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Mécanique du Solide Déformable

Travaux Dirigés

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1.2.1. Problème A

Rappels théoriques

$$\begin{cases} x = x\left(\xi,\eta,\zeta\right) \\ y = x\left(\xi,\eta,\zeta\right) \\ z = x\left(\xi,\eta,\zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x,y,z\right) \\ \eta = \eta\left(x,y,z\right) \\ \zeta = \zeta\left(x,y,z\right) \end{cases} \tag{1}$$

$$f\left(x,y,z\right) = f\left(x\left(\xi,\eta,\zeta\right), y\left(\xi,\eta,\zeta\right), z\left(\xi,\eta,\zeta\right)\right) \tag{2}$$

$$f\left(\xi,\eta,\zeta\right) = f\left(\xi\left(x,y,z\right), \eta\left(x,y,z\right), \zeta\left(x,y,z\right)\right) \tag{3}$$

$$\nabla^2_{xyz} f\left(x,y,z\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{5}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2} & = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial x} + \dfrac{\partial f}{\partial \zeta}\dfrac{\partial \zeta}{\partial x}\right) = \\ & = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} + \dfrac{\partial f}{\partial \xi}\dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial^2 \eta}{\partial x^2} + \dfrac{\partial f}{\partial \zeta}\dfrac{\partial^2 \zeta}{\partial x^2} \\ \dfrac{\partial^2 f}{\partial y^2} & = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial y} + \dfrac{\partial f}{\partial \zeta}\dfrac{\partial \zeta}{\partial y}\right) = \\ & = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial y} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial y} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial y} + \dfrac{\partial f}{\partial \xi}\dfrac{\partial^2 \xi}{\partial y^2} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial^2 \eta}{\partial y^2} + \dfrac{\partial f}{\partial \zeta}\dfrac{\partial^2 \zeta}{\partial y^2} \\ \dfrac{\partial^2 f}{\partial z^2} & = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial z} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial z} + \dfrac{\partial f}{\partial \zeta}\dfrac{\partial \zeta}{\partial z}\right) = \\ & = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial z} + \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial z} + \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial z} + \dfrac{\partial f}{\partial \xi}\dfrac{\partial^2 \xi}{\partial z^2} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial^2 \eta}{\partial z^2} + \dfrac{\partial f}{\partial \zeta}\dfrac{\partial^2 \zeta}{\partial z^2} \end{cases} \tag{6}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right) \\ \dfrac{\partial}{\partial y} = \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y}\dfrac{\partial}{\partial \zeta}\right) \\ \dfrac{\partial}{\partial z} = \left(\dfrac{\partial \xi}{\partial z}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial z}\dfrac{\partial}{\partial \zeta}\right) \end{cases} \tag{7}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} = \\ & = \left(\dfrac{\partial \xi}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial \xi^2} + \left(\dfrac{\partial \eta}{\partial x}\right)\left(\dfrac{\partial \xi}{\partial x}\right)\dfrac{\partial^2 f}{\partial \eta \partial \xi} + \left(\dfrac{\partial \zeta}{\partial x}\right)\left(\dfrac{\partial \xi}{\partial x}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \xi} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} = \\ & = \left(\dfrac{\partial \xi}{\partial x}\right)\left(\dfrac{\partial \eta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \xi \partial \eta} + \left(\dfrac{\partial \eta}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial \eta^2} + \left(\dfrac{\partial \zeta}{\partial x}\right)\left(\dfrac{\partial \eta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \eta} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} = \\ & = \left(\dfrac{\partial \xi}{\partial x}\right)\left(\dfrac{\partial \zeta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\dfrac{\partial \eta}{\partial x}\right)\left(\dfrac{\partial \zeta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\dfrac{\partial \zeta}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial \zeta^2} \end{cases} \tag{8}$$

$$\begin{cases}
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} & = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} = \\
& = \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 f}{\partial \eta\partial \xi}+\left(\frac{\partial \zeta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 f}{\partial \zeta\partial \xi} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} & = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} = \\
& = \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi\partial \eta}+\left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \zeta\partial \eta} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} & = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} = \\
& = \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi\partial \zeta}+\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta\partial \zeta}+\left(\frac{\partial \zeta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases} \tag{9}$$

$$\begin{cases}
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} = \\
& = \left(\frac{\partial \xi}{\partial z}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \eta\partial \xi}+\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta\partial \xi} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} = \\
& = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi\partial \eta}+\left(\frac{\partial \eta}{\partial z}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta\partial \eta} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} = \\
& = \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi\partial \zeta}+\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta\partial \zeta}+\left(\frac{\partial \zeta}{\partial z}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases} \tag{10}$$

$$\frac{\partial^2 f}{\partial \eta\partial \xi}=\frac{\partial^2 f}{\partial \xi\partial \eta} \qquad \frac{\partial^2 f}{\partial \eta\partial \zeta}=\frac{\partial^2 f}{\partial \zeta\partial \eta} \qquad \frac{\partial^2 f}{\partial \xi\partial \zeta}=\frac{\partial^2 f}{\partial \zeta\partial \xi} \tag{11}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = & \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \zeta^2}+ \\
& +2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi\partial \eta}+2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi\partial \zeta}+2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta\partial \zeta}+ \\
& +\frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}
\end{aligned} \tag{12}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} +$$
$$+ 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} +$$
$$+ \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \tag{13}$$

$$\frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} +$$
$$+ 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} +$$
$$+ \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2} \tag{14}$$

$$
\begin{aligned}
\nabla^2_{\xi\eta\zeta} f(\xi,\eta,\zeta) = & \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right] \frac{\partial^2 f}{\partial \xi^2} + \\
& + \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right] \frac{\partial^2 f}{\partial \eta^2} + \\
& + \left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2\right] \frac{\partial^2 f}{\partial \zeta^2} + \\
& + 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\right] \frac{\partial^2 f}{\partial \xi \partial \eta} + \\
& + 2\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right] \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
& + 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right] \frac{\partial^2 f}{\partial \xi \partial \zeta} + \\
& + \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right) \frac{\partial f}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2}\right) \frac{\partial f}{\partial \eta} + \\
& + \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right) \frac{\partial f}{\partial \zeta}
\end{aligned}
\tag{15}
$$

