Bayesian Historical Monte Carlo Simulation

Luca D'Aquanno

Fincite

March 2023

Outline

Fincite

The Bootstrapping method

- Randomly sample returns from a time series and use them as an estimate for future returns
- Bootstrapping is a method used to obtain a return confidence interval for a certain time horizon

The Bootstrapping method: an example

In this framework, users can express their views on the market through a two-step procedure:

- Standardize historical returns
- Incorporate views in the standardized returns:

$$r_{std} = \frac{r_j - \bar{r}}{\hat{\sigma}_r}, \quad r_{new} = (r_{std} \ x \ \sigma_r^{view}) + \mathbb{E}^{view}(r) \tag{1}$$

- r_{std} are standardized returns
- r_{new} are returns incorporating the user's views on the expected value and the volatility ($\mathbb{E}^{view}(r)$, σ_r^{view})

Fincite

Drawbacks of the current methodology

Currently, we are assigning equal probability to every joint realization of returns:

- Probabilities associated with the most recent scenarios should obtain more weight
- Probabilities associated with scenarios closer to the user's views should obtain more weight

Outline

1. The implemented Bootstrapping method
2. The Entropy pooling method
   - Time-conditioning
   - View-conditioning
   - The Entropy-minimization algorithm
3. Practical application: Monte Carlo simulation with time/views-conditional probabilities

Fincite

The Entropy pooling method:

Consider a market driven by an N-dimensional random variable X (stock returns, interest rates, implied volatilities, etc.):

$$X \sim f_X,$$
$$f_X := \{x_t, p_t\}_{t=1,2,\ldots,T}$$

- The distribution of X is defined by a series of historical scenarios x_t with associated probabilities p_t
- Every x_t (N-length vector for every t) is associated with a scalar p_t (a probability for every joint scenario of returns)

Fincite

The initial model (prior)

The initial reference model takes into account the time-decay effect by assigning higher weights to the most recent observations:

- $f_X := \{x_t, p_t\}_{t=1,2,\dots,T}$
- $p = [p_1 \dots p_t \dots p_T]'$
- $p_{t-1} < p_t$

Exponential-decay model for probabilities

The differential equation describing the Exponential-decay[1]:

$$\frac{dp}{dt} = -\lambda p$$

is solved by:

$$p(t) = p_{ew} e^{-\lambda t}$$

- λ is the decay rate
- $\tau_{HL} = \frac{ln(2)}{\lambda}$, $\lambda = \frac{ln(2)}{\tau_{HL}}$

Fincite

[1] Exponential-decay

Time-conditioned probabilities

Each entry p_t of the vector p can be defined as follows:

- $p_t \middle| \tau_{HL} := p_{ew} e^{-\frac{ln(2)}{\tau_{HL}}(|t-T|)}$
- $p_{ew} := 1/ \sum_t e^{-\frac{ln(2)}{\tau_{HL}}(|t-T|)}$
- p_{ew} is the equally-weighted probability
- τ_{HL} can be interpreted as the time required for the probability of a scenario to decrease to half of its maximum value in T
- The lower is the half-life parameter τ_{HL}, the higher is the decay rate λ

Fincite

Time-conditioned probabilities: an example

t	T	$e^{-\lambda(\lvert t-T \rvert)}$	$p_t \lvert \tau_{HL}$
2	2	1	0,666
1	2	0,5	0,333

- $\tau_{HL} = 1$
- $\lambda = 0,693$
- $T = 2$

It is worth noting:
$p_{\tau_{HL}} = \frac{1}{2} p_T$

Defining the user's views

- Views are represented as expressions on the expectation of arbitrary functions of the market[2]

$$V := \left\{ \mathbb{E}_p \Big(v(X) \Big) \geq v_* \right\} \tag{2}$$

- Where v_* is a threshold value that determines the intensity of the view

[2] in [1] there are some examples of how those kinds of functions v(X) can be defined

Defining the user's views: a simple example

The simplest definition of the function $v(X)$ maps the risk drivers X in a portfolio defined by the vector of weights w:

- $v(X) := Xw$
- $\mathbb{E}_p\Big(v(X)\Big) := p'v(X)$

Fincite

Defining the user's views: a simple example

Assuming a bullish outlook for our portfolio with an average return of 6%, we can express our views as follows:

