# EEIGM

## École Européenne d'Ingénieurs en Génie des Matériaux

$2^{`eme}$ Année, $1^{er}$ Semestre

# Mécanique du Solide Déformable

# Travaux Dirigés

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# 1 Systèmes de coordonnées curvilignes

## 1.1 Énoncé

### 1.1.1 Problème A

Déterminer l'expression de

  – les vecteurs du repère local naturel (base covariante),
  – le jacobien de la transformation,
  – le tenseur métrique,
  – le déplacement infinitésimal d'un point,
  – l'élément infinitésimal de ligne,
  – l'élément infinitésimal de volume;

dans un système de coordonnées

1. cylindriques,
2. sphériques.

### 1.1.2 Problème B

Exprimer l'opérateur laplacien $\nabla^2$ en

1. coordonnées cylindriques,
2. coordonnées spheriques.

### 1.1.3 Problème C

Dériver l'expression des vecteurs de la base contravariante et du gradient d'un fonction scalaire $\nabla f$ en

1. coordonnées cylindriques,
2. coordonnées spheriques.

## 1.2 Corrigé

### 1.2.1 Problème A

**Rappels théoriques**

$$
\begin{cases}
x = x\left(\xi, \eta, \zeta\right) \\
y = y\left(\xi, \eta, \zeta\right) \\
z = z\left(\xi, \eta, \zeta\right)
\end{cases}
\longleftrightarrow
\begin{cases}
\xi = \xi\left(x, y, z\right) \\
\eta = \eta\left(x, y, z\right) \\
\zeta = \zeta\left(x, y, z\right)
\end{cases}
\tag{1}
$$

$$
\mathbf{r} =
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}
\tag{2}
$$

$$
d\mathbf{r} =
\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}
\tag{3}
$$

$$
\begin{cases}
dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta \\
dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta \\
dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta
\end{cases}
\tag{4}
$$

$$
d\mathbf{r} =
\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}
=
\underbrace{\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}}_{\mathbf{J}}
\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}
= \mathbf{J}
\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}
\tag{5}
$$

$$
\begin{bmatrix}
\boxed{\frac{\partial x}{\partial \xi}} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\
\boxed{\frac{\partial y}{\partial \xi}} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\
\boxed{\frac{\partial z}{\partial \xi}} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
\qquad
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \boxed{\frac{\partial x}{\partial \eta}} & \frac{\partial x}{\partial \zeta} \\
\frac{\partial y}{\partial \xi} & \boxed{\frac{\partial y}{\partial \eta}} & \frac{\partial y}{\partial \zeta} \\
\frac{\partial z}{\partial \xi} & \boxed{\frac{\partial z}{\partial \eta}} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
\qquad
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \boxed{\frac{\partial x}{\partial \zeta}} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \boxed{\frac{\partial y}{\partial \zeta}} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \boxed{\frac{\partial z}{\partial \zeta}}
\end{bmatrix}
$$

$$
\frac{\partial \mathbf{r}}{\partial \xi} \qquad\qquad\qquad \frac{\partial \mathbf{r}}{\partial \eta} \qquad\qquad\qquad \frac{\partial \mathbf{r}}{\partial \zeta}
$$

$$\mathbf{i}_\xi = \frac{1}{||\frac{\partial \mathbf{r}}{\partial \xi}||}\frac{\partial \mathbf{r}}{\partial \xi} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}}\begin{bmatrix}\frac{\partial x}{\partial \xi}\\\frac{\partial y}{\partial \xi}\\\frac{\partial z}{\partial \xi}\end{bmatrix}$$

$$\mathbf{j}_\eta = \frac{1}{||\frac{\partial \mathbf{r}}{\partial \eta}||}\frac{\partial \mathbf{r}}{\partial \eta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}}\begin{bmatrix}\frac{\partial x}{\partial \eta}\\\frac{\partial y}{\partial \eta}\\\frac{\partial z}{\partial \eta}\end{bmatrix} \qquad (6)$$

$$\mathbf{k}_\zeta = \frac{1}{||\frac{\partial \mathbf{r}}{\partial \zeta}||}\frac{\partial \mathbf{r}}{\partial \zeta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}}\begin{bmatrix}\frac{\partial x}{\partial \zeta}\\\frac{\partial y}{\partial \zeta}\\\frac{\partial z}{\partial \zeta}\end{bmatrix}$$

$$dl^2 = d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix}dx & dy & dz\end{bmatrix}\begin{bmatrix}dx\\dy\\dz\end{bmatrix} = dx^2 + dy^2 + dz^2 \qquad (7)$$

$$dl^2 = d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix}d\xi & d\eta & d\zeta\end{bmatrix}\mathbf{J}^T\mathbf{J}\begin{bmatrix}d\xi\\d\eta\\d\zeta\end{bmatrix} =$$

$$= \begin{bmatrix}d\xi & d\eta & d\zeta\end{bmatrix}\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}\\\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}\\\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta}\\\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta}\\\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}\begin{bmatrix}d\xi\\d\eta\\d\zeta\end{bmatrix} =$$

