Non-commutative algebra

Leandro Vendramin

Contents

Introduction

The notes correspond to the master course *Non-commutative Algebra* of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into thirteen two-hour lectures.

Most of the material is based on standard results on group algebras covered in the VUB course *Associative Algebras*. Lecture notes for this course are freely available here. Basic texts on group algebras are Lam's book [**4**] and Passman's book [**6**].

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Department of Mathematics and Data Science, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel
E-mail address: `Leandro.Vendramin@vub.be`.

Lecture 1. 15/02/2024

§ 1.1. Solvable groups. A subgroup H of G is said to be **characteristic** if $f(H) \subseteq H$ for all $f \in \mathrm{Aut}(G)$. The center and the commutator subgroup are characteristic subgroups. Every characteristic subgroup is normal, as the maps $x \mapsto gxg^{-1}$ are automorphisms.

Exercise 1.1. Prove that if H is characteristic in K and K is normal in G, then H is normal in G.

For a group G, let $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. The **derived series** of G is the sequence

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

Each $G^{(i)}$ is a characteristic subgroup of G. We say that G is **solvable** if $G^{(n)} = \{1\}$ for some n.

Example 1.2. Abelian groups are solvable.

Example 1.3. The group $\mathbf{SL}_2(3)$ is solvable, as the derived series is

$$\mathbf{SL}_2(3) \supseteq Q_8 \supseteq C_4 \supseteq \{1\}.$$

Example 1.4. Non-abelian simple groups cannot be solvable.

Exercise 1.5. Let G be a group. Prove the following statements:

1) A subgroup H of G is solvable, when G is solvable.
2) Let K be a normal subgroup of G. Then G is solvable if and only if K and G/K are solvable.

Example 1.6. For $n \geq 5$ the group $\mathbb{A}_n$ is simple and non-abelian. Hence it is not solvable. It follows that $\mathbb{S}_n$ is not solvable for $n \geq 5$.

Exercise 1.7. Let p, q and r be prime numbers. Prove that groups of order $p^\alpha q$, p^2q^2 and pqr are solvable.

Exercise 1.8. Prove that groups of order < 60 are solvable.

Exercise 1.9. Let p be a prime number. Prove that finite p-groups are solvable.

Theorem 1.10 (Burnside). *Let p be a prime number. If G is a finite group that has a conjugacy class of G with $p^k > 1$ elements, then G is not simple.*

The easiest way to prove Theorem 1.10 is using character theory.

Theorem 1.11 (Burnside). *Let p and q be prime numbers. If G has order p^aq^b, then G is solvable.*

Proof. If G is abelian, then it is solvable. Suppose now G is non-abelian. Let us assume that the theorem is not true. Let G be a group of minimal order p^aq^b that is not solvable. Since $|G|$ is minimal, G is a non-abelian simple group. By the previous theorem, G has no conjugacy classes of size p^k nor conjugacy classes of size q^l with $k, l \geq 1$. The size of every conjugacy class of G is one or divisible by pq. Note that, since G is a non-abelian simple group, the center of G is trivial. Thus there is only one conjugacy class of size one. By the class equation,

$$|G| = 1 + \sum_{C:|C|>1} |C| \equiv 1 \bmod pq,$$

where the sum is taken over all conjugacy classes with more than one element, a contradiction. □

A recent generalization of Burnside's theorem is based on *word maps*. A word map of a group G is a map

$$G^k \to G, \quad (x_1,\dots,x_k) \mapsto w(x_1,\dots,x_k)$$

for some word $w(x_1,\dots,x_k)$ of the free group F_k of rank k. Some word maps are surjective in certain families of groups. For example, Ore's conjecture is precisely the surjectivity of the word map $(x,y) \mapsto [x,y] = xyx^{-1}y^{-1}$ in every finite non-abelian simple group.

THEOREM 1.12 (Guralnick–Liebeck–O'Brien–Shalev–Tiep). *Let $a,b \geq 0$, p and q be prime numbers and $N = p^a q^b$. The map $(x,y) \mapsto x^N y^N$ is surjective in every non-abelian finite simple group.*

The proof appears in [**3**].

The theorem implies Burnside's theorem. Let G be a group of order $N = p^a q^b$. Assume that G is not solvable. Fix a composition series of G. There is a non-abelian factor S of order that divides N. Since S is simple non-abelian and $s^N = 1$, it follows that the word map $(x,y) \mapsto x^N y^N$ has trivial image in S, a contradiction to the theorem.

THEOREM 1.13 (Feit–Thompson). *Groups of odd order are solvable.*

The proof of Feit–Thompson theorem is extremely hard. It occupies a full volume of the *Pacific Journal of Mathematics* [**1**]. A formal verification of the proof (based on the computer software Coq) was announced in [**2**]. This motivates a natural problem: To formally verify the classification of finite simple groups. Will mathematics move away from depending on just humans to verify proofs? Formal verification with computer-proof assistants could become the new standard for rigor in mathematics.

The proof of the Feit-Thompson theorem is notably hard, spanning an entire volume of the *Pacific Journal of Mathematics*[**1**]. The theorem was recently verified using Coq, a computer-proof assistant; see [**2**]. This prompts a natural question: Can we extend the practice of formal verification to the classification of finite simple groups? And this prompts another question: Will mathematics move away from depending on just humans to verify proofs? Formal verification with computer-proof assistants could become the new standard for rigor in mathematics.

Back in the day, it was believed that if a certain divisibility conjecture is true, the proof of Feit–Thompson theorem could be simplified.

CONJECTURE 1.14 (Feit–Thompson). There are no prime numbers p and q such that $\frac{p^q-1}{p-1}$ divides $\frac{q^p-1}{q-1}$.

The conjecture remains open. However, now we know that proving the conjecture will not simplify further the proof of Feit–Thompson theorem.

In 2012, Le proved that the conjecture is true for $q = 3$, see [**5**].

In [**7**] Stephens proved that a certain stronger version of the conjecture does not hold, as the integers $\frac{p^q-1}{p-1}$ and $\frac{q^p-1}{q-1}$ could have common factors. In fact, if $p=17$ and $q=3313$, then

$$\gcd\left(\frac{p^q-1}{p-1},\frac{q^p-1}{q-1}\right)=112643.$$

Nowadays we can check this easily on almost every desktop computer:

```
gap> Gcd((17^3313-1)/16,(3313^17-1)/3312);
112643
```

No other counterexamples have been found of Stephen's stronger version of the conjecture.

Definition 1.15. Let p be a prime number. A p-group P is said to be **elementary abelian** if $x^p=1$ for all $x\in P$.

Definition 1.16. A subgroup M of G is said to be **minimal normal** if $M\neq\{1\}$, M is normal in G and the only normal subgroup of G properly contained in M is $\{1\}$.

Example 1.17. If a normal subgroup M is minimal (with respect to the inclusion), then it is minimal and normal. However, the converse statement does not hold. For example, the subgroup of $\mathbb{A}_4$ generated by $(12)(34)$, $(13)(24)$ and $(14)(23)$ is minimal normal in $\mathbb{A}_4$ but it is not minimal.

Exercise 1.18. Prove that every finite group contains a minimal normal subgroup.

Example 1.19. Let $G=\mathbb{D}_6=\langle r,s:r^6=s^2=1, srs=r^{-1}\rangle$ the dihedral group of twelve elements. The subgroups $S=\langle r^2\rangle$ and $T=\langle r^3\rangle$ are (the only) minimal normal in G.

Example 1.20. Let $G=\mathbf{SL}_2(3)$. The only minimal normal subgroup of G is its center $Z(\mathbf{SL}_2(3))\simeq C_2$.

The following lemma will be very useful later.

Lemma 1.21. *Let M be a minimal normal subgroup of G. If M is solvable and finite, then M is an elementary abelian p-group for some prime number p.*

Proof. Since M is solvable, $[M,M]\subsetneq M$. Moreover, $[M,M]$ is normal in G, as $[M,M]$ is characterisitic in M and M is normal in G. Since M is minimal normal, $[M,M]=\{1\}$. Hence M is abelian.

Since M is finite, there is a prime number p such that $\{1\}\neq P=\{x\in M:x^p=1\}\subseteq M$. Since P is characteristic in M, P is normal in G. By minimality, $P=M$. □

Theorem 1.22. *Let G be a finite non-trivial solvable group. Then every maximal subgroup of G has index p^α for some prime number p.*

Proof. We proceed by induction on $|G|$. If $|G|$ is a prime power, the claim is clear. Assume that $|G|\geq 6$ and let M be a maximal subgroup of G. Let N be a minimal normal subgroup of G and $\pi\colon G\to G/N$ the canonical map. If $N=G$, then $N=G$ is a p-group and we are done. Assume then that $N\neq G$. Since $M\subseteq NM\subseteq G$, either $M=NM$ or $NM=G$ (by the maximality of M). If $M=NM\supseteq N$, then $\pi(M)$ is a maximal subgroup of $\pi(G)=G/N$. Hence

$$(G:M)=(\pi(G):\pi(M))$$

is a prime power by the inductive hypothesis. If $NM = G$, then

$$(G:M) = \frac{|G|}{|M|} = \frac{|NM|}{|N|} = \frac{|N|}{|N \cap M|}$$

is a prime power, because N is a p-group by the previous lemma. $\square$

EXERCISE 1.23. Let G be a finite non-trivial solvable group. Prove that there exists a prime number p such that G contains a minimal normal p-subgroup.

EXAMPLE 1.24. Let $G = \mathbb{S}_4$. The 2-subgroup

$$K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2$$

is minimal normal. Note that G does not have minimal normal 3-subgroups.

THEOREM 1.25. *Let G be a finite non-trivial group. Then G is solvable if and only if every non-trivial quotient of G contains an abelian non-trivial normal subgroup.*

PROOF. Every quotient of G is solvable and therefore contains an abelian minimal normal subgroup. To prove the converse we proceed by induction on $|G|$. Let N be a normal abelian subgroup of G. If $N = G$, then G is solvable (because it is abelian). If $N \neq G$, then $|G/N| < |G|$. Since every quotient of G/N is a quotient of G, the group G/N satisfies the assumptions of the theorem. Hence G/N is solvable by the inductive hypothesis. Now N and G/N are solvable, so is G. $\square$

EXERCISE 1.26. Let G be a group. Prove that G is solvable if and only if there is a sequence

$$\{1\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = G$$

of normal subgroups such that every quotient N_i/N_{i-1} is abelian.

§ 1.2. Hall's theorem. We start with an extremely simple and useful tool.

LEMMA 1.27 (Frattini's argument). *Let G be a finite group and K be a normal subgroup of G. If $P \in \mathrm{Syl}_p(K)$ for some prime number p, then $G = KN_G(P)$.*

PROOF. Let $g \in G$. Since $gPg^{-1} \subseteq gKg^{-1} = K$, because K is normal in G, and $gPg^{-1} \in \mathrm{Syl}_p(K)$, there exists $k \in K$ such that $kPk^{-1} = gPg^{-1}$. Hence $k^{-1}g \in N_G(P)$, as $P = (k^{-1}g)P(k^{-1}g)^{-1}$. Therefore $g = k(k^{-1}g) \in KN_G(P)$. $\square$

THEOREM 1.28 (Hall). *Let G be a finite group such that every maximal subgroup of G has a prime or a prime-square index. Then G is solvable.*

PROOF. We proceed by induction on $|G|$. Let p be the largest prime divisor of $|G|$. Let $S \in \mathrm{Syl}_p(G)$ and $N = N_G(S)$.

If $N = G$, then S is normal in G. Since every maximal subgroup of G/S has prime or a prime-square index, G/S is solvable by the inductive hypothesis. Since S is a p-group, it is solvable. Therefore G is solvable.

Assume now that $N \neq G$. Let H be a maximal subgroup of G containing N. Then

$$N = N_H(S) = N_G(S),$$

the number of Sylow p-subgroups of G is $(G:N) \equiv 1 \bmod p$ and the number of Sylow p-subgroups of H is $(G:H) \equiv 1 \bmod p$ by the third Sylow's theorem. By assumption, there exists a prime number

q such that $(G:H) \in \{q, q^2\}$. Since q divides $|G|$, it follows that $p < q$. If $(G:H) = q$, then p divides $q-1$ and therefore $p \leq q-1 < p$, a contradiction. Thus $q^2 = (G:H) \equiv 1 \bmod p$. From this it follows that $q \equiv -1 \bmod p$ and hence $q = 2$ and $p = q+1 = 3$.

