

**EEIGM**  
**École Européenne d'Ingénieurs en Génie des Matériaux**

*2<sup>ème</sup>* Année, 1<sup>er</sup> Semestre

**MÉCANIQUE DU SOLIDE DÉFORMABLE**

**TRAVAUX DIRIGÉS**

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**6 août 2020**

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## **List of Acronyms**

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## *List of Acronyms*

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## List of Symbols



## **Abstract**





# 1. Systèmes de coordonnées curvilignes

## 1.1. Énoncé

### 1.1.1. Problème A

Exprimer l'opérateur laplacien  $\nabla^2$  en

1. coordonnées cylindriques,
2. coordonnées sphériques.

### 1.1.2. Problème B

## 1.2. Corrigé

### 1.2.1. Problème A

#### Rappels théoriques

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (1)$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \quad (2)$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \quad (3)$$

$$\nabla_{xyz}^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (4)$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \end{cases} \quad (5)$$





$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
&+ 2 \left(\frac{\partial \xi}{\partial y}\right) \left(\frac{\partial \eta}{\partial y}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial y}\right) \left(\frac{\partial \zeta}{\partial y}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial y}\right) \left(\frac{\partial \zeta}{\partial y}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
&+ \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial y^2}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^2} &= \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
&+ 2 \left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \eta}{\partial z}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
&+ \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial z^2}
\end{aligned} \tag{14}$$

$$\begin{aligned}
\nabla_{\xi\eta\zeta}^2 f(\xi, \eta, \zeta) = & \left[ \left( \frac{\partial \xi}{\partial x} \right)^2 + \left( \frac{\partial \xi}{\partial y} \right)^2 + \left( \frac{\partial \xi}{\partial z} \right)^2 \right] \frac{\partial^2 f}{\partial \xi^2} + \\
& + \left[ \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 + \left( \frac{\partial \eta}{\partial z} \right)^2 \right] \frac{\partial^2 f}{\partial \eta^2} + \\
& + \left[ \left( \frac{\partial \zeta}{\partial x} \right)^2 + \left( \frac{\partial \zeta}{\partial y} \right)^2 + \left( \frac{\partial \zeta}{\partial z} \right)^2 \right] \frac{\partial^2 f}{\partial \zeta^2} + \\
& + 2 \left[ \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta}{\partial x} \right) + \left( \frac{\partial \xi}{\partial y} \right) \left( \frac{\partial \eta}{\partial y} \right) + \left( \frac{\partial \xi}{\partial z} \right) \left( \frac{\partial \eta}{\partial z} \right) \right] \frac{\partial^2 f}{\partial \xi \partial \eta} + \\
& + 2 \left[ \left( \frac{\partial \eta}{\partial x} \right) \left( \frac{\partial \zeta}{\partial x} \right) + \left( \frac{\partial \eta}{\partial y} \right) \left( \frac{\partial \zeta}{\partial y} \right) + \left( \frac{\partial \eta}{\partial z} \right) \left( \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
& + 2 \left[ \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \zeta}{\partial x} \right) + \left( \frac{\partial \xi}{\partial y} \right) \left( \frac{\partial \zeta}{\partial y} \right) + \left( \frac{\partial \xi}{\partial z} \right) \left( \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial^2 f}{\partial \xi \partial \zeta} + \\
& + \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right) \frac{\partial f}{\partial \xi} + \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2} \right) \frac{\partial f}{\partial \eta} + \\
& + \left( \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2} \right) \frac{\partial f}{\partial \zeta}
\end{aligned} \tag{15}$$

### Coordonnées cylindriques

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left( \frac{y}{x} \right) \\ \tilde{z} = z \end{cases} \tag{16}$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}} \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{\partial r}{\partial z} = 0 \end{cases} \longleftrightarrow \begin{cases} \frac{\partial \theta}{\partial x} = -\frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{y}{x^2} = -\frac{1}{1 + \tan^2 \theta} \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{\sin \theta}{r} \\ \frac{\partial \theta}{\partial y} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{1}{x} = \frac{1}{1 + \tan^2 \theta} \frac{1}{r \cos \theta} = \frac{\cos \theta}{r} \\ \frac{\partial \theta}{\partial z} = 0 \end{cases} \begin{cases} \frac{\partial \tilde{z}}{\partial x} = 0 \\ \frac{\partial \tilde{z}}{\partial y} = 0 \\ \frac{\partial \tilde{z}}{\partial z} = 1 \end{cases} \tag{17}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x} (\cos \theta) = \\ = -\sin \theta \frac{\partial \theta}{\partial x} = \frac{\sin^2 \theta}{r} \\ \\ \frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y} (\sin \theta) = \\ = \cos \theta \frac{\partial \theta}{\partial y} = \frac{\cos^2 \theta}{r} \\ \\ \frac{\partial^2 r}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left( -\frac{\sin \theta}{r} \right) = \\ = -\frac{1}{r} \frac{\partial}{\partial x} (\sin \theta) - \sin \theta \frac{\partial}{\partial x} \left( \frac{1}{r} \right) = \\ = -\frac{\cos \theta}{r} \frac{\partial \theta}{\partial x} + \frac{\sin \theta}{r^2} \frac{\partial r}{\partial x} = \\ = \frac{2 \sin \theta \cos \theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \\ \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\cos \theta}{r} \right) = \\ = \frac{1}{r} \frac{\partial}{\partial y} (\cos \theta) + \cos \theta \frac{\partial}{\partial y} \left( \frac{1}{r} \right) = \\ = -\frac{\sin \theta}{r} \frac{\partial \theta}{\partial y} - \frac{\cos \theta}{r^2} \frac{\partial r}{\partial y} = \\ = -\frac{2 \sin \theta \cos \theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \\ \frac{\partial^2 \theta}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \tilde{z}}{\partial x^2} = 0 \\ \\ \frac{\partial^2 \tilde{z}}{\partial y^2} = 0 \\ \\ \frac{\partial^2 \tilde{z}}{\partial z^2} = 0 \end{cases} \quad (18)$$

