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Experimental Physics Vb (WS 2023/2024)

Exercise 6

Tutorial: 1 Deadline: 16.12.2024

Task 1: Local gauge invariance of the Schrödinger equation

Applying the transformation to the right hand side yields:

$$\begin{aligned} iD_0'\Psi' &= i\left(\frac{\partial}{\partial t} + iQV'\right)e^{iQ\alpha(t,x)}\Psi \\ &= i\left(\frac{\partial}{\partial t} + iQ\left(V - \partial_t\alpha\right)\right)e^{iQ\alpha(t,x)}\Psi \\ &= i\left(\psi\partial_t + \partial_t\psi + iQ\left(V-\partial_t\alpha\right)\psi\right)e^{iQ\alpha(t,x)} \\ &= i\left(iQ(\partial_t\alpha)\psi + \partial_t\psi + iQ\left(V-\partial_t\alpha\right)\psi\right)e^{iQ\alpha(t,x)} \\ &= e^{iQ\alpha(t,x)}i\left(\partial_t + iQV\right)\psi \end{aligned}$$

$$\boxed{iD_0'\Psi' = e^{iQ\alpha(t,x)}iD_0\psi}$$

And to the left hand side:

$$\begin{aligned} D_x'\psi' &= \left(-\frac{\partial}{\partial x} + iQA'\right)\psi' \\ &= \left(-\frac{\partial}{\partial x} + iQ\left(A + \partial_x\alpha\right)\right)e^{iQ\alpha(t,x)}\psi \\ &= \left(-\partial_x\psi - \psi\partial_x + iQ\left(A+\partial_x\alpha\right)\psi\right)e^{iQ\alpha(t,x)} \\ &= \left(-\partial_x\psi - iQ(\partial_x\alpha)\psi + iQ\left(A+\partial_x\alpha\right)\psi\right)e^{iQ\alpha(t,x)} \\ &= e^{iQ\alpha(t,x)}\left(-\partial_x + iQA\right)\psi \\ &= e^{iQ\alpha(t,x)}D_x\psi \end{aligned}$$

$$\boxed{D_x'^2\psi' = D_x'\left(e^{iQ\alpha(t,x)}D_x\psi\right) = e^{iQ\alpha(t,x)}D_x^2\psi}$$

All in all this implies that the Schrödinger equation is locally gauge invariant:

$$\begin{aligned} \frac{1}{2m}(iD_x')^2\psi' &= iD_0'\Psi' \\ e^{iQ\alpha(t,x)}\frac{1}{2m}(iD_x)^2\psi &= e^{iQ\alpha(t,x)}iD_0'\Psi' \\ \left(e^{iQ\alpha(t,x)} \neq 0\right) \quad \frac{1}{2m}(iD_x)^2\psi &= iD_0\Psi \end{aligned}$$

Task 2: Gauge invariance and photon mass

(a)

$$
\begin{aligned}
j^\mu &= \partial_\nu \partial^\nu \Lambda'^\mu - \partial^\mu \partial^\nu \Lambda'_\nu \\
&= \partial_\nu \partial^\nu (\Lambda^\mu - \partial^\mu \alpha) - \partial^\mu \partial^\nu (\Lambda_\nu - \partial_\nu \alpha) \\
&= \partial_\nu \partial^\nu \Lambda^\mu - \partial^\mu \partial^\nu \Lambda_\nu - \partial_\nu \partial^\nu \partial^\mu \alpha + \partial^\mu \partial^\nu \partial_\nu \alpha \\
&= \partial_\nu \partial^\nu \Lambda^\mu - \partial^\mu \partial^\nu \Lambda_\nu - \partial^\mu \partial_\nu \partial^\nu \alpha + \partial^\mu \partial_\nu \partial^\nu \alpha \\
&= \partial_\nu \partial^\nu \Lambda^\mu - \partial^\mu \partial^\nu \Lambda_\nu
\end{aligned}
$$

i.e the wave equation is invariant under the gauge transformation $A^\mu \to A^\mu - \partial^\mu \alpha$.