$$= \begin{bmatrix}d\xi & d\eta & d\zeta\end{bmatrix}\underbrace{\begin{bmatrix}\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta}\\\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta}\\\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} & \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2\end{bmatrix}}_{\mathbf{g}}\begin{bmatrix}d\xi\\d\eta\\d\zeta\end{bmatrix}$$

$$(8)$$

$$dV = \frac{\partial \mathbf{r}}{\partial \xi}d\xi \cdot \left(\frac{\partial \mathbf{r}}{\partial \eta}d\eta \wedge \frac{\partial \mathbf{r}}{\partial \zeta}d\zeta\right) = det\begin{bmatrix}\left(\frac{\partial \mathbf{r}}{\partial \xi}\right)^T\\\left(\frac{\partial \mathbf{r}}{\partial \eta}\right)^T\\\left(\frac{\partial \mathbf{r}}{\partial \zeta}\right)^T\end{bmatrix}d\xi d\eta d\zeta = det\left(\mathbf{J}^T\right)d\xi d\eta d\zeta \quad (9)$$

## Coordonnées cylindriques

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{10}$$

$$\begin{cases} \frac{\partial x}{\partial r} = \cos\theta \\ \frac{\partial x}{\partial \theta} = -r\sin\theta \\ \frac{\partial x}{\partial \tilde{z}} = 0 \end{cases} \quad \begin{cases} \frac{\partial y}{\partial r} = \sin\theta \\ \frac{\partial y}{\partial \theta} = r\cos\theta \\ \frac{\partial y}{\partial \tilde{z}} = 0 \end{cases} \quad \begin{cases} \frac{\partial z}{\partial r} = 0 \\ \frac{\partial z}{\partial \theta} = 0 \\ \frac{\partial z}{\partial \tilde{z}} = 1 \end{cases} \tag{11}$$

$$\mathbf{i}_r = \frac{1}{||\frac{\partial \mathbf{r}}{\partial r}||}\frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2\theta + \cos^2\theta}}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$\mathbf{j}_\theta = \frac{1}{||\frac{\partial \mathbf{r}}{\partial \theta}||}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}}\begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{12}$$

$$\mathbf{k}_{\tilde{z}} = \frac{1}{||\frac{\partial \mathbf{r}}{\partial \tilde{z}}||}\frac{\partial \mathbf{r}}{\partial \tilde{z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$d\mathbf{r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} dr + \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} d\theta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\tilde{z} = dr\mathbf{i}_r + rd\theta\mathbf{i}_\theta + d\tilde{z}\mathbf{i}_{\tilde{z}} = \begin{bmatrix} dr \\ rd\theta \\ d\tilde{z} \end{bmatrix} \tag{14}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$dl^2 = dr^2 + r^2 d\theta^2 + d\tilde{z}^2 \tag{16}$$

$$dV = det\left(\mathbf{J}^T\right) dr d\theta d\tilde{z} = r dr d\theta d\tilde{z} \tag{17}$$

**Coordonnées spheriques**

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{18}$$

$$\begin{cases} \frac{\partial x}{\partial r} = \sin\theta\cos\varphi \\ \frac{\partial x}{\partial\theta} = r\cos\theta\cos\varphi \\ \frac{\partial x}{\partial\varphi} = -r\sin\theta\sin\varphi \end{cases} \begin{cases} \frac{\partial y}{\partial r} = \sin\theta\sin\varphi \\ \frac{\partial y}{\partial\theta} = r\cos\theta\sin\varphi \\ \frac{\partial y}{\partial\varphi} = r\sin\theta\cos\varphi \end{cases} \begin{cases} \frac{\partial z}{\partial r} = \cos\theta \\ \frac{\partial z}{\partial\theta} = -r\sin\theta \\ \frac{\partial z}{\partial\varphi} = 0 \end{cases} \tag{19}$$

$$\mathbf{i}_r = \frac{1}{||\frac{\partial\mathbf{r}}{\partial r}||}\frac{\partial\mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}}\begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix}$$

$$\mathbf{j}_\varphi = \frac{1}{||\frac{\partial\mathbf{r}}{\partial\varphi}||}\frac{\partial\mathbf{r}}{\partial\varphi} = \frac{1}{\sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi}}\begin{bmatrix}-r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0\end{bmatrix} = \begin{bmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{bmatrix}$$

$$\mathbf{k}_\theta = \frac{1}{||\frac{\partial\mathbf{r}}{\partial\theta}||}\frac{\partial\mathbf{r}}{\partial\theta} = \frac{1}{\sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}}\begin{bmatrix}r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta\end{bmatrix}$$

$$\tag{20}$$

$$\mathbf{J} = \begin{bmatrix}\sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta\end{bmatrix} \tag{21}$$

$$d\mathbf{r} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} dr + r\sin\theta \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} d\varphi + r \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix} d\theta =$$

$$= dr\mathbf{i}_r + r\sin\theta d\varphi\mathbf{j}_\varphi + rd\theta\mathbf{k}_\theta = \begin{bmatrix} dr \\ r\sin\theta d\varphi \\ rd\theta \end{bmatrix} \tag{22}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2 \end{bmatrix} \tag{23}$$

$$dl^2 = dr^2 + r^2\sin^2\theta d\varphi^2 + r^2 d\theta^2 \tag{24}$$

$$dV = det\left(\mathbf{J}^T\right) dr d\theta d\tilde{z} = r^2\sin\theta dr d\varphi d\theta \tag{25}$$

