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Experimentalphysik Vb (WS 2023/2024)

Exercise 4

Tutorial: 1 Deadline: 25.11.2024

Aufgabe 1: $SU(3)$ Quark model of the hadrons

(a)

Let $A \in SU(3)$ and $B \in \mathfrak{su}(3)$. The defining property of $SU(3)$ is that the determinant is one. This reduced the amount of free parameters, from nine in a 3×3 matrix to 8. $SU(3)$ is thereby an 8-dimensional space, where as $\mathfrak{su}(3)$ must be at least 8-dimensional to be able to generate $SU(3)$.

Starting with the definition of $SU(3)$, one can derive a restraint for matrices of $\mathfrak{su}(3)$:

$$
\begin{aligned}
1 &\overset{!}{=} \det A \\
&= \det(\exp iB) \\
&= \exp(i \operatorname{tr} B) \\
\implies \operatorname{tr} B &= 2\pi n \ , \ n \in \mathbb{Z}
\end{aligned}
$$

Arbitrarily choosing $n = 0$, since it is the simplest, one finds $3 \cdot 3 - 1 = 8$ traceless and linearly independant matrices, which will generate eight linearly independent members of $SU(3)$. Considering that $SU(3)$ is an 8-dimensional space, one can say that they generate $SU(3)$ in its entirety. Taking a look at λ_1, one an easily verify, that they indeed span the space of traceless 3×3 matrices, and therefore generate $SU(3)$.

Note: Allowing for $n \neq 0$ will introduce more possible linearly independant generators of $SU(3)$, however the space they span does not generate members of $SU(3)$ in general, for example:

$$
\begin{aligned}
A &= \begin{pmatrix} 2\pi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \operatorname{tr} A = 2\pi \\
e^{iA} &= \mathbb{1} \in SU(3) \\
e^{i\frac{A}{2\pi}} &= \begin{pmatrix} e^{i\pi^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \notin SU(3)
\end{aligned}
$$

The eight traceless matrices are the only ones that do span a space of generators, since $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr} A = 0$ for a traceless matrix A. Choosing them to be hermitian as well is mathematically not necessary but probably convenient in the context of physics.

(b)

The eigenvectors of $\hat{I}_3$ are:

$$\begin{aligned}
\hat{I}_3 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
|I_3, 1/2\rangle &= (1, 0, 0) \\
|I_3, -1/2\rangle &= (0, 1, 0) \\
|I_3, 0\rangle &= (0, 0, 1)
\end{aligned}$$

The eigenvectors of $\hat{Y}$ are:

$$\begin{aligned}
\hat{Y} &= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\
|Y, 1/3\rangle &= (1, 0, 0) \\
|Y, 1/3\rangle &= (0, 1, 0) \\
|Y, -2/3\rangle &= (0, 0, 1)
\end{aligned}$$

$\hat{Y}$ and $\hat{I}_3$ evidently share the same three eigenstates.

(c)

For the operator $V_\pm$:

$$\begin{aligned}
[\hat{I}_3, \hat{V}_+] &= \frac{1}{2}\hat{V}_+ \\
[\hat{I}_3, \hat{V}_-] &= -\frac{1}{2}\hat{V}_- \\
\implies [\hat{I}_3, \hat{V}_\pm] &= \pm\frac{1}{2}\hat{V}_\pm
\end{aligned}$$

$$\begin{aligned}
\hat{I}_3\hat{V}_\pm |I_3, Y\rangle &= (\hat{V}_\pm\hat{I}_3 + [\hat{I}_3, \hat{V}_\pm]) |I_3, Y\rangle \\
&= \left(\hat{V}_\pm\hat{I}_3 \pm \frac{1}{2}\hat{V}_\pm\right) |I_3, Y\rangle \\
&= \hat{V}_\pm\left(\hat{I}_3 \pm \frac{1}{2}\mathbb{1}\right) |I_3, Y\rangle \\
&= \left(I_3 \pm \frac{1}{2}\right) \hat{V}_\pm |I_3, Y\rangle
\end{aligned}$$

