Non-commutative algebra

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Contents

Introduction

The notes correspond to the master course *Non-commutative Algebra* of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into thirteen two-hour lectures.

Most of the material is based on standard results on group algebras covered in the VUB course *Associative Algebras*. Lecture notes for this course are freely available here. Basic texts on group algebras are Lam's book [**4**] and Passman's book [**6**].

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Lecture 1. 15/02/2024

§ 1.1. Solvable groups. A subgroup H of G is said to be **characteristic** if $f(H) \subseteq H$ for all $f \in \mathrm{Aut}(G)$. The center and the commutator subgroup are characteristic subgroups. Every characteristic subgroup is normal, as the maps $x \mapsto gxg^{-1}$ are automorphisms.

Exercise 1.1. Prove that if H is characteristic in K and K is normal in G, then H is normal in G.

For a group G, let $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. The **derived series** of G is the sequence

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

Each $G^{(i)}$ is a characteristic subgroup of G. We say that G is **solvable** if $G^{(n)} = \{1\}$ for some n.

Example 1.2. Abelian groups are solvable.

Example 1.3. The group $\mathbf{SL}_2(3)$ is solvable, as the derived series is

$$\mathbf{SL}_2(3) \supseteq Q_8 \supseteq C_4 \supseteq \{1\}.$$

Example 1.4. Non-abelian simple groups cannot be solvable.

Exercise 1.5. Let G be a group. Prove the following statements:

1) A subgroup H of G is solvable, when G is solvable.
2) Let K be a normal subgroup of G. Then G is solvable if and only if K and G/K are solvable.

Example 1.6. For $n \geq 5$ the group $\mathbb{A}_n$ is simple and non-abelian. Hence it is not solvable. It follows that $\mathbb{S}_n$ is not solvable for $n \geq 5$.

Exercise 1.7. Let p be a prime number. Prove that finite p-groups are solvable.

Exercise 1.8. Let p, q and r be prime numbers. Prove that groups of order $p^\alpha q$, p^2q^2 and pqr are solvable.

Exercise 1.9. Prove that groups of order < 60 are solvable.

Theorem 1.10 (Burnside). *Let p be a prime number. If G is a finite group that has a conjugacy class with $p^k > 1$ elements, then G is not simple.*

The easiest way to prove Theorem 1.10 is using character theory.

Theorem 1.11 (Burnside). *Let p and q be prime numbers. If G has order p^aq^b, then G is solvable.*

Proof. If G is abelian, then it is solvable. Suppose now G is non-abelian. Let us assume that the theorem is not true. Let G be a group of minimal order p^aq^b that is not solvable. Since $|G|$ is minimal, G is a non-abelian simple group. By the previous theorem, G has no conjugacy classes of size p^k nor conjugacy classes of size q^l with $k, l \geq 1$. The size of every conjugacy class of G is one or divisible by pq. Note that, since G is a non-abelian simple group, the center of G is trivial. Thus there is only one conjugacy class of size one. By the class equation,

$$|G| = 1 + \sum_{C:|C|>1} |C| \equiv 1 \bmod pq,$$

where the sum is taken over all conjugacy classes with more than one element, a contradiction. □

A recent generalization of Burnside's theorem is based on *word maps*. A word map of a group G is a map

$$G^k \to G, \quad (x_1,\dots,x_k) \mapsto w(x_1,\dots,x_k)$$

for some word $w(x_1,\dots,x_k)$ of the free group F_k of rank k. Some word maps are surjective in certain families of groups. For example, Ore's conjecture is precisely the surjectivity of the word map $(x,y) \mapsto [x,y] = xyx^{-1}y^{-1}$ in every finite non-abelian simple group.

Theorem 1.12 (Guralnick–Liebeck–O'Brien–Shalev–Tiep). *Let $a,b \geq 0$, p and q be prime numbers and $N = p^a q^b$. The map $(x,y) \mapsto x^N y^N$ is surjective in every non-abelian finite simple group.*

The proof appears in [**3**].

The theorem implies Burnside's theorem. Let G be a group of order $N = p^a q^b$. Assume that G is not solvable. Fix a composition series of G. There is a non-abelian factor S of order dividing N. Since S is simple non-abelian and $s^N = 1$ for all $s \in S$, it follows that the word map $(x,y) \mapsto x^N y^N$ has trivial image in S, a contradiction to the theorem.

Theorem 1.13 (Feit–Thompson). *Groups of odd order are solvable.*

The proof of Feit–Thompson theorem is extremely hard. It occupies a full volume of the *Pacific Journal of Mathematics* [**1**]. A formal verification of the proof (based on the computer software Coq) was announced in [**2**]. This motivates a natural problem: To formally verify the classification of finite simple groups. Will mathematics move away from depending on just humans to verify proofs? Formal verification with computer-proof assistants could become the new standard for rigor in mathematics.

Back in the day, it was believed that if a certain divisibility conjecture is true, the proof of Feit–Thompson theorem could be simplified.

Conjecture 1.14 (Feit–Thompson). There are no prime numbers p and q such that $\frac{p^q-1}{p-1}$ divides $\frac{q^p-1}{q-1}$.

The conjecture remains open. However, now we know that proving the conjecture will not simplify further the proof of Feit–Thompson theorem.

In 2012, Le proved that the conjecture is true for $q = 3$, see [**5**].

In [**7**] Stephens proved that a certain stronger version of the conjecture does not hold, as the integers $\frac{p^q-1}{p-1}$ and $\frac{q^p-1}{q-1}$ could have common factors. In fact, if $p = 17$ and $q = 3313$, then

$$\gcd\left(\frac{p^q-1}{p-1}, \frac{q^p-1}{q-1}\right) = 112643.$$

Nowadays we can check this easily on almost every desktop computer:

```
gap> Gcd((17^3313-1)/16,(3313^17-1)/3312);
112643
```

No other counterexamples have been found of Stephen's stronger version of the conjecture.

Definition 1.15. Let p be a prime number. A p-group P is said to be **elementary abelian** if $x^p = 1$ for all $x \in P$.

Definition 1.16. A subgroup M of G is said to be **minimal normal** if $M \neq \{1\}$ (or $G = \{1\}$), M is normal in G and the only normal subgroup of G properly contained in M is $\{1\}$.

Example 1.17. If a normal subgroup M is minimal (with respect to the inclusion), then it is minimal and normal. However, the converse statement does not hold. For example, the subgroup of $\mathbb{A}_4$ generated by $(12)(34)$, $(13)(24)$ and $(14)(23)$ is minimal normal in $\mathbb{A}_4$ but it is not minimal.

Exercise 1.18. Prove that every finite group contains a minimal normal subgroup.

Example 1.19. Let $G = \mathbb{D}_6 = \langle r, s : r^6 = s^2 = 1, srs = r^{-1} \rangle$ the dihedral group of twelve elements. The subgroups $S = \langle r^2 \rangle$ and $T = \langle r^3 \rangle$ are (the only) minimal normal in G.

Example 1.20. Let $G = \mathbf{SL}_2(3)$. The only minimal normal subgroup of G is its center $Z(\mathbf{SL}_2(3)) \simeq C_2$.

The following lemma will be very useful later.

Lemma 1.21. *Let M be a minimal normal subgroup of G. If M is solvable and finite, then M is an elementary abelian p-group for some prime number p.*

Proof. Since M is solvable, $[M,M] \subsetneq M$. Moreover, $[M,M]$ is normal in G, as $[M,M]$ is characterisitic in M and M is normal in G. Since M is minimal normal, $[M,M] = \{1\}$. Hence M is abelian.

Since M is finite, there is a prime number p such that $\{1\} \neq P = \{x \in M : x^p = 1\} \subseteq M$. Since P is characteristic in M, P is normal in G. By minimality, $P = M$. $\square$

Theorem 1.22. *Let G be a finite non-trivial solvable group. Then every maximal subgroup of G has index p^α for some prime number p.*

Proof. We proceed by induction on $|G|$. If $|G|$ is a prime power, the claim is clear. Assume that $|G| \geq 6$ and let M be a maximal subgroup of G. Let N be a minimal normal subgroup of G and $\pi \colon G \to G/N$ the canonical map. If $N = G$, then $N = G$ is a p-group and we are done. Assume then that $N \neq G$. Since $M \subseteq NM \subseteq G$, either $M = NM$ or $NM = G$ (by the maximality of M). If $M = NM \supseteq N$, then $\pi(M)$ is a maximal subgroup of $\pi(G) = G/N$. Hence

$$(G : M) = (\pi(G) : \pi(M))$$

is a prime power by the inductive hypothesis. If $NM = G$, then

$$(G : M) = \frac{|G|}{|M|} = \frac{|NM|}{|M|} = \frac{|N|}{|N \cap M|}$$

is a prime power, because N is a p-group by the previous lemma. $\square$

Exercise 1.23. Let G be a finite non-trivial solvable group. Prove that there exists a prime number p such that G contains a minimal normal p-subgroup.

Example 1.24. Let $G=\mathbb{S}_4$. The 2-subgroup

$$K=\{\mathrm{id},(12)(34),(13)(24),(14)(23)\}\simeq C_2\times C_2$$

is minimal normal. Note that G does not have minimal normal 3-subgroups.

Theorem 1.25. *Let G be a finite non-trivial group. Then G is solvable if and only if every non-trivial quotient of G contains an abelian non-trivial normal subgroup.*

Proof. Every quotient of G is solvable and therefore contains an abelian minimal normal subgroup. To prove the converse we proceed by induction on $|G|$. Let N be a normal abelian subgroup of G. If $N=G$, then G is solvable (because it is abelian). If $N\neq G$, then $|G/N|<|G|$. Since every quotient of G/N is a quotient of G, the group G/N satisfies the assumptions of the theorem. Hence G/N is solvable by the inductive hypothesis. Now N and G/N are solvable, so is G. □

Exercise 1.26. Let G be a group. Prove that G is solvable if and only if there is a sequence

$$\{1\}=N_0\subseteq N_1\subseteq\cdots\subseteq N_k=G$$

of normal subgroups such that every quotient N_i/N_{i-1} is abelian.

§ 1.2. Hall's theorem. We start with an extremely simple and useful tool.

Lemma 1.27 (Frattini's argument). *Let G be a finite group and K be a normal subgroup of G. If $P\in\mathrm{Syl}_p(K)$ for some prime number p, then $G=KN_G(P)$.*

Proof. Let $g\in G$. Since $gPg^{-1}\subseteq gKg^{-1}=K$, because K is normal in G, and $gPg^{-1}\in\mathrm{Syl}_p(K)$, there exists $k\in K$ such that $kPk^{-1}=gPg^{-1}$. Hence $k^{-1}g\in N_G(P)$, as $P=(k^{-1}g)P(k^{-1}g)^{-1}$. Therefore $g=k(k^{-1}g)\in KN_G(P)$. □

Theorem 1.28 (Hall). *Let G be a finite group such that every maximal subgroup of G has a prime or a prime-square index. Then G is solvable.*

Proof. We proceed by induction on $|G|$. Let p be the largest prime divisor of $|G|$. Let $S\in\mathrm{Syl}_p(G)$ and $N=N_G(S)$.

If $N=G$, then S is normal in G. Since every maximal subgroup of G/S has prime or a prime-square index, G/S is solvable by the inductive hypothesis. Since S is a p-group, it is solvable. Therefore G is solvable.

Assume now that $N\neq G$. Let H be a maximal subgroup of G containing N. Then

$$N=N_H(S)=N_G(S),$$

the number of Sylow p-subgroups of G is $(G:N)\equiv 1 \bmod p$ and the number of Sylow p-subgroups of H is $(H:N)\equiv 1 \bmod p$ by the third Sylow's theorem. So,

$$\underbrace{(G:N)}_{\equiv 1 \bmod p}=(G:H)\underbrace{(H:N)}_{\equiv 1 \bmod p},$$

implies that also $(G:H)\equiv 1 \bmod p$. By assumption, there exists a prime number q such that $(G:H)\in\{q,q^2\}$. Since q divides $|G|$, it follows that $q<p$. If $(G:H)=q$, then p divides $q-1$ and therefore $p\leq q-1<p$, a contradiction. Thus $q^2=(G:H)\equiv 1 \bmod p$. From this it follows that $q\equiv -1 \bmod p$ and hence $q=2$ and $p=q+1=3$.

Therefore G has order $2^\alpha 3^\beta$. If we apply Burnside's theorem, we are done. Instead, we will finish the proof with an elementary argument. Let K be a minimal normal subgroup of G. By Frattini's argument (Lemma 1.27), $G = KN = KH$. Since H is maximal,

$$(K : K \cap H) = (G : H) = 4,$$

as $(G:H) = |G|/|H| = |KH|/|H| = |K|/|K \cap H|$. Since $(K : K \cap H) = 4$, letting K act on $K/K \cap H$ by left multiplication, there exists a non-trivial group homomorphism $\rho \colon K \to \mathbb{S}_4$. Since $[K,K]$ is characteristic in K and K is normal in G, $[K,K] \subseteq K$ is normal in G. Since K is minimal normal in G, there are two possible cases: either $[K,K] = \{1\}$ or $[K,K] = K$. If $[K,K] = K$, since $\mathbb{S}_4$ is solvable, $\rho(K)$ is solvable. Then

$$\rho(K) = \rho([K,K]) = [\rho(K), \rho(K)],$$

a contradiction. Therefore $[K,K] = \{1\}$ and K is solvable (as it is abelian). $\square$

Lecture 2. 22/02/2024

§ 2.1. Wielandt's theorem.

LEMMA 2.1. *Let G be a finite group and H and K be subgroups of G of coprime indices. Then $G = HK$ and $(H : H \cap K) = (G : K)$.*

PROOF. Let $D = H \cap K$. Since

$$(G : D) = \frac{|G|}{|H \cap K|} = (G : H)(H : H \cap K),$$

$(G : H)$ divides $(G : D)$. Similarly, $(G : K)$ divides $(G : D)$. Since $(G : H)$ and $(G : K)$ are coprime, $(G : H)(G : K)$ divides $(G : D)$. In particular,

$$\frac{|G|}{|H|}\frac{|G|}{|K|} = (G : H)(G : K) \leq (G : D) = \frac{|G|}{|H \cap K|}$$

and hence $|G| = |HK|$. Since

$$|G| = |HK| = |H||K|/|H \cap K|,$$

we conclude that $(G : K) = (H : H \cap K)$. □

DEFINITION 2.2. Let G be a group and H be a subgroup of G. The **normal closure** H^G of H in G is the subgroup $H^G = \langle xHx^{-1} : x \in G \rangle$.

EXERCISE 2.3. Let G be a group and H a subgroup of G. Prove that H^G is normal in G and that H^G is the smallest normal subgroup of G containing H.

EXAMPLE 2.4. Let $G = \mathbb{A}_4$ and $H = \{\mathrm{id}, (12)(34)\}$. The normal closure of H in G is

$$H^G = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2.$$

THEOREM 2.5 (Wielandt). *Let G be a finite group and H, K and L be subgroups of G with pairwise coprime indices. If H, K and L are solvable, then G is solvable.*

PROOF. Let G be a minimal counterexample. Then $G \neq \{1\}$. There are two cases to consider. Assume first that $H = \{1\}$. Then $|G| = (G : H)$ is coprime with $(G : K)$ and thus $G = K$ is solvable. Assume now that $H \neq \{1\}$. Let M be a minimal normal subgroup of H. By Lemma 1.21, M is a p-group for some prime number p. Without loss of generality, we may assume that p does not divide $(G : K)$ (otherwise, if p divides $(G : K)$, then p does not divide $(G : L)$ and we just need to change K by L). There exists $P \in \mathrm{Syl}_p(G)$ such that $P \subseteq K$. Sylow subgroups are conjugate, so there exists $g \in G$ such that $M \subseteq gKg^{-1}$. Since $(G : gKg^{-1}) = (G : K)$ is coprime with $(G : H)$, Lemma 2.1 implies that $G = (gKg^{-1})H$.

We claim that all conjugates of M are in gKg^{-1}. If $x \in G$, write $x = uv$ some some $u \in gKg^{-1}$ and $v \in H$. Since M is normal in H,

$$xMx^{-1} = (uv)M(uv)^{-1} = uMu^{-1} \subseteq gKg^{-1}.$$

Let $N = M^G$ be the normal closure of M in G. Then $\{1\} \neq N \subseteq gKg^{-1}$ is solvable, as gKg^{-1} is solvable. We claim that G/N is solvable. Let $\pi \colon G \to G/(M^G)$ be the canonical map. Since H, K and L are solvable, the subgroups $\pi(H)$, $\pi(K)$ and $\pi(L)$ of $\pi(G)$ are solvable. By the correspondence theorem, $\pi(H)$, $\pi(K)$ and $\pi(L)$ have pairwise coprime indices. Moreover, $\pi(G)$ is solvable, as $|\pi(G)| < |G|$. Hence G is solvable. □

§ 2.2. Hall's theorem.

Definition 2.6. Let G be a finite group of order $p^\alpha m$, where p is a prime number such that $\gcd(p,m)=1$. A subgroup H of G is said to be a ***p*-complement** if $|H|=m$.

Example 2.7. Let $G=\mathbb{S}_3$. Then $H=\langle(123)\rangle$ is a 2-complement and $K=\langle(12)\rangle$ is a 3-complement.

Theorem 2.8 (Hall). *Let G be a finite group that admits a p-complement for every prime divisor p of $|G|$. Then G is solvable.*

Proof. Let $|G|=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ with $p_1<\cdots<p_k$ prime numbers. We proceed by induction on k. If $k=1$, then the claim holds, as G is a p-group. If $k=2$, the result holds by Burnside's theorem. Assume now that $k\geq 3$. For $j\in\{1,2,3\}$, let H_j be a p_j-complement in G. Since $|H_j|=|G|/p_j^{\alpha_j}$, the subgroups H_j have pairwise coprime indices.

