EEIGM

École Européenne d'Ingénieurs en Génie des Matériaux

$2^{`eme}$ Année, 1er Semestre

MÉCANIQUE DU SOLIDE DÉFORMABLE

TRAVAUX DIRIGÉS

Luca Di Stasio

6 août 2020

Contents

List of Acronyms

List of Symbols

Abstract

Abstract

1. Systèmes de coordonnées curvilignes

1.1. Énoncé

1.1.1. Problème A

1.1.2. Problème B

1.2. Corrigé

1.2.1. Problème A

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = x\left(\xi, \eta, \zeta\right) \quad \longleftrightarrow \\ z = x\left(\xi, \eta, \zeta\right) \end{cases} \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{1}$$

$$f\left(x, y, z\right) = f\left(x\left(\xi, \eta, \zeta\right), y\left(\xi, \eta, \zeta\right), z\left(\xi, \eta, \zeta\right)\right) \tag{2}$$

$$f\left(\xi, \eta, \zeta\right) = f\left(\xi\left(x, y, z\right), \eta\left(x, y, z\right), \zeta\left(x, y, z\right)\right) \tag{3}$$

$$\nabla_{xyz}^{2} f\left(x, y, z\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\[4pt] \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\[4pt] \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{5}$$

$$
\begin{cases}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) = \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \\[2mm]
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) = \\
&= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \\[2mm]
\frac{\partial^2 f}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) = \\
&= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2}
\end{cases}
\tag{6}
$$

$$
\begin{cases}
\frac{\partial}{\partial x} = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right) \\[2mm]
\frac{\partial}{\partial y} = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right) \\[2mm]
\frac{\partial}{\partial z} = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)
\end{cases}
\tag{7}
$$

$$
\begin{cases}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} &= \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} = \\
&= \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 f}{\partial \zeta \partial \xi} \\[2mm]
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} &= \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \\
&= \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \zeta \partial \eta} \\[2mm]
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} &= \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} = \\
&= \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{8}
$$

$$
\begin{cases}
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right) = \\
\qquad\qquad = \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right) = \\
\qquad\qquad = \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right) = \\
\qquad\qquad =
\end{cases}
\tag{9}
$$

$$
\begin{cases}
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right) = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right) = \\
\qquad\qquad = \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right) = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right) = \\
\qquad\qquad = \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right) = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right) = \\
\qquad\qquad =
\end{cases}
\tag{10}
$$

$$
\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial^2 f}{\partial \xi \partial \eta} \qquad \frac{\partial^2 f}{\partial \eta \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \eta} \qquad \frac{\partial^2 f}{\partial \xi \partial \zeta} = \frac{\partial^2 f}{\partial \zeta \partial \xi}
\tag{11}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = {}& \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
& + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
& + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}
\end{aligned}
\tag{12}
$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ \qquad\quad +2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} \\[2ex] \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ \qquad\quad +2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} \\[2ex] \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\ \qquad\quad +2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} \end{cases} \tag{13}$$

$$\begin{aligned} \nabla^2_{\xi\eta\zeta} f\left(\xi, \eta, \zeta\right) = & \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \xi^2} + \\ & + \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} + \\ & + \left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \zeta^2} + \\ & + 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \eta} + \\ & + 2\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\ & + 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \zeta} \end{aligned} \tag{14}$$

Coordonnées cylindriques

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases} \tag{15}$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} = 0 \end{cases} \tag{16}$$

Coordonnées spheriques

1.2.2. Problème B

$$\begin{cases} x = x\left(\xi,\eta,\zeta\right) \\ y = x\left(\xi,\eta,\zeta\right) \\ z = x\left(\xi,\eta,\zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x,y,z\right) \\ \eta = \eta\left(x,y,z\right) \\ \zeta = \zeta\left(x,y,z\right) \end{cases} \tag{17}$$

$$f\left(x,y,z\right) = f\left(x\left(\xi,\eta,\zeta\right),y\left(\xi,\eta,\zeta\right),z\left(\xi,\eta,\zeta\right)\right) \tag{18}$$

$$f\left(\xi,\eta,\zeta\right) = f\left(\xi\left(x,y,z\right),\eta\left(x,y,z\right),\zeta\left(x,y,z\right)\right) \tag{19}$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{j}_y + \frac{\partial f}{\partial z}\mathbf{k}_z \tag{20}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{21}$$

$$\mathbf{i}_\xi = \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix}$$

$$\mathbf{j}_\eta = \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \qquad (22)$$

$$\mathbf{k}_\zeta = \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

$$\begin{aligned} \nabla_{\xi\eta\zeta} f =& \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \cdot \mathbf{i}_\xi + \\ & + \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \cdot \mathbf{j}_\eta + \\ & + \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \cdot \mathbf{k}_\zeta = \\ =& \begin{bmatrix} \frac{\partial f}{\partial \xi} \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \eta} \sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2} \\ \frac{\partial f}{\partial \zeta} \sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2} \end{bmatrix} \end{aligned} \qquad (23)$$

Coordonnées cylindriques

Coordonnées spheriques

2. Second section

A. First appendix