### 1.2.2 Problème B

**Rappels théoriques**

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = y\left(\xi, \eta, \zeta\right) \\ z = z\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{26}$$

$$f\left(x, y, z\right) = f\left(x\left(\xi, \eta, \zeta\right), y\left(\xi, \eta, \zeta\right), z\left(\xi, \eta, \zeta\right)\right) \tag{27}$$

$$f\left(\xi, \eta, \zeta\right) = f\left(\xi\left(x, y, z\right), \eta\left(x, y, z\right), \zeta\left(x, y, z\right)\right) \tag{28}$$

$$\nabla^2_{xyz} f\left(x, y, z\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{29}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{30}$$

$$\begin{cases} \begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) = \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) = \\ &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \\ \frac{\partial^2 f}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) = \\ &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2} \end{aligned} \end{cases} \tag{31}$$

$$
\begin{cases}
\frac{\partial}{\partial x} = \left( \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \\[2mm]
\frac{\partial}{\partial y} = \left( \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \\[2mm]
\frac{\partial}{\partial z} = \left( \frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right)
\end{cases}
\tag{32}
$$

$$
\begin{cases}
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial x} = \left( \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial x} = \\[1mm]
\quad = \left( \frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left( \frac{\partial \eta}{\partial x} \right) \left( \frac{\partial \xi}{\partial x} \right) \frac{\partial^2 f}{\partial \eta \partial \xi} + \left( \frac{\partial \zeta}{\partial x} \right) \left( \frac{\partial \xi}{\partial x} \right) \frac{\partial^2 f}{\partial \zeta \partial \xi} \\[3mm]
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x} = \left( \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x} = \\[1mm]
\quad = \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left( \frac{\partial \zeta}{\partial x} \right) \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial^2 f}{\partial \zeta \partial \eta} \\[3mm]
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} = \left( \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} = \\[1mm]
\quad = \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + \left( \frac{\partial \eta}{\partial x} \right) \left( \frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \left( \frac{\partial \zeta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{33}
$$

$$
\begin{cases}
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial y} = \left( \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial y} = \\[1mm]
\quad = \left( \frac{\partial \xi}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left( \frac{\partial \eta}{\partial y} \right) \left( \frac{\partial \xi}{\partial y} \right) \frac{\partial^2 f}{\partial \eta \partial \xi} + \left( \frac{\partial \zeta}{\partial y} \right) \left( \frac{\partial \xi}{\partial y} \right) \frac{\partial^2 f}{\partial \zeta \partial \xi} \\[3mm]
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} = \left( \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} = \\[1mm]
\quad = \left( \frac{\partial \xi}{\partial y} \right) \left( \frac{\partial \eta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + \left( \frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left( \frac{\partial \zeta}{\partial y} \right) \left( \frac{\partial \eta}{\partial y} \right) \frac{\partial^2 f}{\partial \zeta \partial \eta} \\[3mm]
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} = \left( \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} = \\[1mm]
\quad = \left( \frac{\partial \xi}{\partial y} \right) \left( \frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + \left( \frac{\partial \eta}{\partial y} \right) \left( \frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \left( \frac{\partial \zeta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{34}
$$

$$
\begin{cases}
\dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial z} &= \left(\dfrac{\partial \xi}{\partial z}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial z}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial z} = \\
&= \left(\dfrac{\partial \xi}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial \xi^2} + \left(\dfrac{\partial \eta}{\partial z}\right)\left(\dfrac{\partial \xi}{\partial z}\right)\dfrac{\partial^2 f}{\partial \eta \partial \xi} + \left(\dfrac{\partial \zeta}{\partial z}\right)\left(\dfrac{\partial \xi}{\partial z}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \xi} \\[2mm]
\dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial z} &= \left(\dfrac{\partial \xi}{\partial z}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial z}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial z} = \\
&= \left(\dfrac{\partial \xi}{\partial x}\right)\left(\dfrac{\partial \eta}{\partial z}\right)\dfrac{\partial^2 f}{\partial \xi \partial \eta} + \left(\dfrac{\partial \eta}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial \eta^2} + \left(\dfrac{\partial \zeta}{\partial z}\right)\left(\dfrac{\partial \eta}{\partial z}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \eta} \\[2mm]
\dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial z} &= \left(\dfrac{\partial \xi}{\partial z}\dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial z}\dfrac{\partial}{\partial \zeta}\right)\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial z} = \\
&= \left(\dfrac{\partial \xi}{\partial z}\right)\left(\dfrac{\partial \zeta}{\partial z}\right)\dfrac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\dfrac{\partial \eta}{\partial z}\right)\left(\dfrac{\partial \zeta}{\partial z}\right)\dfrac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\dfrac{\partial \zeta}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{35}
$$