- $\mathbb{E}_p\Big(v(X)\Big) \geq v_*$
- $v_* = 6\%$

Computing the posterior probability distribution

- Our ultimate goal is to compute a posterior distribution f_X^{post}, that departs from the prior: $f_X := \{x_t, p_t | \tau_{HL}\}_{t=1,2,\ldots,T}$
- To take into account the views, the posterior distribution f_X^{post} is defined by new probabilities p_t^{post} for the same scenario outcomes x_t:

$$f_X^{post} := \{x_t, p_t^{post}\}_{t=1,2,\ldots,T}$$

Computing the posterior probability distribution

To compute p^{post}, we must rely on the relative entropy $\xi(p^{post}, p)$ as a measure of the distance between p and p^{post}:

$$\xi(p^{post}, p) := (p^{post})' \left(ln(p^{post}) - ln(p) \right) \tag{3}$$

Fincite

Computing the posterior probability distribution

We then define the posterior as the distribution that is closest to the prior, as measured by (3), which satisfies the views (2):

$$\underset{p^{post} \in V}{\operatorname{argmin}_{\mathrm{p}^{\mathrm{post}}}} \ \xi(p^{post}, p), \tag{4}$$

using the Exponential-decay model as the prior for probabilities ($\mathrm{p} = p|\tau_{HL}$).

Entropy minimization: defining constraints

- With reference to equation (2), consider two views: one for the expected value and one for the variance of our portfolio.
- Define distinct functions $v_1(X)$ and $v_2(X)$ to map N-dimensional risk drivers X to a generic vector Z for each view:

$$
\begin{aligned}
Z_1 &= [v_1(x_1) \quad \dots \quad v_1(x_t) \quad \dots \quad v_1(x_T)] \\
Z_2 &= [v_2(x_1) \quad \dots \quad v_2(x_t) \quad \dots \quad v_2(x_T)]
\end{aligned}
$$

Fincite

Entropy minimization: defining constraints

To formulate the optimization problem, we must define both inequality and equality constraints:

$$V := \left\{ Fq \geq f, Hq = h \right\},$$

using a vector q as the counterpart of p^{post} posterior probabilities.

Entropy minimization: defining constraints

- $F = \begin{bmatrix} v_1(x_1) & \dots & v_1(x_t) & \dots & v_1(x_T) \\ v_2(x_1) & \dots & v_2(x_t) & \dots & v_2(x_T) \end{bmatrix}$
- $q = [q_1 \quad \dots \quad q_t \quad \dots \quad q_T]'$ is a vector collecting the probability for each scenario x_t at time step t
- H is the counterpart of F for equality constraints: $\sum_t q_t = 1$ is specified, using the vector $H = [1 \quad \dots \quad 1 \quad \dots \quad 1]$ and the scalar $h = 1$
- $f = [f_1 \quad f_2]'$ is a vector collecting the value for each view intensity

Fincite

Entropy minimization

The optimization problem is:

$$\underset{Fq \leq f, Hq = h}{\text{argmin}_{\text{q}}} \sum_{t=1}^{T} q_t(ln(q_t) - ln(p_t)) \tag{5}$$

and the Lagrangian function can be expressed in the vectorial notation as:

$$L(q, \lambda_1, \lambda_2) = q'(ln(q) - ln(p)) + \lambda_1'(Fq - f) + \lambda_2'(Hq - h) \tag{6}$$

Fincite

Entropy minimization

The Lagrange multipliers λ_1' and λ_2' are row vectors where the number of rows equals the number of inequality and equality constraints.

Entropy minimization

The first order condition for q reads:

- $\frac{dL}{dq} = ln(q) - ln(p) + 1 + F'\lambda_1 - H'\lambda_2$
- $\frac{dL}{dq} = [0 \quad \dots \quad 0 \quad \dots \quad 0]'$

and solving for q:

$$q(\lambda_1, \lambda_2) = e^{ln(p) - 1 - F'\lambda_1 - H'\lambda_2} \tag{7}$$

The Duality principle

Given the convexity of $\xi(q,p)$ in (4) for a posterior q and a fixed prior p, then:

$$\xi(q,p) \leq L(q,\lambda_1,\lambda_2), \forall\ \lambda_1 \geq 0.$$

Taking the minimum of both sides with respect to q, we get

$$\min_q \xi(q,p) \leq \min_q L(q,\lambda_1,\lambda_2).$$

Taking the maximum of both sides with respect to λ, we get:

$$\max_{\lambda_1 \geq 0, \lambda_2} \min_q \xi(q,p) \leq \max_{\lambda_1 \geq 0, \lambda_2} \min_q L(q,\lambda_1,\lambda_2)$$

Fincite

The Duality principle

It is worth noting that, according to (5):

- If $\lambda_1 < 0$ then the constraint $Fq \leq f$ is violated
- λ_2 does not need to be constrained in order to satisfy $Hq = h$