We claim that H_1 is solvable. Note that $|H_1|=p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Let p be a prime number dividing $|H_1|$ and Q be a p-complement in G. Since $(G:H_1)$ and $(G:Q)$ are coprime, Lemma 2.1 implies that

$$(H_1:H_1\cap Q)=(G:Q).$$

Then $H_1\cap Q$ is a p-complement in H_1. Therefore H_1 is solvable by the inductive hypothesis. Similarly, both H_2 and H_3 are solvable.

Since H_1, H_2 and H_3 are solvable of pairwise coprime indices, the theorem follows from Wielandt's theorem. □

§ 2.3. Nilpotent groups. For a group G and $x,y,z\in G$, conjugation will be considered as a left action of G on G and we will use the following notation: ${}^xy=xyx^{-1}$. The commutator between x and y will be written as

$$[x,y]=xyx^{-1}y^{-1}=({}^xy)y^{-1}.$$

We will also use the following notation: $[x,y,z]=[x,[y,z]]$. For subgroups X, Y and Z of G, we write $[X,Y,Z]=[X,[Y,Z]]$. Note that $[X,Y]=[Y,X]$.

Exercise 2.9 (The Hall–Witt identity). Let G be a group and $x,y,z\in G$. Prove that

$$\left({}^y[x,y^{-1},z]\right)\left({}^z[y,z^{-1},x]\right)\left({}^x[z,x^{-1},y]\right)=1. \tag{2.1}$$

If G is a group and $[G,G]$ is central in G, then the Hall-Witt identity becomes Jacobi's identity.

Lemma 2.10 (Hall's three subgroups lemma). *Let X, Y and Z be subgroups of G such that $[X,Y,Z]=[Y,Z,X]=\{1\}$. Then $[Z,X,Y]=\{1\}$.*

Proof. Since $[x,y]\in C_G(z)$ implies $[X,Y]\subseteq C_G(Z)$, it is enough to prove that $[z,x^{-1},y]=1$ for all $x\in X$, $y\in Y$ and $z\in Z$. Since $[y^{-1},z]\in[Y,Z]$, $[x,y^{-1},z]\in[X,Y,Z]=\{1\}$. Thus ${}^y[x,y^{-1},z]=1$. Similarly, ${}^z[y,z^{-1},x]=1$. Using the Hall–Witt identity, we conclude that $[z,x^{-1},y]=1$. □

Exercise 2.11. Let N be a normal subgroup of G and X, Y and Z be subgroups of G. If $[X,Y,Z]\subseteq N$ and $[Y,Z,X]\subseteq N$, then $[Z,X,Y]\subseteq N$.

Definition 2.12. Let G be a group. The **lower central series** is the sequence $\gamma_k(G)$ of subgroups defined inductively as

$$\gamma_1(G)=G,\quad \gamma_{i+1}(G)=[G,\gamma_i(G)]\quad i\geq 1.$$

DEFINITION 2.13. A group G is said to be **nilpotent** if there exists a positive integer c such that $\gamma_{c+1}(G) = \{1\}$. The smallest c with $\gamma_{c+1}(G) = \{1\}$ is the **nilpotency class** of G.

EXERCISE 2.14. Prove that every nilpotent group is solvable.

A group is nilpotent of nilpotency class one if and only if it is abelian.

EXAMPLE 2.15. The group $\mathbb{S}_3$ is solvable, as $\mathbb{S}_3 \supseteq \mathbb{A}_3 \supseteq \{1\}$ is a composition series with abelian factors. However, $\mathbb{S}_3$ is not nilpotent, as

$$\gamma_1(\mathbb{S}_3) = \mathbb{A}_3, \quad \gamma_2(\mathbb{S}_3) = [\mathbb{A}_3, \mathbb{S}_3] = \mathbb{A}_3,$$

and therefore $\gamma_i(\mathbb{S}_3) \neq \{1\}$ for all $i \geq 1$.

EXAMPLE 2.16. The group $G = \mathbb{A}_4$ is not nilpotent, as

$$\gamma_1(G) = G, \quad \gamma_j(G) = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2$$

for all $j \geq 2$. We can do this with the computer:

```
gap> IsNilpotent(AlternatingGroup(4));
false
```

Let us do the calculation of the lower central series with the computer:

```
gap> List(LowerCentralSeries(AlternatingGroup(4)),\
StructureDescription);
[ "A4", "C2 x C2" ]
```

Here is an alternative:

```
gap> G := AlternatingGroup(4);;
gap> gamma_1 := G;;
gap> gamma_2 := DerivedSubgroup(G);;
gap> gamma_3 := CommutatorSubgroup(gamma_2,G);;
gap> StructureDescription(gamma_1);
"A4"
gap> StructureDescription(gamma_2);
"C2 x C2"
gap> StructureDescription(gamma_3);
"C2 x C2"
```

EXERCISE 2.17. Let G be a group. Prove the following statements:

1) Each $\gamma_i(G)$ is a characteristic subgroup of G.
2) $\gamma_i(G) \supseteq \gamma_{i+1}(G)$ for all $i \geq 1$.
3) If $f \colon G \to H$ is a surjective group homomorphism, then $f(\gamma_i(G)) = \gamma_i(H)$ for all $i \geq 1$.

EXERCISE 2.18. Prove that if H and K are nilpotent groups, then $H \times K$ is nilpotent.

EXERCISE 2.19. Let G be a nilpotent group. Prove the following statements:

1) Subgroups of G are nilpotent.
2) If $f \colon G \to H$ is a surjective homomorphism, then H is nilpotent.

EXERCISE 2.20. True or false? If G is a nilpotent group and N is normal subgroup of G such that N and G/N are nilpotent, then G is nilpotent.

PROPOSITION 2.21. *Finite p-groups are nilpotent.*

PROOF. We proceed by induction on $|G|$. The case $G = \{1\}$ is trivial. Assume the result holds for p-groups of order $< |G|$. Since G is a p-group, $Z(G) \neq \{1\}$. By the inductive hypothesis, $G/Z(G)$ is nilpotent. There exists c such that $\gamma_{c+1}(G/Z(G)) = \{1\}$.

Let $\pi \colon G \to G/Z(G)$ be the canonical map. By Exercise 2.17,

$$\pi(\gamma_{c+1}(G)) = \gamma_{c+1}(G/Z(G)) = \{1\}.$$

Then $\gamma_{c+1}(G) \subseteq \ker \pi = Z(G)$. Hence G is nilpotent, as

$$\gamma_{c+2}(G) = [G, \gamma_{c+1}(G)] = [G, Z(G)] = \{1\}. \qquad \square$$

THEOREM 2.22. *If G is a group, then $[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G)$ for all $i, j \geq 1$.*

PROOF. We proceed by induction on i. The case $i = 1$ is trivial, as $[G, \gamma_j(G)] = \gamma_{j+1}(G)$ by definition. Assume that the result holds for some $i \geq 1$ and all $j \geq 1$.

First note that

$$[G, \gamma_i(G), \gamma_j(G)] \subseteq [G, \gamma_{i+j}(G)] = \gamma_{i+j+1}(G)$$

by the inductive hypothesis. Moreover, by the inductive hypothesis,

$$[\gamma_i(G), \gamma_j(G), G] = [\gamma_i(G), G, \gamma_j(G)] = [\gamma_i(G), \gamma_{j+1}(G)] \subseteq \gamma_{i+j+1}(G)$$

By using Exercise 2.11 with $X = G$, $Y = \gamma_i(G)$ and $Z = \gamma_j(G)$, we get that

$$[\gamma_j(G), G, \gamma_i(G)] \subseteq \gamma_{i+j+1}(G).$$

Hence

$$[\gamma_{i+1}(G), \gamma_j(G)] = [\gamma_j(G), \gamma_{i+1}(G)] = [\gamma_j(G), G, \gamma_i(G)] \subseteq \gamma_{i+j+1}(G). \qquad \square$$

We consider arbitrary commutators, not necessarily associated on the right. For example, both $[x, y, z] = [x, [y, z]]$ and $[[x, y], z]$ are commutators of **weight** three.

COROLLARY 2.23. *In a group G, every weight n commutator is contained in $\gamma_n(G)$.*

PROOF. We proceed by induction on n. The case $n = 1$ is trivial. Assume that $n \geq 1$ and the result holds for all $j \leq n$. An arbitrary commutator of weight $n+1$ is of the form $[A, B]$, where A is a commutator of weight k, B is a commutator of weight l and $n+1 = k+l$. Since $k \leq n$ and $l \leq n$, the inductive hypothesis implies that $A \subseteq \gamma_k(G)$ and $B \subseteq \gamma_l(G)$. Hence $[A, B] \subseteq [\gamma_k(G), \gamma_l(G)] \subseteq \gamma_{k+l}(G)$ by the previous theorem. $\square$

EXERCISE 2.24. Let G be a group. Prove that $G^{(k)} \subseteq \gamma_{2^k}(G)$ for all $k \geq 1$.

EXERCISE 2.25. Let G be a nilpotent group of class m. Prove that the derived length (i.e. the length of the derived series) of G is $\leq 1 + \log_2 m$.

The following lemma is important. It states that nilpotent groups satisfy the **normalizer condition**.

Lemma 2.26 (normalizer condition). *Let G be a nilpotent group. If H is a proper subgroup of G, then $H \subsetneq N_G(H)$.*

Proof. There exists c such that $G = \gamma_1(G) \supseteq \cdots \supseteq \gamma_{c+1}(G) = \{1\}$. Since $\{1\} = \gamma_{c+1}(G) \subseteq H$ and $\gamma_1(G) \not\subseteq H$, let k be the smallest positive integer such that $\gamma_k(G) \subseteq H$. Since

$$[H, \gamma_{k-1}(G)] \subseteq [G, \gamma_{k-1}(G)] = \gamma_k(G) \subseteq H,$$

we obtain that $xHx^{-1} \subseteq H$ for all $x \in \gamma_{k-1}(G)$, that is $\gamma_{k-1}(G) \subseteq N_G(H)$. If $N_G(H) = H$, then $\gamma_{k-1}(G) \subseteq H$, a contradiction to the minimality of k. □

For a group G, we define the sequence $\zeta_0(G), \zeta_1(G), \ldots$ recursively as follows:

$$\zeta_0(G) = \{1\}, \quad \zeta_{i+1}(G) = \{g \in G : [x,g] \in \zeta_i(G) \text{ for all } x \in G\}, \quad i \geq 0.$$

For example, $\zeta_1(G) = Z(G)$.

Lemma 2.27. *Let G be a group. For every $i \geq 0$, the set $\zeta_i(G)$ is a normal subgroup of G.*

Proof. We proceed by induction on i. The case $i = 0$ is trivial, as $\zeta_0(G) = \{1\}$. Assume the result holds for some i. We claim that $\zeta_{i+1}(G)$ is normal subgroup of G. Let $g, h \in \zeta_{i+1}(G)$ and $x \in G$. By the inductive hypothesis,

$$\begin{aligned}
&[x, g^{-1}] = (xg^{-1})[x^{-1}, g](xg^{-1})^{-1} \in \zeta_i(G)(xg^{-1})^{-1} = \zeta_i(G),\\
&[x, gh] = [x,h][hxh^{-1}, g] \in \zeta_i(G).
\end{aligned}$$

Since $1 \in \zeta_{i+1}(G)$, we conclude that each $\zeta_i(G)$ is a subgroup of G. Moreover, since

$$[xgx^{-1}, y] = x[g, x^{-1}yx]x^{-1} \in \zeta_i(G)$$

for all $y \in G$, we obtain that $xgx^{-1} \in \zeta_{i+1}(G)$. □

Definition 2.28. Let G be a group. The **ascending central series** of G is the sequence

$$\{1\} = \zeta_0(G) \subseteq \zeta_1(G) \subseteq \zeta_2(G) \subseteq \cdots$$

Definition 2.29. A group G is said to be **perfect** if $[G,G] = G$.

Theorem 2.30 (Grün). *If G is a perfect group, then $Z(G/Z(G)) = \{1\}$.*

Proof. By definition, $[G, \zeta_2(G)] \subseteq Z(G)$ and $[\zeta_2(G), G] \subseteq Z(G)$. Then

$$[G, G, \zeta_2(G)] = [G, \zeta_2(G), G] = \{1\}.$$

By using the three subgroups lemma with $X = Y = G$ and $Z = \zeta_2(G)$,

$$[\zeta_2(G), G] = [\zeta_2(G), [G,G]] = [\zeta_2(G), G, G] = \{1\}.$$

Thus $\zeta_2(G) \subseteq Z(G)$.

We aim to prove that $Z(G/Z(G))$ is trivial. Let $\pi \colon G \to G/Z(G)$ be the canonical map and $g \in G$ be such that $\pi(g)$ is central. Since

$$1 = [\pi(x), \pi(g)] = \pi([x,g])$$

for all $x \in G$, $[x,g] \in Z(G) = \zeta_1(G)$ for all $x \in G$. Hence $g \in \zeta_2(G) \subseteq Z(G)$. □

Lecture 3. 29/02/2024

For subgroups H and K of G, let

$$[H,K] = \langle [h,k] : h \in H, k \in K \rangle.$$

Let G be a group and K be a subgroup of G. We say that K **normalizes** H if $K \subseteq N_G(H)$. We say that K **centralizes** H if $K \subseteq C_G(H)$, that is if and only if $[H,K] = \{1\}$.

EXERCISE 3.1. Let K and H be subgroups of G such that $K \subseteq H$ and K is normal in G. Prove that $[H,G] \subseteq K$ if and only if $H/K \subseteq Z(G/K)$.

LEMMA 3.2. *Let G be a group. There exists an integer c such that $\zeta_c(G) = G$ if and only if $\gamma_{c+1}(G) = \{1\}$. In this case,*

$$\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$$

for all $i \in \{0,1,\dots,c\}$.

PROOF. Assume first that $\zeta_c(G) = G$. To prove that $\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$ holds for all i, we proceed by induction. The case $i = 0$ is trivial. So assume that the result holds for some $i \geq 0$. If $g \in \gamma_{i+2}(G) = [G, \gamma_{i+1}(G)]$, then

$$g = \prod_{k=1}^{N} [x_k, g_k],$$

for some $g_1, \dots, g_N \in \gamma_{i+1}(G)$ and $x_1, \dots, x_N \in G$. By the inductive hypothesis,

$$g_k \in \gamma_i(G) \subseteq \zeta_{c-i}(G)$$

for all k. Hence $[x_k, g_k] \in \zeta_{c-i-1}(G)$ for all k. Therefore $g \in \zeta_{c-(i+1)}(G)$.

We now assume that $\gamma_{c+1}(G) = \{1\}$. We aim to prove that $\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$ holds for all i. We proceed by backward induction on i. The case $i = c$ is trivial. So assume the result holds for some $i+1 \leq c$. Let $g \in \gamma_i(G)$. By the inductive hypothesis,

$$[x,g] \in [G, \gamma_i(G)] = \gamma_{i+1}(G) \subseteq \zeta_{c-i}(G).$$

Thus $g \in \zeta_{c-i+1}(G)$ by definition. □

EXAMPLE 3.3. Let $G = \mathbb{S}_3$. Then $\zeta_j(G) = \{1\}$ for all $j \geq 0$:

```
gap> UpperCentralSeries(SymmetricGroup(3));
[ Group(()) ]
```

DEFINITION 3.4. Let G be a group. A **central series** for G is a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups of G such that for each $i \in \{1,\dots,n\}$, $\pi_i(G_{i-1})$ is a subgroup of $Z(G/G_i)$, where $\pi_i \colon G \to G/G_i$ is the canonical map.

LEMMA 3.5. *Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ be a central series of a group G. Then $\gamma_{i+1}(G) \subseteq G_i$ for all i.*

PROOF. We proceed by induction on i. The case $i = 0$ is trivial. So assume the result holds for some $i \geq 0$. Let $\pi_i \colon G \to G/G_i$ be the canonical map. Then

$$\gamma_{i+1}(G) = [G, \gamma_i(G)] \subseteq [G, G_{i-1}].$$

Since $\pi_i(G_{i-1}) \subseteq Z(G/G_i)$,

$$\pi_i([G,G_{i-1}]) = [\pi_i(G),\pi_i(G_{i-1})] = \{1\}.$$

Hence $\gamma_{i+1}(G) = [G,G_{i-1}] \subseteq G_i$. □

THEOREM 3.6. *A group is nilpotent if and only if it admits a central series.*

PROOF. Let G be a group. If G is nilpotent, then the $\gamma_j(G)$ form a central series of G. Conversely, if $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ is a central series of G, then, by the previous lemma,

$$\gamma_{n+1}(G) \subseteq G_n = \{1\}.$$

Hence G is nilpotent. □

EXERCISE 3.7. Let G be a group. Prove that if K is a subgroup of $Z(G)$ such that G/K is nilpotent, then G is nilpotent.

§ 3.1. Hirsch's theorem.

THEOREM 3.8 (Hirsch). *Let G be a nilpotent group. If H is a non-trivial normal subgroup of G, then $H \cap Z(G) \neq \{1\}$. In particular, $Z(G) \neq \{1\}$.*

PROOF. Since $\zeta_0(G) = \{1\}$ and there exists an integer c such that $\zeta_c(G) = G$, there exists

$$m = \min\{k : H \cap \zeta_k(G) \neq \{1\}\}.$$

Since H is normal in G,

$$[G,H \cap \zeta_m(G)] \subseteq H \cap [G,\zeta_m(G)] \subseteq H \cap \zeta_{m-1}(G) = \{1\}.$$

Therefore $\{1\} \neq H \cap \zeta_m(G) \subseteq H \cap Z(G)$. If $H = G$, then $Z(G) \neq \{1\}$. □

EXERCISE 3.9. Let G be a nilpotent group and M be a minimal normal subgroup of G. Prove that $M \subseteq Z(G)$.