$$\begin{aligned} \nabla_{r\theta\tilde{z}}^2 f(r, \theta, \tilde{z}) &= [\cos^2 \theta + \sin^2 \theta] \frac{\partial^2 f}{\partial r^2} + \left[ \frac{\sin^2 \theta}{r^2} + \frac{\cos^2 \theta}{r^2} \right] \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \tilde{z}^2} + \\ &+ 2 \left[ -\frac{\sin \theta \cos \theta}{r} + \frac{\sin \theta \cos \theta}{r} \right] \frac{\partial^2 f}{\partial r \partial \theta} + \\ &+ \left( \frac{\sin^2 \theta}{r} + \frac{\cos^2 \theta}{r} \right) \frac{\partial f}{\partial r} + \left( \frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2} \right) \frac{\partial f}{\partial \theta} = \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \tilde{z}^2} \end{aligned} \quad (19)$$

### Coordonnées sphériques

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \varphi = \tan^{-1} \left( \frac{y}{x} \right) \\ \theta = \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \end{cases} \quad (20)$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}} = \\ = \frac{x}{\sqrt{x^2 + y^2}} \\ \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2}} = \\ = \frac{y}{\sqrt{x^2 + y^2}} \\ \\ \frac{\partial r}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \varphi}{\partial x} = -\frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{y}{x^2} = \\ = -\frac{1}{1 + \tan^2 \theta} \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{\sin \theta}{r} \\ \\ \frac{\partial \varphi}{\partial y} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{1}{x} = \\ = \frac{1}{1 + \tan^2 \theta} \frac{1}{r \cos \theta} = \frac{\cos \theta}{r} \\ \\ \frac{\partial \varphi}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} = 0 \\ \\ \frac{\partial \theta}{\partial y} = 0 \\ \\ \frac{\partial \theta}{\partial z} = 1 \end{cases} \quad (21)$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} = \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}} = \frac{1}{\cancel{r}} \frac{\cos \theta}{\cancel{r}} \\ \frac{\partial^2 r}{\partial y^2} = \frac{1}{2} \frac{2y}{\sqrt{x^2+y^2}} = \frac{1}{\cancel{r}} \frac{\sin \theta}{\cancel{r}} \\ \frac{\partial^2 r}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \varphi}{\partial x^2} = -\frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{y}{x^2} = -\frac{1}{1+\tan^2 \theta} \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{\sin \theta}{r} \\ \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{1}{1+\tan^2 \theta} \frac{1}{r \cos \theta} = \frac{\cos \theta}{r} \\ \frac{\partial^2 \varphi}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \theta}{\partial x^2} = 0 \\ \frac{\partial^2 \theta}{\partial y^2} = 0 \\ \frac{\partial^2 \theta}{\partial z^2} = 1 \end{cases} \quad (22)$$

### 1.2.2. Problème B

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = x(\xi, \eta, \zeta) \\ z = x(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (23)$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \quad (24)$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \quad (25)$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x} \mathbf{i}_x + \frac{\partial f}{\partial y} \mathbf{j}_y + \frac{\partial f}{\partial z} \mathbf{k}_z \quad (26)$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \quad (27)$$



$$\begin{aligned}\mathbf{i}_\xi &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\ \mathbf{j}_\eta &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\ \mathbf{k}_\zeta &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix}\end{aligned}\tag{28}$$

$$\begin{aligned}\nabla_{\xi\eta\zeta} f &= \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}_\xi + \\ &\quad + \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}_\eta + \\ &\quad + \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}_\zeta = \\ &= \begin{bmatrix} \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix}\end{aligned}\tag{29}$$

**Coordonnées cylindriques**

**Coordonnées sphériques**

## **2. Second section**



## **A. First appendix**

