# EEIGM

**École Européenne d'Ingénieurs en Génie des Matériaux**

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# Mécanique du Solide Déformable

# Travaux Dirigés

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# Contents

# List of Acronyms

# List of Symbols

# Abstract

# 1 Systèmes de coordonnées curvilignes

## 1.1 Énoncé

### 1.1.1 Problème A

Déterminer l'expression de

- les vecteurs du repère local naturel (base covariante),
- le jacobien de la transformation,
- le tenseur métrique,
- le déplacement infinitésimal d'un point,
- l'élément infinitésimal de ligne,
- l'élément infinitésimal de volume;

dans un système de coordonnées

1. cylindriques,
2. sphériques.

### 1.1.2 Problème B

Exprimer l'opérateur laplacien $\nabla^2$ en

1. coordonnées cylindriques,
2. coordonnées spheriques.

### 1.1.3 Problème C

## 1.2 Corrigé

### 1.2.1 Problème A

#### Rappels théoriques

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \tag{1}$$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{2}$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{3}$$

$$\begin{cases} dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta \\ dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta \\ dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta \end{cases} \tag{4}$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

$$\frac{\partial \mathbf{r}}{\partial \xi} \qquad\qquad \frac{\partial \mathbf{r}}{\partial \eta} \qquad\qquad \frac{\partial \mathbf{r}}{\partial \zeta}$$

$$\begin{aligned}
\mathbf{i}_\xi =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \xi}||}\frac{\partial \mathbf{r}}{\partial \xi} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{bmatrix} \\
\mathbf{j}_\eta =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \eta}||}\frac{\partial \mathbf{r}}{\partial \eta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{bmatrix} \\
\mathbf{k}_\zeta =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \zeta}||}\frac{\partial \mathbf{r}}{\partial \zeta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{bmatrix}
\end{aligned} \tag{6}$$

$$dl^2 = d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} dx & dy & dz \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = dx^2 + dy^2 + dz^2 \tag{7}$$

$$\begin{aligned}
dl^2 &= d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \mathbf{J}^T \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \underbrace{\begin{bmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} & \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \end{bmatrix}}_{\mathbf{g}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}
\end{aligned} \tag{8}$$

$$dV = \frac{\partial \mathbf{r}}{\partial \xi} d\xi \cdot \left( \frac{\partial \mathbf{r}}{\partial \eta} d\eta \wedge \frac{\partial \mathbf{r}}{\partial \zeta} d\zeta \right) = det \begin{bmatrix} \left(\frac{\partial \mathbf{r}}{\partial \xi}\right)^T \\ \left(\frac{\partial \mathbf{r}}{\partial \eta}\right)^T \\ \left(\frac{\partial \mathbf{r}}{\partial \zeta}\right)^T \end{bmatrix} d\xi d\eta d\zeta = det \left( \mathbf{J}^T \right) d\xi d\eta d\zeta \tag{9}$$

### Coordonnées cylindriques

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{10}$$

$$\begin{cases} \frac{\partial x}{\partial r} &= \cos\theta \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial x}{\partial \tilde{z}} &= 0 \end{cases} \qquad \begin{cases} \frac{\partial y}{\partial r} &= \sin\theta \\ \frac{\partial y}{\partial \theta} &= r\cos\theta \\ \frac{\partial y}{\partial \tilde{z}} &= 0 \end{cases} \qquad \begin{cases} \frac{\partial z}{\partial r} &= 0 \\ \frac{\partial z}{\partial \theta} &= 0 \\ \frac{\partial z}{\partial \tilde{z}} &= 1 \end{cases} \tag{11}$$

$$\begin{aligned} \mathbf{i}_r &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial r}||}\frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2\theta+\cos^2\theta}}\begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} = \begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} \\ \mathbf{j}_\theta &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \theta}||}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2\sin^2\theta+r^2\cos^2\theta}}\begin{bmatrix}-r\sin\theta \\ r\cos\theta \\ 0\end{bmatrix} = \begin{bmatrix}-\sin\theta \\ \cos\theta \\ 0\end{bmatrix} \\ \mathbf{k}_{\tilde{z}} &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \tilde{z}}||}\frac{\partial \mathbf{r}}{\partial \tilde{z}} = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} \end{aligned} \tag{12}$$

