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Non-commutative algebra

Notes

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Lecture 1

§1. Group rings

Let K be a field and G be a group (written multiplicatively). Let $K[G]$ be the vector space with basis $\{g : g \in G\}$. Then $\dim K[G] < \infty$ if and only if G is finite. The vector space $K[G]$ is an algebra with multiplication

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{h\in G}\mu_h h\right) = \sum_{g,h\in G}\lambda_g\mu_h(gh).$$

Exercise 1.1. Prove that $\mathbb{C}[\mathbb{Z}] \simeq \mathbb{C}[X, X^{-1}]$.

For $n \in \mathbb{Z}_{>1}$ let C_n be the cyclic group of order n.

Exercise 1.2. Let $n \in \mathbb{Z}_{>1}$. Prove that $\mathbb{C}[C_n] \simeq \mathbb{C}[X]/(X^n - 1)$.

Exercise 1.3. Prove that if G and H are isomorphic groups, then $K[G] \simeq K[H]$.

In a similar way, if R is a commutative ring (with 1) and G is a group, then one defines the group ring $R[G]$. More precisely, $R[G]$ is the set of finite linear combinations

$$\sum_{g\in G}\lambda_g g$$

where $\lambda_g \in R$ and $\lambda_g = 0$ for all but finitely many $g \in G$. One easily proves that $R[G]$ is a ring with addition

$$\left(\sum_{g\in G}\lambda_g g\right)+\left(\sum_{g\in G}\mu_g g\right) = \sum_{g\in G}(\lambda_g+\mu_g)(g)$$

and multiplication

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{h\in G}\mu_h h\right)=\sum_{g,h\in G}\lambda_g\mu_h(gh).$$

Moreover, $R[G]$ is a left R-module with $\lambda(\sum_{g\in G}\lambda_g g)=\sum_{g\in G}(\lambda\lambda_g)g$.

Exercise 1.4. Let G be a group. Prove that if $\mathbb{Z}[G]\simeq\mathbb{Z}[H]$, then $R[G]\simeq R[H]$ for any commutative ring R.

question:IP *Question 1.1 (Isomorphism problem).* Let G and H be groups. Does $\mathbb{Z}[G]\simeq\mathbb{Z}[H]$ imply $G\simeq H$?

Despite the fact that there are several cases were the isomorphism problem has an affirmative answer (e.g. abelian groups, metabelian groups, nilpotent groups, nilpotent-by-abelian groups, simple groups, abelian-by-nilpotent groups), it is false in general. In 2001 Hertweck found a counterexample of order $2^{21}97^{28}$, see [5].

question:MIP *Question 1.2 (Modular isomorphism problem).* Let p be a prime number. Let G and H be finite p-groups and let K be a field of characteristic p. Does $K[G]\simeq K[H]$ imply $G\simeq H$?

Question 1.2 has an affirmative answer in several cases. However, it is not true in general. This question recently answered by García, Margolis and del Río [2]. They found two non-isomorphic groups G and H both of order 512 such that $K[G]\simeq K[H]$ for all field K of characteristic two.

§2. Kapanskly's problems

Let G be a group and K be a field. If $x\in G\setminus\{1\}$ is such that $x^n=1$, then, since

$$(1-x)(1+x+x^2+\cdots+x^{n-1})=0,$$

it follows that $K[G]$ has non-trivial zero divisors. What happens in the case where G is torsion-free?

example:k[Z] **Example 2.1.** Let $G=\langle x\rangle\simeq\mathbb{Z}$. Then $K[G]$ has no zero divisors. Let $\alpha,\beta\in K[G]\setminus\{0\}$ and write $\alpha=\sum_{i\leq n}a_ix^i$ with $a_n\neq 0$ and $\beta=\sum_{j\leq m}b_jx^j$ with $b_m\neq 0$. Since the coefficient of x^{n+m} of $\alpha\beta$ is non-zero, it follows that $\alpha\beta\neq 0$.

A similar problem concerns units of group algebras. A unit $u\in K[G]$ is said to be **trivial** if $u=\lambda g$ for some $\lambda\in K\setminus\{0\}$ and $g\in G$.

Exercise 2.2. Prove that units of $\mathbb{C}[C_2]$ are trivial.

Exercise 2.3. Prove that $\mathbb{C}[C_5]$ has non-trivial units.

prob:dominio **Open problem 2.1 (Zero divisors).** Let G be a torsion-free group. Is it true that $K[G]$ is a domain?

We mention some intriguing problems, generally known as Kaplansky's problems.

prob:units

Open problem 2.2 (Units). Let G be a torsion-free group. Is it true that all units of $K[G]$ are trivial?

The unit problem is still open for fields of characteristic zero. However, it was recently solved by Gardam [3] in the case of K the field of two elements. We will present Gardam's theorem as a computer calculation. We will use GAP [1].

Lemma 2.4. *The group* $G = \langle a,b : a^{-1}b^2a = b^{-2}, b^{-1}a^2b = a^{-2}\rangle$ *is torsion-free. Moreover, the subgroup* $N = \langle a^2, b^2, (ab)^2\rangle$ *is normal in* G*, free-abelian of rank three and* $G/N \simeq C_2 \times C_2$.

Proof. We first construct the group.

```
gap> F := FreeGroup(2);;
gap> A := F.1;;
gap> B := F.2;;
gap> rels := [(B^2)^A*B^2,(A^2)^B*A^2];;
gap> G := F/rels;;
gap> a := G.1;;
gap> b := G.2;;
```

Now we construct the subgroup N generated by a^2, b^2 and $(ab)^2$. It is easy to check that N is normal in G and that $G/N \simeq C_2 \times C_2$. It is even easier to do this with the computer.

```
gap> N := Subgroup(G, [a^2,b^2,(a*b)^2]);;
gap> IsNormal(G,N);
true
gap> StructureDescription(G/N);
"C2 x C2"
```

It is easy to check by hand that N is abelian, and not so easy to do it with the computer. For example,

$$b^{-2}a^2b^{-2} = b^{-1}a^{-2}b = (b^{-1}a^2b)^{-1} = (a^{-2})^{-1} = a^2.$$

We use the computer to show that N is free abelian of rank three.

```
gap> AbelianInvariants(N);
[ 0, 0, 0 ]
```

Let us prove that G is torsion-free. Let $x = a^2$, $y = b^2$ and $z = (ab)^2$. Since $(G:N) = 4$, the group G decomposes as a disjoint union $G = N \cup aN \cup bN \cup (ab)N$. Let $g \in G$ be a non-trivial element of finite order. Since N is torsion-free, $g \in aN \cup bN \cup (ab)N$. Without loss of generality we may assume that $g \in aN$, so $g = an$ for some $n \in N$. Let $\pi\colon G \to G/N$ be the canonical map. Since $g \notin N$ and $\pi(g) \in G/N \simeq C_2 \times C_2$,

$$\pi(g^2) = \pi(g)^2 = 1$$

so $g^2 \in N$ and hence $g^2 = 1$, as N is torsion-free. Thus

$$1 = g^2 = (an)^2 = (an)(an) = a^2(a^{-1}na)n = x(a^{-1}na)n.$$

Write $n = x^i y^j z^k$ for some $i, j, k \in \mathbb{Z}$. Then

$$a^{-1}na = (a^{-1}x^i a)(a^{-1}y^j a)(a^{-1}z^k a) = x^i t^{-j} z^{-k}$$

and hence $(a^{-1}na)n = x^{2i}$. Then it follows that $1 = g^2 = x(a^{-1}na)n = x^{2i+1}$, a contradiction. □

Let P be the group generated by

$$a = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group P appears in the literature with various names. For us P will be the Promislow group. It is easy to check that there exists a surjective group homomorphism $G \to P$. Prove that $G \simeq P$.

thm:Gardam

Theorem 2.5 (Gardam). *Let $\mathbb{F}_2$ be the field of two elements. Consider the elements $x = a^2$, $y = b^2$ and $z = (ab)^2$ of P and let*

$$p = (1+x)(1+y)(1+z^{-1}), \qquad q = x^{-1}y^{-1} + x + y^{-1}z + z,$$
$$r = 1 + x + y^{-1}z + xyz, \qquad s = 1 + (x + x^{-1} + y + y^{-1})z^{-1}.$$

Then $u = p + qa + rb + sab$ is a non-trivial unit in $\mathbb{F}_2[P]$.