(b)

$$
\begin{aligned}
j^\mu &= (\partial_\nu \partial^\nu + m^2)\Lambda'^\mu - \partial^\mu \partial^\nu \Lambda'_\nu \\
&= \partial_\nu \partial^\nu \Lambda'^\mu - \partial^\mu \partial^\nu \Lambda'_\nu + m^2 \Lambda'^\mu \\
&= \partial_\nu \partial^\nu \Lambda^\mu - \partial^\mu \partial^\nu \Lambda_\nu + m^2 \Lambda'^\mu \\
0 &= m^2 \underbrace{\Lambda'^\mu}_{\neq 0}
\end{aligned}
$$

i.e. wave equation for a massive vector field is only invariant under the same gauge transformation for $m = 0$.

Task 3: Higgs factory

(a)

Electrons have no inner structure, which means that the initial state of a e^+e^- collision is known, where as with a $p\bar{p}$ the quarks constituents have different energy levels resulting in a variety of possible inital states. Another advantage is that e^+e^- colliders produce fewer hadronic backgrounds.

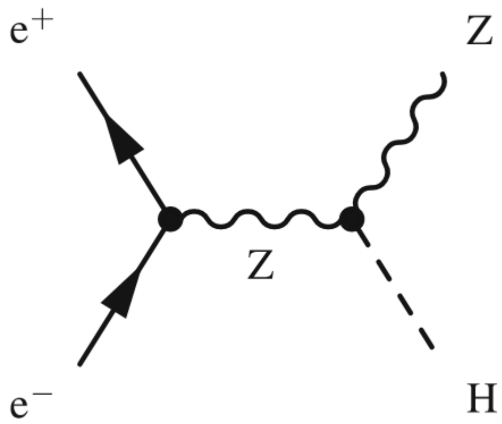
(b)

The minimum center of mass energy has to suffice to produce a non virtual Z and higgs boson:

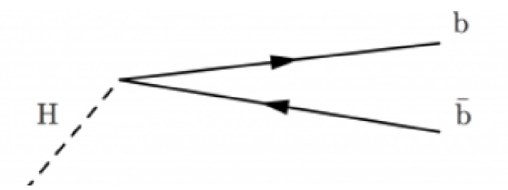
$$\sqrt{s_{\text{min}}} = m_Z + m_H \approx 216\,\text{GeV}$$

(c)

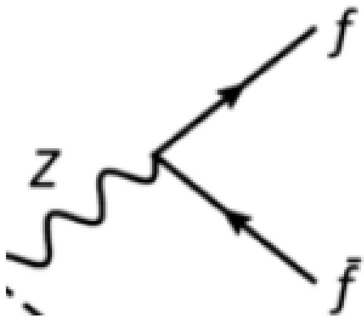
The e^+e^- collision produces a Z which radiates a higgs boson. The Feynman-diagram of this production is:



The heaviest fermion decay that can happen non virtually is $H \to b\bar{b}$, with the following feynman diagram:



While the Z decays either into two leptons $Z \to l\bar{l}$ or two quarks $Z \to q\bar{q}$:



The experimental signature is therefore:

1. a pair of b-jets with invariant mass of m_H, correspoding to the Higgs
2. a pair of leptons or quark jets with invariant masses matching the Z-Bosons mass m_Z.

(d)

It follows from conservation of four-momentum that:

$$p' = p$$

$$p_{e^+} + p_{e^-} = p_Z + p_H$$

$$\boxed{p_H = p_{e^+} + p_{e^-} - p_Z}$$

The mass is then given by:

$$m_H^2 = p_H^2$$

$$
\begin{aligned}
&= (p_{e^+} + p_{e^-} - p_Z)^2 \\
&= (p_{e^+} + p_{e^-})^2 - 2(p_{e^+} + p_{e^-})p_Z + p_Z^2 \\
&= s - 2(p_{e^+} + p_{e^-})p_Z + m_Z^2 \\
\text{(CoM frame)} \qquad &= s - 2(\sqrt{s}, \vec{0}) \cdot (E_Z, \vec{p}_Z) + m_Z^2
\end{aligned}
$$

$$
\boxed{m_H = \sqrt{s - 2E_Z\sqrt{s} + m_Z^2}}
$$

This relation can be used to calculate the Higgs mass without reconstructing the decay product of the Higgs boson. One just has to measure the center-of-mass energy $\sqrt{m}$ along with the energy of the E_Z (assuming m_Z is known with sufficient accuracy).