$$\implies \hat{V}_\pm |I_3\rangle \propto \left| I_3 \pm \frac{1}{2} \right\rangle$$

$$\begin{aligned}
[\hat{Y}, \hat{V}_+] &= \hat{V}_+ \\
[\hat{Y}, \hat{V}_-] &= -\hat{V}_-
\end{aligned}$$

$$\implies [\hat{Y}, \hat{V}_\pm] = \pm \hat{V}_\pm$$

$$\begin{aligned}
\hat{Y}\hat{V}_\pm |Y\rangle &= (\hat{V}_\pm \hat{Y} + [\hat{Y}, \hat{V}_\pm]) |Y\rangle \\
&= \left(\hat{V}_\pm \hat{I}_3 \pm \hat{V}_\pm\right) |Y\rangle \\
&= \hat{V}_\pm \left(\hat{I}_3 \pm \mathbb{1}\right) |Y\rangle \\
&= (I_3 \pm 1) \hat{V}_\pm |Y\rangle \\
\implies \hat{V}_\pm |Y\rangle &= |Y \pm 1\rangle
\end{aligned}$$

$$\implies \boxed{\hat{V}_\pm |I_3, Y\rangle \propto \left| I_3 \pm \frac{1}{2}, Y \pm 1 \right\rangle}$$

For the operator $U_\pm$:

$$\begin{aligned}
[\hat{I}_3, \hat{U}_+] &= -\frac{1}{2}\hat{U}_+ \\
[\hat{I}_3, \hat{U}_-] &= \frac{1}{2}\hat{U}_- \\
\implies [\hat{I}_3, \hat{U}_\pm] &= \mp \frac{1}{2}\hat{U}_\pm
\end{aligned}$$

$$\begin{aligned}
\hat{I}_3\hat{U}_\pm |I_3\rangle &= (\hat{U}_\pm \hat{I}_3 + [\hat{I}_3, \hat{U}_\pm]) |I_3\rangle \\
&= \left(\hat{U}_\pm \hat{I}_3 \mp \frac{1}{2}\hat{U}_\pm\right) |I_3\rangle \\
&= \hat{U}_\pm \left(\hat{I}_3 \mp \frac{1}{2}\mathbb{1}\right) |I_3\rangle \\
&= \left(I_3 \mp \frac{1}{2}\right) \hat{U}_\pm |I_3\rangle \\
\implies \hat{U}_\pm |I_3\rangle &\propto \left| I_3 \mp \frac{1}{2} \right\rangle
\end{aligned}$$

$$\begin{aligned}
[\hat{Y}, \hat{U}_+] &= \hat{U}_+ \\
[\hat{Y}, \hat{U}_-] &= -\hat{U}_- \\
\implies [\hat{Y}, \hat{U}_\pm] &= \pm \hat{U}_\pm
\end{aligned}$$

$$\begin{aligned}
\hat{Y}\hat{U}_\pm |Y\rangle &= (\hat{U}_\pm \hat{Y} + [\hat{Y}, \hat{U}_\pm]) |Y\rangle \\
&= \left(\hat{U}_\pm \hat{Y} \pm \hat{U}_\pm\right) |Y\rangle \\
&= \hat{U}_\pm \left(\hat{Y} \pm \mathbb{1}\right) |Y\rangle \\
&= (Y \pm 1) \hat{U}_\pm |Y\rangle
\end{aligned}$$

$$\implies \boxed{\hat{U}_\pm \left|I_3, Y\right\rangle \propto \left|I_3 \mp \frac{1}{2}, Y \pm 1\right\rangle}$$