The Duality principle

The dual function is given by definition:

$$G(\lambda_1, \lambda_2) := \min_q L(q, \lambda_1, \lambda_2)$$

therefore, we have:

$$\max_{\lambda_1 \geq 0, \lambda_2} G(\lambda_1, \lambda_2) \geq \min_q \xi(q, p)$$

- The solution of the dual problem is an upper bound for the solution of the primal problem
- If the objective function is strictly convex, then the minimization problem has a unique solution:

$$\max_{\lambda_1 \geq 0, \lambda_2} G(\lambda_1, \lambda_2) = \min_q \xi(q, p)$$

Fincite

Entropy minimization

According to (7) the Lagrange dual function can be expressed as:

$$G(\lambda_1, \lambda_2) := L(q(\lambda_1, \lambda_2), \lambda_1, \lambda_2). \tag{8}$$

The two vectors of Lagrange multipliers (λ_1,λ_2) result from maximizing the Lagrange dual function:

$$(\lambda_1^*, \lambda_2^*) := \underset{\lambda_1 \geq 0,}{\text{argmax}}_{\lambda_1,\lambda_2} \left\{ G(\lambda_1, \lambda_2) \right\}$$

Fincite

KKT conditions

To ensure that λ_1, λ_2 are also a solution for the optimization problem (5), we need to check if the KKT conditions are met:

$$\max_{\lambda_1 \geq 0, \lambda_2} \min_{q} L(q(\lambda_1 \lambda_2), p),$$

such that:

- $\frac{dL(q,p)}{dq} = 0$
- $\lambda_1(Fq - f) = 0$
- $Fq - f \leq 0$
- $Hq - h = 0$

KKT conditions

In the minimization case we have:

$$\min_{\lambda_1 \leq 0, \lambda_2} - \min_{q} L(q(\lambda_1 \lambda_2), p)$$

such that:

- $\frac{dL(q,p)}{dq} = 0$
- $\lambda_1(Fq - f) = 0$
- $Fq - f \geq 0$
- $Hq - h = 0$

Entropy minimization

Finally, we can define the set of posterior probabilities as:

$$p^{post} := q(\lambda_1^*, \lambda_2^*) \tag{9}$$

Outline

Fincite

Monte Carlo simulation with historical bootstrapping

Assuming we want to evaluate the expected performance of our portfolio over the next year:

1. Generate $\{x_{n,t}, p_t^{post}\}_{t=1,2,\ldots,252}^{n=1,2,\ldots,N}$ scenarios, for t time steps and n simulations
2. For each time step t, calculate the portfolio return for each simulated scenario n
    1. $r_{n,ptf,t} = x_{n,t} w_t$
    2. with w_t portfolio weights at time step t
3. Compute the cumulative return over the time period, for each simulated scenario:
    1. $\{R_{n,t}\}_{t=1,2\ldots,252}^{n=1,2,\ldots,N} = \sum_{t=1}^{252} \prod_{i=1}^{t} (1 + r_{n,ptf,t}) - 1$

Monte Carlo simulation with historical bootstrapping

Determine the estimate of the confidence interval for portfolio returns by calculating the 1st, 50th, and 99th quantiles.

Fincite

Monte Carlo simulation with historical bootstrapping

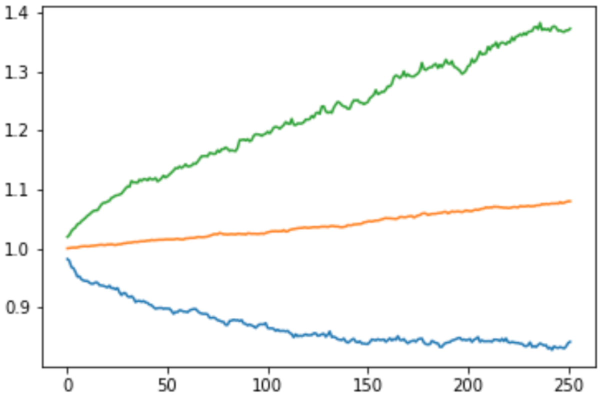

Figure: Evolution range of €1 in one year

Advantages of Entropy pooling approach

The Entropy pooling method offers versatility in incorporating different market views:

- Scenarios can be historical or simulated
- Extension of Black & Litterman Portfolio Optimization:
  - Non-linear constraints
  - Non-normal markets

[1] Attilio Meucci. "Mixing probabilities, priors and kernels via entropy pooling". In: *GARP Risk Professional* (2011), pp. 32–36.