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Mécanique du Solide Déformable

Travaux Dirigés

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1.2.1. Problème A

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = x(\xi, \eta, \zeta) \\ z = x(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \tag{1}$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \tag{2}$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \tag{3}$$

$$\nabla^2_{xyz} f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{5}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) = \\ & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) \\ \frac{\partial^2 f}{\partial z^2} & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) \end{cases} \tag{6}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) \\ \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) \\ \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) \end{cases} \tag{7}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = & \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ & +2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} \\ \frac{\partial^2 f}{\partial y^2} = & \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ & +2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} \\ \frac{\partial^2 f}{\partial z^2} = & \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ & +2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} \end{cases} \tag{8}$$

$$\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial^2 f}{\partial \xi \partial \eta} \qquad \frac{\partial^2 f}{\partial \eta \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \eta} \qquad \frac{\partial^2 f}{\partial \xi \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \xi} \tag{9}$$

$$
\begin{aligned}
\nabla^2_{\xi\eta\zeta} f\left(\xi,\eta,\zeta\right) = & \left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\xi}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial\xi^2} + \\
& + \left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2 + \left(\frac{\partial\eta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial\eta^2} + \\
& + \left[\left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial\zeta^2} + \\
& + 2\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\eta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial\xi\partial\eta} + \\
& + 2\left[\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\zeta}{\partial x}\right) + \left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\zeta}{\partial y}\right) + \left(\frac{\partial\eta}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial\eta\partial\zeta} + \\
& + 2\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\zeta}{\partial x}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\zeta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial\xi\partial\zeta}
\end{aligned} \tag{10}
$$

Coordonnées cylindriques

$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases} \tag{11}
$$

$$
\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} = 0 \end{cases} \tag{12}
$$

Coordonnées spheriques

1.2.2. Problème B

$$
\begin{cases} x = x\left(\xi,\eta,\zeta\right) \\ y = x\left(\xi,\eta,\zeta\right) \\ z = x\left(\xi,\eta,\zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x,y,z\right) \\ \eta = \eta\left(x,y,z\right) \\ \zeta = \zeta\left(x,y,z\right) \end{cases} \tag{13}
$$

$$
f\left(x,y,z\right) = f\left(x\left(\xi,\eta,\zeta\right), y\left(\xi,\eta,\zeta\right), z\left(\xi,\eta,\zeta\right)\right) \tag{14}
$$

$$f\left(\xi,\eta,\zeta\right)=f\left(\xi\left(x,y,z\right),\eta\left(x,y,z\right),\zeta\left(x,y,z\right)\right) \tag{15}$$

$$\nabla f_{xyz}=\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{bmatrix}=\frac{\partial f}{\partial x}\begin{bmatrix}1\\0\\0\end{bmatrix}+\frac{\partial f}{\partial y}\begin{bmatrix}0\\1\\0\end{bmatrix}+\frac{\partial f}{\partial z}\begin{bmatrix}0\\0\\1\end{bmatrix}=\frac{\partial f}{\partial x}\mathbf{i}_x+\frac{\partial f}{\partial y}\mathbf{j}_y+\frac{\partial f}{\partial z}\mathbf{k}_z \tag{16}$$

$$\nabla f=\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{bmatrix}=\begin{bmatrix}\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}+\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z}+\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z}+\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\end{bmatrix}=\frac{\partial f}{\partial \xi}\begin{bmatrix}\frac{\partial \xi}{\partial x}\\ \frac{\partial \xi}{\partial y}\\ \frac{\partial \xi}{\partial z}\end{bmatrix}+\frac{\partial f}{\partial \eta}\begin{bmatrix}\frac{\partial \eta}{\partial x}\\ \frac{\partial \eta}{\partial y}\\ \frac{\partial \eta}{\partial z}\end{bmatrix}+\frac{\partial f}{\partial \zeta}\begin{bmatrix}\frac{\partial \zeta}{\partial x}\\ \frac{\partial \zeta}{\partial y}\\ \frac{\partial \zeta}{\partial z}\end{bmatrix} \tag{17}$$

$$\begin{aligned}
\mathbf{i}_\xi &=\frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2+\left(\frac{\partial \xi}{\partial z}\right)^2}}\begin{bmatrix}\frac{\partial \xi}{\partial x}\\ \frac{\partial \xi}{\partial y}\\ \frac{\partial \xi}{\partial z}\end{bmatrix}\\
\mathbf{j}_\eta &=\frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2+\left(\frac{\partial \eta}{\partial z}\right)^2}}\begin{bmatrix}\frac{\partial \eta}{\partial x}\\ \frac{\partial \eta}{\partial y}\\ \frac{\partial \eta}{\partial z}\end{bmatrix}\\
\mathbf{k}_\zeta &=\frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2+\left(\frac{\partial \zeta}{\partial y}\right)^2+\left(\frac{\partial \zeta}{\partial z}\right)^2}}\begin{bmatrix}\frac{\partial \zeta}{\partial x}\\ \frac{\partial \zeta}{\partial y}\\ \frac{\partial \zeta}{\partial z}\end{bmatrix}
\end{aligned} \tag{18}$$

$$
\begin{aligned}
\nabla_{\xi\eta\zeta} f =& \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}_\xi + \\
&+ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}_\eta + \\
&+ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}_\zeta = \\
&= \begin{bmatrix} \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix}
\end{aligned} \tag{19}
$$

Coordonnées cylindriques

Coordonnées spheriques

2. Second section

A. First appendix