Proof. We claim that the inverse of u is the element $v = p_1 + q_1 a + r_1 b + s_1 ab$, where

$$p_1 = x^{-1}(a^{-1}pa), \qquad q_1 = -x^{-1}q, \qquad r_1 = -y^{-1}r, \qquad s_1 = z^{-1}(a^{-1}sa).$$

We only need to show that $uv = vu = 1$. We will perform this calculation with GAP. We first need to create the group $P = \langle a, b \rangle$.

```
gap> a := [[1,0,0,1/2],[0,-1,0,1/2],[0,0,-1,0],[0,0,0,1]];;
gap> b := [[-1,0,0,0],[0,1,0,1/2],[0,0,-1,1/2],[0,0,0,1]];;
gap> P := Group([a,b]);
```

Now we create the group algebra $F[P]$ and the embedding $P \hookrightarrow F[P]$. The field $\mathbb{F}_2$ will be `GF(2)` and the embedding will be denoted by `f`.

```
gap> R := GroupRing(GF(2),P);;
gap> f := Embedding(P, R);;
```

We first need the elements x, y and z that were defined in the statement.

```
gap> x := Image(f, a^2);;
```

```
gap> y := Image(f, b^2);;
gap> z := Image(f, (a*b)^2);;
```

Now we define the elements p, q, r and s. Note that the identity of the group algebra `R` is `One(R)`.

```
gap> p := (One(R)+x)*(One(R)+y)*(One(R)+Inverse(z));;
gap> r := One(R)+x+Inverse(y)*z+x*y*z;;
gap> q := Inverse(x)*Inverse(y)+x+Inverse(y)*z+z;;
gap> s := One(R)+(x+Inverse(x)+y+Inverse(y))*Inverse(z);
```

Rather than trying to compute the inverse of u we will show that $uv = vu = 1$. For that purpose we need to define p_1, q_1, r_1 and s_1.

```
gap> p1 := Inverse(x)*p^Image(f, a);;
gap> q1 := -Inverse(x)*q;;
gap> r1 := -Inverse(y)*r;;
gap> s1 := Inverse(z)*s^Image(f, a);;
```

Now it is time to prove the theorem.

```
gap> u := p+q*a+r*b+s*a*b;;
gap> v := p1+q1*a+r1*b+s1*a*b;;
gap> IsOne(u*v);
true
gap> IsOne(v*u);
true
```

This completes the proof of the theorem. □

Our proof of Theorem 2.5 is exactly as that of [3].

Exercise 2.6. Let p be a prime number and $\mathbb{F}_p$ be the field of size p. Use the technique for proving Gardam's theorem to prove Murray's theorem on the existence on non-trivial units in $\mathbb{F}_p[P]$. Reference: arXiv:2106.02147.

Lecture 2

We now describe some very-well known open problems in the theory of group rings and the connection between them.

Definition 2.7. A ring R is **reduced** if for all $r \in R$ such that $r^2 = 0$ one has $r = 0$.

Integral domains and boolean rings are reduced. $\mathbb{Z}/8$ and $M_2(\mathbb{R})$ are not reduced.

Example 2.8. $\mathbb{Z}^n$ with $(a_1, \dots, a_n)(b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n)$ is reduced.

The structure of reduced rings is described by Andrunakevic–Rjabuhin's theorem. It states that a ring is reduced if and only if it is a subdirect products of domains. See [4, 3.20.5] for a proof.

prob:reducido **Open problem 2.3.** Let G be a torsion-free group. Is it true that $K[G]$ is reduced?

prob:J **Open problem 2.4 (Semisimplicity).** Let G be a torsion-free group. It is true that $J(K[G]) = \{0\}$ if G is non-trivial?

Recall that an element e of a ring is said to be *idempotent* if $e^2 = e$. Examples of idempotents are 0 and 1 and these are known as the trivial idempotents.

pro:idempotente **Open problem 2.5 (Idempotents).** Let G be a torsion-free group and $\alpha \in K[G]$ be an idempotent. Is it true that $\alpha \in \{0, 1\}$?

Exercise 2.9. Prove that if $K[G]$ has no zero-divisors and $\alpha \in K[G]$ is an idempotent, then $\alpha \in \{0, 1\}$.

Exercise 2.10. Prove that $K[C_4]$ contains non-trivial zero divisors and every idempotent of $K[C_4]$ is trivial.

The problems mentioned are all related. Our goal is the prove the following implications:

$$2.4 \Longleftarrow 2.2 \Longrightarrow 2.3 \Longleftrightarrow 2.1$$

We first prove that an affirmative solution to the Units Problem 2.2 yields a solution to Problem 2.3 about the reducibility of group algebras.

Theorem 2.11. *Let K be a field of characteristic $\neq 2$ and G be a non-trivial group. Assume that $K[G]$ has only trivial units. Then $K[G]$ is reduced.*

Proof. Let $\alpha \in K[G]$ be such that $\alpha^2 = 0$. We claim that $\alpha = 0$. Since $\alpha^2 = 0$,

$$(1-\alpha)(1+\alpha) = 1-\alpha^2 = 1,$$

it follows that $1-\alpha$ is a unit of $K[G]$. Since units of $K[G]$ are trivial, there exist $\lambda \in K \setminus \{0\}$ and $g \in G$ such that $1-\alpha = \lambda g$. If $g \neq 1$, then

$$0 = \alpha^2 = (1-\lambda g)^2 = 1-2\lambda g+\lambda^2 g^2,$$

a contradiction. Therefore $g = 1$ and hence $\alpha = 1-\lambda \in K$. Since K is a field, one concludes that $\alpha = 0$. □

We now prove that an affirmative solution to the Units Problem 2.2 also yields a solution to the Jacobson Semisimplicity Problem 2.4.

Theorem 2.12. *Let G be a non-trivial group. Assume that $K[G]$ has only trivial units. If $|K| > 2$ or $|G| > 2$, then $J(K[G]) = \{0\}$.*

Proof. Let $\alpha \in J(K[G])$. There exist $\lambda \in K \setminus \{0\}$ and $g \in G$ such that $1-\alpha = \lambda g$. Assume that $g \neq 1$. If $|K| \geq 3$, then there exist $\mu \in K \setminus \{0,1\}$ such that

$$1-\alpha\mu = 1-\mu+\lambda\mu g$$

is a non-trivial unit, a contradiction. If $|G| \geq 3$, there exists $h \in G \setminus \{1, g^{-1}\}$ such that $1-\alpha h = 1-h+\lambda g h$ is a non-trivial unit, a contradiction. Thus $g = 1$ and hence $\alpha = 1-\lambda \in K$. Therefore $1+\alpha h$ is a trivial unit for all $h \neq 1$ and hence $\alpha = 0$. □

Exercise 2.13. Prove that if $G = \langle g \rangle \simeq \mathbb{Z}/2$, then $J(\mathbb{F}_2[G]) = \{0, g-1\} \neq \{0\}$.

§3. Passman's theorem

Now we prove that an affirmative solution to the Units Problem (Open Problem 2.2) yields a solution to Open Problem 2.1 about zero divisors in group algebras. The proof is hard and requires some preliminaries. We first need to discuss a group theoretical tool known as the *transfer map*.

If H is a subgroup of G, a **transversal** of H in G is a complete set of coset representatives of G/H.

lem:d **Lemma 3.1.** *Let G be a group and H be a subgroup of G of finite index. Let R and S be transversals of H in G and let $\alpha\colon H \to H/[H,H]$ be the canonical map. Then*

$$d(R,S) = \prod \alpha(rs^{-1}),$$

where the product is taken over all pairs $(r,s) \in R\times S$ *such that* $Hr = Hs$*, is well-defined and satisfies the following properties:*

1) $d(R,S)^{-1} = d(S,R)$.
2) $d(R,S)d(S,T) = d(R,T)$ *for all transversal* T *of* H *in* G.
3) $d(Rg,Sg) = d(R,S)$ *for all* $g \in G$.
4) $d(Rg,R) = d(Sg,S)$ *for all* $g \in G$.