Therefore G has order $2^\alpha 3^\beta$. If we apply Burnside's theorem, we are done. Instead, we will finish the proof with an elementary argument. Let K be a minimal normal subgroup of G. By Frattini's argument (Lemma 1.27), $G = KN = KH$. Since H is maximal,

$$(K : K \cap H)(G:H) = 4,$$

as $(G:H) = |G|/|H| = |KH|/|H| = |K|/|K \cap H|$. Since $(K : K \cap H) = 4$, letting K act on $K/K \cap H$ by left multiplication, there exists a non-trivial group homomorphism $\rho \colon K \to \mathbb{S}_4$. Since $[K,K]$ is characteristic in K and K is normal in G, $[K,K] \subseteq K$ is normal in G. Since K is minimal normal in G, there are two possible cases: either $[K,K] = \{1\}$ or $[K,K] = K$. If $[K,K] = K$, since $\mathbb{S}_4$ is solvable, $\rho(K)$ is solvable. Then

$$\rho(K) = \rho([K,K]) = [\rho(K), \rho(K)],$$

a contradiction. Therefore $[K,K] = \{1\}$ and K is solvable (as it is abelian). □

Lecture 2. 22/02/2024

§ 2.1. Wielandt's theorem.

LEMMA 2.1. *Let G be a finite group and H and K be subgroups of G of coprime indices. Then $G = HK$ and $(H : H\cap K) = (G : K)$.*

PROOF. Let $D = H\cap K$. Since
$$(G:D) = \frac{|G|}{|H\cap K|} = (G:H)(H:H\cap K),$$
$(G:H)$ divides $(G:D)$. Similarly, $(G:K)$ divides $(G:D)$. Since $(G:H)$ and $(G:K)$ are coprime, $(G:H)(G:K)$ divides $(G:D)$. In particular,
$$\frac{|G|}{|H|}\frac{|G|}{|K|} = (G:H)(G:K) \leq (G:D) = \frac{|G|}{|H\cap K|}$$
and hence $|G| = |HK|$. Since
$$|G| = |HK| = |H||K|/|H\cap K|,$$
we conclude that $(G:K) = (H:H\cap K)$. □

DEFINITION 2.2. Let G be a group and H be a subgroup of G. The **normal closure** H^G of H in G is the subgroup $H^G = \langle xHx^{-1} : x\in G\rangle$.

EXERCISE 2.3. Let G be a group and H a subgroup of G. Prove that H^G is normal in G and that H^G is the smallest normal subgroup of G containing H.

EXAMPLE 2.4. Let $G = \mathbb{A}_4$ and $H = \{\mathrm{id}, (12)(34)\}$. The normal closure of H in G is
$$H^G = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2\times C_2.$$

THEOREM 2.5 (Wielandt). *Let G be a finite group and H, K and L be subgroups of G with pairwise coprime indices. If H, K and L are solvable, then G is solvable.*

PROOF. Let G be a minimal counterexample. Then $G\neq\{1\}$. Let N be a minimal normal subgroup of G and $\pi\colon G\to G/N$ be the canonical map. Since H, K and L are solvable, the subgroups $\pi(H) = \pi(HN)$, $\pi(K) = \pi(KN)$ and $\pi(L) = \pi(LN)$ of $\pi(G) = G/N$ are solvable. By the correspondence theorem, $\pi(H)$, $\pi(K)$ and $\pi(L)$ have pairwise coprime indices. By the minimality of G, $\pi(G)$ is solvable. If $H = \{1\}$, then $|G| = (G:H)$ is coprime with $(G:K)$ and thus $G = K$ is solvable. If $H\neq\{1\}$, let M be a minimal normal subgroup of H. By Lemma 1.21, M is a p-group for some prime number p. Without loss of generality, we may assume that p does not divide $(G:K)$ (otherwise, if p divides $(G:K)$, then p does not divide $(G:L)$ and we just need to change K by L). There exists $P\in\mathrm{Syl}_p(G)$ such that $P\subseteq K$. Sylow subgroups are conjugate, so there exists $g\in G$ such that $M\subseteq gKg^{-1}$. Since $(G:gKg^{-1}) = (G:K)$ is coprime with $(G:H)$, Lemma 2.1 implies that $G = (gKg^{-1})H$.

We claim that all conjugates of M are in gKg^{-1}. If $x\in G$, write $x = uv$ some some $u\in gKg^{-1}$ and $v\in H$. Since M is normal in H,
$$xMx^{-1} = (uv)M(uv)^{-1} = uMu^{-1}\subseteq gKg^{-1}.$$
In particular, $\{1\}\neq M^G\subseteq gKg^{-1}$ is solvable, as gKg^{-1} is solvable. By the minimality of M, the group G/M^G is solvable. Hence G is solvable. □

§ 2.2. Hall's theorem.

Definition 2.6. Let G be a finite group of order $p^{\alpha}m$, where p is a prime number such that $\gcd(p,m)=1$. A subgroup H of G is said to be a p**-complement** if $|H|=m$.

Example 2.7. Let $G=\mathbb{S}_3$. Then $H=\langle(123)\rangle$ is a 2-complement and $K=\langle(12)\rangle$ is a 3-complement.

Theorem 2.8 (Hall). *Let G be a finite group that admits a p-complement for every prime divisor p of $|G|$. Then G is solvable.*

Proof. Let $|G|=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ with $p_1<\cdots<p_k$ prime numbers. We proceed by induction on k. If $k=1$, then the claim holds, as G is a p-group. If $k=2$, the result holds by Burnside's theorem. Assume now that $k\geq 3$. For $j\in\{1,2,3\}$, let H_j be a p_j-complement in G. Since $|H_j|=|G|/p_j^{\alpha_j}$, the subgroups H_j have pairwise coprime indices.

We claim that H_1 is solvable. Note that $|H_1|=p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Let p be a prime number dividing $|H_1|$ and Q be a p-complement in G. Since $(G:H_1)$ and $(G:Q)$ are coprime, Lemma 2.1 implies that

$$(H_1:H_1\cap Q)=(G:Q).$$

Then $H_1\cap Q$ is a p-complement in H_1. Therefore H_1 is solvable by the inductive hypothesis. Similarly, both H_2 and H_3 are solvable.

Since H_1, H_2 and H_3 are solvable of pairwise coprime indices, the theorem follows from Wieland's theorem. $\square$

§ 2.3. Nilpotent groups. For a group G and $x,y,z\in G$, conjugation will be considered as a left action of G on G and we will use the following notation: ${}^{x}y=xyx^{-1}$. The commutator between x and y will be written as

$$[x,y]=xyx^{-1}y^{-1}=({}^{x}y)y^{-1}.$$

We will also use the following notation: $[x,y,z]=[x,[y,z]]$. For subgroups X, Y and Z of G, we write $[X,Y,Z]=[X,[Y,Z]]$. Note that $[X,Y]=[Y,X]$.

Exercise 2.9 (The Hall–Witt identity). Let G be a group and $x,y,z\in G$. Prove that

$$\left({}^{y}[x,y^{-1},z]\right)\left({}^{z}[y,z^{-1},x]\right)\left({}^{x}[z,x^{-1},y]\right)=1. \tag{2.1}$$

If G is a group and $[G,G]$ is central in G, then the Hall-Witt becomes Jacobi's identity.

Lemma 2.10 (Hall's three subgroups lemma). *Let X, Y and Z be subgroups of G such that $[X,Y,Z]=[Y,Z,X]=\{1\}$. Then $[Z,X,Y]=\{1\}$.*

Proof. Since $[x,y]\in C_G(z)$ implies $[X,Y]\subseteq C_G(Z)$, it is enough to prove that $[z,x^{-1},y]=1$ for all $x\in X$, $y\in Y$ and $z\in Z$. Since $[y^{-1},z]\in[Y,Z]$, $[x,y^{-1},z]\in[X,Y,Z]=\{1\}$. Thus ${}^{y}[x,y^{-1},z]=1$. Similarly, ${}^{z}[y,z^{-1},x]=1$. Using the Hall–Witt identity, we conclude that $[z,x^{-1},y]=1$. $\square$

Exercise 2.11. Let N be a normal subgroup of G and X, Y and Z be subgroups of G. If $[X,Y,Z]\subseteq N$ and $[Y,Z,X]\subseteq N$, then $[Z,X,Y]\subseteq N$.

Definition 2.12. Let G be a group. The **lower central series** is the sequence $\gamma_k(G)$ of subgroups defined inductively as

$$\gamma_1(G)=G,\quad \gamma_{i+1}(G)=[G,\gamma_i(G)]\quad i\geq 1.$$

DEFINITION 2.13. A group G is said to be **nilpotent** if there exists a positive integer c such that $\gamma_{c+1}(G) = \{1\}$. The smallest c with $\gamma_{c+1}(G) = \{1\}$ is the **nilpotency class** of G.

EXERCISE 2.14. Prove that every nilpotent group is solvable.

A group is nilpotent of nilpotency class one if and only if it is abelian.

EXAMPLE 2.15. The group $\mathbb{S}_3$ is solvable, as $\mathbb{S}_3 \supseteq \mathbb{A}_3 \supseteq \{1\}$ is a composition series with abelian factors. However, $\mathbb{S}_3$ is not nilpotent, as

$$\gamma_1(\mathbb{S}_3) = \mathbb{A}_3, \quad \gamma_2(\mathbb{S}_3) = [\mathbb{A}_3, \mathbb{S}_3] = \mathbb{A}_3,$$

and therefore $\gamma_i(\mathbb{S}_3) \neq \{1\}$ for all $i \geq 1$.

EXAMPLE 2.16. The group $G = \mathbb{A}_4$ is not nilpotent, as

$$\gamma_1(G) = G, \quad \gamma_j(G) = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2$$

for all $j \geq 2$. We can do this with the computer:

```
gap> IsNilpotent(AlternatingGroup(4));
false
```

Let us do the calculation of the lower central series with the computer:

```
gap> List(LowerCentralSeries(AlternatingGroup(4)),\
StructureDescription);
[ "A4", "C2 x C2" ]
```

Here is an alternative:

```
gap> G := AlternatingGroup(4);;
gap> gamma_1 := G;;
gap> gamma_2 := DerivedSubgroup(G);;
gap> gamma_3 := CommutatorSubgroup(gamma_2,G);;
gap> StructureDescription(gamma_1);
"A4"
gap> StructureDescription(gamma_2);
"C2 x C2"
gap> StructureDescription(gamma_3);
"C2 x C2"
```

EXERCISE 2.17. Let G be a group. Prove the following statements:

1) Each $\gamma_i(G)$ is a characteristic subgroup of G.
2) $\gamma_i(G) \supseteq \gamma_{i+1}(G)$ for all $i \geq 1$.
3) If $f\colon G \to H$ is a surjective group homomorphism, then $f(\gamma_i(G)) = \gamma_i(H)$ for all $i \geq 1$.

EXERCISE 2.18. Prove that if H and K are nilpotent groups, then $H \times K$ is nilpotent.

EXERCISE 2.19. Let G be a nilpotent group. Prove the following statements:

1) Subgroups of G are nilpotent.
2) If $f\colon G \to H$ is a surjective homomorphism, then H is nilpotent.

EXERCISE 2.20. True or false? If G is a nilpotent group and N is normal subgroup of G such that N and G/N are nilpotent, then G is nilpotent.

PROPOSITION 2.21. *Finite p-groups are nilpotent.*

PROOF. We proceed by induction on $|G|$. The case $G = \{1\}$ is trivial. Assume the result holds for p-groups of order $< |G|$. Since G is a p-group, $Z(G) \neq \{1\}$. By the inductive hypothesis, $G/Z(G)$ is nilpotent. There exists c such that $\gamma_{c+1}(G/Z(G)) = \{1\}$.

Let $\pi\colon G \to G/Z(G)$ be the canonical map. By Exercise 2.17,

$$\pi(\gamma_{c+1}(G)) = \gamma_{c+1}(G/Z(G)) = \{1\}.$$

Then $\gamma_{c+1}(G) \subseteq \ker\pi = Z(G)$. Hence G is nilpotent, as

$$\gamma_{c+2}(G) = [G, \gamma_{c+1}(G)] = [G, Z(G)] = \{1\}.$$

□

THEOREM 2.22. *If G is a group, then $[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G)$ for all $i, j \geq 1$.*

PROOF. We proceed by induction on i. The case $i = 1$ is trivial, as $[G, \gamma_j(G)] = \gamma_{j+1}(G)$ by definition. Assume that the result holds for some $i \geq 1$ and all $j \geq 1$.