DEFINITION 3.10. Let G be a group. A subgroup M is said to be **maximal normal** in G if $M \neq G$ and M is the only proper normal subgroup of G containing M.

COROLLARY 3.11. *Let G be a non-abelian nilpotent group, and A be a maximal normal and abelian subgroup of G. Then $A = C_G(A)$.*

PROOF. Since A is abelian, $A \subseteq C_G(A)$. Assume that $A \neq C_G(A)$. The centralizer $C_G(A)$ is normal in G, as, since A is normal in G,

$$gC_G(A)g^{-1} = C_G(gAg^{-1}) = C_G(A).$$

for all $g \in G$. Let $\pi\colon G \to G/A$ be the canonical map. Then $\pi(C_G(A))$ is a non-trivial normal subgroup of $\pi(G)$. Since G is nilpotent, $\pi(G)$ is nilpotent. By Hirsch's theorem,

$$\pi(C_G(A)) \cap Z(\pi(G)) \neq \{1\}.$$

Let $x \in C_G(A) \setminus A$ be such that $\pi(x)$ is central in $\pi(G)$. Then $\langle A,x\rangle$ is abelian, as $x \in C_G(A)$. Moreover, $\langle A,x\rangle$ is normal in G, as A is normal in G and $gxg^{-1}x^{-1} \in A$ for all $g \in G$ (because $\pi(x)$ is central). Hence $gxg^{-1} \in \langle A,x\rangle$ and therefore $A \subsetneq \langle A,x\rangle \subsetneq G$, a contradiction. □

THEOREM 3.12. *Let G be a nilpotent group. The following statements hold:*

1) *Every minimal normal subgroup of G has prime order and is central.*
2) *Every maximal subgroup of G is normal of prime index and contains $[G,G]$.*

PROOF.

1) Let N be a minimal normal subgroup of G. Since $N\cap Z(G)\neq\{1\}$ by Hirsch's theorem, $N\cap Z(G)$ is a normal subgroup of G contained in N. Then $N=N\cap Z(G)\subseteq Z(G)$ by the minimality of N. In particular, N is abelian. Since every subgroup of N is normal in G, N is simple. Hence $N\simeq C_p$ for some prime number p.
2) If M is a maximal subgroup, then M is normal in G by the normalizer condition (Lemma 2.26). By the maximality of M, the quotient G/M contains no proper non-trivial subgroups. Thus $G/M\simeq C_p$ for some prime p. Since G/M is abeliano, $[G,G]\subseteq M$. □

The previous theorem does not prove the existence of maximal subgroups. For example, $\mathbb{Q}$ is a nilpotent group (as it is abelian) that contains no maximal subgroups.

PROPOSITION 3.13. *Let G be a nilpotent group and H be a subgroup with $(G:H)=n$. If $g\in G$, then $g^n\in H$.*

PROOF. We proceed by induction on n. The case $n=1$ is trivial. The case $n=2$ follows from the normality of H. So assume the result holds for all groups of index $<n$. Let H be a subgroup of G such that $(G:H)=n$. Let $H_0=H$ and $H_{i+1}=N_G(H_i)$ for all $i\geq 0$. By definition, H_i is normal in H_{i+1}. Since G is nilpotent, $H_i\neq G$ implies that $H_i\subsetneq H_{i+1}$ by the normalizer condition. Since $(G:H)$ is finite, there exists k such that $H_k=G$. Since $(H_j:H_{j-1})<n$ for all j, the inductive hypothesis implies that $x^{(H_j:H_{j-1})}\in H_{j-1}$ for all $x\in H_j$ and all j. Hence

$$g^{(G:H)}=g^{(H_k:H_{k-1})(H_{k-1}:H_{k-2})\cdots(H_1:H_0)}\in H.$$ □

EXERCISE 3.14. Does the previous proposition hold for non-nilpotent groups?

The following lemma is useful for performing induction in the nilpotency index of nilpotent groups.

LEMMA 3.15. *Let G be a nilpotent group of class $c\geq 2$. If $x\in G$, then the subgroup $\langle x,[G,G]\rangle$ is nilpotente of class $<c$.*

PROOF. Let $H=\langle x,[G,G]\rangle$. If $x\in[G,G]$, the there is nothing to prove. So assume that $x\notin[G,G]$. Note that

$$H=\{x^nc:n\in\mathbb{Z},c\in[G,G]\},$$

as $[G,G]$ is normal in G. We need to show that $[H,H]\subseteq\gamma_3(G)$. Let $h=x^nc,k=x^md\in H$ be such that $c,d\in[G,G]$. Since

$$[h,x^m]=[x^n,[c,x^m]][c,x^m]\in\gamma_4(G)\gamma_3(G)\subseteq\gamma_3(G),$$

then

$$\begin{aligned}[h,k]&=[h,x^m][x^m,[h,d]][h,d]\\&=[x^n,[c,x^m]][c,x^m][x^m,[h,d]][h,d]\in\gamma_3(G).\end{aligned}$$ □

EXAMPLE 3.16. Let $G=\mathbb{D}_8=\langle r,s:r^8=s^2=1,srs=r^{-1}\rangle$ the dihedral group of order 16. Then G is nilpotent of class three and $[G,G]=\{1,r^2,r^4,r^6\}\simeq C_4$. The subgroup $\langle s,[G,G]\rangle\simeq\mathbb{D}_4$ is nilpotent of class two.

```
gap> G := DihedralGroup(IsPermGroup,16);;
gap> gens := GeneratorsOfGroup(G);;
gap> r := gens[1];;
gap> s := gens[2];;
gap> D := DerivedSubgroup(G);;
gap> S := Subgroup(G, Concatenation(Elements(D), [s]));;
gap> StructureDescription(S);
"D8"
gap> NilpotencyClassOfGroup(G);
3
gap> NilpotencyClassOfGroup(S);
2
```

Let us discuss a concrete application of Lemma 3.15.

THEOREM 3.17. *If G is a nilpotent group, then*

$$T(G) = \{g \in G : g^n = 1 \text{ for some } n \geq 1\}$$

is a subgroup of G.

PROOF. We proceed by induction on the nilpotency class of G. Let $a,b \in T(G)$ and

$$A = \langle a, [G,G] \rangle, \quad B = \langle b, [G,G] \rangle.$$

Since A and B are nilpotent of class $< c$ by the previous lemma, the inductive hypothesis implies that $T(A)$ is a subgroup of A and $T(B)$ is a subgroup of B. Since $T(A)$ is characteristic in A and A is normal in G, $T(A)$ is normal in G. Similarly, $T(B)$ is normal in G.

We claim that every element of $T(A)T(B)$ has finite order. If $x \in T(A)T(B)$, say $x = a_1b_1$ with a_1 of order m, then x has finite order, as

$$x^m = (a_1b_1)^m = (a_1b_1a_1^{-1})(a_1^2b_1a_1^{-2})\cdots(a^{m-1}b_1a^{-m+1})b_1 \in T(B).$$

To see clearly what is what we did, let us work out a concrete example, say $m = 3$. In this case, we obtain the following formula:

$$\begin{aligned}(a_1b_1)^3 &= (a_1b_1)(a_1b_1)(a_1b_1)\\ &= (a_1b_1a_1^{-1})(a_1^2b_1a_1^{-2})a_1^3b_1 = (a_1b_1a_1^{-1})(a_1^2b_1a_1^{-2})b_1,\end{aligned}$$

as $a_1^3 = 1$.

With this trick, we prove that ab and a^{-1} have finite order. Hence $T(G)$ is a subgroup of G. $\square$

Another application:

THEOREM 3.18. *Let G be a torsion-free nilpotent group and $a,b \in G$. If there exists $n \neq 0$ such that $a^n = b^n$, then $a = b$.*

PROOF. We proceed by induction on the nilpotency order c of G. The result clearly holds for abelian groups. Assume that G is nilpotent of class $c \geq 2$. Since $\langle a, [G,G] \rangle$ is a nilpotent subgroup of G of class $< c$ and $bab^{-1} = [b,a]a \in \langle a, [G,G] \rangle$, the inductive hypotehsis implies that $ba = ab$, as

$$a^n = (bab^{-1})^n = b^n.$$

Thus $(ab^{-1})^n = a^nb^{-n} = 1$. Since G has no torsion, we conclude that $a = b$. $\square$

COROLLARY 3.19. *Let G be a torsion-free nilpotent group. If $x,y \in G$ are such that $x^ny^m = y^mx^n$ for some $n,m \neq 0$, then $xy = yx$.*

Proof. Let $a=x$ and $b=y^nxy^{-n}$. Since $a^m=b^m$, the previous theorem implies that $a=b$. Thus $xy^n=y^nx$. Apply the previous theorem again, this time with $a=y$ and $b=xyx^{-1}$. Then we conclude that $xy=yx$. $\square$

Before proving another theorem, we recall a basic lemma about finitely generated groups.

Lemma 3.20. *Let G be a finitely generated group and H a finite-index subgroup. Then H is finitely generated.*

Proof. Assume that G is generated by $\{g_1,\dots,g_m\}$. Without loss of generality, we may assume that for each i there exists k such that $g_i^{-1}=g_k$.

Let $\{1=t_1,\dots,t_n\}$ be a transversal of H in G, that is a complete set of representatives of G/H. For $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$, write

$$t_ig_j=h(i,j)t_{k(i,j)}.$$

We claim that H is generated by the $h(i,j)$. Let $x\in H$. Then

$$\begin{aligned} x&=g_{i_1}\cdots g_{i_s}\\ &=(t_1g_{i_1})g_{i_2}\cdots g_{i_s}\\ &=h(1,i_1)t_{k_1}g_{i_2}\cdots g_{i_s}\\ &=h(1,i_1)h(k_1,i_2)t_{k_2}g_{i_3}\cdots g_{i_s}\\ &=h(1,i_1)h(k_1,i_2)\cdots h(k_{s-1},i_s)t_{k_s}, \end{aligned}$$

where $k_1,\dots,k_{s-1}\in\{1,\dots,n\}$. Since $t_{k_s}\in H$ (because $x\in H$), $t_{k_s}=1\in H$. Hence x is generated by the $h(i,j)$. $\square$

Now the theorem:

Theorem 3.21. *Let G be a finitely generated torsion group that is nilpotent. Then G is finite*

Proof. We proceed by induction on the nilpotency class c of G. The case $c=1$ is true, as G is abelian. So assume the result holds for groups of class $c\geq 1$. Since $[G,G]$ and $G/[G,G]$ are finitely generated (Lemma 3.20) torsion nilpotent groups of class $<c$, the inductive hypothesis implies that $[G,G]$ and $G/[G,G]$ are finite groups. Thus G is finite. $\square$

§ 3.2. Finite nilpotent groups. Before studying finite nilpotent groups, we need a lemma.

Lemma 3.22. *Let G be a finite group and p a prime number dividing $|G|$. If $P\in\mathrm{Syl}_p(G)$, then*

$$N_G(N_G(P))=N_G(P).$$

Proof. Let $H=N_G(P)$. Since P is normal in H, P is the only Sylow p-subgroup of H. To prove that $N_G(H)=H$, it is enough to see that $N_G(H)\subseteq H$. Let $g\in N_G(H)$. Since

$$gPg^{-1}\subseteq gHg^{-1}=H,$$

$gPg^{-1}\in\mathrm{Syl}_p(H)$ and H has only one Sylow p-subgroup, $P=gPg^{-1}$. Hence $g\in N_G(P)=H$. $\square$

Theorem 3.23. *Let G be a finite group. The following statements are equivalent:*

1) *G is nilpotent.*
2) *Every Sylow subgroup of G is normal in G.*
3) *G is a direct product of its Sylow subgroups.*

Proof. We first prove that $(1) \implies (2)$. Let $P \in \mathrm{Syl}_p(G)$. We aim to prove that P is normal in G, that is $N_G(P) = G$. By Lemma 3.22, $N_G(N_G(P)) = N_G(P)$. Now the normalizer condition (Lemma 2.26) implies that $N_G(P) = G$.

We now prove that $(2) \implies (3)$. Let $p_1, \cdots, p_k$ be the prime factors of $|G|$. For each $i \in \{1, \dots, k\}$, let $P_i \in \mathrm{Syl}_{p_i}(G)$. By assumption, each P_j is normal in G.

We claim that $P_1 \cdots P_j \simeq P_1 \times \cdots \times P_j$ for all j. The case $j = 1$ is trivial. So assume the result holds for some $j \geq 1$. Since

$$N = P_1 \cdots P_j \simeq P_1 \times \cdots \times P_j$$

is normal in G and it has order coprime with $|P_{j+1}|$,

$$N \cap P_{j+1} = \{1\}.$$

Hence

$$NP_{j+1} \simeq N \times P_{j+1} \simeq P_1 \times \cdots \times P_j \times P_{j+1},$$

as P_{j+1} is normal in G. Since now $P_1 \cdots P_k \simeq P_1 \times \cdots \times P_k$ is a subgroup of G of order $|G|$, we conclude that $G = P_1 \times \cdots \times P_k$.

Finally, we prove that $(3) \implies (1)$. We just need to note that every p-group is nilpotent (Proposition 2.21) and that the direct product of nilpotent groups is nilpotent. □

Exercise 3.24. Let G be a finite group. Prove that if $P \in \mathrm{Syl}_p(G)$ and M is a subgroup of G such that $N_G(P) \subseteq M$, then $M = N_G(M)$.

Exercise 3.25. Let G be a finite group. Prove that the following statements are equivalent:

1) G is nilpotent.
2) If $H \subsetneq G$ is a subgroup of G, then $H \subsetneq N_G(H)$.
3) Every maximal subgroup of G is normal in G.

Theorem 3.26. *Let G be a finite nilpotent group. If p is a prime number dividing $|G|$, there exist a minimal normal subgroup of order p and there exists a maximal subgroup of index p.*

Proof. Assume that $|G| = p^\alpha m$ with $\gcd(p,m) = 1$. Write $G = P \times H$, where $P \in \mathrm{Syl}_p(G)$. Since $Z(P)$ is a non-trivial normal subgroup of P, every subgroup of $Z(P)$ that is minimal normal in G has order p (and such subgroups exist because G is finite). Since P contains a subgroup of index p, it is maximal. Hence $P \times H$ contains a maximal subgroup of index p. □

Exercise 3.27. Let p be a prime number and G be a non-trivial group of order p^n. Prove the following statements:

1) G has a normal subgroup of order p.
2) For every $j \in \{0, \dots, n\}$ there exists a normal subgroup of G of order p^j.

Exercise 3.28. Let G be a finite group. Prove that the following statements are equivalent:

1) G is nilpotent.
2) Any two elements of coprime order commute.
3) Every non-trivial quotient of G has a non-trivial center.
4) If d divides $|G|$, then there exists a normal subgroup of G of order d.

§ 3.3. Baumslag–Wiegold theorem. The following result can be proved with elementary tools and was discovered in 2014.

THEOREM 3.29 (Baumslag–Wiegold). *Let G be a finite group such that $|xy| = |x||y|$ for all $x, y \in G$ of coprime orders. Then G is nilpotent.*

PROOF. Let $p_1, \dots, p_n$ be the prime factors of $|G|$. For each $i \in \{1, \dots, n\}$, let $P_i \in \mathrm{Syl}_{p_i}(G)$. We first prove that $G = P_1 \cdots P_n$. To prove the non-trivial inclusion, we need to show that the map

$$\psi \colon P_1 \times \cdots \times P_n \to G, \quad (x_1, \dots, x_n) \mapsto x_1 \cdots x_n$$

is surjective. We first show that ψ is injective: If $\psi(x_1, \dots, x_n) = \psi(y_1, \dots, y_n)$, then

$$x_1 \cdots x_n = y_1 \cdots y_n.$$

If $y_n \neq x_n$, then $x_1 \cdots x_{n-1} = (y_1 \cdots y_{n-1}) y_n x_n^{-1}$. Since $x_1 \cdots x_{n-1}$ has order coprime with p_n and $y_1 \cdots y_{n-1} y_n x_n^{-1}$ has order a multiple of p_n, we get a contradiction. Thus $x_n = y_n$. The same argument shows that ψ is injective. Since $|P_1 \times \cdots \times P_n| = |G|$, we conclude that ψ is bijective. In particular, ψ is surjective.

We now prove that each P_j is normal in G. Let $j \in \{1, \dots, n\}$ and $x_j \in P_j$. Let $g \in G$ and $y_j = g x_j g^{-1}$. Since $y_j \in G$, we can write $y_j = z_1 \cdots z_n$ with $z_k \in P_k$ for all k. Since the order of y_j is a power of p_j, the element $z_1 \cdots z_n$ has order a power of p_j. Thus $z_k = 1$ for all $k \neq j$. Moreover, $y_j = z_j \in P_j$. Since every Sylow subgroup of G is normal in G, we conclude that G is nilpotent. □

§ 3.4. Itô's theorem.

DEFINITION 3.30. A group G is said to be **metabelian** if $[G, G]$ is abelian.

EXERCISE 3.31. Prove that a group G is metabelian if and only if there exists a normal subgroup K of G such that K and G/K are abelian.

EXERCISE 3.32. Let G be a metabelian group. Prove the following statements:

1) If H is a subgroup of G, then H is metabelian.
2) If $f \colon G \to H$ is a group homomorphism, then $f(H)$ is metabelian.

LEMMA 3.33. *In a group, the following formulas hold:*

1) $[a, bc] = [a, b] b [a, c] b^{-1}$.
2) $[ab, c] = a [b, c] a^{-1} [a, c]$.