Proof. The product that defines $d(R,S)$ is well-defined since $H/[H,H]$ is an abelian group. The first three claim are trivial. Let us prove 4). By 2),

$$d(Rg,Sg)d(Sg,S)d(S,R) = d(Rg,S)d(S,R) = d(Rg,R).$$

Since $H/[H,H]$ is abelian, 1) and 3) imply that

$$d(Rg,Sg)d(Sg,S)d(S,R) = d(R,S)d(S,R)d(Sg,S) = d(Sg,S). \qquad \square$$

We are know ready to state and prove the theorem:

thm:transfer

Theorem 3.2. *Let* G *be a group and* H *be a finite-index subgroup of* G*. The map*

$$\nu\colon G \to H/[H,H], \quad g \mapsto d(Rg,R),$$

does not depend on the transversal R *of* H *in* G *and it is a group homomorphism.*

Proof. The lemma implies that the map does not depend on the transversal used. Moreover, ν is a group homomorphism, as

$$\nu(gh) = d(R(gh),R) = d(R(gh),Rh)d(Rh,R) = d(Rg,R)d(Rh,R) = \nu(g)\nu(h). \quad \square$$

The theorem justifies the following definition:

Definition 3.3. Let G be a group and H be a finite-index subgroup of G. The **transfer map** of G in H is the group homomorphism

$$\nu\colon G \to H/[H,H], \quad g \mapsto d(Rg,R),$$

of Theorem 3.2, where R is some transversal of H in G.

We need methods for computing the transfer map. If H is a subgroup of G and $(G:H) = n$, let $T = \{x_1,\dots,x_n\}$ be a transversal of H. For $g \in G$ let

$$\nu(g) = \prod \alpha(xy^{-1}),$$

where the product is taken over all pairs $(x,y) \in (Tg,T)$ such that $Hx = Hy$ and $\alpha\colon H \to H/[H,H]$ is the canonical map. If we write $x = x_ig$ for some $i \in \{1,\dots,n\}$, then $Hx_ig = Hx_{\sigma(i)}$ for some permutation $\sigma \in \mathbb{S}_n$. Thus

$$\nu(g) = \prod_{i=1}^{n} \alpha(x_igx_{\sigma(i)}^{-1}).$$

lem:transfer **Lemma 3.4.** *Let G be a group and H be a subgroup such that $(G:H)=n$. Let T be a transversal of H in G. For each $g\in G$ there exist k and positive integers $n_1,\dots,n_k$ such that $n_1+\cdots+n_k=n$ and elements $t_1,\dots,t_k\in T$ such that*

$$\nu(g)=\prod_{i=1}^{k}\alpha(t_ig^{n_i}t_i^{-1}),$$

where $\alpha\colon H\to H/[H,H]$ is the canonical map.

Proof. There exists $\sigma\in\mathbb{S}_n$ such that

$$\nu(g)=\prod_{i=1}^{n}\alpha(t_igt_{\sigma(i)}^{-1}).$$

Write σ as a product of k disjoint cycles $\sigma=\alpha_1\cdots\alpha_k$, where each α_j is a cycle of length n_j. For every cycle of the form $(i_1\cdots i_{n_j})$ we reorder the product in such a way that

$$\alpha(x_{i_1}gx_{i_2}^{-1})\alpha(x_{i_2}gx_{i_3}^{-1})\cdots\alpha(x_{i_{n_j}}gx_{i_1}^{-1})=\alpha(x_{i_1}g^{n_1}x_{i_1}^{-1}).$$

There exist $t_1,\dots,t_k\in T$ such that $\nu(g)=\prod_{j=1}^{k}\alpha(t_ig^{n_i}t_i^{-1})$. $\square$

An application:

pro:center **Proposition 3.5.** *If G is a group such that $Z(G)$ has finite index n, then $(gh)^n=g^nh^n$ for all $g,h\in G$.*

Proof. Let $g\in G$. By Lemma 3.4 there are positive integers $n_1,\dots,n_k$ such that $n_1+\cdots+n_k=n$ and elements $t_1,\dots,t_k$ of a transversal of $Z(G)$ in G such that

$$\nu(g)=\prod_{i=1}^{k}\alpha(t_ig^{n_1}t_i^{-1}),$$

where $\alpha\colon G\to H/[H,H]$ is the canonical map. Since $g^{n_i}\in Z(G)$ for all $i\in\{1,\dots,k\}$ (as $t_ig^{n_i}t_i^{-1}\in Z(G)$), it follows that $\nu(g)=g^{n_1+\cdots+n_k}=g^n$. Since ν is a group homomorphism by Theorem 3.2, we conclude that

$$(gh)^n=\nu(gh)=\nu(g)\nu(h)=g^nh^n. \qquad\square$$

For a group G we consider

$$\Delta(G)=\{g\in G:(G:C_G(g))<\infty\}.$$

Exercise 3.6. Prove that $\Delta(\Delta(G))=\Delta(G)$.

Lemma 3.7. *If G is a group, then $\Delta(G)$ is a characteristic subgroup of G.*

Proof. We first prove that $\Delta(G)$ is a subgroup of G. If $x,y\in\Delta(G)$ and $g\in G$, then $g(xy^{-1})g^{-1}=(gxg^{-1})(gyg^{-1})^{-1}$. Moreover, $1\in\Delta(G)$. Let us show now that $\Delta(G)$ is characteristic in G. If $f\in\mathrm{Aut}(G)$ and $x\in G$, then, since

$$f(gxg^{-1}) = f(g)f(x)f(g)^{-1},$$

it follows that $f(x) \in \Delta(G)$. $\square$

Exercise 3.8. Prove that if $G = \langle r, s : s^2 = 1, srs = r^{-1} \rangle$ is the infinite dihedral group, then $\Delta(G) = \langle r \rangle$.

Exercise 3.9. Let H and K be finite-index subgroups of G. Prove that

$$(G : H \cap K) \leq (G : H)(G : K).$$

Lecture 3

pro:FCabeliano

Proposition 3.10. *If G is a torsion-free group such that $\Delta(G) = G$, then G is abelian.*

Proof. Let $x, y \in G$ and $S = \langle x, y\rangle$. The group $Z(S) = C_S(x) \cap C_S(y)$ has finite index, say n, in S. By Proposition 3.5, the map $S \to Z(S)$, $s \mapsto s^n$, is a group homomorphism. Thus

$$[x, y]^n = (xyx^{-1}y^{-1})^n = x^n y^n x^{-n} y^{-n} = 1$$

as $x^n \in Z(S)$. Since G is torsion-free, $[x, y] = 1$. □

lem:Neumann

Lemma 3.11 (Neumann). *Let $H_1, \dots, H_m$ be subgroups of G. Assume there are finitely many elements $a_{ij} \in G$, $1 \le i \le m$, $1 \le j \le n$, such that*

$$G = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} H_i a_{ij}.$$

Then some H_i has finite index in G.

Proof. We proceed by induction on m. The case $m = 1$ is trivial. Let us assume that $m \ge 2$. If $(G : H_1) = \infty$, there exists $b \in G$ such that

$$Hb \cap \left(\bigcup_{j=1}^{n} H_1 a_{1j} \right) = \emptyset.$$

Since $H_1 b \subseteq \bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij}$, it follows that

$$H_1 a_{1k} \subseteq \bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H a_{ij} b^{-1} a_{1k}.$$

Hence G can be covered by finitely many cosets of $H_2, \dots, H_m$. By the inductive hypothesis, some of these H_j has finite index in G. □

We now consider a projection operator of group algebras. If G is a group and H is a subgroup of G, let

$$\pi_H : K[G] \to K[H], \quad \pi_H\left(\sum_{g\in G}\lambda_g g\right) = \sum_{g\in H}\lambda_g g.$$

If R and S are rings, a (R,S)-bimodule is an abelian group M that is both a left R-module and a right S-module and the compatibility condition

$$(rm)s = r(ms)$$

holds for all $r \in R$, $s \in S$ and $m \in M$.

Exercise 3.12. Let G be a group and H be a subgroup of G. Prove that if $\alpha \in K[G]$, then π_H is a $(K[H],K[H])$-bimodule homomorphism with usual left and right multiplications,

$$\pi_H(\beta\alpha\gamma) = \beta\pi_H(\alpha)\gamma$$

for all $\beta,\gamma \in K[H]$.

lem:escritura

Lemma 3.13. *Let X be a left transversal of H in G. Every $\alpha \in K[G]$ can be written uniquely as*

$$\alpha = \sum_{x\in X} x\alpha_x,$$

where $\alpha_x = \pi_H(x^{-1}\alpha) \in K[H]$.

