

EEIGM
École Européenne d'Ingénieurs en Génie des Matériaux

2^{ème} Année, 1^{er} Semestre

MÉCANIQUE DU SOLIDE DÉFORMABLE

TRAVAUX DIRIGÉS

Luca Di Stasio



Cette oeuvre est mise à disposition selon les termes de la
Licence Creative Commons Attribution - Pas d'Utilisation Commerciale
4.0 International.

6 août 2020

Contents

List of Acronyms	iii
List of Symbols	v
Abstract	vii
1. Systèmes de coordonnées curvilignes	1
1.1. Énoncé	1
1.1.1. Problème A	1
1.1.2. Problème B	1
1.2. Corrigé	1
1.2.1. Problème A	1
1.2.2. Problème B	5
2. Second section	7
A. First appendix	9

List of Acronyms

List of Acronyms

List of Symbols

Abstract

1. Systèmes de coordonnées curvilignes

1.1. Énoncé

1.1.1. Problème A

1.1.2. Problème B

1.2. Corrigé

1.2.1. Problème A

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (1)$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \quad (2)$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \quad (3)$$

$$\nabla_{xyz}^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (4)$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \end{cases} \quad (5)$$

$$\left\{ \begin{array}{l} \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ \quad + 2 \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} \\ \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial \xi}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ \quad + 2 \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} \\ \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial \xi}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ \quad + 2 \left(\frac{\partial \xi}{\partial z} \right) \left(\frac{\partial \eta}{\partial z} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial z} \right) \left(\frac{\partial \zeta}{\partial z} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial z} \right) \left(\frac{\partial \zeta}{\partial z} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} \end{array} \right. \quad (13)$$

$$\begin{aligned} \nabla_{\xi\eta\zeta}^2 f(\xi, \eta, \zeta) &= \left[\left(\frac{\partial \xi}{\partial x} \right)^2 + \left(\frac{\partial \xi}{\partial y} \right)^2 + \left(\frac{\partial \xi}{\partial z} \right)^2 \right] \frac{\partial^2 f}{\partial \xi^2} + \\ &+ \left[\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 + \left(\frac{\partial \eta}{\partial z} \right)^2 \right] \frac{\partial^2 f}{\partial \eta^2} + \\ &+ \left[\left(\frac{\partial \zeta}{\partial x} \right)^2 + \left(\frac{\partial \zeta}{\partial y} \right)^2 + \left(\frac{\partial \zeta}{\partial z} \right)^2 \right] \frac{\partial^2 f}{\partial \zeta^2} + \\ &+ 2 \left[\left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) + \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) + \left(\frac{\partial \xi}{\partial z} \right) \left(\frac{\partial \eta}{\partial z} \right) \right] \frac{\partial^2 f}{\partial \xi \partial \eta} + \\ &+ 2 \left[\left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \zeta}{\partial y} \right) + \left(\frac{\partial \eta}{\partial z} \right) \left(\frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ &+ 2 \left[\left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \zeta}{\partial y} \right) + \left(\frac{\partial \xi}{\partial z} \right) \left(\frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial^2 f}{\partial \xi \partial \zeta} \end{aligned} \quad (14)$$

Coordonnées cylindriques

$$\left\{ \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{array} \right. \longleftrightarrow \left\{ \begin{array}{l} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left(\frac{y}{x} \right) \\ z = z \end{array} \right. \quad (15)$$

$$\left\{ \begin{array}{l} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}} = \frac{r \cos \theta}{r} \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2}} = \frac{r \sin \theta}{r} \\ \frac{\partial r}{\partial z} = 0 \end{array} \right. \quad (16)$$

Coordonnées spheriques

1.2.2. Problème B

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (17)$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \quad (18)$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \quad (19)$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x} \mathbf{i}_x + \frac{\partial f}{\partial y} \mathbf{j}_y + \frac{\partial f}{\partial z} \mathbf{k}_z \quad (20)$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \quad (21)$$

$$\begin{aligned}
\mathbf{i}_\xi &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\
\mathbf{j}_\eta &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\
\mathbf{k}_\zeta &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix}
\end{aligned} \tag{22}$$

$$\begin{aligned}
\nabla_{\xi\eta\zeta} f &= \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}_\xi + \\
&\quad + \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}_\eta + \\
&\quad + \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}_\zeta = \\
&= \begin{bmatrix} \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix}
\end{aligned} \tag{23}$$

Coordonnées cylindriques

Coordonnées sphériques

2. Second section

A. First appendix