$$
\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial^2 f}{\partial \xi \partial \eta} \qquad \frac{\partial^2 f}{\partial \eta \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \eta} \qquad \frac{\partial^2 f}{\partial \xi \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \xi}
\tag{36}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} =& \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \zeta^2}+ \\
&+ 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta}+ \\
&+ \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} =& \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \zeta^2}+ \\
&+ 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta}+ \\
&+ \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2}
\end{aligned}
\tag{38}
$$

$$\frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} +$$

$$+ 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} +$$

$$+ \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2}$$

$$(39)$$

$$\nabla^2_{\xi \eta \zeta} f(\xi, \eta, \zeta) = \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \xi^2} +$$

$$+ \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} +$$

$$+ \left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \zeta^2} +$$

$$+ 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \eta} +$$

$$+ 2\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \eta \partial \zeta} +$$

$$+ 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \zeta} +$$

$$+ \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right)\frac{\partial f}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\frac{\partial f}{\partial \eta} +$$

$$+ \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right)\frac{\partial f}{\partial \zeta}$$

$$(40)$$

**Coordonnées cylindriques**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{41}$$

$$\begin{cases} \frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= 0 \end{cases} \begin{cases} \frac{\partial\theta}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\ \frac{\partial\theta}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\ \frac{\partial\theta}{\partial z} &= 0 \end{cases} \begin{cases} \frac{\partial\tilde{z}}{\partial x} &= 0 \\ \\ \frac{\partial\tilde{z}}{\partial y} &= 0 \\ \\ \frac{\partial\tilde{z}}{\partial z} &= 1 \end{cases} \tag{42}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x}(\cos\theta) = \\ &= -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\ \\ \frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y}(\sin\theta) = \\ &= \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\ \\ \frac{\partial^2 r}{\partial z^2} &= 0 \end{cases} \begin{cases} \frac{\partial^2\theta}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\ &= -\frac{1}{r}\frac{\partial}{\partial x}(\sin\theta) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\ &= -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\ &= \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2\theta}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\ &= \frac{1}{r}\frac{\partial}{\partial y}(\cos\theta) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\ &= -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\ &= -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2\theta}{\partial z^2} &= 0 \end{cases} \begin{cases} \frac{\partial^2\tilde{z}}{\partial x^2} &= 0 \\ \\ \\ \frac{\partial^2\tilde{z}}{\partial y^2} &= 0 \\ \\ \\ \frac{\partial^2\tilde{z}}{\partial z^2} &= 0 \end{cases} \tag{43}$$

$$\begin{aligned} \nabla^2_{r\theta\tilde{z}}f(r,\theta,\tilde{z}) &= \left[\cos^2\theta + \sin\theta^2\right]\frac{\partial^2 f}{\partial r^2} + \left[\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right]\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\tilde{z}^2} + \\ &\quad + 2\left[-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right]\frac{\partial^2 f}{\partial r\partial\theta} + \\ &\quad + \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial f}{\partial r} + \left(\frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2}\right)\frac{\partial f}{\partial\theta} = \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\tilde{z}^2} \end{aligned} \tag{44}$$

**Coordonnées spheriques**

$$
\begin{cases}
x = r\sin\theta\cos\varphi \\
y = r\sin\theta\sin\varphi \\
z = r\cos\theta
\end{cases}
\longleftrightarrow
\begin{cases}
r = \sqrt{x^2+y^2+z^2} \\
\varphi = \tan^{-1}\left(\frac{y}{x}\right) \\
\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)
\end{cases}
\tag{45}
$$

$$
\begin{cases}
\frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \\
&= \frac{\cancel{r}\sin\theta\cos\varphi}{\cancel{r}} \\
\frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}} = \\
&= \frac{\cancel{r}\sin\theta\sin\varphi}{\cancel{r}} \\
\frac{\partial r}{\partial z} &= \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \\
&= \frac{\cancel{r}\cos\theta}{\cancel{r}}
\end{cases}
\quad
\begin{cases}
\frac{\partial\varphi}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\
&= -\frac{\sin\varphi}{r\sin\theta} \\
\frac{\partial\varphi}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\
&= \frac{\cos\varphi}{r\sin\theta} \\
\frac{\partial\varphi}{\partial z} &= 0
\end{cases}
\quad
\begin{cases}
\frac{\partial\theta}{\partial x} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}x}{\cancel{2}z\sqrt{x^2+y^2}} = \\
&= \frac{1}{r}\cos\theta\cos\varphi \\
\frac{\partial\theta}{\partial y} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}y}{\cancel{2}z\sqrt{x^2+y^2}} = \\
&= \frac{1}{r}\cos\theta\sin\varphi \\
\frac{\partial\theta}{\partial z} &= -\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\sqrt{x^2+y^2}}{z^2} = \\
&= -\frac{1}{r}\sin\theta
\end{cases}
\tag{46}
$$

$$
\begin{cases}
\frac{\partial^2 r}{\partial x^2} &= \cos\theta\cos\varphi\frac{\partial\theta}{\partial x} - \sin\theta\sin\varphi\frac{\partial\varphi}{\partial x} = \\
&= \frac{1}{r}\left(\cos^2\varphi\cos^2\theta + \sin^2\varphi\right) \\
\frac{\partial^2 r}{\partial y^2} &= \cos\theta\sin\varphi\frac{\partial\theta}{\partial y} + \sin\theta\cos\varphi\frac{\partial\varphi}{\partial y} = \\
&= \frac{1}{r}\left(\sin^2\varphi\cos^2\theta + \cos^2\varphi\right) \\
\frac{\partial^2 r}{\partial z^2} &= -\sin\theta\frac{\partial\theta}{\partial z} = \\
&= \frac{1}{r}\sin^2\theta
\end{cases}
\tag{47}
$$