Proof. Let $\alpha \in K[G]$. Since $\operatorname{supp}\alpha$ is finite, $\operatorname{supp}\alpha$ is contained in finitely many cosets of H, say $x_1H,\dots,x_nH$, where each x_j belongs to X. Write $\alpha = \alpha_1+\cdots+\alpha_n$, where $\alpha_i = \sum_{g\in x_iH}\lambda_g g$. If $g \in x_iH$, then $x_i^{-1}g \in H$ and hence

$$\alpha = \sum_{i=1}^{n} x_i(x_i^{-1}\alpha_i) = \sum_{x\in X} x\alpha_x$$

with $\alpha_x \in K[H]$ for all $x \in X$. For the uniqueness, note that for each $x \in X$ the previous exercise implies that

$$\pi_H(x^{-1}\alpha) = \pi_H\left(\sum_{y\in X} x^{-1}y\alpha_y\right) = \sum_{y\in X}\pi_H(x^{-1}y)\alpha_y = \alpha_x,$$

as

$$\pi_H(x^{-1}y) = \begin{cases} 1 & \text{si } x = y,\\ 0 & \text{si } x \neq y.\end{cases}$$ □

lem:ideal_pi

Lemma 3.14. *Let G be a group and H be a subgroup of G. If I is a non-zero left ideal of $K[G]$, then $\pi_H(I) \neq 0$.*

Proof. Let X be a left transversal of H in G and $\alpha \in I\setminus\{0\}$. By Lemma 3.13 we can write $\alpha = \sum_{x\in X} x\alpha_x$ with $\alpha_x = \pi_H(x^{-1}\alpha) \in K[H]$ for all $x \in X$. Since $\alpha \neq 0$, there exists $y \in X$ such that $0 \neq \alpha_y = \pi_H(y^{-1}\alpha) \in \pi_H(I)$ ($y^{-1}\alpha \in I$ since I is a left ideal). □

Another application:

Proposition 3.15. *Let G be a group, H be a subgroup of G and $\alpha \in K[H]$. The following statements hold:*

1) *α is invertible in $K[H]$ if and only if α is invertible in $K[G]$.*
2) *α is a zero divisor of $K[H]$ if and only if α is a zero divisor of $K[G]$.*

Proof. If α is invertible in $K[G]$, there exists $\beta \in K[G]$ such that $\alpha\beta = \beta\alpha = 1$. Apply π_H and use that π_H is a $(K[H], K[H])$-bimodule homomorphism to obtain

$$\alpha\pi_H(\beta) = \pi_H(\alpha\beta) = \pi_H(1) = 1 = \pi_H(1) = \pi_H(\beta\alpha) = \pi_H(\beta)\alpha.$$

Assume now that $\alpha\beta = 0$ for some $\beta \in K[G] \setminus \{0\}$. Let $g \in G$ be such that $1 \in \operatorname{supp}(\beta g)$. Since $\alpha(\beta g) = 0$,

$$0 = \pi_H(0) = \pi_H(\alpha(\beta g)) = \alpha\pi_H(\beta g),$$

where $\pi_H(\beta g) \in K[H] \setminus \{0\}$, as $1 \in \operatorname{supp}(\beta g)$. □

lem:Passman

Lemma 3.16 (Passman). *Let G be a group and $\gamma_1, \gamma_2 \in K[G]$ be such that $\gamma_1 K[G]\gamma_2 = 0$. Then $\pi_{\Delta(G)}(\gamma_1)\pi_{\Delta(G)}(\gamma_2) = 0$.*

Proof. It is enough to show that $\pi_{\Delta(G)}(\gamma_1)\gamma_2 = 0$, as in this case

$$0 = \pi_{\Delta(G)}(\pi_{\Delta(G)}(\gamma_1)\gamma_2) = \pi_\Delta(\gamma_1)\pi_{\Delta(G)}(\gamma_2).$$

Write $\gamma_1 = \alpha_1 + \beta_1$, where

$$\begin{aligned}
\alpha_1 &= a_1u_1 + \cdots + a_ru_r, && u_1, \dots, u_r \in \Delta(G),\\
\beta_1 &= b_1v_1 + \cdots + b_sv_s, && v_1, \dots, v_s \notin \Delta(G),\\
\gamma_2 &= c_1w_1 + \cdots + c_tw_t, && w_1, \dots, w_t \in G.
\end{aligned}$$

The subgroup $C = \bigcap_{i=1}^r C_G(u_i)$ has finite index in G. Assume that

$$0 \neq \pi_\Delta(\gamma_1)\gamma_2 = \alpha_1\gamma_2.$$

Let $g \in \operatorname{supp}(\alpha_1\gamma_2)$. If v_i is a conjugate in G of some gw_j^{-1}, let $g_{ij} \in G$ be such that $g_{ij}^{-1}v_ig_{ij} = gw_j^{-1}$. If v_i and gw_j^{-1} are not conjugate, we take $g_{ij} = 1$.

For every $x \in C$ it follows that $\alpha_1\gamma_2 = (x^{-1}\alpha_1x)\gamma_2$. Since

$$x^{-1}\gamma_1x\gamma_2 \in x^{-1}\gamma_1K[G]\gamma_2 = 0,$$

it follows that

$$\begin{aligned}
(a_1u_1 + \cdots + a_ru_r)\gamma_2 &= \alpha_1\gamma_2 = x^{-1}\alpha_1x\gamma_2 = -x^{-1}\beta_1x\gamma_2\\
&= -x^{-1}(b_1v_1 + \cdots + b_sv_r)x(c_1w_1 + \cdots + c_tw_t).
\end{aligned}$$

Now $g \in \operatorname{supp}(\alpha_1\gamma_2)$ implies that there exist i, j such that $g = x^{-1}v_i x w_j$. Thus v_i and gw_j^{-1} are conjugate and hence $x^{-1}v_i x = gw_j^{-1} = g_{ij}^{-1}v_i g_{ij}$, that is $x \in C_G(v_i)g_{ij}$. This proves that

$$C \subseteq \bigcup_{i,j} C_G(v_i)g_{ij}.$$

Since C has finite index in G, it follows that G can be covered by finitely many cosets of the $C_G(v_i)$. Every $v_i \notin \Delta(G)$, so each $C_G(v_i)$ has infinite index in G, a contradiction to Neumann's lemma. □

Theorem 3.17 (Passman). *Let G be a torsion-free group. If $K[G]$ is reduced, then $K[G]$ is a domain.*

Proof. Assume that $K[G]$ is not a domain. Let $\gamma_1, \gamma_2 \in K[G] \setminus \{0\}$ be such that $\gamma_2\gamma_1 = 0$. If $\alpha \in K[G]$, then

$$(\gamma_1\alpha\gamma_2)^2 = \gamma_1\alpha\gamma_2\gamma_1\alpha\gamma_2 = 0$$

and thus $\gamma_1\alpha\gamma_2 = 0$, as $K[G]$ is reduced. In particular, $\gamma_1 K[G]\gamma_2 = 0$. Let I be the left ideal of $K[G]$ generated by γ_2. Since $I \neq 0$, $\pi_{\Delta(G)}(I) \neq 0$ by Lemma 3.14 and hence $\pi_{\Delta(G)}(\beta\gamma_2) \neq 0$ for some $\beta \in K[G]$. Similarly, $\pi_{\Delta(G)}(\gamma_1\alpha) \neq 0$ for some $\alpha \in K[G]$. Since

$$\gamma_1\alpha K[G]\beta\gamma_2 \subseteq \gamma_1 K[G]\gamma_2 = 0,$$

it follows that $\pi_{\Delta(G)}(\gamma_1\alpha)\pi_{\Delta(G)}(\beta\gamma_2) = 0$ by Passman's lemma. Hence $K[\Delta(G)]$ has zero divisors, a contradictions since $\Delta(G)$ is an abelian group. □

Lecture 4

§4. Bi-ordered groups

Based on Example 2.1 we will study some properties of groups.

Recall that a **total order** is a partial order in which any two elements are comparable. This means that a total order is a binary relation $\leq$ on some set X such that for all $x, y, z \in X$ one has

1) $x \leq x$.
2) $x \leq y$ and $y \leq z$ imply $x \leq z$.
3) $x \leq y$ and $y \leq x$ imply $x = y$.
4) $x \leq y$ or $y \leq x$.

A set equipped with a total order is a **totally ordered set**.

Definition 4.1. A group G is **bi-ordered** if there exists a total order $<$ in G such that $x < y$ implies that $xz < yz$ and $zx < zy$ for all $x, y, z \in G$.

Example 4.2. The group $\mathbb{R}_{>0}$ of positive real numbers is bi-ordered.

Exercise 4.3. Let G be a bi-ordered group and $x, x_1, y, y_1 \in G$. Prove that $x < y$ and $x_1 < y_1$ imply $xx_1 < yy_1$.

Exercise 4.4. Let G be a bi-ordered group and $g, h \in G$. Prove that $g^n = h^n$ for some $n > 0$ implies $g = h$.