$$\mathbf{J} = \begin{bmatrix}\cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{13}$$

$$d\mathbf{r} = \begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} dr + \begin{bmatrix}-r\sin\theta \\ r\cos\theta \\ 0\end{bmatrix} d\theta + \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} d\tilde{z} = dr\mathbf{i}_r + rd\theta\mathbf{i}_\theta + d\tilde{z}\mathbf{i}_{\tilde{z}} = \begin{bmatrix}dr \\ rd\theta \\ d\tilde{z}\end{bmatrix} \tag{14}$$

$$\mathbf{g} = \begin{bmatrix}1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{15}$$

$$dl^2 = dr^2 + r^2 d\theta^2 + d\tilde{z}^2 \tag{16}$$

$$dV = det\left(\mathbf{J}^T\right) dr d\theta d\tilde{z} = r dr d\theta d\tilde{z} \tag{17}$$

**Coordonnées spheriques**

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{18}$$

$$\begin{cases} \frac{\partial x}{\partial r} &= \sin\theta\cos\varphi \\ \frac{\partial x}{\partial \theta} &= r\cos\theta\cos\varphi \\ \frac{\partial x}{\partial \varphi} &= -r\sin\theta\sin\varphi \end{cases} \quad \begin{cases} \frac{\partial y}{\partial r} &= \sin\theta\sin\varphi \\ \frac{\partial y}{\partial \theta} &= r\cos\theta\sin\varphi \\ \frac{\partial y}{\partial \varphi} &= r\sin\theta\cos\varphi \end{cases} \quad \begin{cases} \frac{\partial z}{\partial r} &= \cos\theta \\ \frac{\partial z}{\partial \theta} &= -r\sin\theta \\ \frac{\partial z}{\partial \varphi} &= 0 \end{cases} \tag{19}$$

$$\begin{aligned}
\mathbf{i}_r =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial r}||}\frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}}\begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix} \\
\mathbf{j}_\varphi =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \varphi}||}\frac{\partial \mathbf{r}}{\partial \varphi} = \frac{1}{\sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi}}\begin{bmatrix}-r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0\end{bmatrix} = \begin{bmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{bmatrix} \\
\mathbf{k}_\theta =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \theta}||}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}}\begin{bmatrix}r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta\end{bmatrix}
\end{aligned} \tag{20}$$

$$\mathbf{J} = \begin{bmatrix}\sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta\end{bmatrix} \tag{21}$$

$$\begin{aligned} d\mathbf{r} = &\begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} dr + r\sin\theta \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} d\varphi + r \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix} d\theta = \\ &= dr\mathbf{i}_r + r\sin\theta d\varphi\mathbf{j}_\varphi + rd\theta\mathbf{k}_\theta = \begin{bmatrix} dr \\ r\sin\theta d\varphi \\ rd\theta \end{bmatrix} \end{aligned} \tag{22}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2 \end{bmatrix} \tag{23}$$

$$dl^2 = dr^2 + r^2\sin^2\theta d\varphi^2 + r^2 d\theta^2 \tag{24}$$

$$dV = det\left(\mathbf{J}^T\right) drd\theta d\tilde{z} = r^2\sin\theta drd\varphi d\theta \tag{25}$$

### 1.2.2 Problème B

#### Rappels théoriques

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = y\left(\xi, \eta, \zeta\right) \\ z = z\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{26}$$

$$f\left(x, y, z\right) = f\left(x\left(\xi, \eta, \zeta\right), y\left(\xi, \eta, \zeta\right), z\left(\xi, \eta, \zeta\right)\right) \tag{27}$$

$$f\left(\xi, \eta, \zeta\right) = f\left(\xi\left(x, y, z\right), \eta\left(x, y, z\right), \zeta\left(x, y, z\right)\right) \tag{28}$$

$$\nabla^2_{xyz} f\left(x, y, z\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{29}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{30}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) = \\ & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) = \\ & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \\ \frac{\partial^2 f}{\partial z^2} & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) = \\ & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2} \end{cases} \tag{31}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right) \\ \dfrac{\partial}{\partial y} = \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y}\dfrac{\partial}{\partial \zeta}\right) \\ \dfrac{\partial}{\partial z} = \left(\dfrac{\partial \xi}{\partial z}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial z}\dfrac{\partial}{\partial \zeta}\right) \end{cases} \tag{32}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 f}{\partial \zeta \partial \xi} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \zeta \partial \eta} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} \end{cases} \tag{33}$$