PROOF. This is a straightforward calculation:

$$[a,b]b[a,c]b^{-1} = aba^{-1}b^{-1}baca^{-1}c^{-1}b^{-1} = abca^{-1}c^{-1}b^{-1} = [a, bc],$$
$$a[b,c]a^{-1}[a,c] = abcb^{-1}c^{-1}a^{-1}aca^{-1}c^{-1} = abcb^{-1}a^{-1}c^{-1} = [ab, c].$$

□

EXAMPLE 3.34. The group $\mathbb{S}_3$ is metabelian, as $\mathbb{A}_3 \simeq C_3$ is a normal subgroup and the quotient $\mathbb{S}_3/\mathbb{A}_3 \simeq C_2$ an abelian group.

EXAMPLE 3.35. The group $\mathbb{A}_4$ is metabelian, as the normal subgroup

$$K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$$

is abelian and the quotient $\mathbb{A}_4/K \simeq C_3$ is abelian.

EXAMPLE 3.36. The group $\mathbf{SL}_2(3)$ is not metabelian, as $[\mathbf{SL}_2(3), \mathbf{SL}_2(3)] \simeq Q_8$ is not abelian:

```
gap> IsAbelian(DerivedSubgroup(SL(2,3)));
false
gap> StructureDescription(DerivedSubgroup(SL(2,3)));
"Q8"
```

THEOREM 3.37 (Itô). *Let $G = AB$ be a factorization of G with A and B abelian subgroups of G. Then G is metabelian.*

PROOF. Since $G = AB$ is a group, $AB = BA$. We claim that $[A,B]$ is a normal subgroup of G. Let $a,\alpha \in A$ and $b,\beta \in B$. Let $a_1,a_2 \in A$ and $b_1,b_2 \in B$ be such that $\alpha b\alpha^{-1} = b_1a_1$, $\beta a\beta^{-1} = a_2b_2$. Since

$$\alpha[a,b]\alpha^{-1} = a(\alpha b\alpha^{-1})a^{-1}(\alpha b^{-1}\alpha^{-1}) = ab_1a_1a^{-1}a_1^{-1}b_1^{-1} = [a,b_1] \in [A,B]$$
$$\beta[a,b]\beta^{-1} = (\beta a\beta^{-1})\beta b\beta^{-1}(\beta a^{-1}\beta^{-1})b^{-1} = a_2b_2bb_2^{-1}a_2^{-1}b^{-1} = [a_2,b] \in [A,B],$$

it follows that $[A,B]$ is normal in G.

We now claim that $[A,B]$ is abelian. Since

$$\beta\alpha[a,b]\alpha^{-1}\beta^{-1} = \beta[a,b_1]\beta^{-1} = (\beta a\beta^{-1})b_1(\beta a^{-1}\beta^{-1})b_1^{-1} = [a_2,b_1],$$
$$\alpha\beta[a,b]\beta^{-1}\alpha^{-1} = \alpha[a_2,b]\alpha^{-1} = a_2(\alpha b\alpha^{-1})a_2^{-1}(\alpha b\alpha^{-1}) = [a_2,b_1],$$

a direct calculation shows that

$$[\alpha^{-1},\beta^{-1}][a,b][\alpha^{-1},\beta^{-1}]^{-1} = [a,b].$$

Two arbitrary generators of $[A,B]$ commute, so the group $[A,B]$ is abelian.

To finish the proof, note that $[G,G] = [A,B]$. In fact,

$$[a_1b_1,a_2b_2] = a_1[a_2,b_1]^{-1}a_1^{-1}a_2[a_1,b_2]a_2^{-1} \subseteq [A,B],$$

as $[A,B]$ is normal in G. □

In 1988 Sysak proved the following generalization of Itô's theorem.

THEOREM 3.38 (Sysak). *Let A and B be abelian subgroups of G. If H is a subgroup of G contained in AB, then H is metabelian.*

For the proof, see [**8**].

§ 3.5. Nilpotent groups of class two. The following exercises go over groups of nilpotency class two.

EXERCISE 3.39. Let G be a group. Prove that if $x,y \in G$ are such that $[x,y] \in C_G(x) \cap C_G(y)$, then

$$[x,y]^n = [x^n,y] = [x,y^n]$$

for all $n \in \mathbb{Z}$.

EXERCISE 3.40 (Hall). Let G be a class-two nilpotent group and $x,y \in G$. Prove that

$$(xy)^n = [y,x]^{n(n-1)/2}x^ny^n$$

for all $n \geq 1$.

EXERCISE 3.41. Let p be an odd prime number and P p-group of nilpotency class ≤ 2. Prove that if $[y,x]^p = 1$ for all $x,y \in P$, then $P \to [P,P]$, $x \mapsto x^p$, is a group homomorphism.

EXERCISE 3.42. Let p be an odd prime number and P a p-group of nilpotency class ≤ 2. Prove that $\{x \in P : x^p = 1\}$ is a subgroup of P.

§ 3.6. Frattini subgroup.

DEFINITION 3.43. Let G be a group. If G has maximal subgroups, the **Frattini subgroup** is the intersection $\Phi(G)$ of all the maximal subgroups of G. Otherwise, $\Phi(G) = G$.

EXERCISE 3.44. Prove that $\Phi(G)$ is a characteristic subgroup of G.

EXAMPLE 3.45. Sea $G = \mathbb{S}_3$. The maximal subgroups of G are

$$M_1 = \langle (123) \rangle, \quad M_2 = \langle (12) \rangle, \quad M_3 = \langle (23) \rangle, \quad M_4 = \langle (13) \rangle.$$

Hence $\Phi(G) = \{1\}$.

EXAMPLE 3.46. Let $G = \langle g \rangle \simeq C_{12}$. The subgroups of G are

$$\{1\}, \quad \langle g^6 \rangle \simeq C_2, \quad \langle g^4 \rangle \simeq C_3, \quad \langle g^3 \rangle \simeq C_4, \quad \langle g^2 \rangle \simeq C_6, \quad G.$$

Let us draw a picture:

$$\begin{tikzcd}
 & C_{12} \arrow[dl, dash] \arrow[dr, dash] & & \\
C_4 \arrow[dr, dash] & & C_6 \arrow[dl, dash] \arrow[dr, dash] & \\
 & C_2 \arrow[dr, dash] & & C_3 \arrow[dl, dash] \\
 & & \{1\} &
\end{tikzcd}$$

The maximal subgroups of G are $\langle g^3 \rangle \simeq C_4$ and $\langle g^2 \rangle \simeq C_6$. Hence $\Phi(G) = \langle g^3 \rangle \cap \langle g^2 \rangle = \langle g^6 \rangle \simeq C_2$. Let us see how to do this calculation with the computer:

```
gap> G = CyclicGroup(12);;
gap> StructureDescription(FrattiniSubgroup(G));
"C2"
```

LEMMA 3.47 (Dedekind). *Let H, K and L be subgroups of G such that $H \subseteq L \subseteq G$. Then $HK \cap L = H(K \cap L)$.*

PROOF. One inclusion is trivial. Let us prove then that $HK \cap L \subseteq H(K \cap L)$. If $x = hk \in HK \cap L$ with $x \in L$, $h \in H$ and $k \in K$, then $k = h^{-1}x \in L \cap K$, as $H \subseteq L$. Thus $x = hk \in H(L \cap K)$. $\square$

LEMMA 3.48. *Let G be a finite group and H be a subgroup of G such that $G = H\Phi(G)$. Then $H = G$.*

Proof. If $H\neq G$, let M be a maximal subgroup of G such that $H\subseteq M$. Since $\Phi(G)\subseteq M$, $G=H\Phi(G)\subseteq M$, a contradiction. □

Proposition 3.49. *Let N be a normal subgroup of a finite group G. Then $\Phi(N)\subseteq\Phi(G)$.*

Proof. Since $\Phi(N)$ is characteristic in N and N is normal in G, $\Phi(N)$ is normal in G. If there exists a maximal subgroup M such that $\Phi(N)\not\subseteq M$, then $\Phi(N)M=G$. (This happens because, otherwise, $M=\Phi(N)M\supseteq\Phi(N)$.) By Dedekind's lemma (with $H=\Phi(N)$, $K=M$ and $L=N$),

$$N=G\cap N=(\Phi(N)M)\cap N=\Phi(N)(M\cap N).$$

By Lemma 3.48 (with $G=N$ and $H=M\cap N$), $\Phi(N)\subseteq N\subseteq M$, a contradiction. Hence every maximal subgroup of G contains $\Phi(N)$ and therefore $\Phi(G)\supseteq\Phi(N)$. □

The following proposition states that the elements of the Frattini subgroup are the **non-generators** of the group.

Proposition 3.50. *Let G be a finite group. Then*

$$\Phi(G)=\{x\in G: \text{if } G=\langle x,Y\rangle \text{ for some } Y\subseteq G\text{, then } G=\langle Y\rangle\}.$$

Proof. We first prove $\supseteq$. Let $x\in G$. If M is a maximal subgroup of G such that $x\notin M$, then, since $G=\langle x,M\rangle$, we obtain that $G=\langle M\rangle=M$, a contradiction. Thus $x\in M$ for all maximal subgroup M of G. Hence $x\in\Phi(G)$.

We now prove $\subseteq$. Let $x\in\Phi(G)$ be such that $G=\langle x,Y\rangle$ for some subset Y of G. If $G\neq\langle Y\rangle$, there exists a maximal subgroup M such that $\langle Y\rangle\subseteq M$. Since $x\in M$, $G=\langle x,Y\rangle\subseteq M$, a contradiction. □

Example 3.51. For a prime number p, let G be an elementary p-group, that is $G\simeq C_p^m$ for some $m\geq 1$. Assume that $G=\langle x_1\rangle\times\cdots\times\langle x_m\rangle$ with $\langle x_j\rangle\simeq C_p$. We claim that $\Phi(G)$ is trivial. For $j\in\{1,\dots,m\}$, let $n_j\in\{1,\dots,p-1\}$. Since

$$\{x_1,\dots,x_{j-1},x_j^{n_j},x_{j+1},\dots,x_m\}$$

generates G and $\{x_1,\dots,x_{j-1},x_{j+1},\dots,x_m\}$ does not, $x_j^{n_j}\notin\Phi(G)$ by Proposition 3.50. Hence $\Phi(G)=\{1\}$.

Theorem 3.52 (Frattini). *Let G be a finite group. Then $\Phi(G)$ is nilpotent.*

Proof. Let $P\in\mathrm{Syl}_p(\Phi(G))$ for some prime number p. Since $\Phi(G)$ is normal in G, Lemma 1.27 (Frattini's argument) implies that $G=\Phi(G)N_G(P)$. By Lemma 3.48, $G=N_G(P)$. Since every Sylow subgroup of $\Phi(G)$ is normal in G, $\Phi(G)$ is nilpotent. □

Exercise 3.53. Let G be a group and M be a normal subgroup of G. Prove that if M is maximal, then G/M is cyclic of prime order.

Theorem 3.54 (Gaschütz). *If G is a finite group, then*

$$[G,G]\cap Z(G)\subseteq\Phi(G).$$

Proof. Let $D=[G,G]\cap Z(G)$. Assume that D is not contained in $\Phi(G)$. Since $\Phi(G)$ is contained in every maximal subgroup of G, there is a maximal subgroup M of G not containing D. Then $G=MD$. Since $D\subseteq Z(G)$, M is normal in G, as $g=md\in G=MD$ implies

$$gMg^{-1}=(md)Md^{-1}m^{-1}=mMm^{-1}=M.$$

Since G/M is cyclic of prime order, G/M is, in particular, abelian and hence $[G,G]\subseteq M$. Therefore $D\subseteq[G,G]\subseteq M$, a contradiction. □

Lemma 3.55. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. If H is a subgroup of G such that $N_G(P) \subseteq H$, then $N_G(H) = H$.*

Proof. Let $x \in N_G(H)$. Since $P \in \mathrm{Syl}_p(H)$ and $Q = xPx^{-1} \in \mathrm{Syl}_p(H)$, the second Sylow's theorem implies that there exists $h \in H$ such that $hQh^{-1} = (hx)P(hx)^{-1} = P$. Then $hx \in N_G(P) \subseteq H$ and hence $x \in H$. □

Theorem 3.56 (Wielandt). *A finite group G is nilpotent if and only if $[G,G] \subseteq \Phi(G)$.*

Proof. Assume that $[G,G] \subseteq \Phi(G)$. Let $P \in \mathrm{Syl}_p(G)$. If $N_G(P) \neq G$, then $N_G(P) \subseteq M$ for some maximal subgroup M of G. If $g \in G$ and $m \in M$, then, since

$$gmg^{-1}m^{-1} = [g,m] \in [G,G] \subseteq \Phi(G) \subseteq M,$$

M is normal in G. Furthermore $N_G(P) \subseteq M$. By Lemma 3.55,

$$G = N_G(M) = M,$$

a contradiction. Thus $N_G(P) = G$ and every Sylow subgroup of G si normal in G. Therefore G is nilpotent.

Conversely, assume that G is nilpotent. Let M be a maximal subgroup of G. Since M is normal in G and maximal, G/M has no proper non-trivial subgroups. Then $G/M \simeq C_p$ for some prime number p. In particular, G/M is abelian and $[G,G] \subseteq M$. Since $[G,G]$ is contained in every maximal subgroup of G, $[G,G] \subseteq \Phi(G)$. □

Theorem 3.57. *A finite group G is nilpotent if and only if $G/\Phi(G)$ is nilpotent.*

Proof. If G is nilpotent, then $G/\Phi(G)$ is nilpotent. Conversely, assume that $G/\Phi(G)$ is nilpotent. Let $P \in \mathrm{Syl}_p(G)$. Since $\Phi(G)P/\Phi(G) \in \mathrm{Syl}_p(G/\Phi(G))$ and $G/\Phi(G)$ is nilpotent, $\Phi(G)P/\Phi(G)$ is a normal subgroup of $G/\Phi(G)$. By the correspondence theorem, $\Phi(G)P$ is a normal subgroup of G. Since $P \in \mathrm{Syl}_p(\Phi(G)P)$, Frattini's argument (Lemma 1.27) implies that

$$G = \Phi(G)PN_G(P) = \Phi(G)N_G(P),$$

as $P \subseteq N_G(P)$. Thus $G = N_G(P)$ by Lemma 3.48). Hence P is normal in G and therefore G is nilpotent. □

Theorem 3.58 (Hall). *Let G be a finite group with a normal subgroup N. If both N and $G/[N,N]$ are nilpotent, then G is nilpotent.*

Proof. Since N is nilpotent, $[N,N] \subseteq \Phi(N)$ by Wielandt's theorem 3.56. By Proposition 3.49, $[N,N] \subseteq \Phi(N) \subseteq \Phi(G)$. By the universal property, there exists a surjective group homomorphism $G/[N,N] \to G/\Phi(G)$ such that the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G/\Phi(G) \\
G/[N,N] \arrow[ur, dashed] &
\end{tikzcd}$$

is commutative. Since $G/[N,N]$ is nilpotent, $G/\Phi(G)$ is nilpotent by Exercise 2.19. Thus G is nilpotent by the previous theorem. □

Definition 3.59. A **minimal generating set** of a group G is a set X of generators of G such that no proper subset of X generates G.

Note that a minimal generating set does not necessarily have minimal size.

EXAMPLE 3.60. Let $G = \langle g \rangle \simeq C_6$. If $a = g^2$ and $b = g^3$, then $\{a,b\}$ is a minimal generating set of G that does not have minimal size, as $G = \langle ab \rangle$.

For a prime number p, we write $\mathbb{F}_p$ to denote the field of p elements.

LEMMA 3.61. *Let p be a prime number and G be a finite p-group. Then $G/\Phi(G)$ is a vector space over $\mathbb{F}_p$.*

PROOF. Let K be a maximal subgroup of G. Since G is nilpotent (see Proposition 2.21), K is normal in G (Exercise 3.25). Thus $G/K \simeq C_p$ because it is a simple p-group.

It is enough to prove that $G/\Phi(G)$ is an elementary abelian p-group. It is a p-group because G is a p-group. Let $K_1,\dots,K_m$ be the maximal subgroups of G. If $x \in G$, then $x^p \in K_j$ for all $j \in \{1,\dots,m\}$. Hence $x^p \in \Phi(G) = \cap_{j=1}^m K_j$. Moreover, $G/\Phi(G)$ is abelian, as $[G,G] \subseteq \Phi(G)$ because G is nilpotent (Wielandt's theorem 3.56). $\square$

THEOREM 3.62 (Burnside). *Let p be a prime number and G a finite p-group. If X is a minimal generating set of G, then $|X| = \dim G/\Phi(G)$.*

PROOF. By Lemma 3.61, $G/\Phi(G)$ is a vector space over $\mathbb{F}_p$. Let $\pi\colon G \to G/\Phi(G)$ be the canonical map and $\{x_1,\dots,x_n\}$ be a minimal generating set of G. We claim that $\{\pi(x_1),\dots,\pi(x_n)\}$ is a linearly independent subset of $G/\Phi(G)$. Assume this is not the case. Without loss of generality, let us assume that $\pi(x_1) \in \langle \pi(x_2),\dots,\pi(x_n)\rangle$. There exists $y \in \langle x_2,\dots,x_n\rangle$ such that $x_1y^{-1} \in \Phi(G)$. Since G is generated by $\{x_1y^{-1},x_2,\dots,x_n\}$ and $x_1y^{-1} \in \Phi(G)$, Proposition 3.50 implies that G is generated by $\{x_2,\dots,x_n\}$, a contradiction to the minimality of $\{x_1,\dots,x_n\}$. Therefore $n = \dim G/\Phi(G)$. $\square$

EXERCISE 3.63. Let p be a prime number and G a finite p-group. Prove that if $x \notin \Phi(G)$, then x belongs to some minimal set of generators of G.

Lecture 4. 07/03/2024

§ 4.1. The Fitting subgroup.