Definition 4.5. Let G be a bi-ordered group. The **positive cone** of G is the set $P(G) = \{x \in G : 1 < x\}$.

Let us state some properties of positive cones.

pro:biordenableP1

Proposition 4.6. *Let G be a bi-ordered group and let P be its positive cone.*

1) *P is closed under multiplication, i.e. $PP \subseteq P$.*
2) *$G = P \cup P^{-1} \cup \{1\}$ (disjoint union).*

3) $xPx^{-1} = P$ *for all* $x \in G$.

Proof. If $x, y \in P$ and $z \in G$, then, since $1 < x$ and $1 < y$, it follows that $1 < xy$. Thus $1 = z1z^{-1} < zxz^{-1}$. It remains to prove the second claim. If $g \in G$, then $g = 1$ or $g > 1$ or $g < 1$. Note that $g < 1$ if and only if $1 < g^{-1}$, so the claim follows. □

The previous proposition admits a converse statement.

pro:biordenableP2

Proposition 4.7. *Let* G *be a group and* P *be a subset of* G *such that* P *is closed under multiplication,* $G = P \cup P^{-1} \cup \{1\}$ *(disjoint union) and* $xPx^{-1} = P$ *for all* $x \in G$. *Let* $x < y$ *whenever* $yx^{-1} \in P$. *Then* G *is bi-ordered with positive cone is* P.

Proof. Let $x, y \in G$. Since $yx^{-1} \in G$ and $G = P \cup P^{-1} \cup \{1\}$ (disjoint union), either $yx^{-1} \in P$ or $xy^{-1} = (yx^{-1})^{-1} \in P$ or $yx^{-1} = 1$. Thus either $x < y$ or $y < x$ or $x = y$. If $x < y$ and $z \in G$, then $zx < zy$, as $(zy)(zx)^{-1} = z(yx^{-1})z^{-1} \in P$ and $zPz^{-1} = P$. Moreover, $xz < yz$ since $(yz)(xz)^{-1} = yx^{-1} \in P$. To prove that P is the positive cone of G note that $x1^{-1} = x \in P$ if and only if $1 < x$. □

An important property:

pro:BOsintorsion

Proposition 4.8. *Bi-ordered groups are torsion-free.*

Proof. Let G be a bi-ordered group and $g \in G \setminus \{1\}$. If $g > 1$, then $1 < g < g^2 < \cdots$. If $g < 1$, then $1 > g > g^2 > \cdots$. Hence $g^n \neq 1$ for all $n \neq 0$. □

The converse of the previous proposition does not hold.

Exercise 4.9. Let $G = \langle x, y : yxy^{-1} = x^{-1} \rangle$.

1) Prove that x and y are torsion-free.
2) Prove that G is torsion-free.
3) Prove that $G \simeq \langle a, b : a^2 = b^2 \rangle$.

Example 4.10. The torsion-free group $G = \langle x, y : yxy^{-1} = x^{-1} \rangle$ is not bi-ordered. If not, let P be the positive cone. If $x \in P$, then $yxy^{-1} = x^{-1} \in P$, a contradiction. Hence $x^{-1} \in P$ and $x = y^{-1}x^{-1}y \in P$, a contradiction.

thm:BO

Theorem 4.11. *Let* G *be a bi-ordered group. Then* $K[G]$ *is a domain such that only has trivial units. Moreover, if* G *is non-trivial, then* $J(K[G]) = \{0\}$.

Proof. Let $\alpha, \beta \in K[G]$ be such that

$$\alpha = \sum_{i=1}^{m} a_i g_i, \qquad g_1 < g_2 < \cdots < g_m, \qquad a_i \neq 0 \qquad \forall i \in \{1, \dots, m\},$$
$$\beta = \sum_{j=1}^{n} b_j h_j, \qquad h_1 < h_2 < \cdots < h_n, \qquad b_j \neq 0 \qquad \forall j \in \{1, \dots, n\}.$$

Then

$$g_1 h_1 \leq g_i h_j \leq g_m h_n$$

for all i,j. Moreover, $g_1h_1 = g_ih_j$ if and only if $i = j = 1$. The coefficient of g_1h_1 in $\alpha\beta$ is $a_1b_1 \neq 0$. In particular, $\alpha\beta \neq 0$. If $\alpha\beta = \beta\alpha = 1$, then the coefficient of g_mh_n in $\alpha\beta$ is a_mb_n. Hence $m = n = 1$ and therefore $\alpha = a_1g_1$ and $\beta = b_1h_1$ with $a_1b_1 = b_1a_1 = 1$ in K and $g_1h_1 = 1$ in G. □

thm:Levi

Theorem 4.12 (Levi). *Let A be an abelian group. Then A is bi-ordered if and only if A is torsion-free.*

Proof. If A is bi-ordered, then A is torsion-free. Let us prove the non-trivial implication, so assume that A is torsion-free abelian. Let $\mathcal{S}$ be the class of subsets P of A such that $0 \in P$, are closed under the addition of A and satisfy the following property: if $x \in P$ and $-x \in P$, then $x = 0$. Clearly, $\mathcal{S} \neq \emptyset$, as $\{0\} \in \mathcal{S}$. The inclusion turns $\mathcal{S}$ into a partially ordered set and $\bigcup_{i\in I} P_i$ is an upper bound for the chain $\{P_i : i \in I\}$. By Zorn's lemma, $\mathcal{S}$ admits a maximal element $P \in \mathcal{S}$.

Claim. If $x \in A$ is such that $kx \in P$ for some $k > 0$, then $x \in P$.

Let $Q = \{x \in A : kx \in P \text{ for some } k > 0\}$. We will show that $Q \in \mathcal{S}$. Clearly, $0 \in Q$. Moreover, Q is closed under addition, as $k_1x_1 \in P$ and $k_2x_2 \in P$ imply $k_1k_2(x_1 + x_2) \in P$. Let $x \in A$ be such that $x \in Q$ and $-x \in Q$. Thus $kx \in P$ and $l(-x) \in P$ for some $l > 0$. Since $klx \in P$ and $kl(-x) \in P$, it follows that $klx = 0$, a contradiction since A is torsion-free. Hence $x \in Q \subseteq P$.

Claim. If $x \in A$ is such that $x \notin P$, then $-x \in P$.

Assume that $-x \notin P$ and let $P_1 = \{y + nx : y \in P, n \geq 0\}$. We will show that $P_1 \in \mathcal{S}$. Clearly, $0 \in P_1$ and P_1 is closed under addition. If $P_1 \notin S$, there exists

$$0 \neq y_1 + n_1x = -(y_2 + n_2x),$$

where $y_1, y_2 \in P$ and $n_1, n_2 \geq 0$. Thus $y_1 + y_2 = -(n_1 + n_2)x$. If $n_1 = n_2 = 0$, then $y_1 = -y_2 \in P$ and $y_1 = y_2 = 0$, so it follows that $y_1 + n_1x = 0$, a contradiction. If $n_1 + n_2 > 0$, then, since

$$(n_1 + n_2)(-x) = y_1 + y_2 \in P,$$

it follows from the first claim that $-x \in P$, a contradiction. Let us show that $P_1 \in \mathcal{S}$. Since $P \subseteq P_1$, the maximality of P implies that $x \in P = P_1$.

By Proposition 4.7, $P^* = P \setminus \{0\}$ is the positive cone of a bi-order in A. In fact, P^* is closed under addition, as $x, y \in P^*$ implies that $x + y \in P$ and $x + y = 0$ implies $x = y = 0$, as $x = -y \in P$. Moreover, $G = P^* \cup -P^* \cup \{0\}$ (disjoint union), as the second claim states that $x \notin P^*$ implies $-x \in P$. □

Corollary 4.13. *Let A be a non-trivial torsion-free abelian group. Then $K[A]$ is a domain that only admits trivial units and $J(K[A]) = \{0\}$.*

Proof. Apply Levi's theorem and Theorem 4.11. □

Some exercises.

Exercise 4.14. Let N be a central subgroup of G. If N and G/N are bi-ordered, then G is bi-ordered. Prove with an example that N needs to be central, normal is not enough.

Exercise 4.15. Let G be a group that admits a sequence

$$\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

such that each G_k is normal in G_{k+1} and each quotient G_{k+1}/G_k is torsion-free abelian. Prove that G is bi-ordered.

Exercise 4.16. Prove that torsion-free nilpotent groups are bi-ordered.