For the operator $I_\pm$:

$$\begin{aligned}
&[\hat{I}_3, \hat{I}_+] = \hat{I}_+ \\
&[\hat{I}_3, \hat{I}_-] = -\hat{I}_- \\
\implies &[\hat{I}_3, \hat{I}_\pm] = \pm\hat{I}_\pm
\end{aligned}$$

$$\begin{aligned}
\hat{I}_3\hat{I}_\pm \left|I_3\right\rangle &= (\hat{I}_\pm\hat{I}_3 + [\hat{I}_3, \hat{I}_\pm]) \left|I_3\right\rangle \\
&= \left(\hat{I}_\pm\hat{I}_3 \pm \hat{I}_\pm\right) \left|I_3\right\rangle \\
&= \hat{I}_\pm \left(\hat{I}_3 \pm \mathbb{1}\right) \left|I_3\right\rangle \\
&= (I_3 \pm 1)\, \hat{I}_\pm \left|I_3\right\rangle \\
\implies \hat{I}_\pm \left|I_3\right\rangle &\propto \left|I_3 \pm 1\right\rangle
\end{aligned}$$

$$\begin{aligned}
&[\hat{Y}, \hat{I}_+] = 0 \\
&[\hat{Y}, \hat{I}_-] = 0 \\
\implies &[\hat{Y}, \hat{I}_\pm] = 0
\end{aligned}$$

$$\begin{aligned}
\hat{Y}\hat{I}_\pm \left|Y\right\rangle &= (\hat{I}_\pm\hat{Y} + [\hat{Y}, \hat{I}_\pm]) \left|Y\right\rangle \\
&= \hat{I}_\pm\hat{Y} \left|Y\right\rangle \\
&= Y\hat{I}_\pm \left|Y\right\rangle
\end{aligned}$$

$$\implies \boxed{\hat{I}_\pm \left|I_3, Y\right\rangle \propto \left|I_3 \pm 1, Y\right\rangle}$$

(d)

Isospin and hypercharge of the three basisvectors where shown to be (task (b)):

$$\begin{aligned}
\hat{I}_3 \left|u\right\rangle &= \frac{1}{2} \left|u\right\rangle \\
\hat{I}_3 \left|d\right\rangle &= -\frac{1}{2} \left|d\right\rangle \\
\hat{I}_3 \left|s\right\rangle &= 0 \left|s\right\rangle
\end{aligned}$$

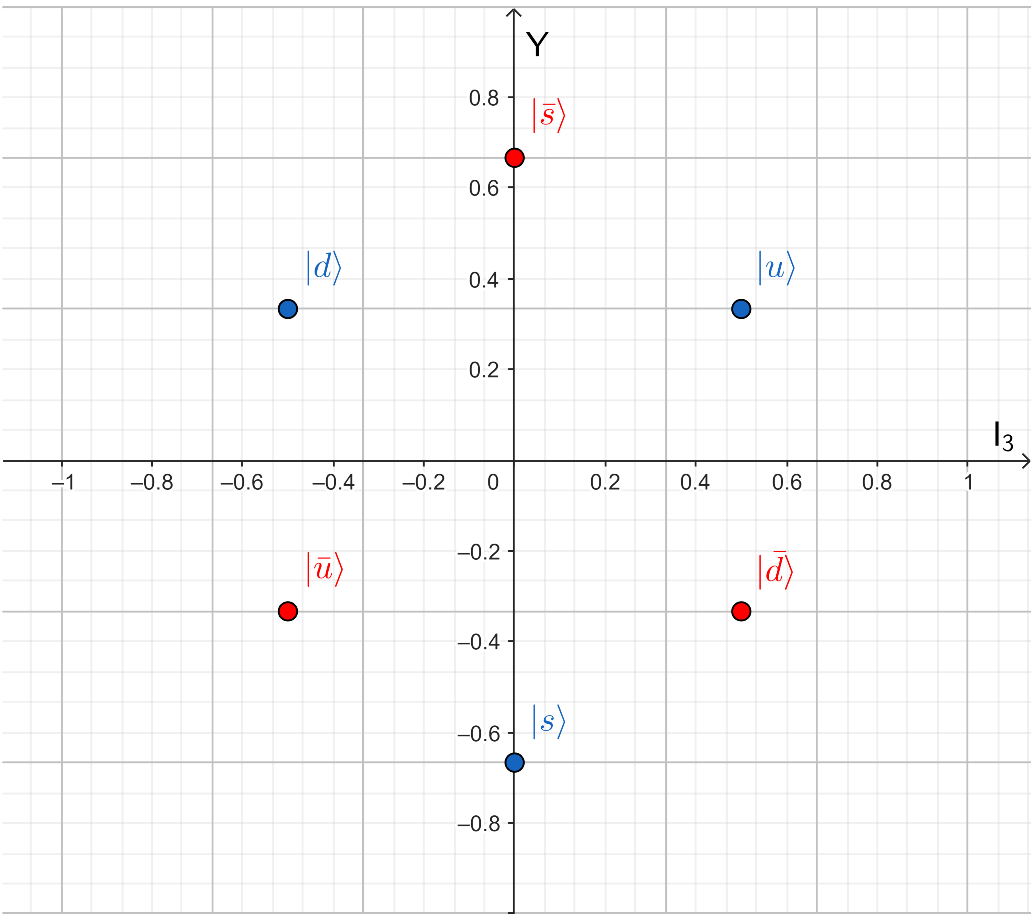
$$\hat{Y}\,|u\rangle = \frac{1}{3}\,|u\rangle$$
$$\hat{Y}\,|d\rangle = \frac{1}{3}\,|d\rangle$$
$$\hat{Y}\,|s\rangle = -\frac{2}{3}\,|s\rangle$$