Definition 4.1. Let G be a finite group and p be a prime number. The p**-radical** of G is the subgroup

$$O_p(G) = \bigcap_{P\in \mathrm{Syl}_p(G)} P.$$

Lemma 4.2. *Let G be a finite group and p be a prime number. The following statements hold:*

1) *$O_p(G)$ is normal in G.*

2) *If N is a normal subgroup of G contained in some $P \in \mathrm{Syl}_p(G)$, then $N \subseteq O_p(G)$.*

Proof. Let $P \in \mathrm{Syl}_p(G)$. Let G act on G/P by left multiplication. There is a group homomorphism $\rho\colon G \to \mathbb{S}_{G/P}$ with kernel

$$\begin{aligned} \ker\rho &= \{x \in G : \rho_x = \mathrm{id}\} \\ &= \{x \in G : xgP = gP \ \forall g \in G\} \\ &= \{x \in G : x \in gPg^{-1} \ \forall g \in G\} \\ &= \bigcap_{g\in G} gPg^{-1} \\ &= O_p(G). \end{aligned}$$

Then $O_p(G)$ is normal in G.

Let N be a normal subgroup of G such that $N \subseteq P$. Since $N = gNg^{-1} \subseteq gPg^{-1}$ for all $g \in G$, we conclude that $N \subseteq O_p(G)$. □

Definition 4.3. Let G be a finite group and $p_1,\dots,p_k$ be the prime divisors of $|G|$. The **Fitting subgroup** of G is the subgroup

$$F(G) = O_{p_1}(G)\cdots O_{p_k}(G)$$

Exercise 4.4. Prove that $F(G)$ is characteristic in G.

Example 4.5. Let $G = \mathbb{S}_3$. Then $O_2(G) = \{1\}$ and $O_3(G) = \langle(123)\rangle$. Hence $F(G) = \langle(123)\rangle$.

Theorem 4.6 (Fitting). *Let G be a finite group. Then $F(G)$ is a nilpotent and normal in G. Moreover, $F(G)$ contains every nilpotent normal subgroup of G.*

Proof. By definition, $|F(G)| = \prod_p |O_p(G)|$. Since $O_p(G) \in \mathrm{Syl}_p(F(G))$, we conclude that $F(G)$ is nilpotent, as it contains a normal Sylow p-subgroup for every prime p. Hence $F(G)$ is nilpotent by Theorem 3.23.

Let N be a nilpotent normal subgroup of G and $P \in \mathrm{Syl}_p(N)$. Since N is nilpotent, P is normal in N and hence P is the only Sylow p-subgroup of N. Thus P is characteristic in N and P is normal in G. Since N is nilpotent, N is a direct product of its Sylow subgroups. Therefore $N \subseteq O_p(G)$ by Lemma 4.2. □

Corollary 4.7. *If G is a finite group, then $Z(G) \subseteq F(G)$.*

Proof. Since $Z(G)$ is nilpotent (in fact, it is abelian) and normal in G, by Fitting's theorem 4.6 we conclude that $Z(G) \subseteq F(G)$ □

COROLLARY 4.8 (Fitting). *Let K and L be nilpotent normal subgroups of a finite group G. Then KL is nilpotent.*

PROOF. By Fitting's theorem 4.6, $K \subseteq F(G)$ and $L \subseteq F(G)$. Then $KL \subseteq F(G)$ and KL is nilpotent, as $F(G)$ is nilpotent. $\square$

COROLLARY 4.9. *Let N be a normal subgroup of a finite group G. Then $N \cap F(G) = F(N)$.*

PROOF. Since $F(N)$ is characteristic in N, $F(N)$ is normal in G. Then $F(N) \subseteq N \cap F(G)$ because $F(N)$ is nilpotent. Conversely, since $F(G)$ is normal in G, the subgroup $F(G) \cap N$ is normal in N. Since $F(G) \cap N$ is nilpotent, $F(G) \cap N \subseteq F(N)$. $\square$

We now discuss an application to finite solvable groups.

THEOREM 4.10. *Let G be a non-trivial solvable group. Every normal non-trivial subgroup N contains a normal abelian non-trivial subgroup. Moreover, this subgroup is contained in $F(N)$.*

PROOF. Note that $N \cap G^{(0)} = N \neq \{1\}$. Since G is solvable, there exists $m \geq 1$ such that $N \cap G^{(m)} = \{1\}$. Let n be the largest positive integer such that $N \cap G^{(n)} \neq \{1\}$. Since $[N,N] \subseteq N$ and $[G^{(n)}, G^{(n)}] = G^{(n+1)}$,

$$[N \cap G^{(n)}, N \cap G^{(n)}] \subseteq N \cap G^{(n+1)} = \{1\}.$$

Then $N \cap G^{(n)}$ is an abelian subgroup of G. Moreover, it is normal and nilpotent. Hence

$$N \cap G^{(n)} \subseteq N \cap F(G) = F(N). \qquad \square$$

THEOREM 4.11. *If N is a minimal normal subgroup of a finite group G, then $F(G) \subseteq C_G(N)$.*

PROOF. By Fitting's theorem 4.6, $F(G)$ is a normal nilpotent group. The subgroup $N \cap F(G)$ is normal in G. Moreover, $[F(G),N] \subseteq N \cap F(G)$. If $N \cap F(G) = \{1\}$, then $[F(G),N] = \{1\}$. Otherwise, $N = N \cap F(G) \subseteq F(G)$ by the minimality of N. By Hirsch's theorem, $N \cap Z(F(G)) \neq \{1\}$ Since $Z(F(G))$ is characteristic in $F(G)$ and $F(G)$ is normal in G, $Z(F(G))$ is normal in G. Since $\{1\} \neq N \cap Z(F(G))$ is normal in G, the minimality of N implies that $N = N \cap Z(F(G)) \subseteq Z(F(G))$. Hence $[F(G),N] = \{1\}$. $\square$

COROLLARY 4.12. *Let G be a finite solvable group. The following statements hold:*

1) *If N is a minimal normal subgroup, then $N \subseteq Z(F(G))$.*
2) *If H is a non-trivial normal subgroup, then $H \cap F(G) \neq \{1\}$.*

PROOF. Let us prove the first claim. Since N is a p-group by Lemma 1.21, N is nilpotent and hence $N \subseteq F(G)$. Moreover, $F(G) \subseteq C_G(N)$ by the previous theorem. Therefore $N \subseteq Z(F(G))$.

Let us prove now the second claim. The subgroup H contains a minimal normal subgroup N and $N \subseteq F(G)$. Then $H \cap F(G) \neq \{1\}$. $\square$

THEOREM 4.13. *Let G be a finite group. The following statements hold:*

1) $\Phi(G) \subseteq F(G)$ *and* $Z(G) \subseteq F(G)$.
2) $F(G)/\Phi(G) \simeq F(G/\Phi(G))$.

PROOF. Let us prove the first claim. Since $\Phi(G)$ is normal in G, nilpotent by Frattini's Theorem 3.52 and $F(G)$ contains every normal nilpotent subgroup of G, $\Phi(G) \subseteq F(G)$. Moreover, $Z(G)$ is normal in G and nilpotent. Hence $Z(G) \subseteq F(G)$.

Let us prove the second claim. Let $\pi\colon G \to G/\Phi(G)$ be the canonical map. Since $F(G)$ is nilpotent, $\pi(F(G))$ is nilpotent. Hence

$$\pi(F(G)) \subseteq F(G/\Phi(G))$$

by Fitting's Theorem 4.6. Let $H = \pi^{-1}(F(G/\Phi(G)))$. By the correspondence theorem, H is a normal subgroup of G containing $\Phi(G)$. If $P \in \mathrm{Syl}_p(H)$, then $\pi(P) \in \mathrm{Syl}_p(\pi(H))$. In fact, $\pi(P) \simeq P/P\cap\Phi(G)$ is a p-group and $(\pi(H):\pi(P))$ is coprime with p because

$$(\pi(H):\pi(P)) = \frac{|\pi(H)|}{|\pi(P)|} = \frac{|H/\Phi(G)|}{|P/P\cap\Phi(G)|} = \frac{(H:P)}{(\Phi(G):P\cap\Phi(G))}$$

divides $(H:P)$, a number coprime with p. Since $\pi(H)$ is nilpotent, $\pi(P)$ is characteristic in $\pi(H)$. Then $\pi(P)$ is normal in $\pi(G) = G/\Phi(G)$ and $P\Phi(G) = \pi^{-1}(\pi(P))$ is normal in G. Since $P \in \mathrm{Syl}_p(P\Phi(G))$, Frattini's argument (Lemma 1.27) implies that $G = \Phi(G)N_G(P)$. Therefore P is normal in G by Lemma 3.48. Since P is nilpotent and normal in G, $P \subseteq F(G)$ by Fitting's theorem 4.6. Hence $H \subseteq F(G)$ and $F(G/\Phi(G)) = \pi(H) \subseteq \pi(F(G))$. $\square$

Lecture 5. 14/03/2024

§ 5.1. Super-solvable groups.

DEFINITION 5.1. A group G is said to be **super-solvable** if there exists a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups of G such that every quotient G_{i-1}/G_i is cyclic.

In the previous definition, we do not require the group to be finite. Hence the quotients could be finite cyclic groups or isomorphic to $\mathbb{Z}$.

EXAMPLE 5.2. The dihedral group $\mathbb{D}_n$ of order $2n$ is super-solvable, as

$$\mathbb{D}_n \supseteq \langle r \rangle \supseteq \{1\}$$

is a sequence of normal subgroups with cyclic factors.

Every solvable group is super-solvable. See Exercise 1.5.

EXAMPLE 5.3. The alternating group $\mathbb{A}_4$ solvable but not super-solvable. The only proper non-trivial normal subgroup of $\mathbb{A}_4$ is

$$\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2.$$

Thus $\mathbb{A}_4$ does not have a sequence of normal subgroups with cyclic factors.

EXERCISE 5.4. Prove that $\mathrm{Aff}(\mathbb{Z})$ is super-solvable.

EXAMPLE 5.5. The group $\mathbf{SL}_2(3)$ is solvable but not super-solvable. Here is a computer verification:

```
gap> IsSolvable(SL(2,3));
true
gap> IsSupersolvable(SL(2,3));
false
```

EXERCISE 5.6. Prove the following statements:

1) Every subgroup of a super-solvable group is super-solvable.
2) Quotients of super-solvable groups are super-solvable.

EXERCISE 5.7. Prove that the direct product of super-solvable groups is super-solvable.

EXERCISE 5.8. Let H and K be normal subgroups of a group G such that G/K and G/H are super-solvable. Prove that $G/H \cap K$ is super-solvable.

EXERCISE 5.9. Let N be a cyclic normal subgroup of G. If G/N is super-solvable, then G is super-solvable.

THEOREM 5.10. *Let G be a super-solvable non-trivial group. Then G admits a sequence*

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups such that every quotient G_{i-1}/G_i is cyclic of prime order or isomorphic to $\mathbb{Z}$.

PROOF. Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ be a sequence of normal subgroups of G such that every quotient G_{i-1}/G_i is cyclic. Let $i \in \{1,\dots,n\}$ be such that $G_{i-1}/G_i \simeq C_n$ for some non-prime n and let $\pi\colon G_{i-1} \to G_{i-1}/G_i$ be the canonical map. Let p be a prime divisor of n and H be a subgroup of G such that $\pi(H)$ is a subgroup of G_{i-1}/G_i of order p. By the correspondence theorem, $G_i \subseteq H \subseteq G_{i-1}$.

We claim that H is normal in G. Let $g \in G$. Since $\pi(gHg^{-1})$ is a subgroup of order p of the cyclic group G_{i-1}/G_i, $\pi(gHg^{-1}) = \pi(H)$. Then $gHg^{-1} = G_iH \subseteq H$ and hence $gHg^{-1} = H$.

Note that H/G_i is cyclic of prime order, as

$$H/G_i = H/H\cap G_i \simeq \pi(H) \simeq C_p.$$

Moreover, G_{i-1}/H is cyclic, as

$$G_{i-1}/H \simeq \frac{G_{i-1}/G_i}{H/G_i}$$

is the quotient of a cyclic group.

We have shown that by adding H to our sequence of normal subgroups, we obtain a sequence with cyclic factors where H/G_i is cylic of prime order. Repeating this procedure, we obtain the desired result. □

Let us discuss an immediate application.

COROLLARY 5.11. *A finite super-solvable group admits a sequence of normal subgroups where each quotient is cyclic of prime order.*

We now discuss other properties of super-solvable groups.

THEOREM 5.12. *Let G be a super-solvable group. The following statement hold:*

1) *If N is minimal normal in G, then $N \simeq C_p$ for some prime number p.*
2) *If M is maximal in G, then $(G:M) = p$ for some prime number p.*
3) *The commutator subgroup $[G,G]$ is nilpotent.*
4) *If G is non-abelian, there exists a normal subgroup $N \neq G$ such that $Z(G) \subsetneq N$.*

PROOF. Let us prove the first claim. Since G is super-solvable, there exists a sequence

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups with cyclic factors. Since each $G_i \cap N$ is a normal subgroup of G contained in N, the minimality implies that each $G_i \cap N$ is either trivial or equal to N. Moreover, $N \cap G_0 = N$ and $N \cap G_n = \{1\}$. Let j be the smallest positive integer such that $N \cap G_j = \{1\}$. Since $N \subseteq G_{j-1}$ (because $N \cap G_{j-1} = N$), the composition

$$N \hookrightarrow G_{j-1} \to G_{j-1}/G_j$$

is an injective group homomorphism, as its kernel is equal $N \cap G_j = \{1\}$. Thus N is cyclic, as it is isomorphic to a subgroup of the cyclic group G_{i-1}/G_i. If $G_{i-1}/G_i \simeq \mathbb{Z}$, then $N \simeq \mathbb{Z}$, a contradiction to the fact that N is minimal normal. (For example, $2\mathbb{Z}$ is characteristic subgroup of $\mathbb{Z}$ and hence it is normal in G. Thus N is cyclic and finite. Hence $N \simeq C_p$.)

We now prove the second claim. Let M be a maximal subgroup of G. If M is normal in G, then G/M does not contain non-trivial proper subgroups. Then $G/M \simeq C_p$ for some prime number p.

Assume that M is not normal in G. Let $H=\cap_{g\in G} gMg^{-1}$ and $\pi\colon G\to G/H$ be the canonical map. Since $\pi(M)$ is maximal in $\pi(G)=G/H$ and

$$(G:M)=(G/H:M/H)=(G/H:M/H\cap M)=(\pi(G):\pi(M)),$$

we may assume that M does not contain non-trivial normal subgroups of G (if needed, we just replace G by G/H). Since G is super-solvable, there exists a sequence $G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=\{1\}$ of normal subgroups of G with factors either cyclic of prime order or isomorphic to $\mathbb{Z}$. Let $N=G_{n-1}$. Since N is cyclic, every subgroup of N is characteristic in N and hence normal in G. In particular, $M\cap N$ is normal in G and therefore $M\cap N=\{1\}$. Since $M\subseteq NM\subseteq G$, the maximality of M implies that either $M=NM$ or $G=NM$. Since $N\subseteq NM=M$ yields a contradiction, we conclude that $G=NM$.

If $N\simeq C_p$ for some prime p, then $(G:M)=p$ and the proof is complete. Assume that $N\simeq\mathbb{Z}$. Let H be a proper subgroup of N. Since H is characteristic in N, H is normal in G. Since $M\subseteq HM\subseteq NM=G$, the maximality of M implies that either $HM=M$ or $HM=G$. Since $HM=M$ implies $H\subseteq M\cap N=\{1\}$, we may assume that $HM=G$. If $n\in N\setminus H$, then $n=hm$ for some $h\in H$ and $m\in M$. Then $h=n$, as $h^{-1}n\in N\cap M=\{1\}$, a contradiction.

We now prove the third claim. Since G is super-solvable, there exists a sequence

$$G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=\{1\}$$

of normal subgroups of G such that each G_i/G_{i+1} is cyclic. For $i\in\{0,\dots,n\}$, let $H_i=[G,G]\cap G_i$. Since $[G,G]$ the each G_i are normal in G, one obtains a sequence

$$[G,G]=H_0\supseteq H_1\supseteq\cdots\supseteq H_n=\{1\}$$

of normal subgroups of G. Since H_i and H_{i+1} are normal in G, the group G acts by conjugation on H_i/H_{i+1}. Thus there exists a group homomorphism $\gamma\colon G\to\operatorname{Aut}(H_i/H_{i+1})$. Since H_i/H_{i+1} is cyclic, $\operatorname{Aut}(H_i/H_{i+1})$ is abelian. Thus $[G,G]\subseteq\ker\gamma$. Therefore $[G,G]$ acts trivially by conjugation on H_i/H_{i+1}. Hence

$$H_i/H_{i+1}\subseteq Z([G,G]/H_{i+1}).$$

Finally, we prove the fourth claim. Since G is non-abelian, $Z(G)\neq G$. Let $\pi\colon G\to G/Z(G)$ be the canonical map. The group $G/Z(G)$ is super-solvable and the sequence

$$G/Z(G)=\pi(G)\supseteq\pi(G_1)\supseteq\cdots\supseteq\pi(1)=\{1\}$$

is a sequence of normal subgroups of $G/Z(G)$ with cyclic quotients. In particular, $1\neq\pi(G_1)$ is a proper normal subgroup of $G/Z(G)$. By the correspondence theorem, $\pi^{-1}(\pi(G_1))\neq G$ is a normal subgroup of G properly containing $Z(G)$. $\square$

There are solvable groups with a non-nilpotent derived subgroup.

Example 5.13. The group $\mathbb{S}_4$ is solvable and $[\mathbb{S}_4,\mathbb{S}_4]=\mathbb{A}_4$ is not nilpotent.

