

**EEIGM**  
**École Européenne d'Ingénieurs en Génie des Matériaux**

*2<sup>ème</sup>* Année, 1<sup>er</sup> Semestre

**MÉCANIQUE DU SOLIDE DÉFORMABLE**

**TRAVAUX DIRIGÉS**

**Luca Di Stasio**



Cette oeuvre est mise à disposition selon les termes de la  
Licence Creative Commons Attribution - Pas d'Utilisation Commerciale  
4.0 International.

**4 septembre 2020**

## Table des matières

<b>1</b>	<b>Systèmes de coordonnées curvilignes</b>	<b>1</b>
1.1	Énoncé . . . . .	1
1.1.1	Problème A . . . . .	1
1.1.2	Problème B . . . . .	1
1.1.3	Problème C . . . . .	1
1.2	Corrigé . . . . .	2
1.2.1	Problème A . . . . .	2
1.2.2	Problème B . . . . .	7
1.2.3	Problème C . . . . .	14
<b>2</b>	<b>Déplacement et mesures de déformation</b>	<b>21</b>
2.1	Énoncé . . . . .	21
2.2	Corrigé . . . . .	22



# 1 Systèmes de coordonnées curvilignes

## 1.1 Énoncé

### 1.1.1 Problème A

Déterminer l'expression de

- les vecteurs du repère local naturel (base covariante),
- le jacobien de la transformation,
- le tenseur métrique,
- le déplacement infinitésimal d'un point,
- l'élément infinitésimal de ligne,
- l'élément infinitésimal de volume ;

dans un système de coordonnées

1. cylindriques,
2. sphériques.

### 1.1.2 Problème B

Exprimer l'opérateur laplacien  $\nabla^2$  en

1. coordonnées cylindriques,
2. coordonnées sphériques.

### 1.1.3 Problème C

Dériver l'expression des vecteurs de la base contravariante et du gradient d'une fonction scalaire  $\nabla f$  en

1. coordonnées cylindriques,
2. coordonnées sphériques.

## 1.2 Corrigé

### 1.2.1 Problème A

#### Rappels théoriques

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (1)$$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad (2)$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \quad (3)$$

$$\begin{cases} dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta \\ dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta \\ dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta \end{cases} \quad (4)$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

$\frac{\partial \mathbf{r}}{\partial \xi}$ 
 $\frac{\partial \mathbf{r}}{\partial \eta}$ 
 $\frac{\partial \mathbf{r}}{\partial \zeta}$

$$\begin{aligned}
\mathbf{i}_\xi &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \xi}\|} \frac{\partial \mathbf{r}}{\partial \xi} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{bmatrix} \\
\mathbf{j}_\eta &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \eta}\|} \frac{\partial \mathbf{r}}{\partial \eta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{bmatrix} \\
\mathbf{k}_\zeta &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \zeta}\|} \frac{\partial \mathbf{r}}{\partial \zeta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{bmatrix}
\end{aligned} \tag{6}$$

$$dl^2 = d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} dx & dy & dz \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = dx^2 + dy^2 + dz^2 \tag{7}$$

$$\begin{aligned}
dl^2 &= d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \mathbf{J}^T \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \underbrace{\begin{bmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \eta} & \frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \xi} + \frac{\partial z}{\partial \eta} \frac{\partial z}{\partial \xi} & \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta} \frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \zeta} \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \zeta} \frac{\partial y}{\partial \xi} + \frac{\partial z}{\partial \zeta} \frac{\partial z}{\partial \xi} & \frac{\partial x}{\partial \zeta} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \zeta} \frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \zeta} \frac{\partial z}{\partial \eta} & \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}
\end{aligned} \tag{8}$$

$$dV = \frac{\partial \mathbf{r}}{\partial \xi} d\xi \cdot \left( \frac{\partial \mathbf{r}}{\partial \eta} d\eta \wedge \frac{\partial \mathbf{r}}{\partial \zeta} d\zeta \right) = \det \begin{bmatrix} \left( \frac{\partial \mathbf{r}}{\partial \xi} \right)^T \\ \left( \frac{\partial \mathbf{r}}{\partial \eta} \right)^T \\ \left( \frac{\partial \mathbf{r}}{\partial \zeta} \right)^T \end{bmatrix} d\xi d\eta d\zeta = \det(\mathbf{J}^T) d\xi d\eta d\zeta \tag{9}$$