Now for the strangeness S:

$$Y = B + S$$
$$S = Y - B = Y - \frac{1}{3}$$
$$\implies \hat{S} = \hat{Y} - \frac{1}{3}\mathbb{1}$$

$$\hat{S}\,|u\rangle = 0\,|u\rangle$$
$$\hat{S}\,|d\rangle = 0\,|d\rangle$$
$$\hat{S}\,|s\rangle = -1\,|s\rangle$$

(e)



(f)

Python-Code 1:

```
import numpy as np
from sympy import *
from sympy.abc import y
import matplotlib.pyplot as plt
plt.rcParams.update({"xtick.top": True , "ytick.right": True,
                     "xtick.minor.visible": True, "ytick.minor.visible": True,
                     "xtick.direction": "in" , "ytick.direction": "in",
                     "axes.labelsize": "large", "text.usetex": True, "font.size": 13
                     })

l1 = Matrix([[0,1,0],[1,0,0],[0,0,0]])
l2 = Matrix([[0,-1j,0],[1j,0,0],[0,0,0]])
l4 = Matrix([[0,0,1],[0,0,0],[1,0,0]])
l5 = Matrix([[0,0,-1j],[0,0,0],[1j,0,0]])
l6 = Matrix([[0,0,0],[0,0,1],[0,1,0]])
l7 = Matrix([[0,0,0],[0,0,-1j],[0,1j,0]])
l3 = Matrix([[1,0,0],[0,-1,0],[0,0,0]])
l8 = 1/sqrt(3) * Matrix([[1,0,0],[0,1,0],[0,0,-2]])

Ip = (l1 + 1j*l2)/2
Im = (l1 - 1j*l2)/2
Vp = (l4 + 1j*l5)/2
Vm = (l4 - 1j*l5)/2
Up = (l6 + 1j*l7)/2
Um = (l6 - 1j*l7)/2
operators = [1,Ip,Im,Vp,Vm,Up,Um]
operator_names = ["", "$I_+$","$I_-$","$V_+$","$V_-$","$U_+$","$U_-$"]

I3 = l3/2
Y = 1/sqrt(3) * l8

u = Matrix([1,0,0])
d = Matrix([0,1,0])
s = Matrix([0,0,1])
quarks = np.array([u,d,s])
quark_names = [r"$|u\rangle$",r"$|d\rangle$",r"$|s\rangle$"]
quark_colors = ["b","g","r"]

com = lambda A,B: A*B - B*A
eigenval = lambda A,x: solve(A*x - y *x,y)[y]
getY = lambda x: eigenval(Y,x)
getI3 = lambda x: eigenval(I3,x)
getX = lambda x: np.array([getI3(x), getY(x)])

fig,ax = plt.subplots()
for quark,name,color in zip(quarks,quark_names,quark_colors):
```

```
    t = 0.1 * getX(quark)
    for op,op_name in zip(operators,operator_names):
        rot = op*quark
        if rot == 0*u: continue
        ax.scatter(*(getX(rot)+t), c=color)
        ax.text(*(getX(rot)+2.5*t-[0.05,0]), op_name+name)
plt.axhline(0,linestyle="--",c="gray",linewidth=0.5)
plt.axvline(0,linestyle="--",c="gray",linewidth=0.5)
ax.set(xlabel="$I_3$",ylabel="$Y$",xlim=(-0.8,0.8),ylim=(-0.9,0.6),title="Effects of
↪  shiftoperators")
plt.savefig("effects_of_shiftoperators.svg")
```