Proposition 5.14. *Let p be a prime number. Every finite p-group is super-solvable.*

Proof. Let G be a minimal counterexample. We may assume that $|G|=p^n$ for some $n>1$ (otherwise, if $n=1$, then G is trivially super-solvable). The group G is nilpotent and contains a normal subgroup N of order p. Moreover, since $|G/N|=p^{n-1}$, the group G/N is super-solvable. Since N is cyclic and G/N is super-solvable, G is super-solvable by Exercise 5.9. $\square$

Exercise 5.15. Prove that finite nilpotent groups are super-solvable.

THEOREM 5.16. *Super-solvable groups have maximal subgroups.*

PROOF. We proceed by induction on the length of the super-solvable series. The claim holds for groups with a super-solvable series of length one, as in this case we are dealing with cyclic groups. So let G be a group admitting a sequence

$$G = G_0 \supseteq \cdots \supseteq G_k = \{1\}$$

and suppose the theorem holds for super-solvable groups with super-solvable series of length $< k$. Each G_{k-1} is normal in G. Let $\pi\colon G \to G/G_{k-1}$ be the canonical map. The sequence

$$G/G_{k-1} = \pi(G) \supseteq \pi(G_1) \supseteq \cdots \supseteq \pi(G_{k-1}) = \{1\}$$

has length $< k$ and proves the super-solvability of $\pi(G)$. By the inductive hypothesis, G/G_{k-1} admits maximal subgroups. By the correspondence theorem, G admits maximal subgroups. $\square$

Solvable or nilpotent groups do not always admit maximal subgroups. Can you give an example?

DEFINITION 5.17. A group G satisfies the **maximal condition on subgroups** if for every non-empty subset $\mathscr{S}$ of subgroups contains a maximal subgroup (i.e. a subgroup not contained in any other subgroup of $\mathscr{S}$).

EXERCISE 5.18. A group satisfies the maximal condition on subgroups if and only if every subgroup of G is finitely generated.

EXERCISE 5.19. Let H be a subgroup of a group G. If G satisfies the maximal condition on subgroups, then so does H.

EXERCISE 5.20. Let G be a group and N be a normal subgroup of G. If G/N and N satisfy the maximal condition on subgroups, then so does G.

PROPOSITION 5.21. *Super-solvable groups satisfy the maximal condition on subgroups. In particular, every super-solvable group is finitely generated.*

PROOF. We proceed by induction on the length of the super-solvable sequence. If the length is one, the result holds as the group is cyclic. So assume the result holds for super-solvable groups with super-solvable series of length $\leq n-1$. Let G be a non-trivial super-solvable group and

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{1\}$$

a sequence of normal subgroups of G with cyclic factors. Since G_1 is super-solvable (Exercise 5.8), G_1 satisfies the maximal condition on subgroups by the inductive hypothesis. By Exercise 5.20, G satisfies the maximal condition on subgroups, as G/G_1 is cyclic. $\square$

EXAMPLE 5.22. The abelian group $\mathbb{Q}$ is nilpotent but not super-solvable, as it is not finitely generated.

If G is a group and $x_1,\dots,x_{n+1} \in G$, let

$$[x_1,x_2\dots,x_{n+1}] = [x_1,[x_2,\dots,x_{n+1}]], \quad n \geq 1.$$

We will prove in Theorem 5.25 that nilpotent groups are super-solvable if and only if they are finitely generated. For this, we need two lemmas.

Lemma 5.23. *Let G be a finite generated group, say $G = \langle X \rangle$ for some finite set X. For $n \geq 2$, let*
$$G_n = \langle g[x_1,\dots,x_n]g^{-1} : x_1,\dots,x_n \in X,\, g \in G\rangle.$$
Then $G_n = \gamma_n(G)$ for all $n \geq 2$.

Proof. Note that each G_n is normal in G. We proceed by induction on n. The case $n = 2$ is trivial. So let us assume that $\gamma_{n-1}(G) = G_{n-1}$ for some $n \geq 2$. Let $x_1,\dots,x_n \in X$. Since $[x_1,\dots,x_n] \in \gamma_n(G)$, $G_{n-1} \subseteq \gamma_n(G)$. Let $N = G_n$ and $\pi\colon G \to G/N$ be the canonical map. The group G/N is finitely generated. Since
$$[\pi(x_1),[\pi(x_2),\dots,\pi(x_n)]] = \pi([x_1,\dots,x_n]) = 1,$$
we obtain that $\pi([x_2,\dots,x_n]) \in Z(G/N)$. Hence $\pi(g[x_2,\dots,x_n]g^{-1}) = 1$ for all $g \in G$. By the inductive hypothesis,
$$\pi(\gamma_{n-1}(G)) = \pi(G_{n-1}) \subseteq Z(G/N).$$
Since
$$\pi(\gamma_n(G)) = \pi([G,\gamma_{n-1}(G)]) = [\pi(G),\pi(\gamma_{n-1}(G))] = \{1\},$$
we conclude that $\gamma_n(G) \subseteq N = G_n$. □

Lemma 5.24. *Let G be a finitely generated group. Then $\gamma_n(G)/\gamma_{n+1}(G)$ is finitely generated.*

Proof. Assume that $G = \langle X \rangle$ for some finite set X. Write
$$g[x_1,\dots,x_n]g^{-1} = [g,[x_1,\dots,x_n]][x_1,\dots,x_n].$$
By Lemma 5.23, $[g,[x_1,\dots,x_n]] \in \gamma_{n+1}(G) = G_{n+1}$. Then
$$g[x_1,\dots,x_n]g^{-1} \equiv [x_1,\dots,x_n] \bmod \gamma_{n+1}(G).$$
Hence $\gamma_n(G)/\gamma_{n+1}(G)$ is generated by the finite set
$$\{[x_1,\dots,x_n]\gamma_{n+1}(G) : x_1,\dots,x_n \in X\}.$$
□

Theorem 5.25. *Let G be a nilpotent group. Then G is super-solvable if and only if G is finitely generated.*

Proof. If G is super-solvable, it is then finitely generated by Proposition 5.21.

Now assume that the nilpotent group G is finitely generated. By Lemma 5.24, each quotient $\gamma_n(G)/\gamma_{n+1}(G)$ is finitely generated, say by the elements $y_1,\dots,y_m$. Let $\pi\colon G \to G/\gamma_{n+1}(G)$ the canonical map. For $j \in \{1,\dots,m\}$, let
$$K_j = \langle \gamma_{n+1}(G), y_1,\dots,y_j\rangle.$$
Since $[G,K_j] \subseteq [G,\gamma_n(G)] = \gamma_{n+1}(G)$, we obtain that $\pi(K_j)$ is central in $\pi(G)$. Thus $\pi(K_j)$ is normal in $\pi(G)$. Hence K_j is normal in G. Each quotient K_j/K_{j-1} is cyclic and generated by y_jK_{j-1}. Therefore, in between $\gamma_n(G)$ and $\gamma_{n+1}(G)$, we have constructed a sequence of normal subgroups of G with cyclic factors. Since G is nilpotent, there exists an integer c such that $\gamma_{c+1}(G) = \{1\}$. Hence G is super-solvable. □

Corollary 5.26. *Every finitely generated nilpotent group satisfies the maximal condition on subgroups.*

Proof. This is an immediate consequence of Proposition 5.21 and Theorem 5.25. □

Theorem 5.27. *Let G be a nilpotent finitely generated group. Then $T(G)$ is finite.*

Proof. Since G is nilpotent, G satisfies the maximal condition on subgroups (Corollary 5.26). Thus every subgroup of G is finitely generated. Since $T(G)$ is a subgroup (Theorem 3.17), it is a torsion finitely generated group. Hence $T(G)$ is finite by Theorem 3.21. $\square$

Lecture 6.

§ 6.1. The Schur–Zassenhaus theorem. Recall that a group Q **acts by automorphisms** on a group K if there exists a map $Q\times K\to K$, $(q,k)\mapsto q\cdot k$, such that

1) $1\cdot a=a$ for all $a\in K$,
2) $x\cdot(y\cdot a)=(xy)\cdot a$ for all $x,y\in Q$ and $a\in K$,
3) $x\cdot 1=1$ for all $x\in Q$, and
4) $x\cdot(ab)=(x\cdot a)(x\cdot b)$ for all $x\in Q$ and $a,b\in K$,

For example, if K is a normal subgroup of G, then G acts by automorphisms on K by conjugation.

Definition 6.1. Let Q and K be groups, where Q acts by automorphisms on K. A map $\varphi\colon Q\to K$ is said to be a **1-cocycle** if

$$\varphi(xy)=\varphi(x)(x\cdot\varphi(y))$$

for all $x,y\in Q$.

Let Q and K be groups, where Q acts by automorphisms on K. The set of 1-cocycles $Q\to K$ will be denoted by

$$Z^1(Q,K)=\{\delta\colon Q\to K:\delta \text{ is a 1-cocycle}\}.$$

Example 6.2. Let Q be a group acting by automorphisms on K. The semidirect product $K\rtimes Q$ is a group G that contains a normal subgroup isomorphic to K and a subgroup isomorphic to such that $G=KQ$ and $K\cap Q=\{1\}$. Under the obvious identifications, Q acts on K by conjugation. For each $k\in K$, the map $Q\to K$, $x\mapsto[k,x]=kxk^{-1}x^{-1}$, is a 1-cocycle.

Exercise 6.3. Let $\varphi\colon Q\to K$ be a 1-cocycle. Prove the following statements:

1) $\varphi(1)=1$.
2) $\varphi(y^{-1})=(y^{-1}\cdot\phi(y))^{-1}=y^{-1}\cdot\phi(y)^{-1}$.
3) The set $\ker\varphi=\{x\in Q:\varphi(x)=1\}$ is a subgroup of Q.

Lemma 6.4. *Let G be a group with a normal subgroup N. If $\varphi\colon G\to N$ is a 1-cocycle (where G acts on N by conjugation) with kernel*

$$K=\ker\varphi=\{g\in G:\varphi(g)=1\},$$

then $\varphi(x)=\varphi(y)$ if and only if $xK=yK$. In particular, $(G:K)=|\varphi(G)|$.

Proof. If $\varphi(x)=\varphi(y)$, then, since

$$\varphi(x^{-1}y)=\varphi(x^{-1})(x^{-1}\cdot\varphi(y))=\varphi(x^{-1})(x^{-1}\cdot\varphi(x))=\varphi(x^{-1}x)=\varphi(1)=1,$$

we obtain that $xK=yK$. Conversely, if $x^{-1}y\in K$, then, since

$$1=\varphi(x^{-1}y)=\varphi(x^{-1})(x^{-1}\cdot\varphi(y)),$$

we obtain that $\varphi(y)=x\cdot\varphi(x^{-1})^{-1}$. We conclude that $\varphi(x)=\varphi(y)$.

The second claim now is clear, as φ is constant in each coclass of K and takes $(G:K)$ different values. □

Lemma 6.5. *Let G be a finite group, N be an abelian normal subgroup of G and S, T and U be transversals of N in G. Let*

$$d(S,T)=\prod st^{-1}\in N,$$

where the product runs over all elements $s \in S$ and $t \in T$ such that $sN = tN$. The following statements hold:

1) $d(S,T)d(T,U) = d(S,U)$.
2) $d(gS,gT) = gd(S,T)g^{-1}$ *for all* $g \in G$.
3) $d(nS,S) = n^{(G:N)}$ *for all* $n \in N$.

Proof. If $s \in S$, $t \in T$ and $u \in U$ are such that $sN = tN = uN$, then, since N is abelian and $(st^{-1})(tu^{-1}) = su^{-1}$, we obtain that

$$d(S,T)d(T,U) = \prod (st^{-1})(tu^{-1}) = \prod su^{-1} = d(S,U).$$

Since $sN = tN$ if and only if $gsN = gtN$ for all $g \in G$,

$$g\left(\prod st^{-1}\right)g^{-1} = \prod gst^{-1}g^{-1} = \prod (gs)(gt)^{-1} = d(gS,gT).$$

Finally, since N is normal in G, $nsN = sN$ for all $n \in N$. Thus

$$d(nS,S) = \prod (ns)s^{-1} = n^{(G:N)}.$$

□

Recall that a subgroup K of G admits a **complement** Q if G factorizes as $G = KQ$ with $K \cap Q = \{1\}$. A typical example is the semidirect product $G = K \rtimes Q$, where K is a normal subgroup of G and Q is a subgroup of G such that $K \cap Q = \{1\}$.

Exercise 6.6. Let Q act by automorphisms on K. Prove that there is a bijection between the set of complements of K in $K \rtimes Q$ and the set $Z^1(Q,K)$.

We are now ready to prove the first version of the Schur–Zassenhaus theorem.

Theorem 6.7 (Schur–Zassenhaus). *Let G be a finite group and N be an abelian normal subgroup of G. If $|N|$ and $(G:N)$ are coprime, then N admits a complement in G. Moreover, all complements of N are conjugate.*

Proof. Let T be a transversal of N in G and $\theta\colon G \to N$, $\theta(g) = d(gT,T)$. Since N is abelian, Lemma 6.5 implies that θ is a 1-cocycle, where G acts on N by conjugation:

$$\begin{aligned}\theta(xy) &= d(xyT,T) = d(xyT,xT)d(xT,T)\\ &= (xd(yT,T)x^{-1})d(xT,T) = (x\cdot\theta(y))\theta(x).\end{aligned}$$

Claim. $\theta|_N\colon N \to N$ is surjective.

If $n \in N$, Lemma 6.5 implies that $\theta(n) = d(nT,T) = n^{(G:N)}$. Since $|N|$ and $(G:N)$ are coprime, there exist $r,s \in \mathbb{Z}$ such that $r|N| + s(G:N) = 1$. Thus

$$n = n^{r|N|+s(G:N)} = (n^s)^{(G:N)} = \theta(n^s).$$

Let $H = \ker\theta$. We prove that H is a complement of N. By Exercise 6.3, H is a subgroup of G. By Lemma 6.4,

$$|N| = |\theta(G)| = (G:H) = \frac{|G|}{|H|}.$$

Since $N \cap H$ is a subgroup of N and a subgroup of H, $N \cap H = \{1\}$, as the numbers $|N|$ and $(G:N) = |H|$ are coprime. Since $|NH| = |N||H| = |G|$, we conclude that $G = NH$. Hence H is a complement of N.

We now prove that two complements of N are conjugate. Let K be a complement of N in G. Since $NK = G$ and $N \cap K = \{1\}$, K is a transversal of N. Let $m = d(T,K) \in N$. Since the restriction map $\theta|_N$ is surjective, there exists $n \in N$ such that $\theta(n) = m$. By Lemma 6.5,

$$kmk^{-1} = kd(T,K)k^{-1} = d(kT,kK) = d(kT,K) = d(kT,T)d(T,K) = \theta(k)m$$

for all $k \in K$. Since N is abelian, $\theta(n^{-1}) = m^{-1}$. Thus

$$\begin{aligned}\theta(nkn^{-1}) &= \theta(n)n\theta(kn^{-1})n^{-1} = m\theta(kn^{-1})\\ &= m\theta(k)k\theta(n^{-1})k^{-1} = m\theta(k)km^{-1}k^{-1} = 1.\end{aligned}$$

Therefore $nKn^{-1} \subseteq H = \ker\theta$. Since $|K| = (G:N) = |H|$, we conclude that $nKn^{-1} = H$. □

Theorem 6.8 (Schur–Zassenhaus). *Let G be a finite group and N be a normal subgroup of G. If $|N|$ and $(G:N)$ are coprime, then N admits a complement in G.*

Proof. We proceed by induction on $|G|$. If there is a proper subgroup K of G such that $NK = G$, then, since $(K : K\cap N) = (G:N)$ and $|N|$ are coprime, $(K : K\cap N) = (G:N)$ is coprime with $|K\cap N|$. Since $K\cap N$ is normal in K, the inductive hypothesis implies that $K\cap N$ admits a complement in K. Thus there exists a subgroup H of K such that $|H| = (K : K\cap N) = (G:N)$.

Assume that there is no proper subgroup K of G such that $NK = G$. We may assume that $N \neq \{1\}$ (otherwise, G would be a complement of N in G). Since N is contained in every maximal subgroup of G (because, if there is a maximal subgroup $M \subsetneq G$ such that $N \not\subseteq M$, then $NM = G$), it follows that $N \subseteq \Phi(G)$. By Frattini's theorem 3.52, $\Phi(G)$ is nilpotent. Thus N is nilpotent and then $Z(N) \neq \{1\}$. Let $\pi\colon G \to G/Z(N)$ be the canonical map. Since N is normal in G and $Z(N)$ is characteristic in N, $Z(N)$ is normal in G. Moreover,

$$(\pi(G):\pi(N)) = \frac{|\pi(G)|}{|\pi(N)|} = \frac{|G/Z(N)|}{|N/N\cap Z(N)|} = (G:N)$$

is coprime with $|N|$. Then $(\pi(G):\pi(N))$ is coprime with $|\pi(N)|$. By the inductive hypothesis, $\pi(N)$ admits a complement in $G/Z(N)$, say $\pi(K)$ for some subgroup K of G. Hence $G = NK$, as $\pi(G) = \pi(N)\pi(K) = \pi(NK)$. Since $K = G$ (because there is no K such that $G = NK$), $\pi(N)$ is abelian, as

$$\pi(Z(N)) = \pi(N)\cap\pi(K) = \pi(N)\cap\pi(G) = \pi(N).$$

Thus $N \subseteq Z(N)$ is abelian. By Theorem 6.7, the subgroup N admits a complement. □

Theorem 6.9. *Let G be a finite group and N be a normal subgroup of G such that $|N|$ and $(G:N)$ are coprime. If either N or G/N is solvable, then all complements of N in G are conjugate.*

Proof. Let G be a minimal counterexample to the theorem, that is there are complements K_1 and K_2 of N in G such that K_1 and K_2 are not conjugate.

