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MÉCANIQUE DU SOLIDE DÉFORMABLE

TRAVAUX DIRIGÉS

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List of Acronyms

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List of Symbols

List of Symbols

Abstract

Abstract

1. Systèmes de coordonnées curvilignes

1.1. Énoncé

1.1.1. Problème A

1.1.2. Problème B

1.2. Corrigé

1.2.1. Problème A

Rappels théoriques

$$\begin{cases} x = x (\xi, \eta, \zeta) \\ y = x (\xi, \eta, \zeta) \\ z = x (\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi (x, y, z) \\ \eta = \eta (x, y, z) \\ \zeta = \zeta (x, y, z) \end{cases}$$
(1)

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta))$$
(2)

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z))$$
(3)

$$\nabla_{xyz}^{2} f(x, y, z) = \frac{\partial^{2} f}{\partial x^{2}} + \frac{\partial^{2} f}{\partial y^{2}} + \frac{\partial^{2} f}{\partial z^{2}}$$

$$\tag{4}$$

$$\begin{cases}
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\
\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z}
\end{cases} (5)$$

$$\begin{cases} \frac{\partial^{2} f}{\partial x^{2}} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) = \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi} \frac{\partial^{2} \xi}{\partial x^{2}} + \frac{\partial f}{\partial \eta} \frac{\partial^{2} \eta}{\partial x^{2}} + \frac{\partial f}{\partial \zeta} \frac{\partial^{2} \zeta}{\partial x^{2}} \\ \frac{\partial^{2} f}{\partial y^{2}} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) = \\ &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi} \frac{\partial^{2} \xi}{\partial y^{2}} + \frac{\partial f}{\partial \eta} \frac{\partial^{2} \eta}{\partial y^{2}} + \frac{\partial f}{\partial \zeta} \frac{\partial^{2} \zeta}{\partial y^{2}} \\ \frac{\partial^{2} f}{\partial z^{2}} &= \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) = \\ &= \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi} \frac{\partial^{2} \xi}{\partial z^{2}} + \frac{\partial f}{\partial \eta} \frac{\partial^{2} \eta}{\partial z^{2}} + \frac{\partial f}{\partial \zeta} \frac{\partial^{2} \zeta}{\partial z^{2}} \\ &= \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi} \frac{\partial^{2} \xi}{\partial z^{2}} + \frac{\partial f}{\partial \eta} \frac{\partial^{2} \gamma}{\partial z^{2}} + \frac{\partial f}{\partial \zeta} \frac{\partial^{2} \zeta}{\partial z^{2}} \end{aligned}$$

$$\begin{cases}
\frac{\partial}{\partial x} = \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta}\right) \\
\frac{\partial}{\partial y} = \left(\frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta}\right) \\
\frac{\partial}{\partial z} = \left(\frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta}\right)
\end{cases} (7)$$

$$\begin{cases}
\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial x} &= \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial x} = \\
&= \left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \xi}{\partial x} \right) \frac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial x} \right) \left(\frac{\partial \xi}{\partial x} \right) \frac{\partial^2 f}{\partial \zeta \partial \xi} \\
\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x} &= \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x} = \\
&= \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) \frac{\partial^2 f}{\partial \zeta \partial \eta} \\
\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} &= \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} = \\
&= \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \zeta^2}
\end{cases}$$
(8)

$$\begin{cases}
\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial y} &= \left(\frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial y} = \\
&= \left(\frac{\partial \xi}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \xi}{\partial y} \right) \frac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial y} \right) \left(\frac{\partial \xi}{\partial y} \right) \frac{\partial^2 f}{\partial \zeta \partial \xi} \\
\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} &= \left(\frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} = \\
&= \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) \frac{\partial^2 f}{\partial \zeta \partial \eta} \\
\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} &= \left(\frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} = \\
&= \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \zeta^2}
\end{cases}$$

$$\begin{cases}
\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} &= \left(\frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} = \\
&= \left(\frac{\partial \xi}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z} \right) \left(\frac{\partial \xi}{\partial z} \right) \frac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial z} \right) \left(\frac{\partial \xi}{\partial z} \right) \frac{\partial^2 f}{\partial \zeta \partial \xi} \\
\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} &= \left(\frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} = \\
&= \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial z} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z} \right) \left(\frac{\partial \eta}{\partial z} \right) \frac{\partial^2 f}{\partial \zeta \partial \eta} \\
\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} &= \left(\frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} = \\
&= \left(\frac{\partial \xi}{\partial z} \right) \left(\frac{\partial \zeta}{\partial z} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial z} \right) \left(\frac{\partial \zeta}{\partial z} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \zeta^2}
\end{cases}$$