First note that

$$[G, \gamma_i(G), \gamma_j(G)] \subseteq [G, \gamma_{i+j}(G)] = \gamma_{i+j+1}(G)$$

by the inductive hypothesis. Moreover, by the inductive hypothesis,

$$[\gamma_i(G), \gamma_j(G), G] = [\gamma_i(G), G, \gamma_j(G)] = [\gamma_i(G), \gamma_{j+1}(G)] \subseteq \gamma_{i+j+1}(G)$$

By using Exercise 2.11 with $X = G$, $Y = \gamma_i(G)$ and $Z = \gamma_j(G)$, we get that

$$[\gamma_j(G), G, \gamma_i(G)] \subseteq \gamma_{i+j+1}(G).$$

Hence

$$[\gamma_{i+1}(G), \gamma_j(G)] = [\gamma_j(G), \gamma_{i+1}(G)] = [\gamma_j(G), G, \gamma_i(G)] \subseteq \gamma_{i+j+1}(G).$$

□

We consider arbitrary commutators, not necessarily associated on the right. For example, both $[G, G, G] = [G, [G, G]]$ and $[[G, G], G]$ are commutators of **weight** three.

COROLLARY 2.23. *In a group G, every weight n commutator is contained in $\gamma_n(G)$.*

PROOF. We proceed by induction on n. The case $n = 1$ is trivial. Assume that $n \geq 1$ and the result holds for all $k \leq n$. An arbitrary commutator of weight $n+1$ is of the form $[A, B]$, where A is a commutator of weight k, B is a commutator of weight l and $n+1 = k+l$. Since $k < n$ and $l < n$, the inductive hypothesis implies that $A \subseteq \gamma_k(G)$ and $B \subseteq \gamma_l(G)$. Hence $[A, B] \subseteq [\gamma_k(G), \gamma_l(G)] \subseteq \gamma_{k+l}(G)$ by the previous theorem. □

EXERCISE 2.24. Let G be a group. Prove that $G^{(k)} \subseteq G^{2^k}$ for all $k \geq 1$.

EXERCISE 2.25. Let G be a nilpotent group of class m. Prove that the derived length of G is $\leq 1 + \log_2 m$.

The following lemma is important. It states that nilpotent groups satisfy the **normalizer condition**.

LEMMA 2.26 (normalizer condition). *Let G be a nilpotent group. If H is a proper subgroup of G, then $H \subsetneq N_G(H)$.*

PROOF. There exists c such that $G = \gamma_1(G) \supseteq \cdots \supseteq \gamma_{c+1}(G) = \{1\}$. Since $\{1\} = \gamma_{c+1}(G) \subseteq H$ and $\gamma_1(G) \not\subseteq H$, let k be the smallest positive integer such that $\gamma_k(G) \subseteq H$. Since

$$[H, \gamma_{k-1}(G)] \subseteq [G, \gamma_{k-1}(G)] = \gamma_k(G) \subseteq H,$$

we obtain that $xHx^{-1} \subseteq H$ for all $x \in \gamma_{k-1}(G)$, that is $\gamma_{k-1}(G) \subseteq N_G(H)$. If $N_G(H) = H$, then $\gamma_{k-1}(G) \subseteq H$, a contradiction to the minimality of k. □

For a group G, we define the sequence $\zeta_0(G), \zeta_1(G), \dots$ recursively as follows:

$$\zeta_0(G) = \{1\}, \quad \zeta_{i+1}(G) = \{g \in G : [x,g] \in \zeta_i(G) \text{ for all } x \in G\}, \quad i \geq 0.$$

For example, $\zeta_1(G) = Z(G)$.

LEMMA 2.27. *Let G be a group. For every $i \geq 0$, the set $\zeta_i(G)$ is a normal subgroup of G.*

PROOF. We proceed by induction on i. The case $i = 0$ is trivial, as $\zeta_0(G) = \{1\}$. Assume the result holds for some i. We claim that $\zeta_{i+1}(G)$ is normal subgroup of G. Let $g, h \in \zeta_{i+1}(G)$ and $x \in G$. By the inductive hypothesis,

$$[x, g^{-1}] = (xg^{-1})[x^{-1}, g](xg^{-1})^{-1} \zeta_i(G)(xg^{-1})^{-1} = \zeta_i(G),$$
$$[x, gh] = [x,h][hxh^{-1}, g] \in \zeta_i(G).$$

Since $1 \in \zeta_{i+1}(G)$, we conclude that each $\zeta_i(G)$ is a subgroup of G. Moreover, $xgx^{-1} \in \zeta_{i+1}(G)$, as

$$[xgx^{-1}, y] = x[g, x^{-1}yx]x^{-1} \in \zeta_i(G)$$

para todo $y \in G$. □

DEFINITION 2.28. Let G be a group. The **ascending central series** of G is the sequence

$$\{1\} = \zeta_0(G) \subseteq \zeta_1(G) \subseteq \zeta_2(G) \subseteq \cdots$$

DEFINITION 2.29. A group G is said to be **perfect** if $[G,G] = G$.

THEOREM 2.30 (Grün). *If G is a perfect group, then $Z(G/Z(G)) = \{1\}$.*

PROOF. By definition, $[G, \zeta_2(G)] \subseteq Z(G)$ and $[\zeta_2(G), G] \subseteq Z(G)$. Then

$$[G, G, \zeta_2(G)] = [G, \zeta_2(G), G] = \{1\}.$$

By using the three subgroups lemma with $X = Y = G$ and $Z = \zeta_2(G)$,

$$[\zeta_2(G), G] = [\zeta_2(G), [G,G]] = [\zeta_2(G), G, G] = \{1\}.$$

Thus $\zeta_2(G) \subseteq Z(G)$.

We aim to prove that $Z(G/Z(G))$ is trivial. Let $\pi \colon G \to G/Z(G)$ be the canonical map and $g \in G$ be such that $\pi(g)$ is central. Since

$$1 = [\pi(x), \pi(g)] = \pi([x,g])$$

for all $x \in G$, $[x,g] \in Z(G) = \zeta_1(G)$ for all $x \in G$. Hence $g \in \zeta_2(G) \subseteq Z(G)$. □

For subgroups H and K of G, let

$$[H,K] = \langle [h,k] : h \in H, k \in K \rangle.$$

Let G be a group and K be a subgroup of G. We say that K **normalizes** H if $K \subseteq N_G(H)$. We say that K **centralizes** H if $K \subseteq C_G(H)$, that is if and only if $[H,K] = \{1\}$.

EXERCISE 2.31. Let K and H be subgroups of G such that $K \subseteq H$ and K is normal in G. Prove that $[H,G] \subseteq K$ if and only if $H/K \subseteq Z(G/K)$.

LEMMA 2.32. *Let G be a group. There exists an integer c such that $\zeta_c(G) = G$ if and only if $\gamma_{c+1}(G) = \{1\}$. In this case,*

$$\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$$

for all $i \in \{0,1,\dots,c\}$.

PROOF. Assume first that $\zeta_c(G) = G$. To prove that $\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$ holds for all i, we proceed by induction. The case $i = 0$ is trivial. So assume that the result holds for some $i \geq 0$. If $g \in \gamma_{i+2}(G) = [G, \gamma_{i+1}(G)]$, then

$$g = \prod_{k=1}^{N} [x_k, g_k],$$

for some $g_1, \dots, g_N \in \gamma_{i+1}(G)$ and $x_1, \dots, x_N \in G$. By the inductive hypothesis,

$$g_k \in \gamma_i(G) \subseteq \zeta_{c-i}(G)$$

for all k. Hence $[x_k, g_k] \in \zeta_{c-i-1}(G)$ for all k. Therefore $g \in \zeta_{c-(i+1)}(G)$.

We now assume that $\gamma_{c+1}(G) = \{1\}$. We aim to prove that $\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$ holds for all i. We proceed by backwards induction on i. The case $i = c$ is trivial. So assume the result holds for some $i+1 \leq c$. Let $g \in \gamma_i(G)$. By the inductive hypothesis,

$$[x,g] \in [G, \gamma_i(G)] = \gamma_{i+1}(G) \subseteq \zeta_{c-i}(G).$$

Thus $g \in \zeta_{c-i+1}(G)$ by definition. $\square$

EXAMPLE 2.33. Let $G = \mathbb{S}_3$. Then $\zeta_j(G) = \{1\}$ for all $j \geq 0$:

```
gap> UpperCentralSeries(SymmetricGroup(3));
[ Group(()) ]
```

DEFINITION 2.34. Let G be a group. A **central series** for G is a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups of G such that for each $i \in \{1,\dots,n\}$, $\pi_i(G_{i-1})$ is a subgroup of $Z(G/G_i)$, where $\pi_i \colon G \to G/G_i$ is the canonical map.

LEMMA 2.35. *Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ be a central series of a group G. Then $\gamma_{i+1}(G) \subseteq G_i$ for all i.*

PROOF. We proceed by induction on i. The case $i = 0$ is trivial. So assume the result holds for some $i \geq 0$. Let $\pi_i \colon G \to G/G_i$ be the canonical map. Then

$$\gamma_{i+1}(G) = [G, \gamma_i(G)] \subseteq [G, G_{i-1}].$$

Since $\pi_i(G_{i-1}) \subseteq Z(G/G_i)$,

$$\pi_i([G, G_{i-1}]) = [\pi_i(G), \pi_i(G_{i-1})] = \{1\}.$$

Hence $\gamma_{i+1}(G) = [G, G_{i-1}] \subseteq G_i$. □

Theorem 2.36. *A group is nilpotent if and only if it admits a central series.*

Proof. Let G be a group. If G is nilpotent, then the $\gamma_j(G)$ form a central series of G. Conversely, if $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ is a central series of G, then, by the previous lemma,

$$\gamma_{n+1}(G) \subseteq G_n = \{1\}.$$

Hence G is nilpotent. □

Exercise 2.37. Let G be a group. Prove that if K is a subgroup of $Z(G)$ such that G/K is nilpotent, then G is nilpotent.

§ 2.4. Hirsch's theorem.

Theorem 2.38 (Hirsch). *Let G be a nilpotent group. If H is a non-trivial normal subgroup of G, then $H \cap Z(G) \neq \{1\}$. In particular, $Z(G) \neq \{1\}$.*

Proof. Since $\zeta_0(G) = \{1\}$ and there exists an integer c such that $\zeta_c(G) = G$, there exists

$$m = \min\{k : H \cap \zeta_k(G) \neq \{1\}\}.$$

Since H is normal in G,

$$[G, H \cap \zeta_m(G)] \subseteq H \cap [G, \zeta_m(G)] \subseteq H \cap \zeta_{m-1}(G) = \{1\}.$$

Therefore $\{1\} \neq H \cap \zeta_m(G) \subseteq H \cap Z(G)$. If $H = G$, then $Z(G) \neq \{1\}$. □

Exercise 2.39. Let G be a nilpotent group and M be a minimal normal subgroup of G. Prove that $M \subseteq Z(G)$.

Definition 2.40. Let G be a group. A subgroup M is said to be **maximal normal** in G if $M \neq G$ and M is the only proper normal subgroup of G containing M.

Corollary 2.41. *Let G be a non-abelian nilpotent group and A be a maximal normal and abelian subgroup un subgrup of G. Then $A = C_G(A)$.*

Proof. Since A is abelian, $A \subseteq C_G(A)$. Assume that $A \neq C_G(A)$. The centralizer $C_G(A)$ is normal in G, as, since A is normal in G,

$$gC_G(A)g^{-1} = C_G(gAg^{-1}) = C_G(A).$$

for all $g \in G$. Let $\pi \colon G \to G/A$ be the canonical map. Then $\pi(C_G(A))$ is a non-trivial normal subgroup of $\pi(G)$. Since G is nilpotent, $\pi(G)$ is nilpotent. By Hirsch's theorem,

$$\pi(C_G(A)) \cap Z(\pi(G)) \neq \{1\}.$$

Let $x \in C_G(A) \setminus A$ be such that $\pi(x)$ is central in $\pi(G)$. Then $\langle A, x\rangle$ is abelian, as $x \in C_G(A)$. Moreover, $\langle A, x\rangle$ is normal in G, as A is normal in G and $gxg^{-1}x^{-1} \in A$ for all $g \in G$ (because $\pi(x)$ is central). Hence $gxg^{-1} \in \langle A, x\rangle$ and therefore $A \subsetneq \langle A, x\rangle \subsetneq G$, a contradiction. □

Theorem 2.42. *Let G be a nilpotent group. The following statements hold:*

1) *Every minimal normal subgroup of G has prime order and is central.*
2) *Every maximal subgroup of G is normal of prime index and contains $[G, G]$.*

Proof.