$$
\begin{cases}
\frac{\partial^2\varphi}{\partial x^2} &= \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial r}{\partial x} - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\varphi}{\partial x} + \frac{1}{r}\frac{\sin\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial x} = \\
&= \frac{1}{r^2}\sin\varphi\cos\varphi + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\
\frac{\partial^2\varphi}{\partial y^2} &= -\frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial r}{\partial y} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\varphi}{\partial y} - \frac{1}{r}\frac{\cos\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial y} = \\
&= -\frac{1}{r^2}\sin\varphi\cos\varphi - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\
\frac{\partial^2\varphi}{\partial z^2} &= 0
\end{cases}
\tag{48}
$$

$$\begin{cases} \frac{\partial^2 \theta}{\partial x^2} & = -\frac{1}{r^2}\cos\varphi\cos\theta\frac{\partial r}{\partial x} - \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\varphi}{\partial x} - \frac{1}{r}\cos\varphi\sin\theta\frac{\partial\theta}{\partial x} = \\ & = -\frac{2}{r^2}\cos^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\sin^2\varphi}{\tan\theta} \\ \frac{\partial^2 \theta}{\partial y^2} & = -\frac{1}{r^2}\sin\varphi\cos\theta\frac{\partial r}{\partial y} - \frac{1}{r}\sin\varphi\sin\theta\frac{\partial\varphi}{\partial y} + \frac{1}{r}\cos\varphi\cos\theta\frac{\partial\theta}{\partial y} = \\ & = -\frac{2}{r^2}\sin^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\cos^2\varphi}{\tan\theta} \\ \frac{\partial^2 \theta}{\partial z^2} & = \frac{1}{r^2}\sin\theta\frac{\partial r}{\partial z} - \frac{1}{r}\cos\theta\frac{\partial\theta}{\partial z} = \\ & = \frac{2}{r^2}\cos\theta\sin\theta \end{cases} \tag{49}$$

$$\nabla^2_{r\varphi\theta} f\left(\xi,\eta,\zeta\right) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{2}{r^2\tan\theta}\frac{\partial f}{\partial\theta} \tag{50}$$

### 1.2.3 Problème C

**Rappels théoriques**

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = x\left(\xi, \eta, \zeta\right) \\ z = x\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{51}$$

$$f\left(x, y, z\right) = f\left(x\left(\xi, \eta, \zeta\right), y\left(\xi, \eta, \zeta\right), z\left(\xi, \eta, \zeta\right)\right) \tag{52}$$

$$f\left(\xi, \eta, \zeta\right) = f\left(\xi\left(x, y, z\right), \eta\left(x, y, z\right), \zeta\left(x, y, z\right)\right) \tag{53}$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{i}^x + \frac{\partial f}{\partial y}\mathbf{j}^y + \frac{\partial f}{\partial z}\mathbf{k}^z \tag{54}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{55}$$

$$\mathbf{i}^\xi = \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix}$$

$$\mathbf{j}^\eta = \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \tag{56}$$

$$\mathbf{k}^\zeta = \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

$$
\begin{aligned}
\nabla_{\xi\eta\zeta} f = &\frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}^\xi + \\
&+ \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}^\eta + \\
&+ \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}^\zeta = \\
= &\begin{bmatrix}
\frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\
\frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\
\frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}
\end{bmatrix}
\end{aligned}
\tag{57}
$$

**Coordonnées cylindriques**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{58}$$

$$\begin{cases} \frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\ &= \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= 0 \end{cases} \begin{cases} \frac{\partial \theta}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\ \frac{\partial \theta}{\partial z} &= 0 \end{cases} \begin{cases} \frac{\partial \tilde{z}}{\partial x} &= 0 \\[1em] \frac{\partial \tilde{z}}{\partial y} &= 0 \\[1em] \frac{\partial \tilde{z}}{\partial z} &= 1 \end{cases} \tag{59}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x}\left(\cos\theta\right) = \\ &= -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\[2em] \frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y}\left(\sin\theta\right) = \\ &= \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\[2em] \frac{\partial^2 r}{\partial z^2} &= 0 \end{cases} \begin{cases} \frac{\partial^2 \theta}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\ &= -\frac{1}{r}\frac{\partial}{\partial x}\left(\sin\theta\right) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\ &= -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\ &= \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\ &= \frac{1}{r}\frac{\partial}{\partial y}\left(\cos\theta\right) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\ &= -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\ &= -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} &= 0 \end{cases} \begin{cases} \frac{\partial^2 \tilde{z}}{\partial x^2} &= 0 \\[3em] \frac{\partial^2 \tilde{z}}{\partial y^2} &= 0 \\[3em] \frac{\partial^2 \tilde{z}}{\partial z^2} &= 0 \end{cases}$$
$$\tag{60}$$

$$\begin{cases} \sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1 \\[1em] \sqrt{\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2} = \frac{1}{r} \\[1em] \sqrt{\left(\frac{\partial \tilde{z}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial y}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial z}\right)^2} = 1 \end{cases} \tag{61}$$

$$\mathbf{i}^r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$\mathbf{j}^\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{62}$$

$$\mathbf{k}^\zeta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\nabla_{r\theta\tilde{z}} f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial\theta} \\ \frac{\partial f}{\partial\tilde{z}} \end{bmatrix} \tag{63}$$

