\left(\sigma_k^{tr}\right) > 0$ and $\eta \frac{p_k^{tr}}{3} - \frac{K\eta^2}{\mu\sqrt{2}}\rho_k^{tr} - c < 0$:
   - $T_k = \sigma_k^{tr} - \frac{\Psi(\sigma_k^{tr})}{\mu+K\eta^2}\left(\sqrt{2}\mu n_k^{tr} + K\eta\mathbb{I}_3\right)$
   - $T_k^0 = \mathcal{D}^{el} - \frac{1}{\mu+K\eta^2}\left(\frac{2\sqrt{2}\mu^2\Psi(\sigma_k^{tr})}{\rho_k^{tr}}\left(\mathcal{I}_D - n_k^{tr}\otimes n_k^{tr}\right) + \left(\sqrt{2}\mu n_k^{tr} + K\eta\mathbb{I}_3\right)\otimes\left(\sqrt{2}\mu n_k^{tr} + K\eta\mathbb{I}_3\right)\right)$
   - $\varepsilon_k^{pl}(u) = \varepsilon_{k-1}^{pl}(u) + \frac{\Psi(\sigma_k^{tr})}{\mu+K\eta^2}\left(\frac{n_k^{tr}}{\sqrt{2}} + \frac{\eta}{3}\mathbb{I}_3\right)$

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3. If $\Psi\left(\sigma_k^{tr}\right) > 0$ and $\eta \frac{p_k^{tr}}{3} - \frac{K\eta^2}{\mu\sqrt{2}}\rho_k^{tr} - c \geq 0$:
   - $T_k = \frac{c}{\eta}\mathbb{I}_3$
   - $T_k^0 = \mathbb{O}$
   - $\varepsilon_k^{pl}(u) = \varepsilon_{k-1}^{pl}(u) - \frac{c}{3K\eta}\mathbb{I}_3$

Some details about the code

$\mathcal{D}^{ep}$ is computed inside the function **Constitutive_problem.m**, in which:

- $\mathbf{V_{ij}} = \sum_{k=1}^{N} \int_{K_k} T_k^0 \varepsilon(\phi_j) : \varepsilon(\phi_i)\, ds$
- $\mathbf{V\_u_i} = \sum_{k=1}^{N} \int_{K_k} T_k : \varepsilon(\phi_i)\, ds$ $\quad (\mathbf{V\_u} \approx \mathbf{V} * \mathbf{u})$
- $\mathbf{IT_{ij}} = \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \{\{T_k^0 \varepsilon(\phi_j)\}\} : [[\tilde{\phi}_i]] dl$
- $\mathbf{IT\_u_i} = \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \{\{T_k\}\} : [[\tilde{\phi}_i]] dl$ $\quad (\mathbf{IT\_u} \approx \mathbf{IT} * \mathbf{u})$
- $\mathbf{K^k} = \mathbf{V} - \mathbf{IT} + \mathbf{S}$
- $\mathbf{K^k\_u} = \mathbf{V\_u} - \mathbf{IT\_u} + \mathbf{S} * \mathbf{u}$ $\quad (\mathbf{K^k\_u} \approx \mathbf{K^k} * \mathbf{u})$

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- $\mathbf{IT_{ij}} = \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \{\{T_k^0 \varepsilon(\phi_j)\}\} : [[\tilde{\phi}_i]] dl$
- $\mathbf{IT\_u_i} = \sum_{F \in \mathcal{F}_h^{i,D}} \int_F \{\{T_k\}\} : [[\tilde{\phi}_i]] dl$ $\quad (\mathbf{IT\_u} \approx \mathbf{IT} * \mathbf{u})$
- $\mathbf{K^k} = \mathbf{V} - \mathbf{IT} + \mathbf{S}$
- $\mathbf{K^k\_u} = \mathbf{V\_u} - \mathbf{IT\_u} + \mathbf{S} * \mathbf{u}$ $\quad (\mathbf{K^k\_u} \approx \mathbf{K^k} * \mathbf{u})$