1) Let N be a minimal normal subgroup of G. Since $N\cap Z(G)\neq\{1\}$ by Hirsch's theorem, $N\cap Z(G)$ is a normal subgroup of G contained in N. Then $N=N\cap Z(G)\subseteq Z(G)$ by the minimality of N. In particular, N is abelian. Since every subgroup of N is normal in G, N is simple. Hence $N\simeq C_p$ for some prime number p.

2) If M is a maximal subgroup, then M is normal in G by the normalizer condition (Lemma 2.26). By the maximality of M, the quotient G/M contains no proper non-trivial subgroups. Thus $G/M\simeq C_p$ for some prime p. Since G/M is abeliano, $[G,G]\subseteq M$. □

The previous theorem does not prove the existence of maximal subgroups. For example, $\mathbb{Q}$ is a nilpotent group (as it is abelian) that contains no maximal subgroups.

Proposition 2.43. *Let G be a nilpotent group and H be a subgroup with $(G:H)=n$. If $g\in G$, then $g^n\in H$.*

Proof. We proceed by induction on n. The case $n=1$ is trivial. The case $n=2$ follows from the normality of H. So assume the result holds for all groups of index $<n$. Let H be a subgroup of G such that $(G:H)=n$. Let $H_0=H$ and $H_{i+1}=N_G(H_i)$ for all $i\geq 0$. By definition, H_i is normal in H_{i+1}. Since G is nilpotent, $H_i\neq G$ implies that $H_i\subsetneq H_{i+1}$ by the normalizer condition. Since $(G:H)$ is finite, there exists k such that $H_k=G$. Since $(H_j:H_{j-1})<n$ for all j, the inductive hypotehsis implies that $x^{(H_j:H_{j-1})}\in H_{j-1}$ for all $x\in H_j$ and all j. Hence

$$g^{(G:H)}=g^{(H_k:H_{k-1})(H_{k-1}:H_{k-2})\cdots(H_1:H_0)}\in H.$$ □

Exercise 2.44. Does the previous proposition hold for non-nilpotent groups?

The following lemma is useful for performing induction in the nilpotency index of nilpotent groups.

Lemma 2.45. *Let G be a nilpotent group of class $c\geq 2$. If $x\in G$, then the subgroup $\langle x,[G,G]\rangle$ is nilpotente of class $<c$.*

Proof. Let $H=\langle x,[G,G]\rangle$. If $x\in[G,G]$, the there is nothing to prove. So assume that $x\notin[G,G]$. Note that

$$H=\{x^nc:n\in\mathbb{Z},c\in[G,G]\},$$

as $[G,G]$ is normal in G. We need to show that $[H,H]\subseteq\gamma_3(G)$. Let $h=x^nc,k=x^md\in H$ be such that $c,d\in[G,G]$. Since

$$[h,x^m]=[x^n,[c,x^m]][c,x^m]\in\gamma_4(G)\gamma_3(G)\subseteq\gamma_3(G),$$

then

$$\begin{aligned}[h,k]&=[h,x^m][x^m,[h,d]][h,d]\\&=[x^n,[c,x^m]][c,x^m][x^m,[h,d]][h,d]\in\gamma_3(G).\end{aligned}$$ □

Example 2.46. Let $G=\mathbb{D}_8=\langle r,s:r^8=s^2=1,srs=r^{-1}\rangle$ the dihedral group of order 16. Then G is nilpotent of class three and $[G,G]=\{1,r^2,r^4,r^6\}\simeq C_4$. The subgroup $\langle s,[G,G]\rangle\simeq\mathbb{D}_4$ is nilpotent of class two.

```
gap> G := DihedralGroup(IsPermGroup,16);;
gap> gens := GeneratorsOfGroup(G);;
gap> r := gens[1];;
gap> s := gens[2];;
gap> D := DerivedSubgroup(G);;
```

```
gap> S := Subgroup(G, Concatenation(Elements(D), [s]));;
gap> StructureDescription(S);
"D8"
gap> NilpotencyClassOfGroup(G);
3
gap> NilpotencyClassOfGroup(S);
2
```

Let us discuss a concrete application of Lemma 2.45.

THEOREM 2.47. *If G is a nilpotent group, then*

$$T(G) = \{g \in G : g^n = 1 \textit{ for some } n \geq 1\}$$

is a subgroup of G.

PROOF. We proceed by induction on the nilpotency class of G. Let $a, b \in T(G)$ and

$$A = \langle a, [G,G] \rangle, \quad B = \langle b, [G,G] \rangle.$$

Since A and B are nilpotent of class $< c$ by the previous lemma, the inductive hypothesis implies that $T(A)$ is a subgroup of A and $T(B)$ is a subgroup of B. Since $T(A)$ is characteristic in A and A is normal in G, $T(A)$ is normal in G. Similarly, $T(B)$ is normal in G.

We claim that every element of $T(A)T(B)$ has finite order. If $x \in T(A)T(B)$, say $x = a_1 b_1$ with a_1 of order m, then x has finite order, as

$$x^m = (a_1 b_1)^m = (a_1 b_1 a_1^{-1})(a_1^2 b_1 a_1^{-2}) \cdots (a^{m-1} b_1 a^{-m+1}) b_1 \in T(B).$$

To see clearly what is what we did, let us work out a concrete example, say $m = 3$. In this case, we obtain the following formula:

$$\begin{aligned}(a_1 b_1)^3 &= (a_1 b_1)(a_1 b_1)(a_1 b_1) \\ &= (a_1 b_1 a_1^{-1})(a_1^2 b_1 a_1^{-2}) a_1^3 b_1 = (a_1 b_1 a_1^{-1})(a_1^2 b_1 a_1^{-2}) b_1,\end{aligned}$$

as $a_1^3 = 1$.

With this trick, we prove that ab and a^{-1} have finite order. Hence $T(G)$ is a subgroup of G. □

Another application:

THEOREM 2.48. *Let G be a torsion-free nilpotent group and $a, b \in G$. If there exists $n \neq 0$ such that $a^n = b^n$, then $a = b$.*

PROOF. We proceed by induction on the nilpotency order c of G. The result clearly holds for abelian groups. Assume that G is nilpotent of class $c \geq 2$. Since $\langle a, [G,G] \rangle$ is a nilpotent subgroup of G of class $< c$ and $bab^{-1} = [b,a]a \in \langle a, [G,G] \rangle$, the inductive hypotehsis implies that $ba = ab$, as

$$a^n = (bab^{-1})^n = b^n.$$

Thus $(ab^{-1})^n = a^n b^{-n} = 1$. Since G has no torsion, we conclude that $a = b$. □

COROLLARY 2.49. *Let G be a torsion-free nilpotent group. If $x, y \in G$ are such that $x^n y^m = y^m x^n$ for some $n, m \neq 0$, then $xy = yx$.*

PROOF. Let $a = x$ and $b = y^n x y^{-n}$. Since $a^m = b^m$, the previous theorem implies that $a = b$. Thus $xy^n = y^n x$. Apply the previous theorem again, this time with $a = y$ and $b = xyx^{-1}$. Then we conclude that $xy = yx$. □

Before proving another theorem, we recall a basic lemma about finitely generated groups.

LEMMA 2.50. *Let G be a finitely generated group and H a finite-index subgroup. Then H is finitely generated.*

PROOF. Assume that G is generated by $\{g_1,\dots,g_m\}$. Without loss of generality, we may assume that for each i there exists k such that $g_i^{-1}=g_k$.

Let $\{1=t_1,\dots,t_n\}$ be a transversal of H in G, that is a complete set of representatives of G/H. For $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$, write

$$t_ig_j=h(i,j)t_{k(i,j)}.$$

We claim that H is generated by the $h(i,j)$. Let $x\in H$. Then

$$\begin{aligned} x&=g_{i_1}\cdots g_{i_s}\\ &=(t_1g_{i_1})g_{i_2}\cdots g_{i_s}\\ &=h(1,i_1)t_{k_1}g_{i_2}\cdots g_{i_s}\\ &=h(1,i_1)h(k_1,i_2)t_{k_2}g_{i_3}\cdots g_{i_s}\\ &=h(1,i_1)h(k_1,i_2)\cdots h(k_{s-1},i_s)t_{k_s}, \end{aligned}$$

where $k_1,\dots,k_{s-1}\in\{1,\dots,n\}$. Since $t_{k_s}\in H$ (because $x\in H$), $t_{k_s}=1\in H$. Hence x is generated by the $h(i,j)$. $\square$

Now the theorem:

THEOREM 2.51. *Let G be a finitely generated torsion group that is nilpotent. Then G is finite*

PROOF. We proceed by induction on the nilpotency class c of G. The case $c=1$ is true, as G is abelian. So assume the result holds for groups of class $c\geq 1$. Since $[G,G]$ and $G/[G,G]$ are finitely generated (Lemma 2.50) torsion nilpotent groups of class $<c$, the inductive hypothesis implies that $[G,G]$ and $G/[G,G]$ are finite groups. Thus G is finite. $\square$

§ 2.5. Finite nilpotent groups. Before studying finite nilpotent groups, we need a lemma.

LEMMA 2.52. *Let G be a finite group and p a prime number dividing $|G|$. If $P\in\mathrm{Syl}_p(G)$, then*

$$N_G(N_G(P))=N_G(P).$$

PROOF. Let $H=N_G(P)$. Since P is normal in H, P is the only Sylow p-subgroup of H. To prove that $N_G(H)=H$, it is enough to see that $N_G(H)\subseteq H$. Let $g\in N_G(H)$. Since

$$gPg^{-1}\subseteq gHg^{-1}=H,$$

$gPg^{-1}\in\mathrm{Syl}_p(H)$ and H has only one Sylow p-subgroup, $P=gPg^{-1}$. Hence $g\in N_G(P)=H$. $\square$

THEOREM 2.53. *Let G be a finite group. The following statements are equivalent:*

1) *G is nilpotent.*
2) *Every Sylow subgroup of G is normal in G.*
3) *G is a direct product of its Sylow subgroups.*

PROOF. We first prove that (1) $\implies$ (2). Let $P\in\mathrm{Syl}_p(G)$. We aim to prove that P is normal in G, that is $N_G(P)=G$. By Lemma 2.52, $N_G(N_G(P))=N_G(P)$. Now the normalizer condition (Lemma 2.26) implies that $N_G(P)=G$.

We now prove that (2) $\implies$ (3). Let $p_1,\cdots,p_k$ be the prime factors of $|G|$. For each $i\in\{1,\dots,k\}$, let $P_i\in\mathrm{Syl}_{p_i}(G)$. By assumption, each P_j is normal in G.

We claim that $P_1 \cdots P_j \simeq P_1 \times \cdots \times P_j$ for all j. The case $j = 1$ is trivial. So assume the result holds for some $j \geq 1$. Since

$$N = P_1 \cdots P_j \simeq P_1 \times \cdots \times P_j$$

is normal in G and it has order coprime with $|P_{j+1}|$,

$$N \cap P_{j+1} = \{1\}.$$

Hence

$$NP_{j+1} \simeq N \times P_{j+1} \simeq P_1 \times \cdots \times P_j \times P_{j+1},$$

as P_{j+1} is normal in G. Since now $P_1 \cdots P_k \simeq P_1 \times \cdots \times P_k$ is a subgroup of G of order $|G|$, we conclude that $G = P_1 \times \cdots \times P_k$.

Finally, we prove that (3) $\implies$ (1). We just need to note that every p-group is nilpotent (Proposition 2.21) and that the direct product of nilpotent groups is nilpotent. □

Exercise 2.54. Let G be a finite group. Prove that if $P \in \mathrm{Syl}_p(G)$ and M is a subgroup of G such that $N_G(P) \subseteq M$, then $M = N_G(M)$.

Exercise 2.55. Let G be a finite group. Prove that the following statements are equivalent:

1) G is nilpotent.
2) If $H \subsetneq G$ is a subgroup of G, then $H \subsetneq N_G(H)$.
3) Every maximal subgroup of G is normal in G.