**Coordonnées cylindriques**

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left( \frac{y}{x} \right) \\ \tilde{z} = z \end{cases} \quad (10)$$

$$\begin{cases} \frac{\partial x}{\partial r} = \cos \theta \\ \frac{\partial x}{\partial \theta} = -r \sin \theta \\ \frac{\partial x}{\partial \tilde{z}} = 0 \end{cases} \quad \begin{cases} \frac{\partial y}{\partial r} = \sin \theta \\ \frac{\partial y}{\partial \theta} = r \cos \theta \\ \frac{\partial y}{\partial \tilde{z}} = 0 \end{cases} \quad \begin{cases} \frac{\partial z}{\partial r} = 0 \\ \frac{\partial z}{\partial \theta} = 0 \\ \frac{\partial z}{\partial \tilde{z}} = 1 \end{cases} \quad (11)$$

$$\begin{aligned} \mathbf{i}_r &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial r}\|} \frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2 \theta + \cos^2 \theta}} \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \\ \mathbf{j}_\theta &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \theta}\|} \frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta}} \begin{bmatrix} -r \sin \theta \\ r \cos \theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix} \\ \mathbf{k}_z &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \tilde{z}}\|} \frac{\partial \mathbf{r}}{\partial \tilde{z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \quad (12)$$

$$\mathbf{J} = \begin{bmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (13)$$

$$d\mathbf{r} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} dr + \begin{bmatrix} -r \sin \theta \\ r \cos \theta \\ 0 \end{bmatrix} d\theta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\tilde{z} = dr \mathbf{i}_r + r d\theta \mathbf{i}_\theta + d\tilde{z} \mathbf{i}_{\tilde{z}} = \begin{bmatrix} dr \\ r d\theta \\ d\tilde{z} \end{bmatrix} \quad (14)$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (15)$$

$$dl^2 = dr^2 + r^2 d\theta^2 + d\tilde{z}^2 \quad (16)$$

$$dV = \det(\mathbf{J}^T) dr d\theta d\tilde{z} = r dr d\theta d\tilde{z} \quad (17)$$

### Coordonnées sphériques

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \varphi = \tan^{-1} \left( \frac{y}{x} \right) \\ \theta = \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \end{cases} \quad (18)$$

$$\begin{cases} \frac{\partial x}{\partial r} = \sin \theta \cos \varphi \\ \frac{\partial x}{\partial \theta} = r \cos \theta \cos \varphi \\ \frac{\partial x}{\partial \varphi} = -r \sin \theta \sin \varphi \end{cases} \quad \begin{cases} \frac{\partial y}{\partial r} = \sin \theta \sin \varphi \\ \frac{\partial y}{\partial \theta} = r \cos \theta \sin \varphi \\ \frac{\partial y}{\partial \varphi} = r \sin \theta \cos \varphi \end{cases} \quad \begin{cases} \frac{\partial z}{\partial r} = \cos \theta \\ \frac{\partial z}{\partial \theta} = -r \sin \theta \\ \frac{\partial z}{\partial \varphi} = 0 \end{cases} \quad (19)$$