Task 4: Neutrino oscillations

(a)

$$
\begin{pmatrix} |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix}
$$

$$
\begin{aligned}
P_{\nu_\mu \to \nu_\mu}(t) &= |\langle \nu_\mu | \psi(t) \rangle|^2 \\
&= \left| \langle \nu_\mu | \, e^{-iHt/\hbar} \, | \nu_\mu \rangle \right|^2 \\
(E := \mathrm{diag}(E_2, E_3)) \qquad &= \left| \langle \nu_\mu | \, R e^{-iEt/\hbar} R^\dagger \, | \nu_\mu \rangle \right|^2 \\
\text{(sympy)} \qquad &= \left(e^{\frac{iE_2 t}{\hbar}} \cos^2(\theta) + e^{\frac{iE_3 t}{\hbar}} \sin^2(\theta) \right) \left(e^{-\frac{iE_3 t}{\hbar}} \sin^2(\theta) + e^{-\frac{iE_2 t}{\hbar}} \cos^2(\theta) \right) \\
&= \left(\cos^2\theta + \sin^2\theta e^{i(E_2 - E_3)t/\hbar} \right) \left(\cos^2\theta + \sin^2\theta e^{-i(E_2 - E_3)t/\hbar} \right) \\
&= \cos^4\theta + \sin^4\theta + \sin^2\theta\cos^2\theta \left(e^{i(E_2 - E_3)t/\hbar} + e^{-i(E_2 - E_3)t/\hbar} \right) \\
&= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta \cos\left((E_2 - E_3)t/\hbar\right) \\
&= \frac{\cos(4\theta) + 3}{4} + \frac{1 - \cos(4\theta)}{4} \cos\left((E_2 - E_3)t/\hbar\right) \\
&= \frac{\cos(4\theta) + 3}{4} + \frac{1 - \cos(4\theta)}{4} \left(1 - 2\sin^2\left(\frac{E_2 - E_3}{2} t/\hbar \right) \right) \\
&= 1 - \frac{1 - \cos(4\theta)}{2} \sin^2\left(\frac{E_2 - E_3}{2} t/\hbar \right) \\
&= 1 - \sin^2(2\theta) \sin^2\left(\frac{E_2 - E_3}{2} t/\hbar \right)
\end{aligned}
$$

(b)

The code is appended on the last pages.

$$
P_{\nu_\mu \to \nu_\mu}(t) = 1 - \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2 L}{4E} \right)
$$

The conversion factor from km to 1/eV can be derived as:

$$\hbar c = 197\,\text{MeV}\cdot\text{fm} = 197\cdot 10^{6}\,\text{eV}\cdot 10^{-18}\,\text{km}$$
$$\text{km} = \frac{\hbar c}{197\cdot 10^{-12}}\frac{1}{\text{eV}} \approx 5.076\cdot 10^{9}\frac{1}{\text{eV}}$$