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial y} & = \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial y} = \\ & = \left(\dfrac{\partial \xi}{\partial y}\right)^2 \dfrac{\partial^2 f}{\partial \xi^2} + \left(\dfrac{\partial \eta}{\partial y}\right)\left(\dfrac{\partial \xi}{\partial y}\right)\dfrac{\partial^2 f}{\partial \eta \partial \xi} + \left(\dfrac{\partial \zeta}{\partial y}\right)\left(\dfrac{\partial \xi}{\partial y}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \xi} \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial y} & = \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial y} = \\ & = \left(\dfrac{\partial \xi}{\partial y}\right)\left(\dfrac{\partial \eta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \xi \partial \eta} + \left(\dfrac{\partial \eta}{\partial y}\right)^2 \dfrac{\partial^2 f}{\partial \eta^2} + \left(\dfrac{\partial \zeta}{\partial y}\right)\left(\dfrac{\partial \eta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \eta} \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial y} & = \left(\dfrac{\partial \xi}{\partial y}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial y} = \\ & = \left(\dfrac{\partial \xi}{\partial y}\right)\left(\dfrac{\partial \zeta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\dfrac{\partial \eta}{\partial y}\right)\left(\dfrac{\partial \zeta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\dfrac{\partial \zeta}{\partial y}\right)^2 \dfrac{\partial^2 f}{\partial \zeta^2} \end{cases} \tag{34}$$

$$\begin{cases} \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} = \\ & = \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta \partial \xi} \\ \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta \partial \eta} \\ \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} = \\ & = \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} \end{cases} \tag{35}$$

$$\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial^2 f}{\partial \xi \partial \eta} \qquad \frac{\partial^2 f}{\partial \eta \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \eta} \qquad \frac{\partial^2 f}{\partial \xi \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \xi} \tag{36}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} = & \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ & + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ & + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \end{aligned} \tag{37}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} = & \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ & + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ & + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \end{aligned} \tag{38}$$

$$\frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2} \tag{39}$$

$$\nabla^2_{\xi\eta\zeta} f(\xi,\eta,\zeta) = \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \xi^2} + \\ + \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} + \\ + \left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \zeta^2} + \\ + 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \eta} + \\ + 2\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ + 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \zeta} + \\ + \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right)\frac{\partial f}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\frac{\partial f}{\partial \eta} + \\ + \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right)\frac{\partial f}{\partial \zeta} \tag{40}$$

**Coordonnées cylindriques**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{41}$$

$$\left\{\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= 0\end{aligned}\right. \qquad \left\{\begin{aligned}\frac{\partial \theta}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\ \frac{\partial \theta}{\partial z} &= 0\end{aligned}\right. \qquad \left\{\begin{aligned}\frac{\partial \tilde{z}}{\partial x} &= 0 \\ \frac{\partial \tilde{z}}{\partial y} &= 0 \\ \frac{\partial \tilde{z}}{\partial z} &= 1\end{aligned}\right. \tag{42}$$

$$\left\{\begin{aligned}\frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x}\left(\cos\theta\right) = \\ &= -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\ \frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y}\left(\sin\theta\right) = \\ &= \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\ \frac{\partial^2 r}{\partial z^2} &= 0\end{aligned}\right. \qquad \left\{\begin{aligned}\frac{\partial^2 \theta}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\ &= -\frac{1}{r}\frac{\partial}{\partial x}\left(\sin\theta\right) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\ &= -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\ &= \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\ &= \frac{1}{r}\frac{\partial}{\partial y}\left(\cos\theta\right) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\ &= -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\ &= -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} &= 0\end{aligned}\right. \qquad \left\{\begin{aligned}\frac{\partial^2 \tilde{z}}{\partial x^2} &= 0 \\ \frac{\partial^2 \tilde{z}}{\partial y^2} &= 0 \\ \frac{\partial^2 \tilde{z}}{\partial z^2} &= 0\end{aligned}\right. \tag{43}$$

$$\begin{aligned}\nabla^2_{r\theta\tilde{z}} f\left(r,\theta,\tilde{z}\right) = &\left[\cos^2\theta + \sin\theta^2\right]\frac{\partial^2 f}{\partial r^2} + \left[\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right]\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\tilde{z}^2} + \\ &+ 2\left[-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right]\frac{\partial^2 f}{\partial r\partial\theta} + \\ &+ \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial f}{\partial r} + \left(\frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2}\right)\frac{\partial f}{\partial\theta} = \\ = &\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\tilde{z}^2}\end{aligned} \tag{44}$$