$$(10)$$

$$\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial^2 f}{\partial \xi \partial \eta} \qquad \frac{\partial^2 f}{\partial \eta \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \eta} \qquad \frac{\partial^2 f}{\partial \xi \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \xi} \tag{11}$$

$$\begin{split} \frac{\partial^2 f}{\partial x^2} &= \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ &+ 2 \left(\frac{\partial \xi}{\partial x}\right) \left(\frac{\partial \eta}{\partial x}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial x}\right) \left(\frac{\partial \zeta}{\partial x}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial x}\right) \left(\frac{\partial \zeta}{\partial x}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ &+ \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial x^2} \end{split}$$
(12)

Introduction

$$\frac{\partial^{2} f}{\partial y^{2}} = \left(\frac{\partial \xi}{\partial y}\right)^{2} \frac{\partial^{2} f}{\partial \xi^{2}} + \left(\frac{\partial \eta}{\partial y}\right)^{2} \frac{\partial^{2} f}{\partial \eta^{2}} + \left(\frac{\partial \zeta}{\partial y}\right)^{2} \frac{\partial^{2} f}{\partial \zeta^{2}} + \\
+ 2\left(\frac{\partial \xi}{\partial y}\right) \left(\frac{\partial \eta}{\partial y}\right) \frac{\partial^{2} f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right) \left(\frac{\partial \zeta}{\partial y}\right) \frac{\partial^{2} f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right) \left(\frac{\partial \zeta}{\partial y}\right) \frac{\partial^{2} f}{\partial \eta \partial \zeta} + \\
+ \frac{\partial f}{\partial \xi} \frac{\partial^{2} \xi}{\partial y^{2}} + \frac{\partial f}{\partial \eta} \frac{\partial^{2} \eta}{\partial y^{2}} + \frac{\partial f}{\partial \zeta} \frac{\partial^{2} \zeta}{\partial y^{2}} \tag{13}$$

$$\frac{\partial^{2} f}{\partial z^{2}} = \left(\frac{\partial \xi}{\partial z}\right)^{2} \frac{\partial^{2} f}{\partial \xi^{2}} + \left(\frac{\partial \eta}{\partial z}\right)^{2} \frac{\partial^{2} f}{\partial \eta^{2}} + \left(\frac{\partial \zeta}{\partial z}\right)^{2} \frac{\partial^{2} f}{\partial \zeta^{2}} + \\
+ 2\left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \eta}{\partial z}\right) \frac{\partial^{2} f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^{2} f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^{2} f}{\partial \eta \partial \zeta} + \\
+ \frac{\partial f}{\partial \xi} \frac{\partial^{2} \xi}{\partial z^{2}} + \frac{\partial f}{\partial \eta} \frac{\partial^{2} \eta}{\partial z^{2}} + \frac{\partial f}{\partial \zeta} \frac{\partial^{2} \zeta}{\partial z^{2}} \tag{14}$$

$$\nabla_{\xi\eta\zeta}^{2}f\left(\xi,\eta,\zeta\right) = \left[\left(\frac{\partial\xi}{\partial x}\right)^{2} + \left(\frac{\partial\xi}{\partial y}\right)^{2} + \left(\frac{\partial\xi}{\partial z}\right)^{2}\right] \frac{\partial^{2}f}{\partial\xi^{2}} + \\
+ \left[\left(\frac{\partial\eta}{\partial x}\right)^{2} + \left(\frac{\partial\eta}{\partial y}\right)^{2} + \left(\frac{\partial\eta}{\partial z}\right)^{2}\right] \frac{\partial^{2}f}{\partial\eta^{2}} + \\
+ \left[\left(\frac{\partial\zeta}{\partial x}\right)^{2} + \left(\frac{\partial\zeta}{\partial y}\right)^{2} + \left(\frac{\partial\zeta}{\partial z}\right)^{2}\right] \frac{\partial^{2}f}{\partial\zeta^{2}} + \\
+ 2\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\eta}{\partial z}\right)\right] \frac{\partial^{2}f}{\partial\xi\partial\eta} + \\
+ 2\left[\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\zeta}{\partial x}\right) + \left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\zeta}{\partial y}\right) + \left(\frac{\partial\eta}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\right)\right] \frac{\partial^{2}f}{\partial\eta\partial\zeta} + \\
+ 2\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\zeta}{\partial x}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\zeta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\zeta}{\partial z}\right)\right] \frac{\partial^{2}f}{\partial\xi\partial\zeta} + \\
+ \left(\frac{\partial^{2}\xi}{\partial x^{2}} + \frac{\partial^{2}\xi}{\partial y^{2}} + \frac{\partial^{2}\xi}{\partial z^{2}}\right) \frac{\partial f}{\partial\xi} + \left(\frac{\partial^{2}\eta}{\partial x^{2}} + \frac{\partial^{2}\eta}{\partial y^{2}} + \frac{\partial^{2}\eta}{\partial z^{2}}\right) \frac{\partial f}{\partial\eta} + \\
+ \left(\frac{\partial^{2}\zeta}{\partial x^{2}} + \frac{\partial^{2}\zeta}{\partial y^{2}} + \frac{\partial^{2}\zeta}{\partial z^{2}}\right) \frac{\partial f}{\partial\zeta}$$
(15)