Claim. Every subgroup U of G satisfies the assumptions of the theorem with respect to the normal subgroup $U\cap N$.

Since N is normal in G, $U\cap N$ is normal in U. Moreover, $|U\cap N|$ and $(U : U\cap N)$ are coprime, as $|U\cap N|$ divides $|N|$ and $(U : U\cap N) = (UN : N)$ divides $(G:N)$. If G/N is solvable, then $U/U\cap N$ is solvable, as $U/U\cap N$ is isomorphic to a subgroup of G/N. If N is solvable, then so is $U\cap N$.

Claim. If there is a normal subgroup L of G such that $\pi(N)$ is normal in $\pi(G)$, where $\pi\colon G \to G/L$ is the canonical map, then $\pi(G)$ satisfies the theorem's assumptions with respect to $\pi(N)$. In this case, if H is a complement of N in G, then $\pi(H)$ is a complement of $\pi(N)$ in $\pi(G)$.

If N is solvable, then so is $\pi(N)$. If G/N is solvable, then so is $\pi(G)/\pi(N) \simeq G/NL$. Moreover, $(\pi(G):\pi(N)) = \frac{|G/L|}{|N/N\cap L|}$ divides $(G:N)$.

If H is a complement of N in G, $|\pi(H)|$ and $|\pi(N)|$ are coprime. Then $\pi(H)$ is a complement of $\pi(N)$, as $\pi(G)=\pi(N)\pi(H)=\pi(NH)$ and $\pi(N)\cap\pi(H)=\{1\}$.

Claim. N is minimal normal in G.

Let $M\neq\{1\}$ be a normal subgroup of G such that $M\subseteq N$. Let $\pi\colon G\to G/M$ be the canonical map. Then $\pi(G)$ satisfies the theorem's assumptions with respect to the normal subgroup $\pi(N)$. By the minimality of $|G|$, there exists $x\in G$ such that $\pi(xK_1x^{-1})=\pi(K_2)$. The subgroup $U=MK_2$ satisfies the theorem's assumptions with respect to the normal subgroup $U\cap N$. Since $xK_1x^{-1}\cup K_2\subseteq U$, we conclude that both xK_1x^{-1} and K_2 complement $U\cap N$ in U. Hence $MK_2=G$, as xK_1x^{-1} and K_2 are not conjugate and G is a minimal counterexample. minimal. Therefore $M=N$, as

$$\frac{|K_2|}{|M\cap K_2|}=(MK_2:M)=(G:M)=\frac{|NK_2|}{|M|}=(N:M)|K_2|.$$

Claim. N is not solvable and G/N is solvable.

Otherwise, by Lemma 1.21, N is abelian (because it is minimal normal). This contradicts Theorem 6.7, as it states that K_1 and K_2 are conjugate.

Let $p\colon G\to G/N$ be the canonical map and S be a subgroup such that $p(S)$ is minimal normal in $p(G)=G/N$. By Lemma 1.21, $p(S)$ is a p-group for some prime number p. Since $G=NK_1=NK_2$ and $N\subseteq S$, Dedekind's lemma 3.47 implies that

$$S=N(S\cap K_1)=N(S\cap K_2).$$

Hence both $S\cap K_1$ and $S\cap K_2$ complement N in S. Since $p(S)=p(S\cap K_1)=p(S\cap K_2)$ is a p-group, p divides $|S|$. The theorem's assumptions hold for S with respect to the normal subgroup N, so $|N|$ and $(S:N)$ are coprime. If $p\mid |N|$, then $p\nmid (S:N)=|S\cap K_1|=|S\cap K_2|$, a contradiction. Thus $p\nmid |N|$ and hence $p\nmid |S|$. This implies that both $S\cap K_1$ and $S\cap K_2$ are Sylow p-subgroups of S, as

$$|S\cap K_1|=(S:N)=|S\cap K_2|.$$

By Sylow's theorem, there exists $s\in S$ such that

$$S\cap sK_1s^{-1}=S\cap K_2.$$

In particular, $S\neq G$ by the minimality of G. Let

$$L=S\cap K_2=S\cap sK_1s^{-1}\neq\{1\}.$$

Since S is normal in G, $sK_1s^{-1}\cup K_2\subseteq N_G(L)$ (because L is both normal in sK_1s^{-1} and in K_2). The subgroups $sK_1s^{-1}\subseteq N_G(L)$ and $K_2\subseteq N_G(L)$ complement $N\cap N_G(L)$ in $N_G(L)$. Hence $N_G(L)=G$ by the minimality of G (if $N_G(L)\neq G$, then both sK_1s^{-1} and K_2 are conjugate in G because they are conjugate in $N_G(L)$). Therefore L is normal in G.

Let $\pi_L\colon G\to G/L$ be the canonical map. Since both $\pi_L(K_1)$ and $\pi_L(K_2)$ complement $\pi_L(N)$ in $\pi_L(G)$, the minimality of $|G|$ implies that there exists $g\in G$ such that $\pi_L(gK_1g^{-1})=\pi_L(K_2)$, that is there exists $g\in G$ such that $(gK_1g^{-1})L=K_2L$. Hence $gK_1g^{-1}\cup K_2\subseteq\langle K_2,L\rangle=K_2$, because $L\subseteq K_2$. In conclusion, $gK_1g^{-1}=K_2$, a contradiction to the minimality of $|G|$. □

By the Feit–Thompson theorem, int he previous theorem we do not need to assume that either N or G/N is solvable. Since every group of odd order is solvable and $|N|$ and $(G:N)$ are coprime, one of these groups should have odd order.

THEOREM 6.10. *Let G be a finite solvable group and p a prime number dividing $|G|$. There exists a maximal subgroup M of G of index a power of p.*

PROOF. We proceed by induction on $|G|$. If G is a p-group, the result clearly holds. So we may assume that $|G|$ is divisible by at least two different prime numbers. Let p be a prime dividing $|G|$, N be a minimal normal subgroup of G and $\pi\colon G \to G/N$ be the canonical map. Since G is solvable, by Lemma 1.21, N is a q-group for some prime q. Since G/N is solvable, if p divides $(G:N)$, then, by the inductive hypothesis, G/N has a maximal subgroup M_1 of index a power of p. By the correspondence theorem, $M = \pi^{-1}(M_1)$ is a maximal subgroup of G of index a power of p. p. If p does not divide $(G:N)$, then p divides $|N|$. Thus $N \in \mathrm{Syl}_p(G)$. Since N is normal in G and $|N|$ and $|G/N|$ are coprime, by Schur–Zassenhaus theorem 6.8, there exists a complement K of N in G, that is $G = NK$ and $N \cap K = \{1\}$. Let M be a maximal subgroup containing K. Then $(G:M)$ is a power of p. □

We now discuss an application to finite super-solvable groups.

DEFINITION 6.11. A finite group G is said to be **lagrangian** if for each d dividing $|G|$ there exists a subgroup of G of order d.

The group $\mathbb{A}_4$ is not lagrangian, as it has no subgroups of order six.

THEOREM 6.12. *Every finite super-solvable group is lagrangian.*

PROOF. Let p be a prime number dividing $|G|$. Since subgroups of super-solvable groups are super-solvable, it is enough to show that there exists a subgroup of index p. Since G is solvable, there exists a maxima subgroup M of index p^α by Theorem 6.10. Since maximal subgroups of super-solvable groups have prime index by Theorem 5.12, we conclude that $\alpha = 1$. □

Lecture 7.

§ 7.1. Subnormality.

Definition 7.1. Let G be a group. A subgroup H of G is said to be subnormal in G if there is a sequence of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_k = G$$

with H_i normal in H_{i+1} for all $i \in \{0, \dots, k-1\}$.

Example 7.2. Let $G = \mathbb{S}_4$. Then $K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$ is normal in G. The subgroup $L = \{\mathrm{id}, (12)(34)\}$ is subnormal in G (and not normal). es subnormal.

Exercise 7.3. Prove that the correspondence theorem preserves subnormality.

Theorem 7.4. *Let G be a finite group. Then G is nilpotent if and only if every subgroup of G is subnormal in G.*

Proof. Assume first that every subgroup of G is subnormal in G. Let H be a subnormal subgroup of G, where

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_k = G$$

with H_i normal in H_{i+1}. Without loss of generality, we may assume that $H \subsetneq H_1$. Since $H \subsetneq H_1 \subseteq N_G(H)$, G is nilpotent by Exercise 3.25.

Assume now that G is nilpotent. Let H be a subgroup of G. We proceed by induction on $(G:H)$. If $(G:H) = 1$, then $H = G$ and the theorem holds. If $H \neq G$, since $H \subsetneq N_G(H)$ by Lemma 2.26,

$$(G : N_G(H)) < (G : H).$$

By the inductive hypothesis, $N_G(H)$ is subnormal in G. Since H is normal in $N_G(H)$, we conclude that H is subnormal in G. □

Corollary 7.5. *Let G be a group and K be a central subgroup of G (that is, $K \subseteq Z(G)$). Then G is nilpotent if and only if G/K is nilpotent.*

Proof. If G is nilpotent, then so is G/K. Conversely, let $\pi \colon G \to G/K$ be the canonical map and U be a subgroup of G. Since G/K is nilpotent, Theorem 7.4 implies that $\pi(U)$ is a subnormal subgroup of G/K. By the correspondence theorem, UK is a subnormal subgroup of G. Since K is central, U is normal in UZ. Hence U is subnormal in G and therefore G is nilpotent by Theorem 7.4. □

Theorem 7.6. *Let G be a finite group and H be a subgroup of G. Then H is nilpotent and subnormal in G if and only if $H \subseteq F(G)$.*

Proof. Assume first that $H \subseteq F(G)$. Since $F(G)$ is nilpotent by Theorem 4.6, so is H. Moreover, since H is subnormal in $F(G)$ (Theorem 7.4) and $F(G)$ is normal in G, H is subnormal in G.

Assume now that H is nilpotent and subnormal in G. We proceed by induction on $|G|$. If $H = G$, then the result holds. Assume then that $H \neq G$. Since H is subnormal in G, there is a sequence

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_k = G$$

of subgroups of G with H_i normal in H_{i+1} for all i. Let $M = H_{k-1}$. Since $M \neq G$ and M is normal in G, $H \subseteq F(M)$ by the inductive hypothesis. Thus $H \subseteq F(M) = M \cap F(G) \subseteq F(G)$ by Corollary 4.9. □

Before proving another important theorem of Wielandt, we need a lemma.

Lemma 7.7. *Let M and N be normal subgroups of G such that $M \cap N = \{1\}$. Then $M \subseteq C_G(N)$.*

Proof. Let $m \in M$ and $n \in N$. Then $[n,m] = (nmn^{-1})m \in M$, since M is normal in G and Moreover, $[n,m] = n(mn^{-1}m^{-1}) \in N$, since N is normal in G. Thus $[n,m] \in M \cap N = \{1\}$. □

Exercise 7.8. A group G is said to be **characteristically simple** if G is non-trivial and has no proper characteristic subgroups. Prove that any minimal normal subgroup of G is characteristically simple.

Definition 7.9. Let G be a group. If G admits minimal normal subgroups, the **socle** of G is defined as the subgroup $\mathrm{Soc}(G)$ of G generated by all minimal normal subgroups of G. If G admits no minimal normal subgroups, then $\mathrm{Soc}(G) = \{1\}$.

For example, $\mathrm{Soc}(\mathbb{Z}) = \{0\}$ and $\mathrm{Soc}(\mathbf{SL}_2(3)) \simeq C_2$.

Exercise 7.10. Prove that the socle of a group is a direct product of minimal normal subgroups.

Exercise 7.11. Prove the following statements:

1) A direct product of isomorphic simple groups is characteristically simple.
2) A characteristically simple group with at least one minimal normal subgroup is a direct product of isomorphic simple groups.

Theorem 7.12 (Wielandt). *Let G be a finite group. If S is a subnormal group of G and M is a minimal normal subgroup of G, then $M \subseteq N_G(S)$.*

Proof. We proceed by induction on $|G|$. If $S = G$ the result holds. So assume that $S \neq G$. Since S is subnormal in G, there exists a sequence

$$S = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{k-1} \subseteq S_k = G$$

of subgroups of G such that S_i is normal in S_{i+1} for all i. Let $N = S_{k-1}$.

If $M \cap N \neq \{1\}$, then $M \subseteq N$ (because since M and N are both normal in G, $M \cap N = M$ by the minimality of M). We claim that $M \subseteq \mathrm{Soc}(N)$. Since $M \neq \{1\}$ and M is normal in N, $M \cap \mathrm{Soc}(N) \neq \{1\}$. Moreover, since $\mathrm{Soc}(N)$ is characteristic in N and N is normal in G, it follows that $\mathrm{Soc}(N)$ is normal in G. Hence $M \cap \mathrm{Soc}(N)$ is a normal subgroup of G. Since $\{1\} \neq M \cap \mathrm{Soc}(N) \subseteq M$, we conclude that $M \cap \mathrm{Soc}(N) = M$ by the minimality of M. By the inductive hypothesis, every minimal normal subgroup of N normalizes S. Thus $\mathrm{Soc}(N) \subseteq N_N(S) \subseteq N_G(S)$ and therefore

$$M \subseteq \mathrm{Soc}(N) \subseteq N_G(S).$$

If $M \cap N = 1$, Lemma 7.7 implies that

$$M \subseteq C_G(N) \subseteq C_G(S) \subseteq N_G(S).$$ □

COROLLARY 7.13. *Let G be a finite group and S be a subnormal subgroup of G. Then*

$$\operatorname{Soc}(G) \subseteq N_G(S).$$

PROOF. By Theorem 7.12, every minimal normal subgroup of G is contained in $N_G(S)$. Then $\operatorname{Soc}(G) = \langle M : M \text{ minimal normal subgroup of } G\rangle \subseteq N_G(S)$. □

THEOREM 7.14 (Wielandt). *Let G be a finite group and S and T be subnormal subgroups of G. Then $S \cap T$ and $\langle S, T\rangle$ are subnormal in G.*

PROOF. We first prove that $S \cap T$ is subnormal in G. Since subnormality is a transitive relation, it is enough to see that $S \cap T$ is subnormal in T. Since S is subnormal in G, there exists a sequence

$$S = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = G$$

of subgroups of G such that S_i is normal in S_{i+1} for all i. Each $S_{j-1} \cap T$ is normal in $S_j \cap T$. Then $S \cap T$ is subnormal in T.

We now prove that $\langle S, T\rangle$ is subnormal in G We proceed by induction on $|G|$. Assume that $G \neq \{1\}$. Let M be a minimal normal subgroup of G and $\pi \colon G \to G/M$ be the canonical map. Since both $\pi(S)$ and $\pi(T)$ subnormal in G/M and $|G/M| < |G|$, the inductive hypothesis implies that

$$\pi(\langle S, T\rangle M) = \pi(\langle S, T\rangle) = \langle \pi(S), \pi(T)\rangle$$

is subnormal in G/M. By the correspondence theorem, $\langle S, T\rangle M$ is subnormal in G. Theorem 7.12 implies that $M \subseteq N_G(S)$ and $M \subseteq N_G(T)$. Hence $M \subseteq N_G(\langle S, T\rangle)$. Since $\langle S, T\rangle$ is normal in $\langle S, T\rangle M$ and $\langle S, T\rangle M$ is subnormal in G, we conclude that $\langle S, T\rangle$ is subnormal in G. □

Lecture 8.

§ 8.1. Wielandt's zipper theorem.

THEOREM 8.1 (Wielandt). *Let G be a finite group and S be a subgroup of G subnormal in every proper subgroup of G containing S. If S is not subnormal in G, then there exists a unique maximal subgroup of G containing S.*

PROOF. We proceed by induction on $(G:S)$. If S is not subnormal in G, then $S \neq G$ and the case where $(G:S)=1$ holds.

Since S is not subnormal in G, $N_G(S) \neq G$. Then $S \subseteq N_G(S) \subseteq M$ for some maximal subgroup M of G. Assume that $S \subseteq K$ for some maximal subgroup K of G. We claim that que $K = M$. Since $S \subseteq K \neq G$, S is subnormal in K. If S is normal in K, then $K \subseteq N_G(S) \subseteq M$. Hence $K = M$ by the maximality of K. If S is not normal in K, there exist a sequence $S_0, \dots, S_m$ of subgroups of K such that

$$S = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m = K,$$

where S_i is normal in S_{i+1} for all i and S is not normal in S_2. Let $x \in S_2$ be such that $xSx^{-1} \neq S$ and $T = \langle S, xSx^{-1} \rangle \subseteq K$.

Since $xSx^{-1} \subseteq xS_1x^{-1} = S_1 \subseteq N_G(S)$, we obtain that $T \subseteq N_G(S) \subseteq M$. Moreover, S is normal in T. Thus $T \neq G$.

We claim that T satisfies the theorem's assumptions. If T is subnormal in G, then, since S is normal in T, S is subnormal in G. If H is a proper subgroup of G containing T, then, since $S \subseteq H$, S is subnormal in H. Moreover, xSx^{-1} is subnormal in H. Hence T is subnormal in H by Theorem 7.14.

Since $S \subsetneq T$, $(G:T) < (G:S)$. By the inductive hypothesis, T is contained in a unique maximal subgroup of G. Therefore $K = M$, since $T \subseteq M$ and $T \subseteq K$. □

Before giving an application, we need a lemma.

LEMMA 8.2. *Let G be a group and H be a subgroup of G. If $(xHx^{-1})H = G$ for some $x \in G$, then $H = G$.*

PROOF. Write $x = uv$ for some $u \in xHx^{-1}$ and $v \in H$. Since $u \in xHx^{-1}$ and $u^{-1}x = v \in H$, we obtain that $H = vHv^{-1} = u^{-1}(xHx^{-1})u = xHx^{-1}$. Thus $G = H$. □

Recall that two subgroups S and T of a group G are said to be **permutable** if $ST = TS$.