**Coordonnées spheriques**

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{45}$$

$$
\left\{
\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \\
&= \frac{\cancel{r}\sin\theta\cos\varphi}{\cancel{r}} \\
\frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}} = \\
&= \frac{\cancel{r}\sin\theta\sin\varphi}{\cancel{r}} \\
\frac{\partial r}{\partial z} &= \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \\
&= \frac{\cancel{r}\cos\theta}{\cancel{r}}
\end{aligned}
\right.
\qquad
\left\{
\begin{aligned}
\frac{\partial \varphi}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\
&= -\frac{\sin\varphi}{r\sin\theta} \\
\frac{\partial \varphi}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\
&= \frac{\cos\varphi}{r\sin\theta} \\
\frac{\partial \varphi}{\partial z} &= 0
\end{aligned}
\right.
\qquad
\left\{
\begin{aligned}
\frac{\partial \theta}{\partial x} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}x}{\cancel{2}z\sqrt{x^2+y^2}} = \\
&= \frac{1}{r}\cos\theta\cos\varphi \\
\frac{\partial \theta}{\partial y} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}y}{\cancel{2}z\sqrt{x^2+y^2}} = \\
&= \frac{1}{r}\cos\theta\sin\varphi \\
\frac{\partial \theta}{\partial z} &= -\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\sqrt{x^2+y^2}}{z^2} = \\
&= -\frac{1}{r}\sin\theta
\end{aligned}
\right.
\tag{46}
$$

$$
\left\{
\begin{aligned}
\frac{\partial^2 r}{\partial x^2} &= \cos\theta\cos\varphi\frac{\partial\theta}{\partial x} - \sin\theta\sin\varphi\frac{\partial\varphi}{\partial x} = \\
&= \frac{1}{r}\left(\cos^2\varphi\cos^2\theta + \sin^2\varphi\right) \\
\frac{\partial^2 r}{\partial y^2} &= \cos\theta\sin\varphi\frac{\partial\theta}{\partial y} + \sin\theta\cos\varphi\frac{\partial\varphi}{\partial y} = \\
&= \frac{1}{r}\left(\sin^2\varphi\cos^2\theta + \cos^2\varphi\right) \\
\frac{\partial^2 r}{\partial z^2} &= -\sin\theta\frac{\partial\theta}{\partial z} = \\
&= \frac{1}{r}\sin^2\theta
\end{aligned}
\right.
\tag{47}
$$

$$
\left\{
\begin{aligned}
\frac{\partial^2 \varphi}{\partial x^2} &= \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial r}{\partial x} - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\varphi}{\partial x} + \frac{1}{r}\frac{\sin\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial x} = \\
= &\quad \frac{1}{r^2}\sin\varphi\cos\varphi + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\
\frac{\partial^2 \varphi}{\partial y^2} &= -\frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial r}{\partial y} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\varphi}{\partial y} - \frac{1}{r}\frac{\cos\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial y} = \\
= &\quad -\frac{1}{r^2}\sin\varphi\cos\varphi - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\
\frac{\partial^2 \varphi}{\partial z^2} &= 0
\end{aligned}
\right.
\tag{48}
$$