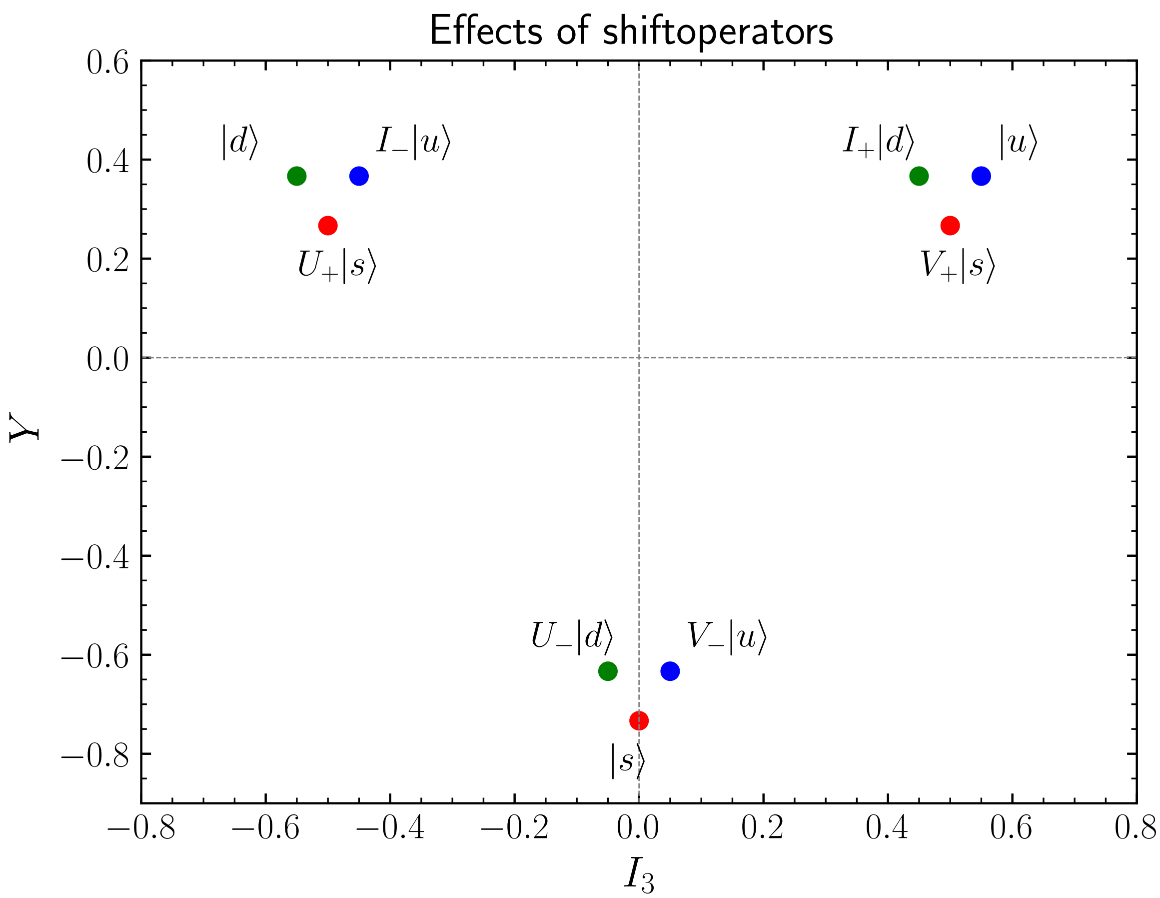


Figure 1: In the graphic, the triplet from the previous task is displayed, along with all states affected by the shift operators. Most states get mapped to (0,0,0) and are therefore omitted. Additionally, states that are mapped to the same point have been slightly separated for better clarity.

(g)

i.

Using the graphics on the problem sheet, it is obvious that $[3] \otimes [\bar{3}]$ decomposes to $[1] \oplus [8]$.

ii.

To decompose $[3] \otimes [3] \otimes [3]$, it is easiest to decompose $[3] \otimes [3]$ first. Again taking a look at the graphical decomposition $[3] \otimes [3]$ decomposes to $[\bar{3}] \oplus [6]$.

$$\begin{aligned}
[3] \otimes [3] \otimes [3] &= [3] \otimes ([\bar{3}] \oplus [6]) \\
&= [3] \otimes [\bar{3}] \oplus [3] \otimes [6] \\
&= [1] \oplus [8] \oplus [3] \otimes [6]
\end{aligned}$$

The remaining decomposition is $[3] \otimes [6]$ and it decomposes to $[10] \oplus [8]$. The composition of $[3] \otimes [3] \otimes [3]$ therefore is:

$$\begin{aligned}
[3] \otimes [3] \otimes [3] &= [1] \oplus [8] \oplus [3] \otimes [6] \\
&= [1] \oplus [8] \oplus [8] \oplus [10]
\end{aligned}$$

(h)

Python-Code 2:

```python
def add_particle(ax, x, y, label, offset=(0, 0), color='black', size=10):
    ax.scatter(x, y, c=color, s=size)
    ax.text(x + offset[0], y + offset[1], label, ha='center', fontsize=9, color=color)

shifts = {
    'ru': (0.1, 0.15), 'rd': (0.1, -0.2),
    'lu': (-0.1, 0.15), 'ld': (-0.2, -0.1),
    'l': (-0.15, 0), 'r': (0.15, 0),
    'u': (0, 0.15), 'd': (0, -0.15)
}

particles = [
    # Mesons - Octet
    (-1, 0.5, r"$K^+$", 'green', 'u'), (-1, -0.5, r"$K^0$", 'green', 'l'),  # Kaons (Y = 1)
    (0, 1, r"$\pi^+$", 'green', 'd'), (0, 0, r"$\pi^0$", 'green', 'd'), (0, -1, r"$\pi^-$",