$$\begin{aligned} \mathbf{i}_r &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial r}\|} \frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta}} \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} \\ \mathbf{j}_\varphi &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \varphi}\|} \frac{\partial \mathbf{r}}{\partial \varphi} = \frac{1}{\sqrt{r^2 \sin^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta \cos^2 \varphi}} \begin{bmatrix} -r \sin \theta \sin \varphi \\ r \sin \theta \cos \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} \\ \mathbf{k}_\theta &= \frac{1}{\|\frac{\partial \mathbf{r}}{\partial \theta}\|} \frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2 \cos^2 \theta \cos^2 \varphi + r^2 \cos^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta}} \begin{bmatrix} r \cos \theta \cos \varphi \\ r \cos \theta \sin \varphi \\ -r \sin \theta \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} \end{aligned} \quad (20)$$

$$\mathbf{J} = \begin{bmatrix} \sin \theta \cos \varphi & -r \sin \theta \sin \varphi & r \cos \theta \cos \varphi \\ \sin \theta \sin \varphi & r \sin \theta \cos \varphi & r \cos \theta \sin \varphi \\ \cos \theta & 0 & -r \sin \theta \end{bmatrix} \quad (21)$$



$$\begin{aligned}
d\mathbf{r} &= \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} dr + r \sin \theta \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} d\varphi + r \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} d\theta = \\
&= dr \mathbf{i}_r + r \sin \theta d\varphi \mathbf{j}_\varphi + rd\theta \mathbf{k}_\theta = \begin{bmatrix} dr \\ r \sin \theta d\varphi \\ rd\theta \end{bmatrix}
\end{aligned} \tag{22}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & r^2 \end{bmatrix} \tag{23}$$

$$dl^2 = dr^2 + r^2 \sin^2 \theta d\varphi^2 + r^2 d\theta^2 \tag{24}$$

$$dV = \det(\mathbf{J}^T) drd\theta d\tilde{z} = r^2 \sin \theta drd\varphi d\theta \tag{25}$$

### 1.2.2 Problème B

#### Rappels théoriques

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (26)$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \quad (27)$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \quad (28)$$

$$\nabla_{xyz}^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (29)$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \end{cases} \quad (30)$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) = \\ = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) = \\ = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial y^2} \\ \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) = \\ = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial z^2} \end{cases} \quad (31)$$

(32)

(33)

(34)

$$\left\{ \begin{array}{l} \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} = \left( \frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \xi} \right) \frac{\partial \xi}{\partial z} = \\ \quad = \left( \frac{\partial \xi}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left( \frac{\partial \eta}{\partial z} \right) \left( \frac{\partial \xi}{\partial z} \right) \frac{\partial^2 f}{\partial \eta \partial \xi} + \left( \frac{\partial \zeta}{\partial z} \right) \left( \frac{\partial \xi}{\partial z} \right) \frac{\partial^2 f}{\partial \zeta \partial \xi} \\ \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} = \left( \frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial z} = \\ \quad = \left( \frac{\partial \xi}{\partial z} \right) \left( \frac{\partial \eta}{\partial z} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + \left( \frac{\partial \eta}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left( \frac{\partial \zeta}{\partial z} \right) \left( \frac{\partial \eta}{\partial z} \right) \frac{\partial^2 f}{\partial \zeta \partial \eta} \\ \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} = \left( \frac{\partial \xi}{\partial z} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial f}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} = \\ \quad = \left( \frac{\partial \xi}{\partial z} \right) \left( \frac{\partial \zeta}{\partial z} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + \left( \frac{\partial \eta}{\partial z} \right) \left( \frac{\partial \zeta}{\partial z} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \left( \frac{\partial \zeta}{\partial z} \right)^2 \frac{\partial^2 f}{\partial \zeta^2} \end{array} \right. \quad (35)$$