Coordonnées cylindriques

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases}$$
 (16)

$$\begin{cases}
\frac{\partial r}{\partial x} &= \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}} = \\
&= \frac{r \cos \theta}{r} \\
\frac{\partial r}{\partial y} &= \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2}} = \\
&= \frac{r \sin \theta}{r} \\
\frac{\partial r}{\partial z} &= 0
\end{cases}$$

$$\begin{cases}
\frac{\partial \theta}{\partial x} &= -\frac{1}{1 + (\frac{y}{x})^2} \frac{y}{x^2} = \\
&= -\frac{1}{1 + \tan^2 \theta} \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{\sin \theta}{r} \\
\frac{\partial \theta}{\partial y} &= \frac{1}{1 + (\frac{y}{x})^2} \frac{1}{x} = \\
&= \frac{1}{1 + \tan^2 \theta} \frac{1}{r \cos \theta} = \frac{\cos \theta}{r} \\
\frac{\partial \tilde{z}}{\partial z} &= 0
\end{cases}$$

$$\begin{cases}
\frac{\partial \tilde{z}}{\partial x} &= 0 \\
\frac{\partial \tilde{z}}{\partial y} &= 0 \quad (17) \\
\frac{\partial \tilde{z}}{\partial z} &= 1
\end{cases}$$

Introduction

$$\begin{cases} \frac{\partial^{2}r}{\partial x^{2}} &= \frac{\partial}{\partial x} (\cos \theta) = \\ &= -\sin \theta \frac{\partial \theta}{\partial x} = \frac{\sin^{2} \theta}{r} \end{cases} \qquad \begin{cases} \frac{\partial^{2}\theta}{\partial x^{2}} &= \frac{\partial}{\partial x} \left(-\frac{\sin \theta}{r} \right) = \\ &= -\frac{1}{r} \frac{\partial}{\partial x} (\sin \theta) - \sin \theta \frac{\partial}{\partial x} \left(\frac{1}{r} \right) = \\ &= -\frac{\cos \theta}{r} \frac{\partial \theta}{\partial x} + \frac{\sin \theta}{r^{2}} \frac{\partial r}{\partial x} = \\ &= \frac{2 \sin \theta \cos \theta}{r^{2}} = \frac{\sin 2\theta}{r^{2}} \end{cases} \\ \begin{cases} \frac{\partial^{2}r}{\partial y^{2}} &= \frac{\partial}{\partial y} (\sin \theta) = \\ &= \cos \theta \frac{\partial \theta}{\partial y} = \frac{\cos^{2} \theta}{r} \end{cases} \qquad \begin{cases} \frac{\partial^{2}\theta}{\partial y^{2}} &= \frac{\partial}{\partial y} \left(\frac{\cos \theta}{r} \right) = \\ &= -\frac{\sin \theta}{r} \frac{\partial \theta}{\partial y} \left(\frac{\cos \theta}{r} \right) = \\ &= -\frac{\sin \theta}{r} \frac{\partial \theta}{\partial y} - \frac{\cos \theta}{r^{2}} \frac{\partial r}{\partial y} = \\ &= -\frac{2 \sin \theta \cos \theta}{r^{2}} = -\frac{\sin 2\theta}{r^{2}} \end{cases} \end{cases} \qquad \begin{cases} \frac{\partial^{2}\tilde{z}}{\partial x^{2}} &= 0 \end{cases}$$