Theorem 2.56. *Let G be a finite nilpotent group. If p is a prime number dividing $|G|$, there exist a minimal normal subgroup of order p and there exists a maximal subgroup of index p.*

Proof. Assume that $|G| = p^\alpha m$ with $\gcd(p,m) = 1$. Write $G = P \times H$, where $P \in \mathrm{Syl}_p(G)$. Since $Z(P)$ is a non-trivial normal subgroup of P, every subgroup of $Z(P)$ that is minimal normal in G has order p (and such subgroups exist because G is finite). Since P contains a subgroup of index p, it is maximal. Hence $P \times H$ contains a maximal subgroup of index p. □

Exercise 2.57. Let p be a prime number and G be a non-trivial group of order p^n. Prove the following statements:

1) G has a normal subgroup of order p.
2) For every $j \in \{0, \dots, n\}$ there exists a normal subgroup of G of order p^j.

Exercise 2.58. Let G be a finite group. Prove that the following statements are equivalent:

1) G is nilpotent.
2) Any two elements of coprime order commute.
3) Every non-trivial quotient of G has a non-trivial center.
4) If d divides $|G|$, then there exists a normal subgroup of G of order d.

§ 2.6. Baumslag–Wiegold theorem. The following result can be proved with elementary tools and was discovered in 2014.

Theorem 2.59 (Baumslag–Wiegold). *Let G be a finite group such that $|xy| = |x||y|$ for all $x, y \in G$ of coprime orders. Then G is nilpotent.*

Proof. Let $p_1,\dots,p_n$ be the prime factors of $|G|$. For each $i\in\{1,\dots,n\}$, let $P_i\in \mathrm{Syl}_{p_i}(G)$. We first prove that $G=P_1\cdots P_n$. To prove the non-trivial inclusion, we need to show that the map

$$\psi\colon P_1\times\cdots\times P_n\to G,\quad (x_1,\dots,x_n)\mapsto x_1\cdots x_n$$

is surjective. We first show that ψ is injective: If $\psi(x_1,\dots,x_n)=\psi(y_1,\dots,y_n)$, then

$$x_1\cdots x_n=y_1\cdots y_n.$$

If $y_n\neq x_n$, then $x_1\cdots x_{n-1}=(y_1\cdots y_{n-1})y_nx_n^{-1}$. Since $x_1\cdots x_{n-1}$ has order coprime with p_n and $y_1\cdots y_{n-1}y_nx_n^{-1}$ has order a multiple of p_n, we get a contradiction. Thus $x_n=y_n$. The same argument shows that ψ is injective. Since $|P_1\times\cdots\times P_n|=|G|$, we conclude that ψ is bijective. In particular, ψ is surjective.

We now prove that each P_j is normal in G. Let $j\in\{1,\dots,n\}$ and $x_j\in P_j$. Let $g\in G$ and $y_j=gx_jg^{-1}$. Since $y_j\in G$, we can write $y_j=z_1\cdots z_n$ with $z_k\in P_k$ for all k. Since the order of y_j is a power of p_j, the element $z_1\cdots z_n$ has order a power of p_j. Thus $z_k=1$ for all $k\neq j$. Moreover, $y_j=z_j\in P_j$. Since every Sylow subgroup of G is normal in G, we conclude that G is nilpotent. $\square$

§ 2.7. Itô's theorem.

Definition 2.60. A group G is said to be **metabelian** if $[G,G]$ is abelian.

Exercise 2.61. Prove that a group G is metabelian if and only if there exists a normal subgroup K of G such that K and G/K are abelian.

Exercise 2.62. Let G be a metabelian group. Prove the following statements:

1) If H is a subgroup of G, then H is metabelian.
2) If $f\colon G\to H$ is a group homomorphism, then $f(H)$ is metabelian.

Lemma 2.63. *In a group, the following formulas hold:*

1) $[a,bc]=[a,b]b[a,c]b^{-1}$.
2) $[ab,c]=a[b,c]a^{-1}[a,c]$.

Proof. This is a straightforward calculation:

$$\begin{aligned}[a,b]b[a,c]b^{-1}&=aba^{-1}b^{-1}baca^{-1}c^{-1}b^{-1}=abca^{-1}c^{-1}b^{-1}=[a,bc],\\ a[b,c]a^{-1}[a,c]&=abcb^{-1}c^{-1}a^{-1}aca^{-1}c^{-1}=abcb^{-1}a^{-1}c^{-1}=[ab,c].\end{aligned}$$ $\square$

Example 2.64. The group $\mathbb{S}_3$ is metabelian, as $\mathbb{A}_3\simeq C_3$ is a normal subgroup and the quotient $\mathbb{S}_3/\mathbb{A}_3\simeq C_2$ an abelian group.

Example 2.65. The group $\mathbb{A}_4$ is metabelian, as the normal subgroup

$$K=\{\mathrm{id},(12)(34),(13)(24),(14)(23)\}$$

is abelian and the quotient $\mathbb{A}_4/K\simeq C_3$ is abelian.

Example 2.66. The group $\mathbf{SL}_2(3)$ is not metabelian, as $[\mathbf{SL}_2(3),\mathbf{SL}_2(3)]\simeq Q_8$ is not abelian:

```
gap> IsAbelian(DerivedSubgroup(SL(2,3)));
false
gap> StructureDescription(DerivedSubgroup(SL(2,3)));
"Q8"
```

THEOREM 2.67 (Itô). *Let $G=AB$ be a factorization of G with A and B abelian subgroups of G. Then G is metabelian.*

PROOF. Since $G=AB$ is a group, $AB=BA$. We claim that $[A,B]$ is a normal subgroup of G. Let $a,\alpha\in A$ and $b,\beta\in B$. Let $a_1,a_2\in A$ and $b_1,b_2\in B$ be such that $\alpha b\alpha^{-1}=b_1a_1$, $\beta a\beta^{-1}=a_2b_2$. Since

$$\alpha[a,b]\alpha^{-1}=a(\alpha b\alpha^{-1})a^{-1}(\alpha b^{-1}\alpha^{-1})=ab_1a_1a^{-1}a_1^{-1}b_1^{-1}=[a,b_1]\in[A,B]$$
$$\beta[a,b]\beta^{-1}=(\beta a\beta^{-1})\beta b\beta^{-1}(\beta a^{-1}\beta^{-1})b^{-1}=a_2b_2bb_2^{-1}a_2^{-1}b^{-1}=[a_2,b]\in[A,B],$$

it follows that $[A,B]$ is normal in G.

We now claim that $[A,B]$ is abelian. Since

$$\beta\alpha[a,b]\alpha^{-1}\beta^{-1}=\beta[a,b_1]\beta^{-1}=(\beta a\beta^{-1})b_1(\beta a^{-1}\beta^{-1})b_1^{-1}=[a_2,b_1],$$
$$\alpha\beta[a,b]\beta^{-1}\alpha^{-1}=\alpha[a_2,b]\alpha^{-1}=a_2(\alpha b\alpha^{-1})a_2^{-1}(\alpha b\alpha^{-1})=[a_2,b_1],$$

a direct calculation shows that

$$[\alpha^{-1},\beta^{-1}][a,b][\alpha^{-1},\beta^{-1}]^{-1}=[a,b].$$

Two arbitrary generators of $[A,B]$ commute, so the group $[A,B]$ is abelian.

To finish the proof, note that $[G,G]=[A,B]$. In fact,

$$[a_1b_1,a_2b_2]=a_1[a_2,b_1]^{-1}a_1^{-1}a_2[a_1,b_2]a_2^{-1}\subseteq[A,B],$$

as $[A,B]$ is normal in G. □

In 1988 Sysak proved the following generalization of Itô's theorem.

THEOREM 2.68 (Sysak). *Let A and B be abelian subgroups of G. If H is a subgroup of G contained in AB, then H is metabelian.*

For the proof, see [**8**].

§ 2.8. Nilpotent groups of class two. The following exercises go over groups of nilpotency class two.

EXERCISE 2.69. Let G be a group. Prove that if $x,y\in G$ are such that $[x,y]\in C_G(x)\cap C_G(y)$, then

$$[x,y]^n=[x^n,y]=[x,y^n]$$

for all $n\in\mathbb{Z}$.

EXERCISE 2.70 (Hall). Let G be a class-two nilpotent group and $x,y\in G$. Prove that

$$(xy)^n=[y,x]^{n(n-1)/2}x^ny^n$$

for all $n\geq 1$.

EXERCISE 2.71. Let p be an odd prime number and P p-group of nilpotency class ≤ 2. Prove that if $[y,x]^p=1$ for all $x,y\in P$, then $P\to[P,P]$, $x\mapsto x^p$, is a group homomorphism.

EXERCISE 2.72. Let p be an odd prime number and P a p-group of nilpotency class ≤ 2. Prove that $\{x \in P : x^p = 1\}$ is a subgroup of P.

Lecture 3. 29/02/2024

§ 3.1. Frattini subgroup.

Definition 3.1. Let G be a group. If G has maximal subgroups, the **Frattini subgroup** is the intersection $\Phi(G)$ of all the maximal subgroups of G. Otherwise, $\Phi(G) = G$.

Exercise 3.2. Prove that $\Phi(G)$ is a characteristic subgroup of G.

Example 3.3. Sea $G = \mathbb{S}_3$. The maximal subgroups of G are

$$M_1 = \langle (123) \rangle, \quad M_2 = \langle (12) \rangle, \quad M_3 = \langle (23) \rangle, \quad M_4 = \langle (13) \rangle.$$

Hence $\Phi(G) = \{1\}$.

Example 3.4. Let $G = \langle g \rangle \simeq C_{12}$. The subgroups of G are

$$\{1\}, \quad \langle g^6 \rangle \simeq C_2, \quad \langle g^4 \rangle \simeq C_3, \quad \langle g^3 \rangle \simeq C_4, \quad \langle g^2 \rangle \simeq C_6, \quad G.$$

Let us draw a picture:

$$\begin{tikzcd}
 & C_{12} \arrow[dl, dash] \arrow[dr, dash] & & \\
C_4 \arrow[dr, dash] & & C_6 \arrow[dl, dash] \arrow[dr, dash] & \\
 & C_2 \arrow[dr, dash] & & C_3 \arrow[dl, dash] \\
 & & \{1\} &
\end{tikzcd}$$

The maximal subgroups of G are $\langle g^3 \rangle \simeq C_4$ and $\langle g^2 \rangle \simeq C_6$. Hence $\Phi(G) = \langle g^3 \rangle \cap \langle g^2 \rangle = \langle g^6 \rangle \simeq C_2$. Let us see how to do this calculation with the computer:

```
gap> G = CyclicGroup(12);;
gap> StructureDescription(FrattiniSubgroup(G));
"C2"
```

Lemma 3.5 (Dedekind). *Let H, K and L be subgroups of G such that $H \subseteq L \subseteq G$. Then $HK \cap L = H(K \cap L)$.*

Proof. One inclusion is trivial. Let us prove then that $HK \cap L \subseteq H(K \cap L)$. If $x = hk \in HK \cap L$ with $x \in L$, $h \in H$ and $k \in K$, then $k = h^{-1}x \in L \cap K$, as $H \subseteq L$. Thus $x = hk \in H(L \cap K)$. $\square$

Lemma 3.6. *Let G be a finite group and H be a subgroup of G such that $G = H\Phi(G)$. Then $H = G$.*

Proof. If $H \neq G$, let M be a maximal subgroup of G such that $H \subseteq M$. Since $\Phi(G) \subseteq M$, $G = H\Phi(G) \subseteq M$, a contradiction. $\square$

Proposition 3.7. *Let N be a normal subgroup of a finite group G. Then $\Phi(N) \subseteq \Phi(G)$.*

PROOF. Since $\Phi(N)$ is characteristic in N and N is normal in G, $\Phi(N)$ is normal in G. If there exists a maximal subgroup M such that $\Phi(N) \not\subseteq M$, then $\Phi(N)M = G$. (This happens because, otherwise, $M = \Phi(N)M \supseteq \Phi(N)$.) By Dedekind's lemma (with $H = \Phi(N)$, $K = M$ and $L = N$),

$$N = G \cap N = (\Phi(N)M) \cap N = \Phi(N)(M \cap N).$$

By Lemma 3.6 (with $G = N$ and $H = M \cap N$), $\Phi(N) \subseteq N \subseteq M$, a contradiction. Hence every maximal subgroup of G contains $\Phi(N)$ and therefore $\Phi(G) \supseteq \Phi(N)$. □

The following proposition states that the elements of the Frattini subgroup are the **non-generators** of the group.

PROPOSITION 3.8. *Let G be a finite group. Then*

$$\Phi(G) = \{x \in G : \textit{if } G = \langle x, Y\rangle \textit{ for some } Y \subseteq G, \textit{ then } G = \langle Y\rangle\}.$$

PROOF. We first prove $\supseteq$. Let $x \in G$. If M is a maximal subgroup of G such that $x \notin M$, then, since $G = \langle x, M\rangle$, we obtain that $G = \langle M\rangle = M$, a contradiction. Thus $x \in M$ for all maximal subgroup M of G. Hence $x \in \Phi(G)$.