$$\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial^2 f}{\partial \xi \partial \eta} \quad \frac{\partial^2 f}{\partial \eta \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \eta} \quad \frac{\partial^2 f}{\partial \xi \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \xi} \quad (36)$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \left( \frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left( \frac{\partial \zeta}{\partial x} \right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ &+ 2 \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left( \frac{\partial \eta}{\partial x} \right) \left( \frac{\partial \zeta}{\partial x} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ &+ \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial x^2} \end{aligned} \quad (37)$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} &= \left( \frac{\partial \xi}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left( \frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left( \frac{\partial \zeta}{\partial y} \right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ &+ 2 \left( \frac{\partial \xi}{\partial y} \right) \left( \frac{\partial \eta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left( \frac{\partial \xi}{\partial y} \right) \left( \frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left( \frac{\partial \eta}{\partial y} \right) \left( \frac{\partial \zeta}{\partial y} \right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ &+ \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial y^2} \end{aligned} \quad (38)$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^2} &= \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
&+ 2 \left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \eta}{\partial z}\right) \frac{\partial^2 f}{\partial \xi \partial \eta} + 2 \left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial \eta}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
&+ \frac{\partial f}{\partial \xi} \frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta} \frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta} \frac{\partial^2 \zeta}{\partial z^2}
\end{aligned} \tag{39}$$

$$\begin{aligned}
\nabla_{\xi\eta\zeta}^2 f(\xi, \eta, \zeta) &= \left[ \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2 \right] \frac{\partial^2 f}{\partial \xi^2} + \\
&+ \left[ \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2 \right] \frac{\partial^2 f}{\partial \eta^2} + \\
&+ \left[ \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2 \right] \frac{\partial^2 f}{\partial \zeta^2} + \\
&+ 2 \left[ \left(\frac{\partial \xi}{\partial x}\right) \left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right) \left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \eta}{\partial z}\right) \right] \frac{\partial^2 f}{\partial \xi \partial \eta} + \\
&+ 2 \left[ \left(\frac{\partial \eta}{\partial x}\right) \left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right) \left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \eta}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \right] \frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
&+ 2 \left[ \left(\frac{\partial \xi}{\partial x}\right) \left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right) \left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right) \left(\frac{\partial \zeta}{\partial z}\right) \right] \frac{\partial^2 f}{\partial \xi \partial \zeta} + \\
&+ \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right) \frac{\partial f}{\partial \xi} + \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2} \right) \frac{\partial f}{\partial \eta} + \\
&+ \left( \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2} \right) \frac{\partial f}{\partial \zeta}
\end{aligned} \tag{40}$$

**Coordonnées cylindriques**

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left( \frac{y}{x} \right) \\ \tilde{z} = z \end{cases} \quad (41)$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}} = \frac{\cos \theta}{r} \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2}} = \frac{\sin \theta}{r} \\ \frac{\partial r}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} = -\frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{y}{x^2} = -\frac{1}{1 + \tan^2 \theta} \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{\sin \theta}{r} \\ \frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{1}{1 + \tan^2 \theta} \frac{1}{r \cos \theta} = \frac{\cos \theta}{r} \\ \frac{\partial \theta}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \tilde{z}}{\partial x} = 0 \\ \frac{\partial \tilde{z}}{\partial y} = 0 \\ \frac{\partial \tilde{z}}{\partial z} = 1 \end{cases} \quad (42)$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x} (\cos \theta) = -\sin \theta \frac{\partial \theta}{\partial x} = -\frac{\sin^2 \theta}{r} \\ \frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y} (\sin \theta) = \cos \theta \frac{\partial \theta}{\partial y} = \frac{\cos^2 \theta}{r} \\ \frac{\partial^2 r}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left( -\frac{\sin \theta}{r} \right) = -\frac{1}{r} \frac{\partial}{\partial x} (\sin \theta) - \sin \theta \frac{\partial}{\partial x} \left( \frac{1}{r} \right) = -\frac{\cos \theta}{r} \frac{\partial \theta}{\partial x} + \frac{\sin \theta}{r^2} \frac{\partial r}{\partial x} = \frac{2 \sin \theta \cos \theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\cos \theta}{r} \right) = \frac{1}{r} \frac{\partial}{\partial y} (\cos \theta) + \cos \theta \frac{\partial}{\partial y} \left( \frac{1}{r} \right) = -\frac{\sin \theta}{r} \frac{\partial \theta}{\partial y} - \frac{\cos \theta}{r^2} \frac{\partial r}{\partial y} = -\frac{2 \sin \theta \cos \theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \tilde{z}}{\partial x^2} = 0 \\ \frac{\partial^2 \tilde{z}}{\partial y^2} = 0 \\ \frac{\partial^2 \tilde{z}}{\partial z^2} = 0 \end{cases} \quad (43)$$

