EEIGM

École Européenne d'Ingénieurs en Génie des Matériaux

2ème Année, 1er Semestre

MÉCANIQUE DU SOLIDE DÉFORMABLE

NOTES DU COURS

Luca Di Stasio

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Table des matières

1 Systèmes de coordonnées curvilignes

1.1 Formulation analytique

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = y\left(\xi, \eta, \zeta\right) \\ z = z\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{1}$$

1.2 Le déplacement infinitésimal d'un point et le jacobien de la transformation

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{2}$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{3}$$

$$\begin{cases} dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta \\ dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta \\ dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta \end{cases} \tag{4}$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{5}$$

1.3 Les vecteurs du repère local naturel (base covariante)

$$
\begin{array}{ccc}
\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} &
\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} &
\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \\
\frac{\partial \mathbf{r}}{\partial \xi} & \frac{\partial \mathbf{r}}{\partial \eta} & \frac{\partial \mathbf{r}}{\partial \zeta}
\end{array}
$$

$$
\begin{aligned}
\mathbf{i}_\xi &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \xi}||} \frac{\partial \mathbf{r}}{\partial \xi} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{bmatrix} \\
\mathbf{j}_\eta &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \eta}||} \frac{\partial \mathbf{r}}{\partial \eta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{bmatrix} \\
\mathbf{k}_\zeta &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \zeta}||} \frac{\partial \mathbf{r}}{\partial \zeta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{bmatrix}
\end{aligned} \tag{6}
$$

1.4 L'élément infinitésimal de ligne et le tenseur métrique

$$
dl^2 = d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} dx & dy & dz \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = dx^2 + dy^2 + dz^2 \tag{7}
$$

$$
\begin{aligned}
dl^2 = d\mathbf{r}^T d\mathbf{r} &= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \mathbf{J}^T \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \underbrace{\begin{bmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} & \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \end{bmatrix}}_{\mathbf{g}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}
\end{aligned} \tag{8}
$$

1.5 L'élément infinitésimal de volume

$$
dV = \frac{\partial \mathbf{r}}{\partial \xi} d\xi \cdot \left(\frac{\partial \mathbf{r}}{\partial \eta} d\eta \wedge \frac{\partial \mathbf{r}}{\partial \zeta} d\zeta \right) = det \begin{bmatrix} \left(\frac{\partial \mathbf{r}}{\partial \xi}\right)^T \\ \left(\frac{\partial \mathbf{r}}{\partial \eta}\right)^T \\ \left(\frac{\partial \mathbf{r}}{\partial \zeta}\right)^T \end{bmatrix} d\xi d\eta d\zeta = det\left(\mathbf{J}^T\right) d\xi d\eta d\zeta \tag{9}
$$

1.6 Les vecteurs de la base contravariante et le gradient d'un fonction scalaire ∇f

$$
f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \tag{10}
$$

$$
f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \tag{11}
$$

$$
\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{i}^x + \frac{\partial f}{\partial y}\mathbf{j}^y + \frac{\partial f}{\partial z}\mathbf{k}^z \tag{12}
$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{13}$$

$$\begin{aligned} \mathbf{i}^{\xi} &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\ \mathbf{j}^{\eta} &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\ \mathbf{k}^{\zeta} &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \end{aligned} \tag{14}$$

$$\begin{aligned} \nabla_{\xi\eta\zeta} f =& \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}^{\xi} + \\ &+ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}^{\eta} + \\ &+ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}^{\zeta} = \\ =& \begin{bmatrix} \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix} \end{aligned} \tag{15}$$

1.7 L'opérateur laplacien ∇^2

$$f(x,y,z) = f\left(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)\right) \tag{16}$$

$$f(\xi,\eta,\zeta) = f\left(\xi(x,y,z), \eta(x,y,z), \zeta(x,y,z)\right) \tag{17}$$

$$\nabla^2_{xyz} f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{18}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{19}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) = \\ & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) = \\ & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \\ \frac{\partial^2 f}{\partial z^2} & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) = \\ & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2} \end{cases} \tag{20}$$

$$\begin{cases} \frac{\partial}{\partial x} = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right) \\ \frac{\partial}{\partial y} = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right) \\ \frac{\partial}{\partial z} = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right) \end{cases} \tag{21}$$

$$
\begin{cases}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} & = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} = \\
 & = \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \xi}+\left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 f}{\partial \zeta \partial \xi} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} & = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \\
 & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta}+\left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \zeta \partial \eta} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} & = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} = \\
 & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta}+\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta}+\left(\frac{\partial \zeta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{22}
$$

$$
\begin{cases}
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} & = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} = \\
 & = \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \xi}+\left(\frac{\partial \zeta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 f}{\partial \zeta \partial \xi} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} & = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} = \\
 & = \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \eta}+\left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \zeta \partial \eta} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} & = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} = \\
 & = \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta}+\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta}+\left(\frac{\partial \zeta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{23}
$$

$$
\begin{cases}
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} = \\
 & = \left(\frac{\partial \xi}{\partial z}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \xi}+\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta \partial \xi} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} = \\
 & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta}+\left(\frac{\partial \eta}{\partial z}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta \partial \eta} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} = \\
 & = \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta}+\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta}+\left(\frac{\partial \zeta}{\partial z}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{24}
$$

$$
\frac{\partial^2 f}{\partial \eta \partial \xi}=\frac{\partial^2 f}{\partial \xi \partial \eta} \qquad \frac{\partial^2 f}{\partial \eta \partial \zeta}=\frac{\partial^2 f}{\partial \zeta \partial \eta} \qquad \frac{\partial^2 f}{\partial \xi \partial \zeta}=\frac{\partial^2 f}{\partial \zeta \partial \xi}
\tag{25}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = & \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
& + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
& + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} = & \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
& + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial y}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
& + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial z^2} = & \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
& + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
& + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\nabla^2_{\xi\eta\zeta} f\left(\xi,\eta,\zeta\right) = & \left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\xi}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial\xi^2} + \\
& + \left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2 + \left(\frac{\partial\eta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial\eta^2} + \\
& + \left[\left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial\zeta^2} + \\
& + 2\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\eta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial\xi\partial\eta} + \\
& + 2\left[\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\zeta}{\partial x}\right) + \left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\zeta}{\partial y}\right) + \left(\frac{\partial\eta}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial\eta\partial\zeta} + \\
& + 2\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\zeta}{\partial x}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\zeta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial\xi\partial\zeta} + \\
& + \left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right)\frac{\partial f}{\partial\xi} + \left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial z^2}\right)\frac{\partial f}{\partial\eta} + \\
& + \left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + \frac{\partial^2\zeta}{\partial z^2}\right)\frac{\partial f}{\partial\zeta}
\end{aligned}
\tag{29}
$$