We now prove $\subseteq$. Let $x \in \Phi(G)$ be such that $G = \langle x, Y\rangle$ for some subset Y of G. If $G \neq \langle Y\rangle$, there exists a maximal subgroup M such that $\langle Y\rangle \subseteq M$. Since $x \in M$, $G = \langle x, Y\rangle \subseteq M$, a contradiction. □

EXAMPLE 3.9. For a prime number p, let G be an elementary p-group, that is $G \simeq C_p^m$ for some $m \geq 1$. Assume that $G = \langle x_1\rangle \times \cdots \times \langle x_m\rangle$ with $\langle x_j\rangle \simeq C_p$. We claim that $\Phi(G)$ is trivial. For $j \in \{1, \dots, m\}$, let $n_j \in \{1, \dots, p-1\}$. Since

$$\{x_1, \dots, x_{j-1}, x_j^{n_j}, x_{j+1}, \dots, x_m\}$$

generates G and $\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m\}$ does not, $x_j^{n_j} \notin \Phi(G)$ by Proposition 3.8. Hence $\Phi(G) = \{1\}$.

THEOREM 3.10 (Frattini). *Let G be a finite group. Then $\Phi(G)$ is nilpotent.*

PROOF. Let $P \in \mathrm{Syl}_p(\Phi(G))$ for some prime number p. Since $\Phi(G)$ is normal in G, Lemma 1.27 (Frattini's argument) implies that $G = \Phi(G)N_G(P)$. By Lemma 3.6, $G = N_G(P)$. Since every Sylow subgroup of $\Phi(G)$ is normal in G, $\Phi(G)$ is nilpotent. □

EXERCISE 3.11. Let G be a group and M be a normal subgroup of G. Prove that if M is maximal, then G/M is cyclic of prime order.

THEOREM 3.12 (Gaschütz). *If G is a finite group, then*

$$[G, G] \cap Z(G) \subseteq \Phi(G).$$

PROOF. Let $D = [G, G] \cap Z(G)$. Assume that D is not contained in $\Phi(G)$. Since $\Phi(G)$ is contained in every maximal subgroup of G, there is a maximal subgroup M of G not containing D. Then $G = MD$. Since $D \subseteq Z(G)$, M is normal in G, as $g = md \in G = MD$ implies

$$gMg^{-1} = (md)Md^{-1}m^{-1} = mMm^{-1} = M.$$

Since G/M is cyclic of prime order, G/M is, in particular, abelian and hence $[G, G] \subseteq M$. Therefore $D \subseteq [G, G] \subseteq M$, a contradiction. □

LEMMA 3.13. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. If H is a subgroup of G such that $N_G(P) \subseteq H$, then $N_G(H) = H$.*

PROOF. Let $x \in N_G(H)$. Since $P \in \mathrm{Syl}_p(H)$ and $Q = xPx^{-1} \in \mathrm{Syl}_p(H)$, the second Sylow's theorem implies that there exists $h \in H$ such that $hQh^{-1} = (hx)P(hx)^{-1} = P$. Then $hx \in N_G(P) \subseteq H$ and hence $x \in H$. □

THEOREM 3.14 (Wielandt). *A finite group G is nilpotent if and only if $[G,G] \subseteq \Phi(G)$.*

PROOF. Assume that $[G,G] \subseteq \Phi(G)$. Let $P \in \mathrm{Syl}_p(G)$. If $N_G(P) \neq G$, then $N_G(P) \subseteq M$ for some maximal subgroup M of G. If $g \in G$ and $m \in M$, then, since

$$gmg^{-1}m^{-1} = [g,m] \in [G,G] \subseteq \Phi(G) \subseteq M,$$

M is normal in G. Furthermore $N_G(P) \subseteq M$. By Lemma 3.13,

$$G = N_G(M) = M,$$

a contradiction. Thus $N_G(P) = G$ and every Sylow subgroup of G si normal in G. Therefore G is nilpotent.

Conversely, assume that G is nilpotent. Let M be a maximal subgroup of G. Since M is normal in G and maximal, G/M has no proper non-trivial subgroups. Then $G/M \simeq C_p$ for some prime number p. In particular, G/M is abelian and $[G,G] \subseteq M$. Since $[G,G]$ is contained in every maximal subgroup of G, $[G,G] \subseteq \Phi(G)$. □

THEOREM 3.15. *A finite group G is nilpotent if and only if $G/\Phi(G)$ is nilpotent.*

PROOF. If G is nilpotent, then $G/\Phi(G)$ is nilpotent. Conversely, assume that $G/\Phi(G)$ is nilpotent. Let $P \in \mathrm{Syl}_p(G)$. Since $\Phi(G)P/\Phi(G) \in \mathrm{Syl}_p(G/\Phi(G))$ and $G/\Phi(G)$ is nilpotent, $\Phi(G)P/\Phi(G)$ is a normal subgroup of $G/\Phi(G)$. By the correspondence theorem, $\Phi(G)P$ is a normal subgroup of G. Since $P \in \mathrm{Syl}_p(\Phi(G)P)$, Frattini's argument (Lemma 1.27) implies that

$$G = \Phi(G)PN_G(P) = \Phi(G)N_G(P),$$

as $P \subseteq N_G(P)$. Thus $G = N_G(P)$ by Lemma 3.6). Hence P is normal in G and therefore G is nilpotent. □

THEOREM 3.16 (Hall). *Let G be a finite group with a normal subgroup N. If both N and $G/[N,N]$ are nilpotent, then G is nilpotent.*

PROOF. Since N is nilpotent, $[N,N] \subseteq \Phi(N)$ by Wielandt's theorem 3.14. By Proposition 3.7, $[N,N] \subseteq \Phi(N) \subseteq \Phi(G)$. By the universal property, there exists a surjective group homomorphism $G/[N,N] \to G/\Phi(G)$ such that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/\Phi(G) \\ G/[N,N] \arrow[ur, dashed] & \end{tikzcd}$$

is commutative. Since $G/[N,N]$ is nilpotent, $G/\Phi(G)$ is nilpotent by Exercise 2.19. Thus G is nilpotent by the previous theorem. □

DEFINITION 3.17. A **minimal generating set** of a group G is a set X of generators of G such that no proper subset of X generates G.

Note that a minimal generating set does not necessarily have minimal size.

EXAMPLE 3.18. Let $G = \langle g \rangle \simeq C_6$. If $a = g^2$ and $b = g^3$, then $\{a,b\}$ is a minimal generating set of G that does not have minimal size, as $G = \langle ab \rangle$.

For a prime number p, we write $\mathbb{F}_p$ to denote the field of p elements.

LEMMA 3.19. *Let p be a prime number and G be a finite p-group. Then $G/\Phi(G)$ is a vector space over $\mathbb{F}_p$.*

PROOF. Let K be a maximal subgroup of G. Since G is nilpotent (see Proposition 2.21), K is normal in G (Exercise 2.55). Thus $G/K \simeq C_p$ because it is a simple p-group.

It is enough to prove that $G/\Phi(G)$ is an elementary abelian p-group. It is a p-group because G is a p-group. Let $K_1,\dots,K_m$ be the maximal subgroups of G. If $x \in G$, then $x^p \in K_j$ for all $j \in \{1,\dots,m\}$. Hence $x^p \in \Phi(G) = \cap_{j=1}^m K_j$. Moreover, $G/\Phi(G)$ is abelian, as $[G,G] \subseteq \Phi(G)$ because G is nilpotent (Wielandt's theorem 3.14). □

THEOREM 3.20 (Burnside). *Let p be a prime number and G a finite p-group. If X is a minimal generating set of G, then $|X| = \dim G/\Phi(G)$.*

PROOF. By Lemma 3.19, $G/\Phi(G)$ is a vector space over $\mathbb{F}_p$. Let $\pi\colon G \to G/\Phi(G)$ be the canonical map and $\{x_1,\dots,x_n\}$ be a minimal generating set of G. We claim that $\{\pi(x_1),\dots,\pi(x_n)\}$ is a linearly independent subset of $G/\Phi(G)$. Assume this is not the case. Without loss of generality, let us assume that $\pi(x_1) \in \langle \pi(x_2),\dots,\pi(x_n)\rangle$. There exists $y \in \langle x_2,\dots,x_n\rangle$ such that $x_1y^{-1} \in \Phi(G)$. Since G is generated by $\{x_1y^{-1},x_2,\dots,x_n\}$ and $x_1y^{-1} \in \Phi(G)$, Proposition 3.8 implies that G is generated by $\{x_2,\dots,x_n\}$, a contradiction to the minimality of $\{x_1,\dots,x_n\}$. Therefore $n = \dim G/\Phi(G)$. □

EXERCISE 3.21. Let p be a prime number and G a finite p-group. Prove that if $x \notin \Phi(G)$, then x belongs to some minimal set of generators of G.

Lecture 4. 07/03/2024

§ 4.1. The Fitting subgroup.

DEFINITION 4.1. Let G be a finite group and p be a prime number. The p**-radical** of G is the subgroup

$$O_p(G) = \bigcap_{P \in \mathrm{Syl}_p(G)} P.$$

LEMMA 4.2. *Let G be a finite group and p be a prime number. The following statements hold:*

1) *$O_p(G)$ is normal in G.*
2) *If N is a normal subgroup of G contained in some $P \in \mathrm{Syl}_p(G)$, then $N \subseteq O_p(G)$.*

PROOF. Let $P \in \mathrm{Syl}_p(G)$. Let G act on G/P by left multiplication. There is a group homomorphism $\rho \colon G \to \mathbb{S}_{G/P}$ with kernel

$$\begin{aligned} \ker \rho &= \{x \in G : \rho_x = \mathrm{id}\} \\ &= \{x \in G : xgP = gP \; \forall g \in G\} \\ &= \{x \in G : x \in gPg^{-1} \; \forall g \in G\} \\ &= \bigcap_{g \in G} gPg^{-1} \\ &= O_p(G). \end{aligned}$$

Then $O_p(G)$ is normal in G.

Let N be a normal subgroup of G such that $N \subseteq P$. Since $N = gNg^{-1} \subseteq gPg^{-1}$ for all $g \in G$, we conclude that $N \subseteq O_p(G)$. □

DEFINITION 4.3. Let G be a finite group and $p_1, \dots, p_k$ be the prime divisors of $|G|$. The **Fitting subgroup** of G is the subgroup

$$F(G) = O_{p_1}(G) \cdots O_{p_k}(G)$$

EXERCISE 4.4. Prove that $F(G)$ is characteristic in G.

EXAMPLE 4.5. Let $G = \mathbb{S}_3$. Then $O_2(G) = \{1\}$ and $O_3(G) = \langle (123) \rangle$. Hence $F(G) = \langle (123) \rangle$.

THEOREM 4.6 (Fitting). *Let G be a finite group. Then $F(G)$ is a nilpotent and normal in G. Moreover, $F(G)$ contains every nilpotent normal subgroup of G.*

PROOF. By definition, $|F(G)| = \prod_p |O_p(G)|$. Since $O_p(G) \in \mathrm{Syl}_p(F(G))$, we conclude that $F(G)$ is nilpotent, as it contains a normal Sylow p-subgroup for every prime p. Hence $F(G)$ is nilpotent by Theorem 2.53.

Let N be a nilpotent normal subgroup of G and $P \in \mathrm{Syl}_p(N)$. Since N is nilpotent, P is normal in N and hence P is the only Sylow p-subgroup of N. Thus P is characteristic in N and P is normal in G. Since N is nilpotent, N is a direct product of its Sylow subgroups. Therefore $N \subseteq O_p(G)$ by Lemma 4.2. □

COROLLARY 4.7. *If G is a finite group, then $Z(G) \subseteq F(G)$.*

PROOF. Since $Z(G)$ is nilpotent (in fact, it is abelian) and normal in G, by Fitting's theorem 4.6 we conclude that $Z(G) \subseteq F(G)$ □

COROLLARY 4.8 (Fitting). *Let K and L be nilpotent normal subgroups of a finite group G. Then KL is nilpotent.*

PROOF. By Fitting's theorem 4.6, $K \subseteq F(G)$ and $L \subseteq F(G)$. Then $KL \subseteq F(G)$ and KL is nilpotent, as $F(G)$ is nilpotent. □

COROLLARY 4.9. *Let N be a normal subgroup of a finite group G. Then $N \cap F(G) = F(N)$.*

PROOF. Since $F(N)$ is characteristic in N, $F(N)$ is normal in G. Then $F(N) \subseteq N \cap F(G)$ because $F(N)$ is nilpotent. Conversely, since $F(G)$ is normal in G, the subgroup $F(G) \cap N$ is normal in N. Since $F(G) \cap N$ is nilpotent, $F(G) \cap N \subseteq F(N)$. □

We now discuss an application to finite solvable groups.