$$\begin{aligned} \nabla_{r\theta\tilde{z}}^2 f(r, \theta, \tilde{z}) &= [\cos^2 \theta + \sin^2 \theta] \frac{\partial^2 f}{\partial r^2} + \left[ \frac{\sin^2 \theta}{r^2} + \frac{\cos^2 \theta}{r^2} \right] \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \tilde{z}^2} + \\ &\quad + 2 \left[ -\frac{\sin \theta \cos \theta}{r} + \frac{\sin \theta \cos \theta}{r} \right] \frac{\partial^2 f}{\partial r \partial \theta} + \\ &\quad + \left( \frac{\sin^2 \theta}{r} + \frac{\cos^2 \theta}{r} \right) \frac{\partial f}{\partial r} + \left( \frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2} \right) \frac{\partial f}{\partial \theta} = \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \tilde{z}^2} \end{aligned} \quad (44)$$

**Coordonnées sphériques**

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \varphi = \tan^{-1} \left( \frac{y}{x} \right) \\ \theta = \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \end{cases} \quad (45)$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \cos \theta \cos \varphi \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \cos \theta \sin \varphi \\ \frac{\partial r}{\partial z} = \frac{1}{2} \frac{2z}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \cos \theta \end{cases} \quad \begin{cases} \frac{\partial \varphi}{\partial x} = -\frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{y}{x^2} = -\frac{\sin \varphi}{r \sin \theta} \\ \frac{\partial \varphi}{\partial y} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{1}{x} = \frac{\cos \varphi}{r \sin \theta} \\ \frac{\partial \varphi}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} = \frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \frac{z}{z^2 \sqrt{x^2 + y^2}} = \frac{1}{r} \cos \theta \cos \varphi \\ \frac{\partial \theta}{\partial y} = \frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \frac{z}{z^2 \sqrt{x^2 + y^2}} = \frac{1}{r} \cos \theta \sin \varphi \\ \frac{\partial \theta}{\partial z} = -\frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \frac{\sqrt{x^2 + y^2}}{z^2} = -\frac{1}{r} \sin \theta \end{cases} \quad (46)$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} = \cos \theta \cos \varphi \frac{\partial \theta}{\partial x} - \sin \theta \sin \varphi \frac{\partial \varphi}{\partial x} = \frac{1}{r} (\cos^2 \varphi \cos^2 \theta + \sin^2 \varphi) \\ \frac{\partial^2 r}{\partial y^2} = \cos \theta \sin \varphi \frac{\partial \theta}{\partial y} + \sin \theta \cos \varphi \frac{\partial \varphi}{\partial y} = \frac{1}{r} (\sin^2 \varphi \cos^2 \theta + \cos^2 \varphi) \\ \frac{\partial^2 r}{\partial z^2} = -\sin \theta \frac{\partial \theta}{\partial z} = \frac{1}{r} \sin^2 \theta \end{cases} \quad (47)$$