Coordonnées cylindriques

$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{16}
$$

$$
\begin{cases}
\frac{\partial r}{\partial x} & = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\
& = \frac{\cancel{r}\cos\theta}{\cancel{r}} \\
\frac{\partial r}{\partial y} & = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\
& = \frac{\cancel{r}\sin\theta}{\cancel{r}} \\
\frac{\partial r}{\partial z} & = 0
\end{cases}
\qquad
\begin{cases}
\frac{\partial \theta}{\partial x} & = -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\
& = -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\
\frac{\partial \theta}{\partial y} & = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\
& = \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\
\frac{\partial \theta}{\partial z} & = 0
\end{cases}
\qquad
\begin{cases}
\frac{\partial \tilde{z}}{\partial x} & = 0 \\
\frac{\partial \tilde{z}}{\partial y} & = 0 \\
\frac{\partial \tilde{z}}{\partial z} & = 1
\end{cases}
\tag{17}
$$

$$
\begin{cases}
\frac{\partial^2 r}{\partial x^2} & = \frac{\partial}{\partial x}\left(\cos\theta\right) = \\
& = -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\
\frac{\partial^2 r}{\partial y^2} & = \frac{\partial}{\partial y}\left(\sin\theta\right) = \\
& = \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\
\frac{\partial^2 r}{\partial z^2} & = 0
\end{cases}
\quad
\begin{cases}
\frac{\partial^2 \theta}{\partial x^2} & = \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\
& = -\frac{1}{r}\frac{\partial}{\partial x}\left(\sin\theta\right) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\
& = -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\
& = \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\
\frac{\partial^2 \theta}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\
& = \frac{1}{r}\frac{\partial}{\partial y}\left(\cos\theta\right) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\
& = -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\
& = -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\
\frac{\partial^2 \theta}{\partial z^2} & = 0
\end{cases}
\quad
\begin{cases}
\frac{\partial^2 \tilde{z}}{\partial x^2} & = 0 \\
\frac{\partial^2 \tilde{z}}{\partial y^2} & = 0 \\
\frac{\partial^2 \tilde{z}}{\partial z^2} & = 0
\end{cases}
\tag{18}
$$

$$
\begin{aligned}
\nabla^2_{r\theta\tilde{z}} f\left(r,\theta,\tilde{z}\right) = & \left[\cos^2\theta + \sin\theta^2\right]\frac{\partial^2 f}{\partial r^2} + \left[\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right]\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\tilde{z}^2} + \\
& + 2\left[-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right]\frac{\partial^2 f}{\partial r\partial\theta} + \\
& + \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial f}{\partial r} + \left(\frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2}\right)\frac{\partial f}{\partial\theta} = \\
= & \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\tilde{z}^2}
\end{aligned}
\tag{19}
$$

Coordonnées spheriques

$$
\begin{cases}
x = r\sin\theta\cos\varphi \\
y = r\sin\theta\sin\varphi \\
z = r\cos\theta
\end{cases}
\longleftrightarrow
\begin{cases}
r = \sqrt{x^2+y^2+z^2} \\
\varphi = \tan^{-1}\left(\frac{y}{x}\right) \\
\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)
\end{cases}
\tag{20}
$$

$$
\begin{cases}
\frac{\partial r}{\partial x} & = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\
& = \frac{\cancel{r}\cos\theta}{\cancel{r}} \\
\frac{\partial r}{\partial y} & = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\
& = \frac{\cancel{r}\sin\theta}{\cancel{r}} \\
\frac{\partial r}{\partial z} & = 0
\end{cases}
\quad
\begin{cases}
\frac{\partial\varphi}{\partial x} & = -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\
& = -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\
\frac{\partial\varphi}{\partial y} & = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\
& = \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\
\frac{\partial\varphi}{\partial z} & = 0
\end{cases}
\quad
\begin{cases}
\frac{\partial\theta}{\partial x} & = 0 \\
\frac{\partial\theta}{\partial y} & = 0 \\
\frac{\partial\theta}{\partial z} & = 1
\end{cases}
\tag{21}
$$

1.2.2. Problème B

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = x(\xi, \eta, \zeta) \\ z = x(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \tag{22}$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \tag{23}$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \tag{24}$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{j}_y + \frac{\partial f}{\partial z}\mathbf{k}_z \tag{25}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{26}$$

$$\begin{aligned} \mathbf{i}_\xi &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\ \mathbf{j}_\eta &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\ \mathbf{k}_\zeta &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \end{aligned} \tag{27}$$

$$
\begin{aligned}
\nabla_{\xi\eta\zeta} f =& \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}_\xi + \\
&+ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}_\eta + \\
&+ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}_\zeta = \\
&= \begin{bmatrix} \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix}
\end{aligned}
\tag{28}
$$

Coordonnées cylindriques

Coordonnées spheriques

2. Second section

A. First appendix