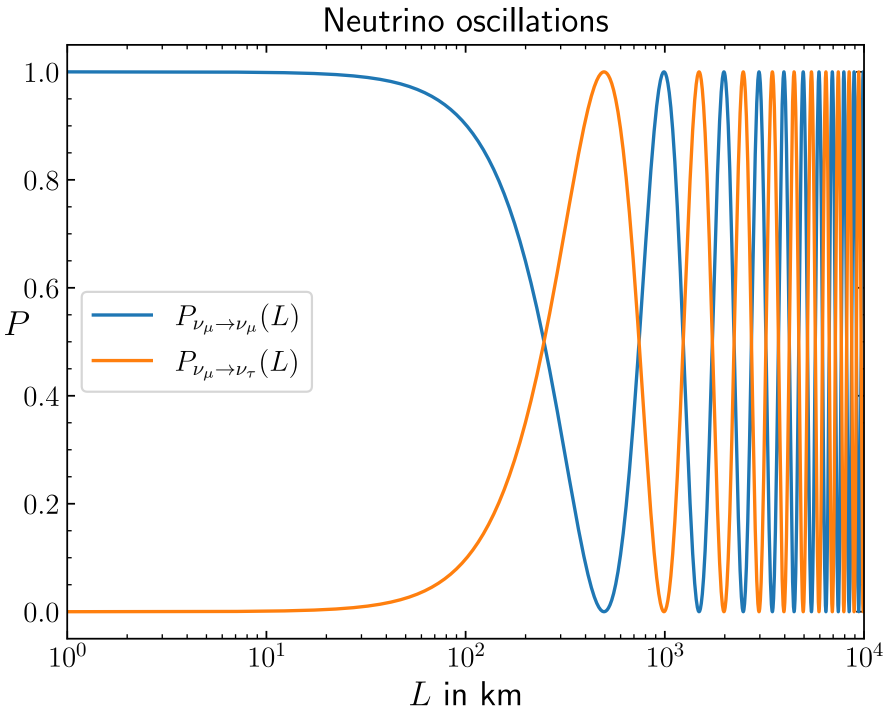


Figure 1: resulting plot

jupyter

December 31, 2024

[35]:
```
import numpy as np
from sympy import *
init_printing(use_latex="mathjax")
```

[10]:
```
from sympy import *
theta,E2,E3,t,hbar = symbols("theta E2 E3 t hbar", real=True)

# Nr.4
# (a)
psi = Matrix([1,0])
R = Matrix([[cos(theta), sin(theta)],[-sin(theta), cos(theta)]])
Rinv = R.adjoint()
E = Matrix([[E2,0],[0,E3]])

A = psi.T * R * exp(-I*E*t/hbar) * Rinv * psi
P = conjugate(A) * A
simplify(P[0])
```

[10]:

$$\left(e^{\frac{iE_2t}{\hbar}}\cos^2(\theta) + e^{\frac{iE_3t}{\hbar}}\sin^2(\theta)\right)\left(e^{-\frac{iE_3t}{\hbar}}\sin^2(\theta) + e^{-\frac{iE_2t}{\hbar}}\cos^2(\theta)\right)$$

[73]:
```
# (b)
import scipy.constants as c
import matplotlib.pyplot as plt
plt.rcParams.update({"xtick.top": True , "ytick.right": True,
                     "xtick.minor.visible": True, "ytick.minor.visible": True,
                     "xtick.direction": "in" , "ytick.direction": "in",
                     "axes.labelsize": "large", "text.usetex": True, "font.
 ↪size": 13
                     })

theta = np.pi/4
Delta_m_sq = 0.0025 / c.c**2 * c.e # J
Delta_E = c.c * Delta_m_sq
E = 1e9 # eV
L = np.logspace(0,4,1000) # km
t = L / c.c

P_mu = 1 - np.sin(2*theta)**2 * np.sin(Delta_E / 2 * t / c.hbar)**2
```

```
P_tau = 1-P_mu

plt.plot(L, P_mu, label="$P_{\\nu_\\mu \\to \\nu_\\mu}(L)$")
plt.plot(L, P_tau, label="$P_{\\nu_\\mu \\to \\nu_\\tau}(L)$")
plt.xscale("log")
plt.xlim(min(L), max(L))
plt.xlabel("$L$ in km")
plt.ylabel("$P$",rotation=0)
plt.legend()
plt.title("Neutrino oscillations")
plt.savefig("neutrino_oscillations.pdf")
```