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Experimental physics Vb (particle and astro-physics)

Exercise 04

Task 1 *SU(3) Quark model of the hadrons* (25 points)

By the early 1960s, the number of known particles had grown to more than 100. The question of the underlying ordering principle was answered when Gell-Mann and Zweig introduced the quark model of the hadrons in 1964. They arranged those mesons and baryons that are known today to be composed of u, d, and s quarks in multiplets of the symmetry group $SU(3)$. At the same time, their model hinted at the underlying quark structure of the hadrons, which was not known at the time. It also allows rough statements about the masses of the quarks involved and about the binding energies inside the hadrons.

Let us first consider the case of spin-$SU(2)$ to understand the basic principle: It is well known that two spin-$\frac{1}{2}$ states ($S = \frac{1}{2}$) may couple to form a triplet ($S = 1$) or a singlet ($S = 0$) state. This solution can be obtained graphically as sketched in Fig. 1. The

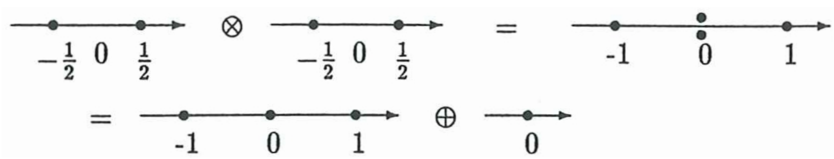

Figure 1: Coupling of two $\frac{1}{2}$-doublets.

two factors on the left side of the equation each represent a spin-$\frac{1}{2}$ multiplet with the two states corresponding to the two possible spin orientations $S_z = \pm\frac{1}{2}$. The states of the first factor are taken as points of origin that the states of the second factor are "attached" to. To decompose the result to summands one uses the known multiplets belonging to spin 1 and spin 0 states, respectively. The symbolic notation is:

$$[2] \otimes [2] = [3] \oplus [1].$$

Now, it is a key experimental observation that the strong interaction does not distinguish between protons and neutrons, similar to the force on an electron in an electric field not depending on whether the electron is in the state $|\uparrow\rangle$ or $|\downarrow\rangle$. Therefore, protons and neutrons are grouped together in a so-called isospin doublet:

$$|p\rangle = |I = \frac{1}{2}, I_3 = +\frac{1}{2}\rangle \quad |n\rangle = |I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle.$$

For a particle with baryon number B and strangeness S, we define the hypercharge

$$Y = B + S.$$

Then the charge Q is given by the Gell-Mann-Nishijima relation

$$Q = e\left(I_3 + \frac{Y}{2}\right).$$

Now we want to consider the structure of $SU(3)$ more closely and then derive a hands-on understanding of the quark model without delving to deep into group theory.

A unitary matrix $U \in SU(3)$ with $\det U = 1$ can be written in a general way as

$$U = \exp\left(i\sum_{a=1}^{8}\alpha_a\lambda_a\right),$$

where the λ_a are a set of eight linearly independent, Hermitian, and traceless matrices called the *generators*.

We choose the following set of generators of $SU(3)$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \sqrt{\tfrac{1}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

We define $F_i \equiv \frac{\lambda_i}{2}$ and hence three sets of shift operators,

$$I_\pm = F_1 \pm \mathrm{i}F_2 \quad V_\pm = F_4 \pm \mathrm{i}F_5 \quad U_\pm = F_6 \pm \mathrm{i}F_7,$$

as well as two diagonal operators,

$$\hat{I}_3 = F_3 \quad \text{and} \quad \hat{Y} = \frac{2}{\sqrt{3}}F_8.$$

a. Justify briefly why the $\{\lambda_i\}_{i=1..8}$ are indeed a set of generators of $SU(3)$. (*1 point*)

b. Show that there are three common eigenstates $|I_3, Y\rangle$ such that

$$\hat{I}_3|I_3, Y\rangle = I_3|I_3, Y\rangle \qquad \text{and} \qquad \hat{Y}|I_3, Y\rangle = Y|I_3, Y\rangle.$$

(*2 points*)

c. Examine the effect of the shift operators. Show that

$$V_{\pm}|I_3, Y\rangle \propto |I_3 \pm \frac{1}{2}, Y \pm 1\rangle, \quad U_{\pm}|I_3, Y\rangle \propto |I_3 \mp \frac{1}{2}, Y \pm 1\rangle, \quad I_{\pm}|I_3, Y\rangle \propto |I_3 \pm 1, Y\rangle.$$

(*4 points*)

Now we identify eigenstates of I_3 and Y as follows,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \equiv |u\rangle \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \equiv |d\rangle \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv |s\rangle,$$

and call them up quark, down quark, and strange quark, and assign them the baryon number $B = \frac{1}{3}$.

