1.

a)

$$D_d(E)\,dE = 2L^d \frac{O_d(k)\,dk}{(2\pi)^d} = 2\left(\frac{L}{2\pi}\right)^d \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} k^{d-1}dk$$

$$E_F = \frac{\hbar^2}{2m}k_F^2 \Rightarrow k_F = \frac{(2mE_F)^{\frac{1}{2}}}{\hbar} \qquad dk = \left(\frac{m}{2E\hbar^2}\right)^{\frac{1}{2}} dE$$

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(1) = 1 \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$D_d(E)\,dE = 2\left(\frac{L}{2\pi}\right)^d \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} k^{d-1}dk$$

$$D_d(E)\,dE = 2\left(\frac{L}{2\pi}\right)^d \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \left(\frac{2mE}{\hbar^2}\right)^{\frac{d-1}{2}} \left(\frac{m}{2E\hbar^2}\right)^{\frac{1}{2}} dE$$

$$= 2\left(\frac{L}{2\pi}\right)^d \frac{2(\pi m)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{(2E)^{\frac{d}{2}-1}}{\hbar^d}\, dE$$

$$\Rightarrow \quad D_d(E) \propto E^{\frac{d}{2}-1}$$

$$D_1(E)\,dE = \frac{L}{\pi}\sqrt{\frac{2m}{E}}\, dE \qquad D_2(E)\,dE = \frac{mL^2}{\pi\hbar^2}\, dE$$

$$D_3(E)\,dE = \frac{L^3 m\sqrt{2mE}}{\pi^2\hbar^3}\, dE$$

b)

$$n = \frac{N}{L^d} \Rightarrow N = nL^d \qquad D(E) = \frac{dN}{dE}$$

$$dN = D(E)\,dE = 2\left(\frac{L}{2\pi}\right)^d \frac{2(\pi m)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{(2E)^{\frac{d}{2}-1}}{\hbar^d}\, dE$$

$$nL^d = N = 2\left(\frac{L}{2\pi}\right)^d \frac{2(\pi m)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{2^{\frac{d}{2}-1}}{\hbar^d} \int E^{\frac{d}{2}-1} dE = 2\left(\frac{L}{2\pi}\right)^d \frac{2(\pi m)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{2^{\frac{d}{2}-1}}{\hbar^d} \frac{2}{d} E^{\frac{d}{2}}$$

$$\Rightarrow E^{\frac{d}{2}} = \frac{1}{4} n \left(\frac{2\pi}{m}\right)^{\frac{d}{2}} \hbar^d \Gamma(\tfrac{d}{2})\, d \qquad \Rightarrow E = \frac{2\pi\hbar^2}{m}\left(\frac{d\,\Gamma(\frac{d}{2})\,n}{4}\right)^{2/d}$$

$$d=1: \; E = \frac{\pi^2\hbar^2 n^2}{8m} \qquad d=2: \; E = \frac{\pi\hbar^2 n}{m} \qquad d=3: \; E = \frac{\pi^{4/3}\hbar^2(3n)^{2/3}}{2m}$$