$$\begin{cases} \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{r^2} \sin \varphi \frac{\partial r}{\partial x} - \frac{1}{r} \cos \varphi \frac{\partial \varphi}{\partial x} + \frac{1}{r} \frac{\sin \varphi}{\sin^2 \theta} \cos \theta \frac{\partial \theta}{\partial x} = \frac{1}{r^2} \sin \varphi \cos \varphi + \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\sin^2 \theta} + \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\tan^2 \theta} \\ \frac{\partial^2 \varphi}{\partial y^2} = -\frac{1}{r^2} \cos \varphi \frac{\partial r}{\partial y} - \frac{1}{r} \sin \varphi \frac{\partial \varphi}{\partial y} - \frac{1}{r} \frac{\cos \varphi}{\sin^2 \theta} \cos \theta \frac{\partial \theta}{\partial y} = -\frac{1}{r^2} \sin \varphi \cos \varphi - \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\sin^2 \theta} - \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\tan^2 \theta} \\ \frac{\partial^2 \varphi}{\partial z^2} = 0 \end{cases} \quad (48)$$

$$\left\{ \begin{array}{l}
\frac{\partial^2 \theta}{\partial x^2} = -\frac{1}{r^2} \cos \varphi \cos \theta \frac{\partial r}{\partial x} - \frac{1}{r} \sin \varphi \cos \theta \frac{\partial \varphi}{\partial x} - \frac{1}{r} \cos \varphi \sin \theta \frac{\partial \theta}{\partial x} = \\
= -\frac{2}{r^2} \cos^2 \varphi \cos \theta \sin \theta + \frac{1}{r^2} \frac{\sin^2 \varphi}{\tan \theta} \\
\frac{\partial^2 \theta}{\partial y^2} = -\frac{1}{r^2} \sin \varphi \cos \theta \frac{\partial r}{\partial y} - \frac{1}{r} \sin \varphi \sin \theta \frac{\partial \varphi}{\partial y} + \frac{1}{r} \cos \varphi \cos \theta \frac{\partial \theta}{\partial y} = \\
= -\frac{2}{r^2} \sin^2 \varphi \cos \theta \sin \theta + \frac{1}{r^2} \frac{\cos^2 \varphi}{\tan \theta} \\
\frac{\partial^2 \theta}{\partial z^2} = \frac{1}{r^2} \sin \theta \frac{\partial r}{\partial z} - \frac{1}{r} \cos \theta \frac{\partial \theta}{\partial z} = \\
= \frac{2}{r^2} \cos \theta \sin \theta
\end{array} \right. \quad (49)$$

$$\nabla_{r\varphi\theta}^2 f(\xi, \eta, \zeta) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{2}{r^2 \tan \theta} \frac{\partial f}{\partial \theta} \quad (50)$$



### 1.2.3 Probl me C

#### Rappels th oriques

$$\begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad (51)$$

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \quad (52)$$

$$f(\xi, \eta, \zeta) = f(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)) \quad (53)$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x} \mathbf{i}^x + \frac{\partial f}{\partial y} \mathbf{j}^y + \frac{\partial f}{\partial z} \mathbf{k}^z \quad (54)$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \quad (55)$$

$$\begin{aligned} \mathbf{i}^\xi &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\ \mathbf{j}^\eta &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\ \mathbf{k}^\zeta &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \end{aligned} \quad (56)$$

$$\begin{aligned}
\nabla_{\xi\eta\zeta} f &= \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}^\xi + \\
&\quad + \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}^\eta + \\
&\quad + \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}^\zeta = \\
&= \begin{bmatrix} \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix}
\end{aligned} \tag{57}$$