THEOREM 4.10. *Let G be a non-trivial solvable group. Every normal non-trivial subgroup N contains a normal abelian non-trivial subgroup. Moreover, this subgroup is contained in $F(N)$.*

PROOF. Note that $N \cap G^{(0)} = N \neq \{1\}$. Since G is solvable, there exists $m \geq 1$ such that $N \cap G^{(m)} = \{1\}$. Let n be the largest positive integer such that $N \cap G^{(n)} \neq \{1\}$. Since $[N,N] \subseteq N$ and $[G^{(n)}, G^{(n)}] = G^{(n+1)}$,

$$[N \cap G^{(n)}, N \cap G^{(n)}] \subseteq N \cap G^{(n+1)} = \{1\}.$$

Then $N \cap G^{(n)}$ is an abelian subgroup of G. Moreover, it is normal and nilpotent. Hence

$$N \cap G^{(n)} \subseteq N \cap F(G) = F(N).$$ □

THEOREM 4.11. *If N is a minimal normal subgroup of a finite group G, then $F(G) \subseteq C_G(N)$.*

PROOF. By Fitting's theorem 4.6, $F(G)$ is a normal nilpotent group. The subgroup $N \cap F(G)$ is normal in G. Moreover, $[F(G),N] \subseteq N \cap F(G)$. If $N \cap F(G) = \{1\}$, then $[F(G),N] = \{1\}$. Otherwise, $N = N \cap F(G) \subseteq F(G)$ by the minimality of N. By Hirsch's theorem, $N \cap Z(F(G)) \neq \{1\}$ Since $Z(F(G))$ is characteristic in $F(G)$ and $F(G)$ is normal in G, $Z(F(G))$ is normal in G. Since $\{1\} \neq N \cap Z(F(G))$ is normal in G, the minimality of N implies that $N = N \cap Z(F(G)) \subseteq Z(F(G))$. Hence $[F(G),N] = \{1\}$. □

COROLLARY 4.12. *Let G be a finite solvable group. The following statements hold:*

1) *If N is a minimal normal subgroup, then $N \subseteq Z(F(G))$.*
2) *If H is a non-trivial normal subgroup, then $H \cap F(G) \neq \{1\}$.*

PROOF. Let us prove the first claim. Since N is a p-group by Lemma 1.21, N is nilpotent and hence $N \subseteq F(G)$. Moreover, $F(G) \subseteq C_G(N)$ by the previous theorem. Therefore $N \subseteq Z(F(G))$.

Let us prove now the second claim. The subgroup H contains a minimal normal subgroup N and $N \subseteq F(G)$. Then $H \cap F(G) \neq \{1\}$. □

THEOREM 4.13. *Let G be a finite group. The following statements hold:*

1) *$\Phi(G) \subseteq F(G)$ and $Z(G) \subseteq F(G)$.*
2) *$F(G)/\Phi(G) \simeq F(G/\Phi(G))$.*

PROOF. Let us prove the first claim. Since $\Phi(G)$ is normal in G, nilpotent by Frattini's Theorem 3.10 and $F(G)$ contains every normal nilpotent subgroup of G, $\Phi(G) \subseteq F(G)$. Moreover, $Z(G)$ is normal in G and nilpotent. Hence $Z(G) \subseteq F(G)$.

Let us prove the second claim. Let $\pi\colon G\to G/\Phi(G)$ be the canonical map. Since $F(G)$ is nilpotent, $\pi(F(G))$ is nilpotent. Hence

$$\pi(F(G))\subseteq F(G/\Phi(G))$$

by Fitting's Theorem 4.6. Let $H=\pi^{-1}(F(G/\Phi(G)))$. By the correspondence theorem, H is a normal subgroup of G containing $\Phi(G)$. If $P\in\mathrm{Syl}_p(H)$, then $\pi(P)\in\mathrm{Syl}_p(\pi(H))$. In fact, $\pi(P)\simeq P/P\cap\Phi(G)$ is a p-group and $(\pi(H):\pi(P))$ is coprime with p because

$$(\pi(H):\pi(P))=\frac{|\pi(H)|}{|\pi(P)|}=\frac{|H/\Phi(G)|}{|P/P\cap\Phi(G)|}=\frac{(H:P)}{(\Phi(G):P\cap\Phi(G))}$$

divides $(H:P)$, a number coprime with p. Since $\pi(H)$ is nilpotent, $\pi(P)$ is characteristic in $\pi(H)$. Then $\pi(P)$ is normal in $\pi(G)=G/\Phi(G)$ and $P\Phi(G)=\pi^{-1}(\pi(P))$ is normal in G. Since $P\in\mathrm{Syl}_p(P\Phi(G))$, Frattini's argument (Lemma 1.27) implies that $G=\Phi(G)N_G(P)$. Therefore P is normal in G by Lemma 3.6. Since P is nilpotent and normal in G, $P\subseteq F(G)$ by Fitting's theorem 4.6. Hence $H\subseteq F(G)$ and $F(G/\Phi(G))=\pi(H)\subseteq\pi(F(G))$. $\square$

Lecture 5. 14/03/2024

§ 5.1. Super solvable groups.

DEFINITION 5.1. A group G is said to be **super solvable** if there exists a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups of G such that every quotient G_{i-1}/G_i is cyclic.

In the previous definition, we do not require the group to be finite. Hence the quotients could be finite cyclic groups or isomorphic to $\mathbb{Z}$.

EXAMPLE 5.2. The dihedral group $\mathbb{D}_n$ of order $2n$ is super solvable, as

$$\mathbb{D}_n \supseteq \langle r \rangle \supseteq \{1\}$$

is a sequence of normal subgroups with cyclic factors.

Every solvable group is super solvable. See Exercise 1.5.

EXAMPLE 5.3. The alternating group $\mathbb{A}_4$ solvable but not super solvable. The only proper non-trivial normal subgroup of $\mathbb{A}_4$ is

$$\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2.$$

Thus $\mathbb{A}_4$ does not have a sequence of normal subgroups with cyclic factors.

EXERCISE 5.4. Prove that $\mathrm{Aff}(\mathbb{Z})$ is super solvable.

EXAMPLE 5.5. The group $\mathbf{SL}_2(3)$ is solvable but not super solvable. Here is a computer verification:

```
gap> IsSolvable(SL(2,3));
true
gap> IsSupersolvable(SL(2,3));
false
```

EXERCISE 5.6. Prove the following statements:

1) Every subgroup of a super solvable group is super solvable.
2) Quotients of super solvable groups are super solvable.

EXERCISE 5.7. Prove that the direct product of super solvable groups is super solvable.

EXERCISE 5.8. Let H and K be normal subgroups of a group G such that G/K and G/H are super solvable. Prove that $G/H \cap K$ is super solvable.

EXERCISE 5.9. Let N be a cyclic normal subgroup of G. If G/N is super solvable, then G is super solvable.

THEOREM 5.10. *Let G be a super solvable non-trivial group. Then G admits a sequence*

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups such that every quotient G_{i-1}/G_i is cyclic of prime order or isomorphic to $\mathbb{Z}$.

PROOF. Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ be a sequence of normal subgroups of G such that every quotient G_{i-1}/G_i is cyclic. Let $i \in \{1,\dots,n\}$ be such that $G_{i-1}/G_i \simeq C_n$ for some non-prime n and let $\pi\colon G_{i-1} \to G_{i-1}/G_i$ be the canonical map. Let p be a prime divisor of n and H be a subgroup of G such that $\pi(H)$ is a subgroup of G_{i-1}/G_i of order p. By the correspondence theorem, $G_i \subseteq H \subseteq G_{i-1}$.

We claim that H is normal in G. Let $g \in G$. Since $\pi(gHg^{-1})$ is a subgroup of order p of the cyclic group G_{i-1}/G_i, $\pi(gHg^{-1}) = \pi(H)$. Then $gHg^{-1} = G_iH \subseteq H$ and hence $gHg^{-1} = H$.

Note that H/G_i is cyclic of prime order, as

$$H/G_i = H/H \cap G_i \simeq \pi(H) \simeq C_p.$$

Moreover, G_{i-1}/H is cyclic, as

$$G_{i-1}/H \simeq \frac{G_{i-1}/G_i}{H/G_i}$$

is the quotient of a cyclic group.

We have shown that by adding H to our sequence of normal subgroups, we obtain a sequence with cyclic factors where H/G_i is cylic of prime order. Repeating this procedure, we obtain the desired result. □

Let us discuss an immediate application.

COROLLARY 5.11. *A finite super solvable group admits a sequence of normal subgroups where each quotient is cyclic of prime order.*

We now discuss other properties of super solvable groups.

THEOREM 5.12. *Let G be a super solvable group. The following statement hold:*

1) *If N is minimal normal in G, then $N \simeq C_p$ for some prime number p.*
2) *If M is maximal in G, then $(G : M) = p$ for some prime number p.*
3) *The commutator subgroup $[G,G]$ is nilpotent.*
4) *If G is non-abelian, there exists a normal subgroup $N \neq G$ such that $Z(G) \subsetneq N$.*

PROOF. Let us prove the first claim. Since G is super solvable, there exists a sequence

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups with cyclic factors. Since each $G_i \cap N$ is a normal subgroup of G contained in N, the minimality implies that each $G_i \cap N$ is either trivial or equal to N. Moreover, $N \cap G_0 = N$ and $N \cap G_n = \{1\}$. Let j be the smallest positive integer such that $N \cap G_j = \{1\}$. Since $N \subseteq G_{j-1}$ (because $N \cap G_{j-1} = N$), the composition

$$N \hookrightarrow G_{j-1} \to G_{j-1}/G_j$$

is an injective group homomorphism, as its kernel is equal $N \cap G_j = \{1\}$. Thus N is cyclic, as it is isomorphic to a subgroup of the cyclic group G_{i-1}/G_i. If $G_{i-1}/G_i \simeq \mathbb{Z}$, then $N \simeq \mathbb{Z}$, a contradiction to the fact that N is minimal normal. (For example, $2\mathbb{Z}$ is characteristic subgroup of $\mathbb{Z}$ and hence it is normal in G. Thus N is cyclic and finite. Hence $N \simeq C_p$.)

We now prove the second claim. Let M be a maximal subgroup of G. If M is normal in G, then G/M does not contain non-trivial proper subgroups. Then $G/M \simeq C_p$ for some prime number p.

Assume that M is not normal in G. Let $H=\cap_{g\in G}gMg^{-1}$ and $\pi\colon G\to G/H$ be the canonical map. Since $\pi(M)$ is maximal in $\pi(G)=G/H$ and

$$(G:M)=(G/H:M/H)=(G/H:M/H\cap M)=(\pi(G):\pi(M)),$$

we may assume that M does not contain non-trivial normal subgroups of G (if needed, we just replace G by G/H). Since G is super solvable, there exists a sequence $G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=\{1\}$ of normal subgroups of G with factors either cyclic of prime order or isomorphic to $\mathbb{Z}$. Let $N=G_{n-1}$. Since N is cyclic, every subgroup of N is characteristic in N and hence normal in G. In particular, $M\cap N$ is normal in G and therefore $M\cap N=\{1\}$. Since $M\subseteq NM\subseteq G$, the maximality of M implies that either $M=NM$ or $G=NM$. Since $N\subseteq NM=M$ yields a contradiction, we conclude that $G=NM$.

If $N\simeq C_p$ for some prime p, then $(G:M)=p$ and the proof is complete. Assume that $N\simeq\mathbb{Z}$. Let H be a proper subgroup of N. Since H is characteristic in N, H is normal in G. Since $M\subseteq HM\subseteq NM=G$, the maximality of M implies that either $HM=M$ or $HM=G$. Since $HM=M$ implies $H\subseteq M\cap N=\{1\}$, we may assume that $HM=G$. If $n\in N\setminus H$, then $n=hm$ for some $h\in H$ and $m\in M$. Then $h=n$, as $h^{-1}n\in N\cap M=\{1\}$, a contradiction.