    ↪ 'green', 'l'),  # Pions (Y = 0)
    (1, 0.5, r"$\overline{K}^0$", 'green', 'l'), (1, -0.5, r"$K^-$", 'green', 'd'),  # Kaons
    ↪ (Y = -1)
    (0, 0, r"$\eta$", 'green', 'ru'),

    # Baryons - Octet
    (1, 0.5, r"$p$", 'blue', 'u'), (1, -0.5, r"$n$", 'blue', 'r'),  # Proton & Neutron (Y = 1)
    (0, 1, r"$\Sigma^+$", 'blue', 'u'), (0, 0, r"$\Sigma^0$", 'blue', 'r'), (0, -1,
    ↪ r"$\Sigma^-$", 'blue', 'd'),  # Sigmas (Y = 0)
    (-1, 0.5, r"$\Xi^0$", 'blue', 'l'), (-1, -0.5, r"$\Xi^-$", 'blue', 'u'),  # Xis (Y = -1)
    (0, 0, r"$\Lambda$", 'blue', 'lu'),
```

```
    # Baryons - Decuplet
    (1, 1.5, r"$\Delta^{++}$", 'red', 'u'), (1, 0.5, r"$\Delta^+$", 'red', 'r'),
    (1, -0.5, r"$\Delta^0$", 'red', 'l'), (1, -1.5, r"$\Delta^-$", 'red', 'd'),  # Deltas (Y =
    ↪  1)
    (0, 1, r"$\Sigma^{*+}$", 'red', 'l'), (0, 0, r"$\Sigma^{*0}$", 'red', 'l'),
    (0, -1, r"$\Sigma^{*-}$", 'red', 'r'),  # Sigma stars (Y = 0)
    (-1, 0.5, r"$\Xi^{*0}$", 'red', 'r'), (-1, -0.5, r"$\Xi^{*-}$", 'red', 'd'),  # Xi stars
    ↪  (Y = -1)
    (-2, 0, r"$\Omega^-$", 'red', 'l')  # Omega (Y = -2)
]

fig, ax = plt.subplots(figsize=(8, 8))

ax.axhline(0, color='black', linewidth=0.5)
ax.axvline(0, color='black', linewidth=0.5)
ax.set_xlim(-3, 3)
ax.set_ylim(-2, 2)
ax.set_xlabel(r"$Y$ (Hypercharge)", fontsize=12)
ax.set_ylabel(r"$I_3$ (Isospin projection)", fontsize=12)
ax.set_title(r"$SU(3)$ Multiplets for Lightest Mesons and Baryons", fontsize=14)

for x, y, label, color, shift_key in particles:
    offset = shifts.get(shift_key, (0, 0))
    add_particle(ax, x, y, label, offset=offset, color=color)

ax.grid(color='gray', linestyle='--', linewidth=0.5)
plt.tight_layout()
plt.show()
fig.savefig("tmp.pdf")
```