**Coordonnées spheriques**

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{64}$$

$$\begin{cases} \frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\sin\theta\cos\varphi}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\sin\theta\sin\varphi}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \end{cases} \quad \begin{cases} \frac{\partial\varphi}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{\sin\varphi}{r\sin\theta} \\ \frac{\partial\varphi}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{\cos\varphi}{r\sin\theta} \\ \frac{\partial\varphi}{\partial z} &= 0 \end{cases} \quad \begin{cases} \frac{\partial\theta}{\partial x} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}x}{\cancel{2}z\sqrt{x^2+y^2}} = \\ &= \frac{1}{r}\cos\theta\cos\varphi \\ \frac{\partial\theta}{\partial y} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}y}{\cancel{2}z\sqrt{x^2+y^2}} = \\ &= \frac{1}{r}\cos\theta\sin\varphi \\ \frac{\partial\theta}{\partial z} &= -\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\sqrt{x^2+y^2}}{z^2} = \\ &= -\frac{1}{r}\sin\theta \end{cases} \tag{65}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} &= \cos\theta\cos\varphi\frac{\partial\theta}{\partial x} - \sin\theta\sin\varphi\frac{\partial\varphi}{\partial x} = \\ &= \frac{1}{r}\left(\cos^2\varphi\cos^2\theta + \sin^2\varphi\right) \\ \frac{\partial^2 r}{\partial y^2} &= \cos\theta\sin\varphi\frac{\partial\theta}{\partial y} + \sin\theta\cos\varphi\frac{\partial\varphi}{\partial y} = \\ &= \frac{1}{r}\left(\sin^2\varphi\cos^2\theta + \cos^2\varphi\right) \\ \frac{\partial^2 r}{\partial z^2} &= -\sin\theta\frac{\partial\theta}{\partial z} = \\ &= \frac{1}{r}\sin^2\theta \end{cases} \tag{66}$$

$$\begin{cases} \frac{\partial^2\varphi}{\partial x^2} &= \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial r}{\partial x} - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\varphi}{\partial x} + \frac{1}{r}\frac{\sin\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial x} = \\ &= \frac{1}{r^2}\sin\varphi\cos\varphi + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\ \frac{\partial^2\varphi}{\partial y^2} &= -\frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial r}{\partial y} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\varphi}{\partial y} - \frac{1}{r}\frac{\cos\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial y} = \\ &= -\frac{1}{r^2}\sin\varphi\cos\varphi - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\ \frac{\partial^2\varphi}{\partial z^2} &= 0 \end{cases} \tag{67}$$

$$\begin{cases} \frac{\partial^2 \theta}{\partial x^2} & = -\frac{1}{r^2}\cos\varphi\cos\theta\frac{\partial r}{\partial x} - \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\varphi}{\partial x} - \frac{1}{r}\cos\varphi\sin\theta\frac{\partial\theta}{\partial x} = \\ & = -\frac{2}{r^2}\cos^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\sin^2\varphi}{\tan\theta} \\ \frac{\partial^2 \theta}{\partial y^2} & = -\frac{1}{r^2}\sin\varphi\cos\theta\frac{\partial r}{\partial y} - \frac{1}{r}\sin\varphi\sin\theta\frac{\partial\varphi}{\partial y} + \frac{1}{r}\cos\varphi\cos\theta\frac{\partial\theta}{\partial y} = \\ & = -\frac{2}{r^2}\sin^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\cos^2\varphi}{\tan\theta} \\ \frac{\partial^2 \theta}{\partial z^2} & = \frac{1}{r^2}\sin\theta\frac{\partial r}{\partial z} - \frac{1}{r}\cos\theta\frac{\partial\theta}{\partial z} = \\ & = \frac{2}{r^2}\cos\theta\sin\theta \end{cases} \quad (68)$$

$$\begin{cases} \sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1 \\ \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2} = \frac{1}{r\sin\theta} \\ \sqrt{\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2} = \frac{1}{r} \end{cases} \quad (69)$$

$$\begin{aligned} \mathbf{i}^r &= \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} \\ \mathbf{j}^\varphi &= r\sin\theta \begin{bmatrix} -\frac{\sin\varphi}{r\sin\theta} \\ \frac{\cos\varphi}{r\sin\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} \\ \mathbf{k}^\theta &= r \begin{bmatrix} \frac{1}{r}\cos\theta\cos\varphi \\ \frac{1}{r}\cos\theta\sin\varphi \\ -\frac{1}{r}\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix} \end{aligned} \quad (70)$$

$$\nabla_{r\varphi\theta} f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi} \\ \frac{1}{r}\frac{\partial f}{\partial\theta} \end{bmatrix} \quad (71)$$