**Coordonnées cylindriques**

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left( \frac{y}{x} \right) \\ \tilde{z} = z \end{cases} \quad (58)$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{r} \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{r} \\ \frac{\partial r}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} = -\frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{y}{x^2} = -\frac{y}{r^2} \\ \frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{r^2} \\ \frac{\partial \theta}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \tilde{z}}{\partial x} = 0 \\ \frac{\partial \tilde{z}}{\partial y} = 0 \\ \frac{\partial \tilde{z}}{\partial z} = 1 \end{cases} \quad (59)$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x} (\cos \theta) = -\sin \theta \frac{\partial \theta}{\partial x} = -\frac{\sin^2 \theta}{r} \\ \frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y} (\sin \theta) = \cos \theta \frac{\partial \theta}{\partial y} = \frac{\cos^2 \theta}{r} \\ \frac{\partial^2 r}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left( -\frac{\sin \theta}{r} \right) = -\frac{1}{r} \frac{\partial}{\partial x} (\sin \theta) - \sin \theta \frac{\partial}{\partial x} \left( \frac{1}{r} \right) = -\frac{\cos \theta}{r} \frac{\partial \theta}{\partial x} + \frac{\sin \theta}{r^2} \frac{\partial r}{\partial x} = \frac{2 \sin \theta \cos \theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\cos \theta}{r} \right) = \frac{1}{r} \frac{\partial}{\partial y} (\cos \theta) + \cos \theta \frac{\partial}{\partial y} \left( \frac{1}{r} \right) = -\frac{\sin \theta}{r} \frac{\partial \theta}{\partial y} - \frac{\cos \theta}{r^2} \frac{\partial r}{\partial y} = -\frac{2 \sin \theta \cos \theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \tilde{z}}{\partial x^2} = 0 \\ \frac{\partial^2 \tilde{z}}{\partial y^2} = 0 \\ \frac{\partial^2 \tilde{z}}{\partial z^2} = 0 \end{cases} \quad (60)$$

$$\begin{cases} \sqrt{\left( \frac{\partial r}{\partial x} \right)^2 + \left( \frac{\partial r}{\partial y} \right)^2 + \left( \frac{\partial r}{\partial z} \right)^2} = 1 \\ \sqrt{\left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial y} \right)^2 + \left( \frac{\partial \theta}{\partial z} \right)^2} = \frac{1}{r} \\ \sqrt{\left( \frac{\partial \tilde{z}}{\partial x} \right)^2 + \left( \frac{\partial \tilde{z}}{\partial y} \right)^2 + \left( \frac{\partial \tilde{z}}{\partial z} \right)^2} = 1 \end{cases} \quad (61)$$

$$\begin{aligned}\mathbf{i}^r &= \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \\ \mathbf{j}^\theta &= \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix} \\ \mathbf{k}^\zeta &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}\tag{62}$$

$$\nabla_{r\theta\bar{z}}f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \bar{z}} \end{bmatrix}\tag{63}$$

**Coordonnées sphériques**

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \varphi = \tan^{-1} \left( \frac{y}{x} \right) \\ \theta = \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \end{cases} \quad (64)$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \cos \theta \cos \varphi \\ \frac{\partial r}{\partial y} = \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \cos \theta \sin \varphi \\ \frac{\partial r}{\partial z} = \frac{1}{2} \frac{2z}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \cos \theta \end{cases} \quad \begin{cases} \frac{\partial \varphi}{\partial x} = -\frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{y}{x^2} = -\frac{\sin \varphi}{r \sin \theta} \\ \frac{\partial \varphi}{\partial y} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{1}{x} = \frac{\cos \varphi}{r \sin \theta} \\ \frac{\partial \varphi}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} = \frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \frac{z}{\sqrt{x^2 + y^2}} = \frac{1}{r} \cos \theta \cos \varphi \\ \frac{\partial \theta}{\partial y} = \frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \frac{z}{\sqrt{x^2 + y^2}} = \frac{1}{r} \cos \theta \sin \varphi \\ \frac{\partial \theta}{\partial z} = -\frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \frac{\sqrt{x^2 + y^2}}{z^2} = -\frac{1}{r} \sin \theta \end{cases} \quad (65)$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} = \cos \theta \cos \varphi \frac{\partial \theta}{\partial x} - \sin \theta \sin \varphi \frac{\partial \varphi}{\partial x} = \frac{1}{r} (\cos^2 \varphi \cos^2 \theta + \sin^2 \varphi) \\ \frac{\partial^2 r}{\partial y^2} = \cos \theta \sin \varphi \frac{\partial \theta}{\partial y} + \sin \theta \cos \varphi \frac{\partial \varphi}{\partial y} = \frac{1}{r} (\sin^2 \varphi \cos^2 \theta + \cos^2 \varphi) \\ \frac{\partial^2 r}{\partial z^2} = -\sin \theta \frac{\partial \theta}{\partial z} = \frac{1}{r} \sin^2 \theta \end{cases} \quad (66)$$