§5. Left-ordered groups

Definition 5.1. Un grupo G se dice **ordenable a derecha** si existe un orden total $<$ en G tal que si $x < y$ entonces $xz < yz$ para todo $x, y, z \in G$.

Si G es un grupo ordenable a derecha, se define el cono positivo de G como el subconjunto $P(G) = \{x \in G : 1 < x\}$.

Exercise 5.2. Sea G un grupo ordenable a derecha con cono positivo P. Demuestre las siguientes afirmaciones:

1) P es cerrado por multiplicación.
2) $G = P \cup P^{-1} \cup \{1\}$ (unión disjunta).

Exercise 5.3. Sea G un grupo y sea P un subconjunto cerrado por multiplicación y tal que $G = P \cup P^{-1} \cup \{1\}$ (unión disjunta). Demuestre que si se define $x < y$ si y sólo si $yx^{-1} \in P$, entonces G es ordenable a derecha con cono positivo P.

Lemma 5.4. *Sea G un grupo y sea N un subgrupo normal de G. Si N y G/N son ordenables a derecha, entonces G también lo es.*

Proof. Como N y G/N son ordenables a derecha, existen los conos positivos $P(N)$ y $P(G/N)$. Sea $\pi\colon G \to G/N$ el morfismo canónico y sea

$$P(G) = \{x \in G : \pi(x) \in P(G/N) \text{ o bien } x \in N\}.$$

Dejamos como ejercicio demostrar que $P(G)$ es cerrado por la multiplicación y que $G = P(G) \cup P(G)^{-1} \cup \{1\}$ (unión disjunta). Luego G es ordenable a derecha. □

Para dar un criterio de ordenabilidad necesitamos un lema:

lemma:fg **Lemma 5.5.** *Sea G un grupo finitamente generado y sea H un subgrupo de índice finito. Entonces H es finitamente generado.*

Proof. Supongamos que G está generado por $\{g_1, \dots, g_m\}$ y supongamos que para cada i existe k tal que $g_i^{-1} = g_k$. Sea $t_1, \dots, t_n$ un conjunto de representantes de G/H. Para $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, escribimos

$$t_i g_j = h(i,j) t_{k(i,j)}.$$

Vamos a demostrar que H está generado por los $h(i,j)$. Sea $x \in H$. Escribamos

$$\begin{aligned} x &= g_{i_1} \cdots g_{i_s} \\ &= (t_1 g_{i_1}) g_{i_2} \cdots g_{i_s} \\ &= h(1,i_1) t_{k_1} g_{i_2} \cdots g_{i_s} \\ &= h(1,i_1) h(k_1,i_2) t_{k_2} g_{i_3} \cdots g_{i_s} \\ &= h(1,i_1) h(k_1,i_2) \cdots h(k_{s-1}, g_{i_s}) t_{k_s}, \end{aligned}$$

donde $k_1, \dots, k_{s-1} \in \{1, \dots, n\}$. Como $t_{k_s} \in H$, $t_{k_s} = t_1 \in H$ y luego $x \in H$. □

El siguiente teorema nos da un criterio de ordenabilidad a derecha:

Theorem 5.6. *Sea G un grupo libre de torsión y finitamente generado. Si A es un subgrupo normal abeliano tal que G/A es finito y cíclico, entonces G es ordenable a derecha.*

Proof. Primero observemos que si A es trivial, entonces G también es trivial. Supongamos entonces que $A \neq 1$. Como A tiene índice finito, es finitamente generado. Procederemos por inducción en la cantidad de generadores de A. Como G/A es cíclico, existe $x \in G$ tal que $G = \langle A, x \rangle$. Luego $[x, A] = \langle [x, a] : a \in A \rangle$ es un subgrupo normal de G tal que $A/C_A(x) \simeq [x, A]$ (pues $a \mapsto [x, a]$ es un morfismo de grupos $A \to A$ con imagen $[x, A]$ y núcleo $C_A(x)$). Si $\pi \colon G \to G/[x, A]$, entonces $G/[x, A] = \langle \pi(A), \pi(x) \rangle$ y luego $G/[x, A]$ es abeliano pues $[\pi(x), \pi(A)] = \pi[x, A] = 1$. Además $G/[x, A]$ es finitamente generado pues G es finitamente generado. Como $(G : A) = n$ y G no tiene torsión, $1 \neq x^n \in A$. Luego $x^n \in C_A(x)$ y entonces $1 \leq \operatorname{rank} C_A(x) < \operatorname{rank} A$ (si $\operatorname{rank} C_A(x) = \operatorname{rank} A$, entonces $[x, A]$ sería un subgrupo de torsión de A, una contradicción pues $x \notin A$). Luego

$$\operatorname{rank}[x, A] = \operatorname{rank}(A/C_A(x)) \leq \operatorname{rank} A - 1$$

y entonces $\operatorname{rank}(A/[x, A]) \geq 1$. Demostramos así que $A/[x, A]$ es infinito y luego $G/[x, A]$ es también infinito.

Como $G/[x, A]$ es un grupo abeliano finitamente generado e infinito, existe un un subgrupo normal H de G tal que $[x, A] \subseteq H$ y $G/H \simeq \mathbb{Z}$. El subgrupo $B = A \cap H$ es abeliano, normal en H y cumple que H/B es cíclico (pues puede identificarse con un subgrupo de G/A). Como $\operatorname{rank} B < \operatorname{rank} A$, la hipótesis inductiva implica que H es ordenable a derecha y luego G también es ordenable a derecha. □

Exercise 5.7 (Malcev–Neumann). Sea G un grupo ordenable a derecha. Demuestre que $K[G]$ no tiene divisores de cero ni unidades no triviales.

En 1973 Formanek demostró que la conjetura de los divisores de cero es verdadera para grupos super resolubles sin torsión. En 1976 Brown e independientemente Farkas y Snider demostraron que la conjetura es verdadera para grupos policíclicos-por-finitos sin torsión.

Lecture 5

§6. Unique product groups

Sea G un grupo y sean $A, B \subseteq G$ subconjuntos no vacíos. Diremos que un elemento $g \in G$ es un producto único en AB si $g = ab = a_1 b_1$ con $a, a_1 \in A$ y $b, b_1 \in B$ implica que $a = a_1$ y $b = b_1$.

Definition 6.1. Se dice que un grupo G tiene la **propiedad del producto único** si dados dos subconjuntos $A, B \subseteq G$ finitos y no vacíos existe al menos un producto único en AB.

Proposition 6.2. *Si un grupo G es ordenable a derecha, entonces G tiene la propiedad del producto único.*

Proof. Sean $A = \{a_1, \dots, a_n\} \subseteq G$ y $B \subseteq G$ ambos finitos y no vacíos. Supongamos que $a_1 < a_2 < \cdots < a_n$. Sea $c \in B$ tal que $a_1 c$ es el mínimo del conjunto $a_1 B = \{a_1 b : b \in B\}$. Veamos que $a_1 c$ admite una única representación de la forma $\alpha\beta$ con $\alpha \in A$ y $\beta \in B$. Si $a_1 c = ab$, entonces, como $ab = a_1 c \leq a_1 b$, se tiene que $a \leq a_1$ y luego $a = a_1$ y $b = c$. □

Exercise 6.3. Demuestre que un grupo que satisface la propiedad del producto único es libre de torsión.

The converse does not hold. Promislow's group is a celebrated counterexample.

Theorem 6.4 (Promislow). *The group $G = \langle a, b : a^{-1}b^2a = b^{-2}, b^{-1}a^2b = a^{-2}\rangle$ does not have the unique product property.*

Proof. Let

$$S = \{a^2b, b^2a, aba^{-1}, (b^2a)^{-1}, (ab)^{-2}, b, (ab)^2x, (ab)^2, (aba)^{-1}, bab, b^{-1}, a, aba, a^{-1}\}. \quad (5.1)$$

eq:Promislow

We use GAP and the representation $G \to \mathbf{GL}(4, \mathbb{Q})$ given by

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to check that G does not have unique product property, as each

$$s \in S^2 = \{s_1 s_2 : s_1, s_2 \in S\}$$

admits at least two different decompositions of the form $s = xy = uv$ for $x, y, u, v \in S$. We first create the matrix representations of a and b.

```
gap> a := [[1,0,0,1/2],[0,-1,0,1/2],[0,0,-1,0],[0,0,0,1]];;
gap> b := [[-1,0,0,0],[0,1,0,1/2],[0,0,-1,1/2],[0,0,0,1]];;
```

Now we create a function that produces the set S.

```
gap> Promislow := function(x, y)
> return Set([
> x^2*y,
> y^2*x,
> x*y*Inverse(x),
> (y^2*x)^(-1),
> (x*y)^(-2),
> y,
> (x*y)^2*x,
> (x*y)^2,
> (x*y*x)^(-1),
> y*x*y,
> y^(-1),
> x,
> x*y*x,
> x^(-1)
]);
end;;
```

So the set S of (5.1) will be `Promislow(a,b)`. We now create a function that checks whether every element of a Promislow subset admits more than one representation.

```
gap> is_UPP := function(S)
> local l,x,y;
> l := [];
> for x in S do
> for y in S do
> Add(l,x*y);
> od;
> od;
> if ForAll(Collected(l), x->x[2] <> 1) then
> return false;
> else
> return fail;
> fi;
> end;;
```

Finally, we check whether every element of S admits more than one representation.

```
gap> S := Promislow(a,b);;
gap> is_UPP(S);
false
```

This completes the proof. □

There are other examples.