# 2 Déplacement et mesures de déformation

## 2.1 Énoncé

## 2.2 Corrigé

### 2.2.1 Rappels théoriques

$$\mathbf{y} = \mathbf{y}\left(\mathbf{x}\right) = \mathbf{x} + \mathbf{u}\left(\mathbf{x}\right) \tag{72}$$

$$\mathbf{y} = \mathbf{x} + \mathbf{u}\left(\mathbf{x}\right) \longleftrightarrow \mathbf{u}\left(\mathbf{x}\right) = \mathbf{y} - \mathbf{x} \tag{73}$$

$$\underline{\underline{\mathbf{F}}} = \nabla\mathbf{y} \longleftrightarrow F_{ij} = \frac{\partial y_i}{\partial x_j} \tag{74}$$

$$d\mathbf{y} = \underline{\underline{\mathbf{F}}}d\mathbf{x} \tag{75}$$

$$d\mathbf{u} = d\mathbf{y} - d\mathbf{x} = \underline{\underline{\mathbf{F}}}d\mathbf{x} - d\mathbf{x} = \left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}}\right)d\mathbf{x} \tag{76}$$

$$d\mathbf{u} = \nabla_{\mathbf{x}}\mathbf{u}d\mathbf{x} \tag{77}$$

$$\nabla_{\mathbf{x}}\mathbf{u} = \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} \tag{78}$$

$$dS^2 = d\mathbf{x}^T d\mathbf{x} \tag{79}$$

$$ds^2 = d\mathbf{y}^T d\mathbf{y} = \left(\underline{\underline{\mathbf{F}}}d\mathbf{x}\right)^T \underline{\underline{\mathbf{F}}}d\mathbf{x} = d\mathbf{x}^T \underbrace{\underline{\underline{\mathbf{F}}}^T \underline{\underline{\mathbf{F}}}}_{\underline{\underline{\mathbf{C}}}} d\mathbf{x} = d\mathbf{x}^T \underline{\underline{\mathbf{C}}}d\mathbf{x} \tag{80}$$

$$ds^2 = d\mathbf{y}^T d\mathbf{y} =$$
$$= \left( d\mathbf{x} + d\mathbf{u}\left(\mathbf{x}\right) \right)^T \left( d\mathbf{x} + d\mathbf{u}\left(\mathbf{x}\right) \right) =$$
$$= d\mathbf{x}^T d\mathbf{x} + d\mathbf{x}^T d\mathbf{u}\left(\mathbf{x}\right) + d\mathbf{u}\left(\mathbf{x}\right)^T d\mathbf{x} + d\mathbf{u}\left(\mathbf{x}\right)^T d\mathbf{u}\left(\mathbf{x}\right) =$$
$$= d\mathbf{x}^T d\mathbf{x} + d\mathbf{x}^T \nabla_{\mathbf{x}}\mathbf{u} d\mathbf{x} + d\mathbf{x}^T \nabla_{\mathbf{x}}\mathbf{u}^T d\mathbf{x} + d\mathbf{x}^T \nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u} d\mathbf{x} =$$
$$= d\mathbf{x}^T \left( \underline{\underline{\mathbf{I}}} + \underbrace{\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u}}_{2\underline{\underline{\mathbf{E}}}} \right) d\mathbf{x} =$$
$$= d\mathbf{x}^T \left( \underline{\underline{\mathbf{I}}} + 2\underline{\underline{\mathbf{E}}} \right) d\mathbf{x} \tag{81}$$

$$\underline{\underline{\mathbf{E}}} = \frac{1}{2} \left( \nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u} \right) =$$
$$= \frac{1}{2} \left( \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{I}}} + \left( \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} \right)^T \left( \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} \right) \right) =$$
$$= \frac{1}{2} \left( \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{I}}} \right) =$$
$$= \frac{1}{2} \left( \underline{\underline{\mathbf{F}}}^T \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} \right) = \frac{1}{2} \left( \underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}} \right) \tag{82}$$

$$ds^2 - dS^2 = d\mathbf{y}^T d\mathbf{y} - d\mathbf{x}^T d\mathbf{x} =$$
$$= d\mathbf{x}^T \left( \underline{\underline{\mathbf{I}}} + 2\underline{\underline{\mathbf{E}}} \right) d\mathbf{x} - d\mathbf{x}^T d\mathbf{x} =$$
$$= d\mathbf{x}^T \left( \underline{\underline{\mathbf{I}}} + 2\underline{\underline{\mathbf{E}}} - \underline{\underline{\mathbf{I}}} \right) d\mathbf{x} =$$
$$= d\mathbf{x}^T 2\underline{\underline{\mathbf{E}}} d\mathbf{x} =$$
$$= d\mathbf{x}^T \left( \underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}} \right) d\mathbf{x} \tag{83}$$

$$\nabla_{\mathbf{x}}\mathbf{u} \ll 1 \longleftrightarrow \underline{\underline{\mathbf{E}}} = \frac{1}{2} \left( \nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \cancel{\nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u}}^{\sim 0} \right) \sim$$
$$\sim \frac{1}{2} \left( \nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T \right) = \underline{\underline{\epsilon}} \tag{84}$$

$$\underline{\underline{\epsilon}} = \frac{1}{2} \left( \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} \mathbf{u}^T \right) =$$
$$= \frac{1}{2} \left( \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{I}}} \right) =$$
$$= \frac{1}{2} \left( \underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{F}}}^T - 2\underline{\underline{\mathbf{I}}} \right) =$$
$$= \frac{1}{2} \left( \underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{F}}}^T \right) - \underline{\underline{\mathbf{I}}}$$