d. Calculate I_3, Y, and S for the three quarks. (*1 point*)

e. Draw the three quark states in the Y-I_3-plane (i.e., I_3 on the x-axis and Y on the y-axis). This gives you the triplet of the fundamental representation [3]. In the same way, draw the triplet for the representation $[\bar{3}]$ corresponding to the antiquarks. This is done by a simple sign flip of the quantum numbers. (*3 points*)

f. Display the effect of the shift operators on the [3] multiplet graphically. (*2 points*)

g. Calculate the decomposition of the meson and baryon states, i.e. use a graphic method to decompose (as explained above for the example of $SU(2)$) the products

$$[3] \otimes [\bar{3}] \quad \text{and} \quad [3] \otimes [3] \otimes [3]$$

into its summands. Use Figure 2, showing the higher-dimensional multiplets of $SU(3)$, which can be calculated in the context of group theory (and which are intuitively clear from the effect of the shift operators). (*6 points*)

h. Using the Review of Particle Physics, assign the lightest mesons (π, K, η, η') and baryons (p, n, Σ, Ξ, Λ, as well as Δ, Σ^*, Ξ^*, and Ω, each with various charge numbers) to the correct multiplets, and draw them in the Y-I_3-plane. (*3 points*)

i. Which effect causes the $SU(3)$ symmetry to be broken? (*1 point*)

j. Based on the quark model, one can derive, e.g., the mass relation

$$\frac{m_N + m_\Xi}{2} = \frac{3m_\Lambda + m_\Sigma}{4},$$

with m_N the nucleon mass. Derive this mass relation, by assuming $m_u = m_d$ for the masses of the up and down quarks and assuming the binding energy W_B between the quarks to be identical for all particle states. Test how well the mass relation is satisfied experimentally. (*2 points*)

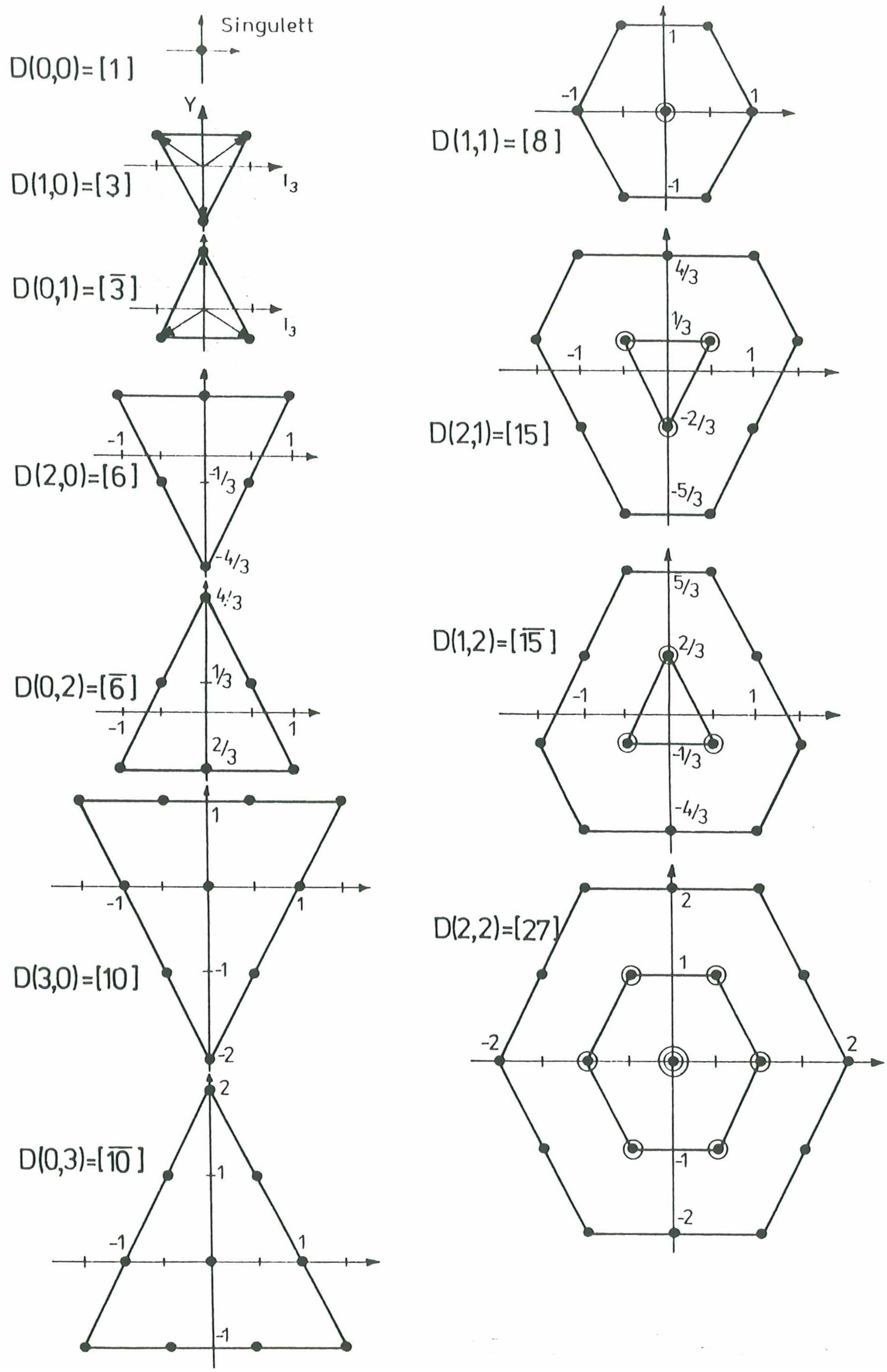

Figure 2: The lowest SU(3) multiplets.