Definition 6.5. Se dice que un grupo G tiene la **propiedad del doble producto único** si dados dos subconjuntos $A,B\subseteq G$ finitos y no vacíos tales que $|A|+|B|>2$ existen al menos dos productos únicos en AB.

theorem:Strojnowski

Theorem 6.6 (Strojnowski). *Sea G un grupo. Las siguientes afirmaciones son equivalentes:*

1) *G tiene la propiedad del doble producto único.*
2) *Para todo subconjunto $A\subseteq G$ finito y no vacío, existe al menos un producto único en $AA=\{a_1a_2:a_1,a_2\in A\}$.*
3) *G tiene la propiedad del producto único.*

Proof. La implicación (1) $\implies$ (2) es trivial. Demostremos que vale (2) $\implies$ (3). Si G no tiene la propiedad del producto único, existen subconjuntos $A,B\subseteq G$ finitos y no vacíos tales que todo elemento de AB admite al menos dos representaciones. Sea $C=AB$. Todo $c\in C$ es de la forma $c=(a_1b_1)(a_2b_2)$ con $a_1,a_2\in A$ y $b_1,b_2\in B$. Como $a_2^{-1}b_1^{-1}\in AB$, existen $a_3\in A\setminus\{a_2\}$ y $b_3\in B\setminus\{b_1\}$ tales que $a_2^{-1}b_1^{-1}=a_3^{-1}b_3^{-1}$. Luego $b_1a_2=b_3a_3$ y entonces

$$c=(a_1b_1)(a_2b_2)=(a_1b_3)(a_3b_2)$$

son dos representaciones distintas de c en AB. pues $a_2\neq a_3$ y $b_1\neq b_3$.

Demostremos ahora que (3) $\implies$ (1). Si G tiene la propiedad del producto único pero no tiene la propiedad del doble producto único, existen subconjuntos $A,B\subseteq G$ finitos y no vacíos con $|A|+|B|>2$ tales que en AB existe un único elemento ab con una única representación en AB. Sean $C=a^{-1}A$ y $D=Bb^{-1}$. Entonces $1\in C\cap D$ y el elemento neutro 1 admite una única representación en CD (pues si $1=cd$ con $c=a^{-1}a_1\neq 1$ y $d=b_1b^{-1}\neq 1$, entonces $ab=a_1b_1$ con $a\neq a_1$ y $b\neq b_1$). Sean $E=D^{-1}C$ y $F=DC^{-1}$. Todo elemento de EF se escribe como $(d_1^{-1}c_1)(d_2c_2^{-1})$. Si $c_1\neq 1$ o $d_2\neq 1$ entonces $c_1d_2=c_3d_3$ para algún $c_3\in C\setminus\{c_1\}$ y algún $d_3\in D\setminus\{d_2\}$. Entonces $(d_1^{-1}c_1)(d_2c_2^{-1})=(d_1^{-1}c_3)(d_3c_2^{-1})$ son dos representaciones distintas para $(d_1^{-1}c_1)(d_2c_2^{-1})$. Si $c_2\neq 1$ o $d_1\neq 1$ entonces $c_2d_1=c_4d_4$ para algún $d_4\in D\setminus\{d_1\}$ y algún $c_4\in C\setminus\{c_2\}$ y entonces, como $d_1^{-1}c_2^{-1}=d_4^{-1}c_4^{-1}$, $(d_1^{-1}1)(1c_2^{-1})=(d_4^{-1}1)(1c_4^{-1})$. Como $|C|+|D|>2$, C o D contienen algún $c\neq 1$, y entonces $(1\cdot 1)(1\cdot 1)=(1\cdot c)(1\cdot c^{-1})$. Demostramos entonces que todo elemento de EF tiene al menos dos representaciones. □

Exercise 6.7. Demuestre que si G es un grupo que satisface la propiedad del producto único, entonces $K[G]$ tiene solamente unidades triviales.

En general es muy difícil verificar si un grupo posee la propiedad del producto único. Una propiedad similar es la de ser un grupo difuso. Si G es un grupo libre de torsión y $A \subseteq G$ es un subconjunto, diremos que A es antisimétrico si $A \cap A^{-1} \subseteq \{1\}$, donde $A^{-1} = \{a^{-1} : a \in A\}$. El conjunto de **elementos extremales** de A se define como $\Delta(A) = \{a \in A : Aa^{-1} \text{ es antisimétrico}\}$. Luego

$$a \in A \setminus \Delta(A) \Longleftrightarrow \text{existe } g \in G \setminus \{1\} \text{ tal que } ga \in A \text{ y } g^{-1}a \in A.$$

Definition 6.8. Un grupo G se dice **difuso** si para todo subconjunto $A \subseteq G$ tal que $2 \leq |A| < \infty$ se tiene $|\Delta(A)| \geq 2$.

Lemma 6.9. *Si G es ordenable a derecha, entonces G es difuso.*

Proof. Supongamos que $A = \{a_1, \dots, a_n\}$ y $a_1 < a_2 < \cdots < a_n$. Vamos a demostrar que $\{a_1, a_n\} \subseteq \Delta(A)$. Si $a_1 \in A \setminus \Delta(A)$, existe $g \in G \setminus \{1\}$ tal que $ga_1 \in A$ y $g^{-1}a_1 \in A$. Esto implica que $a_1 \leq ga_1$ y $a_1 \leq g^{-1}a_1$, de donde se concluye que $1 \leq g$ y $1 \leq g^{-1}$, una contradicción. De la misma forma se demuestra que $a_n \in \Delta(A)$. □

lemma:difuso=>2up **Lemma 6.10.** *Si G es difuso, entonces G tiene la propiedad del doble producto único.*

Proof. Supongamos que G no tiene la propiedad del doble producto único. Existen entonces subconjuntos finitos $A, B \subseteq G$ con $|A| + |B| > 2$ tales que $C = AB$ tiene a lo sumo un producto único. Luego $|C| \geq 2$. Como G es difuso, $|\Delta(C)| \geq 2$. Si $c \in \Delta(C)$, entonces c tiene una única expresión como $c = ab$ con $a \in A$ y $b \in B$ (de lo contrario, si $c = a_0b_0 = a_1b_1$ con $a_0 \neq a_1$ y $b_0 \neq b_1$. Si $g = a_0a_1^{-1}$, entonces $g \neq 1$, $gc = a_0a_1^{-1}a_1b_1 = a_0b_1 \in C$ y además $g^{-1}c = a_1a_0^{-1}a_0b_0 = a_1b_0 \in C$. Luego $c \notin \Delta(c)$, una contradicción. □

Open problem 6.1. Find a non-diffuse group with the unique product property.

§7. Connel's theorem

When $K[G]$ is prime? Connel's theorem gives a full answer to this natural question in the case where K is of characteristic zero.