THEOREM 8.3. *Let G be a finite group and S be a subgroup of G permutable with any of its conjugates. Then S is subnormal in G.*

PROOF. We proceed by induction on $|G|$. Assume that S is subnormal in every subgroup H such that $S \subseteq H \subsetneq G$. If S is not subnormal in G, then, by Theorem 8.1, there exists a unique maximal subgroup M of G such that $S \subseteq M$. Let $x \in G$ and $T = xSx^{-1}$. By Lemma 8.2, $ST \neq G$ (because $S \neq G$). Thus ST is contained in some maximal subgroup of G. Since $S \subseteq ST$ and S is contained in a unique maximal subgroup of G, we conclude that $T \subseteq ST \subseteq M$. Since $S^G = \langle xSx^{-1} : x \in G \rangle \subseteq M \neq G$, the inductive hypothesis implies that S is subnormal in S^G. Hence S is subnormal in G since S^G is normal in G, a contradiction. □

§ 8.2. Baer's theorem.

THEOREM 8.4 (Baer). *Let G be a finite group and H be a subgroup of G. Then $H \subseteq F(G)$ if and only if $\langle H, xHx^{-1} \rangle$ is nilpotent for all $x \in G$.*

PROOF. If $H \subseteq F(G)$, then $xHx^{-1} \subseteq F(G)$ for all $x \in G$, since $F(G)$ is normal in G. Thus $\langle H, xHx^{-1}\rangle$ is nilpotent, as it is a subgroup of $F(G)$.

Conversely, assume that $\langle H, xHx^{-1}\rangle$ is nilpotent for all $x \in G$. Since $H \subseteq \langle H, xHx^{-1}\rangle$, H is nilpotent. By Theorem 7.6, it is enough to see that H is subnormal in G. We proceed by induction on $|G|$. Suppose that H is not subnormal in G. If H is properly contained in some subgroup K, then, since $\langle H, kHk^{-1}\rangle$ is nilpotent for all $k \in K$, H is subnormal in K by the inductive hypothesis. By Theorem 8.1, there exists a unique maximal subgroup M of G containing H. There are two cases to consider.

Assume first that $G = \langle H, xHx^{-1}\rangle$ for some $x \in G$. Since G is nilpotent, H subnormal in G by Theorem 7.4, a contradiction.

Assume now that $\langle H, xHx^{-1}\rangle \neq G$ for all $x \in G$. For each $x \in G$, there exists a maximal subgroup containing $\langle H, xHx^{-1}\rangle$. Since $H \subseteq \langle H, xHx^{-1}\rangle$ and H is contained in a unique maximal subgroup, we conclude that $\langle H, xHx^{-1}\rangle \subseteq M$ for all $x \in G$. In particular, the normal closure H^G of H is properly contained in G. By the inductive hypothesis, H is subnormal in H^G and H^G is normal in G, we conclude that H is subnormal in G, a contradiction. □

§ 8.3. Zenkov's theorem.

THEOREM 8.5 (Zenkov). *Sean G un grupo finito y A,B subgrupos abelianos de G. Sea $M \in \{A \cap gBg^{-1} : g \in G\}$ tal que ningún $A \cap gBg^{-1}$ está propiamente contenido en M. Entonces $M \subseteq F(G)$.*

PROOF. Sin pérdida de generalidad podemos suponer que $M = A \cap B$. Demostraremos por inducción en $|G|$ que $M \subseteq F(G)$.

Supongamos que $G = \langle A, gBg^{-1}\rangle$ para algún $g \in G$. Como A y B son abelianos, $A \cap gBg^{-1} \subseteq Z(G)$. Luego

$$A \cap gBg^{-1} = g^{-1}(A \cap gBg^{-1})g \subseteq A \cap B = M.$$

Por la minimalidad de M, $M = A \cap gBg^{-1} \subseteq Z(G) \subseteq F(G)$ por el corolario **??**.

Supongamos ahora que $G \neq \langle A, gBg^{-1}\rangle$ para todo $g \in G$. Fijemos $g \in G$. Sean $H = \langle A, gBg^{-1}\rangle \neq G$ y $C = B \cap H$. Al usar que $A \subseteq H$ se obtiene fácilmente que $M = A \cap B = A \cap C$ y que $A \cap hCh^{-1} = A \cap hBh^{-1}$ para todo $h \in H$. Esto implica que ningún $A \cap hCh^{-1}$ está propiamente contenido en $A \cap C$. Al aplicar la hipótesis inductiva al subgrupo H obtenemos entonces

$$M = A \cap B = A \cap C \subseteq F(H).$$

Vamos a demostrar ahora que todo p-subgrupo de Sylow P de M está contenido en $F(G)$. Como M está generado por sus subgrupos de Sylow, esto implica que $M \subseteq F(G)$. Si $P \in \mathrm{Syl}_p(M)$ entonces $P \subseteq M \subseteq F(H)$. Como $O_p(H)$ es el único p-subgrupo de Sylow de $F(H)$, $P \subseteq O_p(H)$. Como $P \subseteq M \subseteq B$,

$$gPg^{-1} \subseteq gBg^{-1} \subseteq H$$

para todo $g \in G$. Entonces $O_p(H)(gPg^{-1})$ es un p-subgrupo de H que contiene a $\langle P, gPg^{-1}\rangle$. Luego $\langle P, gPg^{-1}\rangle$ es nilpotente para todo $g \in G$ por ser un p-grupo. Por el teorema de Baer **??**, $P \subseteq F(G)$ para todo p-subgrupo de Sylow P de M. □

COROLLARY 8.6. *Sea G un grupo finito no trivial y sea A un subgrupo abeliano tal que $|A| \geq (G:A)$. Entonces $A \cap F(G) \neq 1$.*

PROOF. Sea $g \in G$. Podemos suponer que $G \neq A$ y luego $(gAg^{-1})A \neq G$ por el lema **??**. Como $|gAg^{-1}||A| = |A|^2 \geq |A|(G:A) = |G|$,

$$|G| > |gAg^{-1}A| = \frac{|A||gAg^{-1}|}{|A \cap gAg^{-1}|} \geq \frac{|G|}{|A \cap gAg^{-1}|}.$$

Luego $A \cap gAg^{-1} \neq 1$ para todo $g \in G$. En particular, ningún $A \cap gAg^{-1}$ está propiamente contenido en A y luego, por el teorema de Zenkov 8.5, $A \subseteq F(G)$. □

COROLLARY 8.7. *Sea $G = NA$ un grupo finito, donde N es normal en G, A es un subgrupo abeliano y $C_A(N) = 1$. Si $F(N) = 1$, entonces $|A| < |N|$.*

PROOF. Como N es normal en G, $N \cap F(G) = F(N) = 1$ por el corolario **??**. Luego $[N, F(G)] = 1$ porque N y $F(G)$ son ambos normales en G. Como

$$|A| \geq |N| \geq \frac{|N|}{|N \cap A|} = (NA : A) = (G : A),$$

$A \cap F(G) \neq 1$ por el corolario 8.6. Si $1 \neq a \in A \cap F(G)$, entonces $a \in C_A(N) = 1$, una contradicción. □

§ 8.4. Brodkey's theorem.

THEOREM 8.8 (Brodkey). *Sea G un grupo finito tal que existe $P \in \mathrm{Syl}_p(G)$ abeliano. Entonces existen $S, T \in \mathrm{Syl}_p(G)$ tales que $S \cap T = O_p(G)$.*

PROOF. Al aplicar el teorema de Zenkov 8.5 con $A = B = P$ se tiene que $P \cap gPg^{-1} \subseteq F(G)$ para algún $g \in G$. Como $O_p(G)$ es el único p-subgrupo de Sylow de $F(G)$, $P \cap gPg^{-1} \subseteq O_p(G)$. Luego $P \cap gPg^{-1} = P_p(G)$ pues $O_p(G)$ está contenido en todo p-subgrupo de Sylow de G. □

COROLLARY 8.9. *Sea G un grupo finito. Si existe $P \in \mathrm{Syl}_p(G)$ abeliano,*

$$(G : O_p(G)) \leq (G : P)^2.$$

PROOF. Por el teorema de Brodkey 8.8, existen $S, T \in \mathrm{Syl}_p(G)$ tales que $S \cap T = O_p(G)$. Entonces

$$|G| \geq |ST| = \frac{|S||T|}{|S \cap T|} = \frac{|P|^2}{|O_p(G)|},$$

que implica el corolario. □

COROLLARY 8.10. *Sea G un grupo finito. Si existe un subgrupo $P \in \mathrm{Syl}_p(G)$ abeliano tal que $|P| < \sqrt{|G|}$ entonces $O_p(G) \neq 1$.*

PROOF. Como $(G : P)^2 < |G|$, el corolario 8.9 implica que $O_p(G) \neq 1$. □

EXERCISE 8.11. Sea G un grupo y sea Sea $K \subseteq Z(G)$. Demuestre que si G/K es cíclico entonces G es abeliano.

Sean $g, h \in G$ y sea $\pi \colon G \to G/K$ el morfismo canónico. Como G/K es cíclico, existe $x \in G$ tal que $G/K = \langle xK \rangle$. Sean k, l tales que $\pi(g) = x^k$, $\pi(h) = x^l$. Entonces existen $z_1, z_2 \in K$ tales que $g = x^k z_1$, $h = x^l z_2$. Luego $[g, h] = [x^k, x^l] = 1$.

Exercise 8.12. Sea G un grupo y sea A un subgrupo de G. Demuestre que ${}_G(A) = \bigcap_{x\in G} xAx^{-1}$ es el mayor subgrupo normal de G contenido en A.

Hagamos actuar a G por multiplicación en las coclases de A: $g\cdot xA = gxA$. Esta acción induce un morfismo $\rho\colon G\to \mathbb{S}_{G/A}$ con núcleo

$$\ker\rho = \bigcap_{x\in G} xAx^{-1} =_G (A).$$

Es claro entonces que ${}_G(A)$ es un subgrupo normal de G contenido en A. Si K es un subgrupo normal de G tal que $K\subseteq A$, entonces $K = xKx^{-1}\subseteq xAx^{-1}$ para todo $x\in G$. Luego $K \subseteq_G (A)$.

§ 8.5. Lucchini's theorem.

Theorem 8.13 (Lucchini). *Sea G un grupo finito y sea A un subgrupo cíclico propio. Si $K =_G (A)$ entonces $(A:K) < (G:A)$.*

Proof. Procederemos por inducción en $|G|$. Sea $\pi\colon G\to G/K$ el morfismo canónico. Observemos que ${}_{G/K}\pi(A)$ es trivial.

Supongamos primero que $K\neq 1$. Como $\pi(A)$ es un subgrupo cíclico propio de G/K y $K\subseteq A$, la hipótesis inductiva implica que

$$(A:K) = |\pi(A)| = (\pi(A):\pi(K)) < (\pi(G):\pi(A)) = \frac{(G:K)}{(A:K)} = (G:A).$$

Supongamos ahora que $K=1$. Queremos demostrar que $|A| < (G:A)$. Supongamos entonces que $|A|\geq (G:A)$. Como $A\neq G$, $A\cap F(G)\neq 1$ por el corolario 8.6. En particular, $F(G)\neq 1$. Sea E un subgrupo minimal-normal de G tal que $E\subseteq F(G)$. Por el teorema **??**, $E\cap Z(F(G))\neq 1$. Luego, como $E\cap Z(F(G))$ es normal en G y E es minimal, $E\cap Z(F(G)) = E$, es decir $E\subseteq Z(F(G))$. En particular, E es abeliano y luego, por la minimalidad de E, existe un primo p tal que $x^p = 1$ para todo $x\in E$.

Claim. $A\cap F(G)$ es un subgrupo normal de EA.

Como E es normal en G, EA es un subgrupo de G. Como $A\cap F(G)\subseteq A$, $A\cap F(G)$ es un subgrupo de EA. Como $F(G)$ es normal en G, $a(A\cap F(G))a^{-1} = A\cap F(G)$ para todo $a\in A$. Por otro lado $E\subseteq Z(F(G))$ y $A\cap F(G)\subseteq F(G)$ implican que $x(A\cap F(G))x^{-1} = A\cap F(G)$ para todo $x\in E$.

Claim. $EA\neq G$.

Si $G = EA$ entonces, como $A\cap F(G)$ es un subgrupo normal de G contenido en A, se concluye que $1\neq A\cap F(G)\subseteq K = 1$, una contradicción. para todo $g\in G$. Luego $1\neq A\cap F(G)\subseteq K$, una contradicción pues $K=1$.

Sea $p\colon G\to G/E$ el morfismo canónico. Por la correspondencia, existe un subgrupo normal M de G con $E\subseteq M$ tal que $p(M) =_{G/E} (p(A))$. Como $EA\neq G$, $p(A)$ es un subgrupo cíclico propio de $p(G)$. Como $p(A)\simeq A/A\cap E\simeq EA/E$ y $p(M)\simeq M/E$, la hipótesis inductiva implica que $(EA:M) < (G:EA)$ pues

$$\frac{|EA/E|}{|M/E|} = (p(A):p(M)) < (p(G):p(A)) = \frac{|G/E|}{|EA/E|}.$$

Claim. $MA = EA$.

Como $E \subseteq M$ entonces $EA \subseteq MA$. Recíprocamente, si $m \in M$ entonces, como $p(m) \in_{G/E} (p(A))$, en particular $p(m) \in p(A)$. Luego $m \in EA$.

Sea $B = A \cap M$. Al usar que $(AE : M) < (G : EA)$, que

$$(A : B) = |A/A \cap M| = |AM/M| = (EA : M)$$

y la hipótesis inductiva obtenemos

$$\begin{aligned} (M : B) &= (M : A \cap M) = (MA : A) \\ &= (EA : A) = \frac{(G : A)}{(G : EA)} < \frac{(G : A)}{(AE : M)} = \frac{(G : A)}{(A : B)} \leq |B| \end{aligned} \tag{8.1}$$

pues $|A| \geq (G : A)$.

Claim. $M = EB$.

Como $E \cup B \subseteq M$ entonces $EB \subseteq M$. Recíprocamente, si $m \in M$ entonces $m = ea$ para algún $e \in E$, $a \in A$. Como $e^{-1}m = a \in A \cap M = B$ pues $E \subseteq M$ entonces $m \in EB$.

Claim. M es no abeliano.

Supongamos que M es abeliano. La función $f \colon M \to M$, $m \mapsto m^p$, es un morfismo de grupos tal que $E \subseteq \ker f$. Como $M = EB$, $f(M) \subseteq f(B) \subseteq B \subseteq A$. Como M es normal en G, $f(M)$ es normal en G. Luego $f(M) = 1$ pues $K =_G (A) = 1$ es el mayor subgrupo normal de G contenido en A; en particular, como B es normal en $M = EB$, M/B es un p-grupo. Como $B \subseteq M$, $f(B) = 1$; además como $B \subseteq A$ es cíclico, $|B| \leq p$. Luego, por la fórmula (8.1), $(M : B) < |B| \leq p$. Esto implica que $M = B \subseteq A$ y que $M = E = 1$ (porque M es normal en G y ${}_G(A) = K = 1$ es el mayor subgrupo normal de G que contiene a A), una contradicción.

Claim. $Z(M)$ es cíclico.

Como M es no abeliano y $M/E = EB/E \simeq B/E \cap B$ es ciclico, $E \not\subseteq Z(M)$ (ejercicio 8.11), es decir $E \cap Z(M) \subsetneq E$. Luego

$$E \cap Z(M) = 1 \tag{8.2}$$

por la minimalidad de E. Entonces

$$Z(M) = Z(M)/Z(M) \cap E \simeq p(Z(M)) \subseteq p(M) =_{G/E} p(A) \subseteq p(A)$$

y luego $Z(M)$ es cíclico pues $p(A)$ es cíclico.

Como $B \subseteq A$ es abeliano y $(M : B) < |B|$, $B \cap F(M) \neq 1$ por el corolario 8.6. Entonces $[E, F(M)] = 1$ (pues $E \subseteq Z(F(G))$ y $F(M) \subseteq F(G)$ por el corolario **??**). Luego $B \cap F(M) \subseteq Z(M)$ pues $M = BE$, $[B \cap F(M), E] \subseteq [F(M), E] = 1$ y también $[B \cap F(M), B] = 1$ porque B es abeliano. Como $Z(M)$ es cíclico, $B \cap F(M)$ es característico en $Z(M)$. Luego, como $Z(M)$ es normal en G, $1 \neq B \cap F(M)$ es un subgrupo normal de G contenido en A, una contradicción. □

Para terminar la sección veamos una aplicación del teorema de Lucchini.

Corollary 8.14 (Horosevskii). *Sea $G \neq 1$ un grupo finito y sea $\sigma \in \mathrm{Aut}(G)$. Entonces $|\sigma| < |G|$.*

PROOF. Sea $A = \langle\sigma\rangle$. Como A actúa por automorfismos en G, podemos considerar el grupo $\Gamma = G \rtimes A$ con la operación

$$(g,\sigma^k)(h,\sigma^l) = (g\sigma^k(h),\sigma^{k+l}).$$

Identificamos A con $1\times A$ y G con $G\times 1$. Como $K\cap G \subseteq A\cap G = 1$ y $A\cap C_\Gamma(G) = 1$,

$$K \subseteq A\cap C_\Gamma(G) = 1$$

pues si $k\in K$ y $g\in G$ entonces $gkg^{-1}k^{-1} \in G\cap K = 1$ (porque K y G son normales en Γ). Por el teorema de Lucchini 8.13, $(A : K) < (\Gamma : A)$, es decir

$$|\sigma| = |A| = (A : K) < (\Gamma : A) = |G|.$$

$\square$

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