$$\begin{cases} \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{r^2} \sin \varphi \frac{\partial r}{\partial x} - \frac{1}{r} \cos \varphi \frac{\partial \varphi}{\partial x} + \frac{1}{r} \frac{\sin \varphi}{\sin^2 \theta} \cos \theta \frac{\partial \theta}{\partial x} = \frac{1}{r^2} \sin \varphi \cos \varphi + \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\sin^2 \theta} + \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\tan^2 \theta} \\ \frac{\partial^2 \varphi}{\partial y^2} = -\frac{1}{r^2} \cos \varphi \frac{\partial r}{\partial y} - \frac{1}{r} \sin \varphi \frac{\partial \varphi}{\partial y} - \frac{1}{r} \frac{\cos \varphi}{\sin^2 \theta} \cos \theta \frac{\partial \theta}{\partial y} = -\frac{1}{r^2} \sin \varphi \cos \varphi - \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\sin^2 \theta} - \frac{1}{r^2} \frac{\sin \varphi \cos \varphi}{\tan^2 \theta} \\ \frac{\partial^2 \varphi}{\partial z^2} = 0 \end{cases} \quad (67)$$

$$\left\{ \begin{array}{l} \frac{\partial^2 \theta}{\partial x^2} = -\frac{1}{r^2} \cos \varphi \cos \theta \frac{\partial r}{\partial x} - \frac{1}{r} \sin \varphi \cos \theta \frac{\partial \varphi}{\partial x} - \frac{1}{r} \cos \varphi \sin \theta \frac{\partial \theta}{\partial x} = \\ \quad = -\frac{2}{r^2} \cos^2 \varphi \cos \theta \sin \theta + \frac{1}{r^2} \frac{\sin^2 \varphi}{\tan \theta} \\ \frac{\partial^2 \theta}{\partial y^2} = -\frac{1}{r^2} \sin \varphi \cos \theta \frac{\partial r}{\partial y} - \frac{1}{r} \sin \varphi \sin \theta \frac{\partial \varphi}{\partial y} + \frac{1}{r} \cos \varphi \cos \theta \frac{\partial \theta}{\partial y} = \\ \quad = -\frac{2}{r^2} \sin^2 \varphi \cos \theta \sin \theta + \frac{1}{r^2} \frac{\cos^2 \varphi}{\tan \theta} \\ \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{r^2} \sin \theta \frac{\partial r}{\partial z} - \frac{1}{r} \cos \theta \frac{\partial \theta}{\partial z} = \\ \quad = \frac{2}{r^2} \cos \theta \sin \theta \end{array} \right. \quad (68)$$

$$\left\{ \begin{array}{l} \sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1 \\ \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2} = \frac{1}{r \sin \theta} \\ \sqrt{\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2} = \frac{1}{r} \end{array} \right. \quad (69)$$

$$\begin{aligned} \mathbf{i}^r &= \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} \\ \mathbf{j}^\varphi &= r \sin \theta \begin{bmatrix} -\frac{\sin \varphi}{r \sin \theta} \\ \frac{\cos \varphi}{r \sin \theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} \\ \mathbf{k}^\theta &= r \begin{bmatrix} \frac{1}{r} \cos \theta \cos \varphi \\ \frac{1}{r} \cos \theta \sin \varphi \\ -\frac{1}{r} \sin \theta \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} \end{aligned} \quad (70)$$

$$\nabla_{r\varphi\theta} f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \\ \frac{1}{r} \frac{\partial f}{\partial \theta} \end{bmatrix} \quad (71)$$



## **2 Déplacement et mesures de déformation**

### **2.1 Énoncé**



## **2.2 Corrigé**