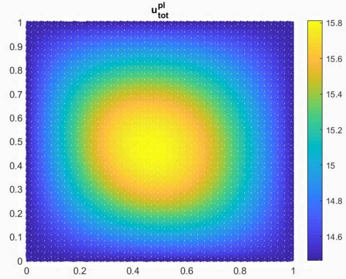
And then proceed with the update rule:

1. Compute $r_i = F_i - \mathbf{K^k\_u}$
2. Find **du** s.t. $\mathbf{K^k} * \mathbf{du} = r_i$
3. Update the solution: $\mathbf{u} = \mathbf{u} + \mathbf{du}$

Comparison with the elastic solution

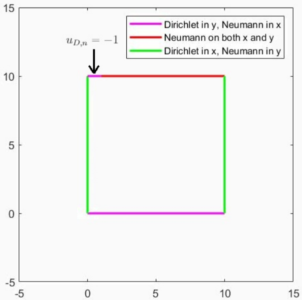
Step	Nº iter	Load incr	Total load	Criterion	Super conv
1	5	0.001	0.001	$9.72 * 10^{-11}$	yes
2	3	0.002	0.003	$2.016 * 10^{-12}$	yes
3	3	0.004	0.007	$7.40 * 10^{-11}$	yes
4	2	0.008	0.015	$8.5904 * 10^{-13}$	yes
5	2	0.016	0.031	$3.0309 * 10^{-13}$	yes
6	2	0.032	0.063	$1.421 * 10^{-12}$	yes
7	2	0.064	0.127	$1.176 * 10^{-12}$	yes
8	1	0.128	0.255	$2.2091 * 10^{-13}$	yes
9	1	0.256	0.511	$3.5034 * 10^{-13}$	yes
10	1	0.512	1.023	$1.402 * 10^{-12}$	yes