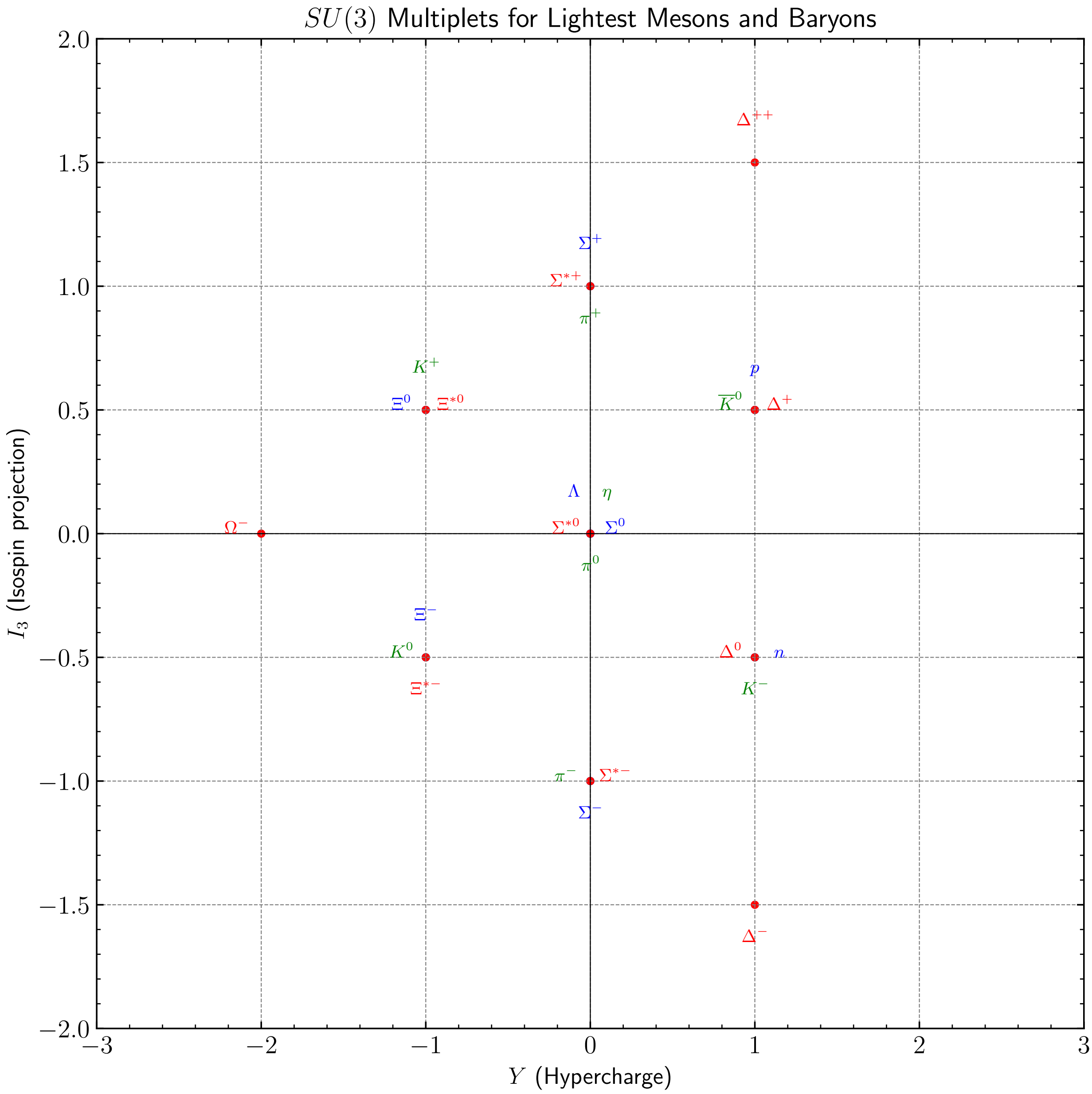


Figure 2: What a tedious task... Mesons, including the pions, kaons, and the η meson, are shown in green, while baryons from the $SU(3)$ octet, such as protons, neutrons, and Λ baryons, are represented in blue. Baryons from the $SU(3)$ decuplet, including the Δ, Σ^P, Ξ^* resonances, and the Ω^- baryon, are displayed in red.

(i)

The breaking of the $SU(3)$ symmetrie is caused by the fact that the different quarks dont have the same mass and interact differently under the weak interaction. The symmetrie would be exact if only the strong force is looked at.

(j)

The relevant particles, their constituents and their masses are:

- Proton/Neutron (N): uud or $udd \implies m_N = 3m_u + W_B$
- Xi (Ξ): uss or $dss \implies m_\Xi = m_u + 2m_s + W_B$

- Lambda (Λ): $uds \implies m_\Lambda = 2m_u + m_s + W_B$
- Sigma (Σ): uus, uds, or $dds \implies m_\Sigma = 2m_u + m_s + W_B$

Using this the mass relation can be shown:

$$\begin{aligned}
\frac{m_N + m_\Xi}{2} &= \frac{3m_\Lambda + m_\Sigma}{4} \\
\frac{(3m_u + W_B) + (m_u + 2m_s + W_B)}{2} &= \frac{3(2m_u + m_s + W_B) + (2m_u + m_s + W_B)}{4} \\
2m_u + m_s + W_B &= 2m_u + m_s + W_B
\end{aligned}$$

The experimental masses are approximatly:

$$\begin{aligned}
m_N &\approx 938\,\mathrm{MeV} \\
m_\Xi &\approx 1315\,\mathrm{MeV} \\
m_\Lambda &\approx 1115\,\mathrm{MeV} \\
m_\Sigma &\approx 1192\,\mathrm{MeV}
\end{aligned}$$

Plugging these values into the mass relation yields:

$$\begin{aligned}
\frac{938 + 1315}{2} &= \frac{3 \cdot 1115 + 1192}{4} \\
1126.5 &= 1134.25
\end{aligned}$$

The relation holds quite well, with a small discrepancy, which can be attributed to the simplified assumptions made.