Demostremos ahora la tercera afirmación. Como G es súper-resoluble, existe una sucesión

$$G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=\{1\}$$

de subgrupos normales tal que cada G_i/G_{i+1} es cíclico. Para cada $i\in\{0,\dots,n\}$ sea $H_i=[G,G]\cap G_i$. Como $[G,G]$ y los G_i son normales en G, se tiene una sucesión

$$[G,G]=H_0\supseteq H_1\supseteq\cdots\supseteq H_n=\{1\}$$

de subgrupos normales de G. Como H_i y H_{i+1} es normal en G, el grupo G actúa por conjugación en H_i/H_{i+1}. Esto induce un morfismo $\gamma\colon G\to\operatorname{Aut}(H_i/H_{i+1})$. Como H_i/H_{i+1} es cíclico, $\operatorname{Aut}(H_i/H_{i+1})$ es abeliano y luego $[G,G]\subseteq\ker\gamma$. Luego $[G,G]$ actúa trivialmente por conjugación en H_i/H_{i+1} y entonces

$$H_i/H_{i+1}\subseteq Z([G,G]/H_{i+1}).$$

Por último demostremos la cuarta afirmación. Como G es no abeliano, $Z(G)\neq G$. Sea $\pi\colon G\to G/Z(G)$ el morfismo canónico. El cociente $G/Z(G)$ es súper-resoluble y la sucesión

$$G/Z(G)=\pi(G)\supseteq\pi(G_1)\supseteq\cdots\supseteq\pi(1)=1$$

es una sucesión de subgrupos normales de $G/Z(G)$ con cocientes cíclicos. En particular, $1\neq\pi(G_1)$ es propio y normal en $G/Z(G)$. Por el teorema de la correspondencia, $\pi^{-1}(\pi(G_1))\neq G$ es un subgrupo normal de G que contiene propiamente a $Z(G)$. □

Example 5.13. Si G es un grupo resoluble, no necesariamente $[G,G]$ es un grupo nilpotente. El grupo $\mathbb{S}_4$ es resoluble pero $[\mathbb{S}_4,\mathbb{S}_4]=\mathbb{A}_4$ no es nilpotente.

Proposition 5.14. *Sea p un número primo. Todo p-grupo finito es súper-resoluble.*

Proof. Sea G un contraejemplo de orden minimal. Podemos suponer que $|G|=p^n$ con $n>1$ (pues si $n=1$ el grupo G es trivialmente súper-resoluble). Como G es un p-grupo, es nilpotente y existe un subgrupo normal N de orden p. El cociente G/N tiene orden p^{n-1} entonces es súper-resoluble pues $|G/N|<|G|$. Como N es cíclico y G/N es súper-resoluble, G es súper-resoluble por la proposición **??**. □

Como todo grupo finito nilpotente es producto directo de (finitos) subgrupos de Sylow, cada p-grupo es súper-resoluble y el producto directo de súper-resolubles es súper-resoluble, se obtiene el siguiente resultado:

PROPOSITION 5.15. *Todo grupo finito nilpotente es súper-resoluble.*

THEOREM 5.16. *Todo grupo súper-resoluble tiene subgrupos maximales.*

PROOF. Procederemos por inducción en la longitud de la sucesión de superresolubilidad. Si la longitud es uno, el teorema es cierto pues en este caso el grupo es cíclico. Supongamos entonces que G admite una sucesión

$$G = G_0 \supseteq \cdots \supseteq G_k = \{1\}$$

y que la afirmación es cierta para grupos súper-resolubles con sucesiones de longitud $< k$. Como G_{k-1} es normal en G, sea $\pi\colon G \to G/G_{k-1}$ el morfismo canónico. Entonces la sucesión

$$G/G_{k-1} = \pi(G) \supseteq \pi(G_1) \supseteq \cdots \supseteq \pi(G_{k-1}) = \{1\}$$

prueba la resolubilidad de $\pi(G)$ y tiene longitud $< k$. Por hipótesis inductiva, G/G_{k-1} admite subgrupos maximales y luego, por el teorema de la correspondencia, G también admite subgrupos maximales. $\square$

Los grupos resolubles o nilpotentes no siempre admiten subgrupos maximales, ver por ejemplo $\mathbb{Q}$.

DEFINITION 5.17. Se dice que un grupo G satisface la **condición maximal para subgrupos** si todo subconjunto $\mathscr{S}$ no vacío de subgrupos tiene un subgrupo maximal (es decir, no contenido en ningún otro subgrupo de $\mathscr{S}$).

PROPOSITION 5.18. *Sea G un grupo. Entonces G satisface la condición maximal para subgrupos si y sólo si todo subgrupo de G es finitamente generado.*

PROOF. Supongamos que G satisface la condición maximal para subgrupos y sea H un subgrupo de G. Sea $\mathscr{S}$ el conjunto de subgrupos de H finitamente generados. Como $\mathscr{S}$ es no vacío (pues $1 \in \mathscr{S}$), existe un elemento maximal $M \in \mathscr{S}$. Sea $x \in H$. Como $\langle M, x\rangle \in \mathscr{S}$, $M = \langle M, x\rangle$ y luego $x \in M$. Como entonces $H = M$, H es finitamente generado.

Supongamos ahora que todo subgrupo de G es finitamente generado. Si $\mathscr{S}$ es un subconjunto no vacío de subgrupos de G sin elemento maximal, podemos construir una sucesión de subgrupos $S_1 \subseteq S_2 \subseteq \cdots$ que no se estabiliza (acá necesitamos utilizar el axioma de elección). Como la unión

$$S = \bigcup_{j \geq 1} S_j$$

es un subgrupo de G, es finitamente generado y luego $S \subseteq S_k$ para algún k suficientemente grande, una contradicción. $\square$

Una consecuencia inmediata.

PROPOSITION 5.19. *Sean G un grupo y H un subgrupo de G. Si G satisface la condición maximal para subgrupos entonces H también.*

PROPOSITION 5.20. *Sea G un grupo y sea N un subgrupo normal de G. Si G/N y N satisfacen la condición maximal para subgrupos entonces G también.*

PROOF. Sea $\pi\colon G \to G/N$ el morfismo canónico. Sea $\mathscr{S}$ un subconjunto no vacío de subgrupos de G. El conjunto $\{S \cap N : S \in \mathscr{S}\}$ tiene un elemento maximal A y el conjunto $\{\pi(S) : S \in \mathscr{S}, S \cap N = A\}$ tiene un elemento maximal B. Sea $S \in \mathscr{S}$ tal que $\pi(S) = B$ y $S \cap N = A$. Si S no es maximal en $\mathscr{S}$, existe $T \in \mathscr{S}$ tal que $S \subseteq T$, $N \cap T = A$ y $\pi(T) = B$. Sea $x \in T \setminus S$. Como $\pi(xN) = \pi(x) \in \pi(T) = B$, existe $y \in S$ tal que $xN = yN$. Luego $y^{-1}x \in N \cap T = A = N \cap S$, una contradicción pues $x \notin S$. $\square$

Proposition 5.21. *Todo grupo súper-resoluble satisface la condición maximal para subgrupos. En particular, todo grupo súper-resoluble es finitamente generado.*

Proof. Procederemos por inducción en la longitud n de la sucesión de súper-resolubilidad. El caso $n=1$ es trivial pues entonces G es cíclico. Supongamos entonces que el resultado vale para grupos súper-resolubles con serie de longitud $\leq n-1$. Sea G un grupo súper-resoluble no trivial y sea

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{1\}$$

una sucesión de subgrupos normales de G con factores cíclicos. Como G_1 es súper-resoluble por el ejercicio 5.8, G_1 satisface la condición maximal para subgrupos por hipótesis inductiva. Luego, por la proposición 5.20, G satisface la condición maximal para subgrupos porque G/G_1 es un grupo cíclico. $\square$

Example 5.22. El grupo abeliano $\mathbb{Q}$ es nilpotente pero no es súper-resoluble porque no es finitamente generado.

Si G es un grupo y $x_1,\dots,x_{n+1} \in G$ se define

$$[x_1,\dots,x_{n+1}] = [[x_1,\dots,x_n],x_{n+1}], \quad n \geq 1.$$

Lemma 5.23. *Sea G un grupo finitamente generado, digamos $G = \langle X \rangle$ con X finito. Para cada $n \geq 2$ se define*

$$G_n = \langle g[x_1,\dots,x_n]g^{-1} : x_1,\dots,x_n \in X,\, g \in G \rangle.$$

Entonces $G_n = \gamma_n(G)$ para todo $n \geq 2$.

Proof. Observemos que cada G_n es normal en G. Procederemos por inducción en n. El caso $n=2$ es trivial. Supongamos entonces que $\gamma_{n-1}(G) = G_{n-1}$. Sean $x_1,\dots,x_n \in X$. Como $[x_1,\dots,x_n] \in \gamma_n(G)$, $G_{n-1} \subseteq \gamma_n(G)$. Sea $N = G_n$ y sea $\pi\colon G \to G/N$ el morfismo canónico. El grupo G/N es finitamente generado. Como

$$[\pi([x_1,\dots,x_{n-1}]),\pi(x_n)] = \pi([x_1,\dots,x_n]) = 1,$$

se tiene que $\pi([x_1,\dots,x_{n-1}]) \in Z(G/N)$. Luego $\pi(g[x_1,\dots,x_n]g^{-1}) = 1$ para todo $g \in G$ y, por hipótesis inductiva, se concluye que

$$\pi(\gamma_{n-1}(G)) = \pi(G_{n-1}) \subseteq Z(G/N).$$

Como entonces

$$\pi(\gamma_n(G)) = \pi([\gamma_{n-1}(G),G]) = [\pi(\gamma_{n-1}(G)),\pi(G)] = \{1\},$$

se concluye que $\gamma_n(G) \subseteq N = G_n$. $\square$

Lemma 5.24. *Sea G un grupo finitamente generado. Entonces $\gamma_n(G)/\gamma_{n+1}(G)$ es finitamente generado.*

Proof. Supongamos que $G = \langle X \rangle$ con X finito. Al escribir

$$g[x_1,\dots,x_n]g^{-1} = [g,[x_1,\dots,x_n]][x_1,\dots,x_n]$$

y usar el lema 5.23 para obtener que $[g,[x_1,\dots,x_n]] \in \gamma_{n+1}(G) = G_{n+1}$,

$$g[x_1,\dots,x_n]g^{-1} \equiv [x_1,\dots,x_n] \bmod \gamma_{n+1}(G).$$

Luego $\gamma_n(G)/\gamma_{n+1}(G)$ está generado por el conjunto finito

$$\{[x_1,\dots,x_n]\gamma_{n+1}(G) : x_1,\dots,x_n \in X\}.$$

$\square$

THEOREM 5.25. *Sea G un grupo nilpotente. Entonces G es súper-resoluble si y sólo si G es finitamente generado.*

PROOF. Si G es súper-resoluble, es finitamente generado por la proposición 5.21. Supongamos que G es finitamente generado y nilpotente. Como por el lema 5.24 cada $\gamma_n(G)/\gamma_{n+1}(G)$ es finitamente generado, digamos por $y_1,\dots,y_m$. Sea $\pi\colon G\to G/\gamma_{n+1}(G)$ el morfismo canónico. Para cada $j\in\{1,\dots,m\}$ sea

$$K_j=\langle\gamma_{n+1}(G),y_1,\dots,y_j\rangle.$$

Como $[K_j,G]\subseteq[\gamma_n(G),G]=\gamma_{n+1}(G)$, se tiene que $\pi(K_j)$ es central en $\pi(G)$. Luego $\pi(K_j)$ es normal en $\pi(G)$ y por lo tanto K_j es normal en G. Como cada K_j/K_{j-1} es cíclico generado por y_jK_{j-1}, entre $\gamma_n(G)$ y $\gamma_{n+1}(G)$ pudimos construir una sucesión de subgrupos normales de G con factores cíclicos. Como G es nilpotente, existe c tal que $\gamma_{c+1}(G)=1$ y luego G es súper-resoluble. $\square$

COROLLARY 5.26. *Todo grupo nilpotente finitamente generado satisface la condición maximal en subgrupos.*

PROOF. Es consecuencia del teorema 5.25 y la proposición 5.21. $\square$

THEOREM 5.27. *Sea G un grupo nilpotente y finitamente generado. Entonces $T(G)$ es finito.*

PROOF. Como G es nilpotente, G satisface la condición maximal para subgrupos por el corolario 5.26 y entonces todo subgrupo de G es finitamente generado. Como $T(G)$ es un subgrupo por el teorema **??**, es finitamente generado y de torsión. Luego $T(G)$ es finito por el teorema **??**. $\square$

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