Newton algorithm details for each loading step



Total displacement

Comparison with a reference solution

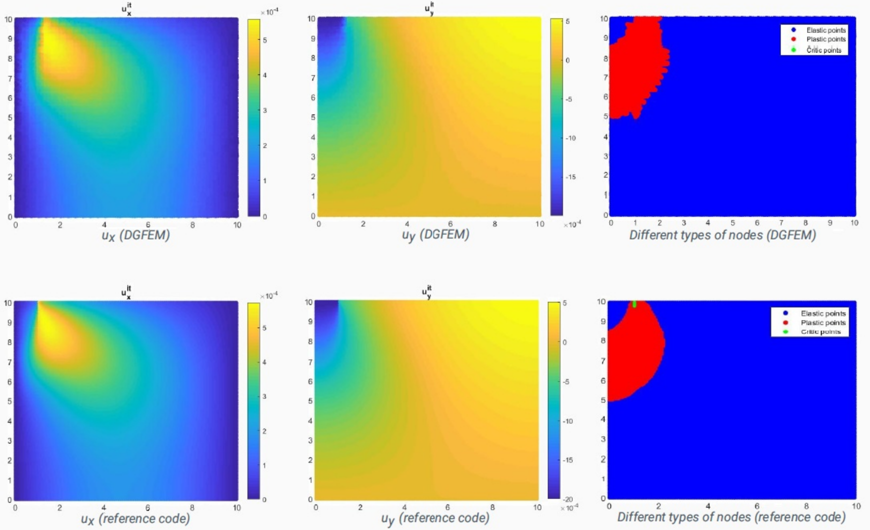


Domain with boundary conditions

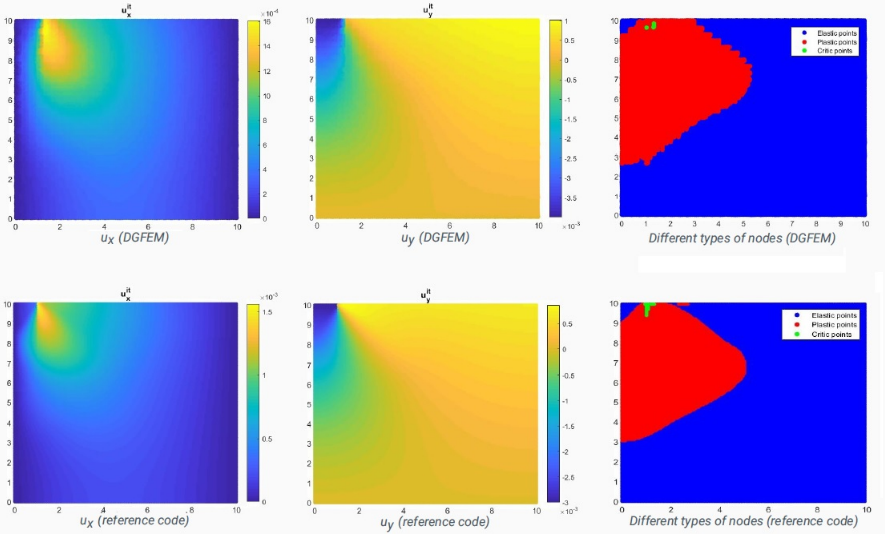
Step	Nº iter	Load incr	Total load	Criterion	Super conv
1	9	0.001	0.001	$2.2292 * 10^{-13}$	yes
2	9	0.001	0.002	$7.7999 * 10^{-13}$	yes
3	12	0.001	0.003	$5.7398 * 10^{-13}$	yes

Newton algorithm details for the first three loading step

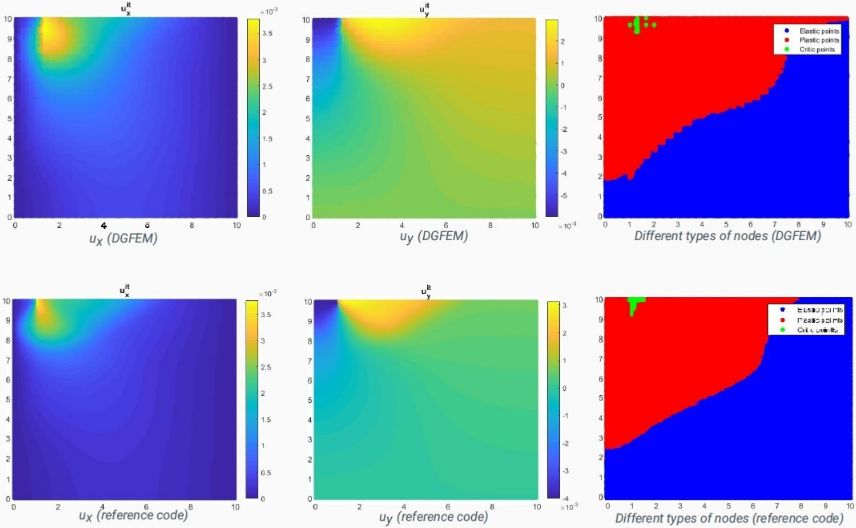
Comparison with a reference solution - 1^{st} loading step



Comparison with a reference solution - 2^{nd} loading step



Comparison with a reference solution - 3^{rd} loading step



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- Try to replace Newton with some inexact method (e.g. Quasi-Newton)
- Tune the hyperparameters to solve optimally the reference problem
- Make the code computationally more efficient (e.g. exploit vectorization in Matlab)
- Different yield criteria could be implemented in order to study the behaviour of different materials.

Conclusions and further developments

- Try to replace Newton with some inexact method (e.g. Quasi-Newton)
- Tune the hyperparameters to solve optimally the reference problem
- Make the code computationally more efficient (e.g. exploit vectorization in Matlab)
- Different yield criteria could be implemented in order to study the behaviour of different materials.
- Extend the model to time dependent problems. For example one application could be wave propagation, in which the plastic behaviour could be used to represent the correct response for soil materials.

Essential bibliography

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