(85)

### 2.2.2 Problème A

$$
\begin{cases}
y_1 = kx_1 \\
y_2 = kx_2 \\
y_3 = kx_3
\end{cases}
\longleftrightarrow
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
= k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\quad k \in \mathbb{R} \tag{86}
$$

$$
\begin{cases}
u_1 = y_1 - x_1 = kx_1 - x_1 = (k-1)\,x_1 \\
u_2 = y_2 - x_2 = kx_2 - x_2 = (k-1)\,x_2 \\
u_3 = y_3 - x_3 = kx_3 - x_3 = (k-1)\,x_3
\end{cases}
\longleftrightarrow
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
= (k-1) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{87}
$$

$$
\underline{\underline{\mathbf{F}}} =
\begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\
\frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3}
\end{bmatrix}
=
\begin{bmatrix}
k & 0 & 0 \\
0 & k & 0 \\
0 & 0 & k
\end{bmatrix} \tag{88}
$$

$$
\begin{aligned}
\nabla_{\mathbf{x}}\mathbf{u} &=
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{bmatrix} = \\
&= \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} = \\
&=
\begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\
\frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3}
\end{bmatrix}
-
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \\
&=
\begin{bmatrix}
k & 0 & 0 \\
0 & k & 0 \\
0 & 0 & k
\end{bmatrix}
-
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
=
\begin{bmatrix}
k-1 & 0 & 0 \\
0 & k-1 & 0 \\
0 & 0 & k-1
\end{bmatrix}
\end{aligned} \tag{89}
$$

$$\underline{\underline{\mathbf{E}}} = \frac{1}{2} \left( \nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \nabla_{\mathbf{x}}\mathbf{u}^T\nabla_{\mathbf{x}}\mathbf{u} \right) =$$

$$= \frac{1}{2} \left( \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} + \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \right) +$$

$$+ \frac{1}{2} \left( \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \right) = \tag{90}$$

$$= \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (k-1)^2 & 0 & 0 \\ 0 & (k-1)^2 & 0 \\ 0 & 0 & (k-1)^2 \end{bmatrix}$$

$$\underline{\underline{\epsilon}} = \frac{1}{2} \left( \nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T \right) =$$

$$= \frac{1}{2} \left( \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} + \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \right) = \tag{91}$$

$$= \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}$$

$$\underline{\underline{\mathbf{E}}} \sim \underline{\underline{\epsilon}} \longleftrightarrow \frac{\frac{1}{2}(k-1)^2}{(k-1) + \frac{1}{2}(k-1)^2} \le e \tag{92}$$

$$(k-1)\left( \frac{1}{2}(1-e)(k-1) - e \right) \le 0 \tag{93}$$

$$1 \le k \le \frac{1+e}{1-e} \tag{94}$$

$$dV = dy_1 dy_2 dy_3 - dx_1 dx_2 dx_3 =$$
$$= k^3 dx_1 dx_2 dx_3 - dx_1 dx_2 dx_3 = \tag{95}$$
$$= \left(k^3 - 1\right) dx_1 dx_2 dx_3$$

$$\frac{dV}{V} = \frac{dy_1 dy_2 dy_3 - dx_1 dx_2 dx_3}{dx_1 dx_2 dx_3} =$$
$$= \frac{\left(k^3 - 1\right) dx_1 dx_2 dx_3}{dx_1 dx_2 dx_3} = k^3 - 1 \tag{96}$$

| e | $\frac{1+e}{1-e}$ | $\varepsilon_{xx}\big|_{k=\frac{1+e}{1-e}}$ | $\varepsilon_{yy}\big|_{k=\frac{1+e}{1-e}}$ | $\varepsilon_{zz}\big|_{k=\frac{1+e}{1-e}}$ | $\frac{dV}{V}$ |
|---|---|---|---|---|---|
| 0.001 (0.1%) | 1.002 | 0.002 (0.2%) | 0.002 (0.2%) | 0.002 (0.2%) | 0.006 (0.6%) |
| 0.005 (0.5%) | 1.01 | 0.01 (1%) | 0.01 (1%) | 0.01 (1%) | 0.03 (3%) |
| 0.01 (1%) | 1.02 | 0.02 (2%) | 0.02 (2%) | 0.02 (2%) | 0.06 (6%) |
| 0.05 (5%) | 1.11 | 0.11 (11%) | 0.11 (11%) | 0.11 (11%) | 0.35 (35%) |

### 2.2.3 Problème B

$$\begin{cases} y_1 = x_1 + \gamma x_2 \\ y_2 = x_2 \\ y_3 = x_3 \end{cases} \qquad \gamma \in \mathbb{R} \tag{97}$$