If S is a finite subset of a group G, then we define $\widehat{S} = \sum_{x \in S} x$.

lemma:sumN **Lemma 7.1.** *Let N be a finite normal subgroup of G. Then $\widehat{N} = \sum_{x \in N} x$ is central in $K[G]$ and $\widehat{N}(\widehat{N} - |N|1) = 0$.*

Proof. Assume that $N = \{n_1, \dots, n_k\}$. Let $g \in G$. Since $N \to N$, $n \mapsto gng^{-1}$, is bijective,

$$g\widehat{N}g^{-1} = g(n_1 + \cdots + n_k)g^{-1} = gn_1g^{-1} + \cdots + gn_kg^{-1} = \widehat{N}.$$

Since $nN = N$ if $n \in N$, it follows that $n\widehat{N} = \widehat{N}$. Thus $\widehat{N}\widehat{N} = \sum_{j=1}^{k} n_j\widehat{N} = |N|\widehat{N}$. □

Before proving Connel's theorem we need to prove two group theoretical results. The first one goes to Dietzman:

theorem:Dietzmann

Theorem 7.2 (Dietzmann). *Let G be a group and $X \subseteq G$ be a finite subset of G closed by conjugation. If there exists n such that $x^n = 1$ for all $x \in X$, then $\langle X \rangle$ is a finite subgroup of G.*

Proof. Let $S = \langle X \rangle$. Since $x^{-1} = x^{n-1}$, every element of S can be written as a finite product of elements of X. Fix $x \in X$. We claim that if $x \in X$ appears $k \geq 1$ times in the word s, then we can write s as a product of m elements of X, where the first k elements are equal to x. Suppose that

$$s = x_1 x_2 \cdots x_{t-1} x x_{t+1} \cdots x_m,$$

where $x_j \neq x$ for all $j \in \{1, \dots, t-1\}$. Then

$$s = x(x^{-1}x_1 x)(x^{-1}x_2 x) \cdots (x^{-1}x_{t-1}x)x_{t+1} \cdots x_m$$

is a product of m elements of X since X is closed under conjugation and the first element is x. The same argument implies that s can be written as

$$s = x^k y_{k+1} \cdots y_m,$$

where each y_j belongs to $X \setminus \{x\}$.

Let $s \in S$ and write s as a product of m elements of X, where m is minimal. We need to show that $m \leq (n-1)|X|$. If $m > (n-1)|X|$, then at least $x \in X$ appear n times in the representation of s. Without loss of generality, we write

$$s = x^n x_{n+1} \cdots x_m = x_{n+1} \cdots x_m,$$

a contradiction to the minimality of m. □

The second result goes back to Schur:

thm:Schur

Theorem 7.3 (Schur). *Let G be a group. If $Z(G)$ has finite index in G, then $[G,G]$ is finite.*

Proof. Let $n = (G : Z(G))$ and X be the set of commutators of G. We claim that X is finite, in fact $|X| \leq n^2$. The map

$$\varphi \colon X \to G/Z(G) \times G/Z(G), \quad [x,y] \mapsto (xZ(G), yZ(G)),$$

is injective: if $(xZ(G), yZ(G)) = (uZ(G), vZ(G))$, then $u^{-1}x \in Z(G)$, $v^{-1}y \in Z(G)$. Thus

$$[u,v] = uvu^{-1}v^{-1} = uv(u^{-1}x)x^{-1}v^{-1} = xvx^{-1}(v^{-1}y)y^{-1} = xyx^{-1}y^{-1} = [x,y].$$

Moreover, X is closed under conjugation, as

$$g[x,y]g^{-1} = [gxg^{-1}, gyg^{-1}]$$

for all $g,x,y \in G$. Since $G \to Z(G)$, $g \mapsto g^n$ is a group homomorphism, Lemma **??** implies that $[x,y]^n = [x^n, y^n] = 1$ for all $[x,y] \in X$. The theorem follows from applying Dietzmann's theorem. □

Si G es un grupo, consideramos el subconjunto

$$\Delta^+(G) = \{x \in \Delta(G) : x \text{ tiene orden finito}\}.$$

lem:DcharG **Lemma 7.4.** *Si G es un grupo, entonces $\Delta^+(G)$ es un subgrupo característico de G.*

Proof. Claramente $1 \in \Delta^+(G)$. Sean $x,y \in \Delta^+(G)$ y sea H el subgrupo de G generado por el conjunto C formado por los finitos conjugados de x e y. Si $|x| = n$ y $|y| = m$, entonces $c^{nm} = 1$ para todo $c \in C$. Como C es finito y cerrado por conjugación, el teorema de Dietzmann implica que H es finito. Luego $H \subseteq \Delta^+(G)$ y en particular $xy^{-1} \in \Delta^+(G)$. Es evidente que $\Delta^+(G)$ es un subgrupo característico pues para todo $f \in \mathrm{Aut}(G)$ se tiene que $f(x) \in \Delta^+(G)$ si $x \in \Delta^+(G)$. □

La segunda aplicación del teorema de Dietzmann es el siguiente resultado:

lem:Connel **Lemma 7.5.** *Sea G un grupo y sea $x \in \Delta^+(G)$. Existe entonces un subgrupo finito H normal en G tal que $x \in H$.*

Dejamos la demostración como ejercicio ya que el muy similar a lo que hicimos en la demostración del lema 7.4.

thm:Connel **Theorem 7.6 (Connell).** *Supongamos que el cuerpo K es de característica cero. Sea G un grupo. Las siguientes afirmaciones son equivalentes:*

1) *$K[G]$ es primo.*
2) *$Z(K[G])$ es primo.*
3) *G no tiene subgrupos finitos normales no triviales.*
4) *$\Delta^+(G) = 1$.*

Proof. Demostremos que (1) $\implies$ (2). Como $Z(K[G])$ es un anillo conmutativo, probar que es primo es equivalente a probar que no existen divisores de cero no triviales. Sean $\alpha,\beta \in Z(K[G])$ tales que $\alpha\beta = 0$. Sean $A = \alpha K[G]$ y $B = \beta K[G]$. Como α y β son centrales, A y B son ideales de $K[G]$. Como $AB = 0$, entonces $A = \{0\}$ o $B = \{0\}$ pues $K[G]$ es primo. Luego $\alpha = 0$ o $\beta = 0$.

Demostremos ahora que (2) $\implies$ (3). Sea N un subgrupo normal finito. Por el lema 7.1, $\widehat{N} = \sum_{x \in N} x$ es central en $K[G]$ y $\widehat{N}(\widehat{N} - |N|1) = 0$. Como $\widehat{N} \neq 0$ (pues K tiene característica cero) y $Z(K[G])$ es un dominio, $\widehat{N} = |N|1$, es decir: $N = \{1\}$.

Demostremos que (3) $\implies$ (4). Sea $x \in \Delta^+(G)$. Por el lema 7.5 sabemos que existe un subgrupo finito H normal en G que contiene a x. Como por hipótesis H es trivial, se concluye que $x = 1$.

Finalmente demostramos que (4) $\implies$ (1). Sean A y B ideales de $K[G]$ tales que $AB = 0$. Supongamos que $B \neq 0$ y sea $\beta \in B \setminus \{0\}$. Si $\alpha \in A$, entonces, como

$\alpha K[G]\beta \subseteq \alpha B \subseteq AB = 0$, el lema 3.16 de Passman implica que $\pi_{\Delta(G)}(\alpha)\pi_{\Delta(G)}(\beta) = 0$. Como por hipótesis $\Delta^+(G)$ es trivial, sabemos que $\Delta(G)$ es libre de torsión y luego $\Delta(G)$ es abeliano por el lema **??**. Esto nos dice que $K[\Delta(G)]$ no tiene divisores de cero y luego $\alpha = 0$. Demostramos entonces que $B \neq 0$ implica que $A = 0$. □

Theorem 7.7 (Connel). *Sea K un cuerpo de característica cero y sea G un grupo. Entonces $K[G]$ es artiniano a izquierda si y sólo si G es finito.*

Proof. Si G es finito, $K[G]$ es un álgebra de dimensión finita y luego es artiniano a izquierda. Supongamos entonces que $K[G]$ es artiniano a izquierda.

Primero observemos que si $K[G]$ es un álgebra prima, entonces por el teorema de Wedderburn $K[G]$ es simple y luego G es el grupo trivial (pues si G no es trivial, $K[G]$ no es simple ya que el ideal de aumentación es un ideal no nulo de $K[G]$).

Como $K[G]$ es artiniano a izquierda, es noetheriano a izquierda por Hopkins–Levitzky y entonces, $K[G]$ admite una serie de composición por el teorema **??**. Para demostrar el teorema procederemos por inducción en la longitud de la serie de composición de $K[G]$. Si la longitud es uno, $\{0\}$ es el único ideal de $K[G]$ y luego $K[G]$ es prima y el resultado está demostrado. Si suponemos que el resultado vale para longitud n y además $K[G]$ no es prima, entonces, por el teorema de Connel, G posee un subgrupo normal H finito y no trivial. Al considerar el morfismo canónico $K[G] \to K[G/H]$ vemos que $K[G/H]$ es artiniano a izquierda y tiene longitud $< n$. Por hipótesis inductiva, G/H es un grupo finito y luego, como H también es finito, G es finito. □

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