$$
\left\{
\begin{aligned}
\frac{\partial^2 \theta}{\partial x^2} &= -\frac{1}{r^2}\cos\varphi\cos\theta\frac{\partial r}{\partial x} - \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\varphi}{\partial x} - \frac{1}{r}\cos\varphi\sin\theta\frac{\partial\theta}{\partial x} = \\
&= -\frac{2}{r^2}\cos^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\sin^2\theta}{\tan\theta} \\
\frac{\partial^2 \varphi}{\partial y^2} &= -\frac{1}{r^2}\sin\varphi\cos\theta\frac{\partial r}{\partial y} - \frac{1}{r}\sin\varphi\sin\theta\frac{\partial\varphi}{\partial y} + \frac{1}{r}\cos\varphi\cos\theta\frac{\partial\theta}{\partial y} = \\
&= -\frac{2}{r^2}\sin^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\cos^2\varphi}{\tan\theta} \\
\frac{\partial^2 \theta}{\partial z^2} &= \frac{1}{r^2}\sin\theta\frac{\partial r}{\partial z} - \frac{1}{r}\cos\theta\frac{\partial\theta}{\partial z} = \\
&= \frac{2}{r^2}\cos\theta\sin\theta
\end{aligned}
\right.
\tag{49}
$$

### 1.2.3 Problème C

$$\begin{cases} x = x(\xi,\eta,\zeta) \\ y = x(\xi,\eta,\zeta) \\ z = x(\xi,\eta,\zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x,y,z) \\ \eta = \eta(x,y,z) \\ \zeta = \zeta(x,y,z) \end{cases} \tag{50}$$

$$f(x,y,z) = f(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)) \tag{51}$$

$$f(\xi,\eta,\zeta) = f(\xi(x,y,z), \eta(x,y,z), \zeta(x,y,z)) \tag{52}$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{j}_y + \frac{\partial f}{\partial z}\mathbf{k}_z \tag{53}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{54}$$

$$\begin{aligned} \mathbf{i}_\xi &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\ \mathbf{j}_\eta &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\ \mathbf{k}_\zeta &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \end{aligned} \tag{55}$$

$$
\begin{aligned}
\nabla_{\xi\eta\zeta} f =& \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}_\xi + \\
&+ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}_\eta + \\
&+ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}_\zeta = \\
=& \begin{bmatrix} \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix}
\end{aligned} \tag{56}
$$

**Coordonnées cylindriques**

$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{57}
$$

$$
\begin{cases} \frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= 0 \end{cases}
\qquad
\begin{cases} \frac{\partial \theta}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\ \frac{\partial \theta}{\partial z} &= 0 \end{cases}
\qquad
\begin{cases} \frac{\partial \tilde{z}}{\partial x} &= 0 \\ \frac{\partial \tilde{z}}{\partial y} &= 0 \\ \frac{\partial \tilde{z}}{\partial z} &= 1 \end{cases} \tag{58}
$$

$$\begin{cases}\frac{\partial^2 r}{\partial x^2} & = \frac{\partial}{\partial x}\left(\cos\theta\right) = \\ & = -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\ \\ \frac{\partial^2 r}{\partial y^2} & = \frac{\partial}{\partial y}\left(\sin\theta\right) = \\ & = \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\ \\ \frac{\partial^2 r}{\partial z^2} & = 0\end{cases} \quad \begin{cases}\frac{\partial^2 \theta}{\partial x^2} & = \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\ & = -\frac{1}{r}\frac{\partial}{\partial x}\left(\sin\theta\right) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\ & = -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\ & = \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\ & = \frac{1}{r}\frac{\partial}{\partial y}\left(\cos\theta\right) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\ & = -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\ & = -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} & = 0\end{cases} \quad \begin{cases}\frac{\partial^2 \tilde{z}}{\partial x^2} & = 0 \\ \\ \\ \\ \frac{\partial^2 \tilde{z}}{\partial y^2} & = 0 \\ \\ \\ \\ \frac{\partial^2 \tilde{z}}{\partial z^2} & = 0\end{cases} \tag{59}$$

$$\begin{cases}\sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1 \\ \sqrt{\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2} = \frac{1}{r} \\ \sqrt{\left(\frac{\partial \tilde{z}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial y}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial z}\right)^2} = 1\end{cases} \tag{60}$$

$$\begin{aligned}\mathbf{i}_r &= \begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} \\ \mathbf{j}_\theta &= \begin{bmatrix}-\sin\theta \\ \cos\theta \\ 0\end{bmatrix} \\ \mathbf{k}_\zeta &= \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}\end{aligned} \tag{61}$$

$$\nabla_{r\theta\tilde{z}} f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \tilde{z}} \end{bmatrix} \tag{62}$$

### Coordonnées spheriques

# 2 Second section

# A First appendix