$$\nabla_{r\theta\tilde{z}}^{2}f\left(r,\theta,\tilde{z}\right) = \left[\cos^{2}\theta + \sin\theta^{2}\right] \frac{\partial^{2}f}{\partial r^{2}} + \left[\frac{\sin^{2}\theta}{r^{2}} + \frac{\cos^{2}\theta}{r^{2}}\right] \frac{\partial^{2}f}{\partial \theta^{2}} + \frac{\partial^{2}f}{\partial\tilde{z}^{2}} +$$

$$+ 2\left[-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right] \frac{\partial^{2}f}{\partial r\partial\theta} +$$

$$+ \left(\frac{\sin^{2}\theta}{r} + \frac{\cos^{2}\theta}{r}\right) \frac{\partial f}{\partial r} + \left(\frac{\sin2\theta}{r^{2}} - \frac{\sin2\theta}{r^{2}}\right) \frac{\partial f}{\partial\theta} =$$

$$= \frac{\partial^{2}f}{\partial r^{2}} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^{2}}\frac{\partial^{2}f}{\partial\theta^{2}} + \frac{\partial^{2}f}{\partial\tilde{z}^{2}}$$

$$(19)$$

Coordonnées spheriques

1.2.2. Problème B

$$\begin{cases} x = x (\xi, \eta, \zeta) \\ y = x (\xi, \eta, \zeta) \\ z = x (\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi (x, y, z) \\ \eta = \eta (x, y, z) \\ \zeta = \zeta (x, y, z) \end{cases}$$
(20)

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta))$$
(21)

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z))$$

$$(22)$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x} \mathbf{i}_x + \frac{\partial f}{\partial y} \mathbf{j}_y + \frac{\partial f}{\partial z} \mathbf{k}_z$$
 (23)

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta} \end{bmatrix} = \frac{\partial f}{\partial \xi} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix}$$
(24)

$$\mathbf{i}_{\xi} = \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^{2} + \left(\frac{\partial \xi}{\partial y}\right)^{2} + \left(\frac{\partial \xi}{\partial z}\right)^{2}}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix}
\mathbf{j}_{\eta} = \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^{2} + \left(\frac{\partial \eta}{\partial y}\right)^{2} + \left(\frac{\partial \eta}{\partial z}\right)^{2}}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix}
\mathbf{k}_{\zeta} = \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^{2} + \left(\frac{\partial \zeta}{\partial y}\right)^{2} + \left(\frac{\partial \zeta}{\partial z}\right)^{2}}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix}$$
(25)

$$\nabla_{\xi\eta\zeta}f = \frac{\partial f}{\partial\xi}\sqrt{\left(\frac{\partial\xi}{\partial x}\right)^{2} + \left(\frac{\partial\xi}{\partial y}\right)^{2} + \left(\frac{\partial\xi}{\partial z}\right)^{2}} \cdot \mathbf{i}_{\xi} +
+ \frac{\partial f}{\partial\eta}\sqrt{\left(\frac{\partial\eta}{\partial x}\right)^{2} + \left(\frac{\partial\eta}{\partial y}\right)^{2} + \left(\frac{\partial\eta}{\partial z}\right)^{2}} \cdot \mathbf{j}_{\eta} +
+ \frac{\partial f}{\partial\zeta}\sqrt{\left(\frac{\partial\zeta}{\partial x}\right)^{2} + \left(\frac{\partial\zeta}{\partial y}\right)^{2} + \left(\frac{\partial\zeta}{\partial z}\right)^{2}} \cdot \mathbf{k}_{\zeta} =
= \begin{bmatrix} \frac{\partial f}{\partial\xi}\sqrt{\left(\frac{\partial\xi}{\partial x}\right)^{2} + \left(\frac{\partial\xi}{\partial y}\right)^{2} + \left(\frac{\partial\xi}{\partial z}\right)^{2}} \\ \frac{\partial f}{\partial\eta}\sqrt{\left(\frac{\partial\eta}{\partial x}\right)^{2} + \left(\frac{\partial\eta}{\partial y}\right)^{2} + \left(\frac{\partial\eta}{\partial z}\right)^{2}} \\ \frac{\partial f}{\partial\zeta}\sqrt{\left(\frac{\partial\zeta}{\partial x}\right)^{2} + \left(\frac{\partial\zeta}{\partial y}\right)^{2} + \left(\frac{\partial\zeta}{\partial z}\right)^{2}} \end{bmatrix}$$
(26)

Coordonnées cylindriques

Coordonnées spheriques

2. Second section

Second section

Appendix A

A. First appendix

Appendix A