Exercice optionnel TD 4

0.1 Énoncé

0.1.1 Problème A

Déterminer l'expression de

- les vecteurs du repère local naturel (base covariante),
- le jacobien de la transformation,
- le tenseur métrique,
- le déplacement infinitésimal d'un point,
- l'élément infinitésimal de ligne,
- l'élément infinitésimal de volume ;

dans un système de coordonnées

1. cylindriques,
2. sphériques.

0.1.2 Problème B

Exprimer l'opérateur laplacien ∇^2 en

1. coordonnées cylindriques,
2. coordonnées spheriques.

0.1.3 Problème C

Dériver l'expression des vecteurs de la base contravariante et du gradient d'un fonction scalaire ∇f en

1. coordonnées cylindriques,
2. coordonnées spheriques.

0.2 Corrigé

0.2.1 Problème A

Rappels théoriques

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = y\left(\xi, \eta, \zeta\right) \\ z = z\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{1}$$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{2}$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{3}$$

$$\begin{cases} dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta \\ dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta \\ dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta \end{cases} \tag{4}$$

$$d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{5}$$

$$\begin{array}{c} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \\ \frac{\partial \mathbf{r}}{\partial \xi} \end{array} \quad \begin{array}{c} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \\ \frac{\partial \mathbf{r}}{\partial \eta} \end{array} \quad \begin{array}{c} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \\ \frac{\partial \mathbf{r}}{\partial \zeta} \end{array}$$

$$\begin{aligned}
\mathbf{i}_\xi =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \xi}||}\frac{\partial \mathbf{r}}{\partial \xi} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{bmatrix} \\
\mathbf{j}_\eta =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \eta}||}\frac{\partial \mathbf{r}}{\partial \eta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{bmatrix} \\
\mathbf{k}_\zeta =& \frac{1}{||\frac{\partial \mathbf{r}}{\partial \zeta}||}\frac{\partial \mathbf{r}}{\partial \zeta} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}} \begin{bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{bmatrix}
\end{aligned} \tag{6}$$

$$dl^2 = d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} dx & dy & dz \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = dx^2 + dy^2 + dz^2 \tag{7}$$

$$\begin{aligned}
dl^2 &= d\mathbf{r}^T d\mathbf{r} = \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \mathbf{J}^T \mathbf{J} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \\
&= \begin{bmatrix} d\xi & d\eta & d\zeta \end{bmatrix} \underbrace{\begin{bmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \zeta} & \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \end{bmatrix}}_{\mathbf{g}} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}
\end{aligned} \tag{8}$$

$$dV = \frac{\partial \mathbf{r}}{\partial \xi} d\xi \cdot \left(\frac{\partial \mathbf{r}}{\partial \eta} d\eta \wedge \frac{\partial \mathbf{r}}{\partial \zeta} d\zeta \right) = det \begin{bmatrix} \left(\frac{\partial \mathbf{r}}{\partial \xi}\right)^T \\ \left(\frac{\partial \mathbf{r}}{\partial \eta}\right)^T \\ \left(\frac{\partial \mathbf{r}}{\partial \zeta}\right)^T \end{bmatrix} d\xi d\eta d\zeta = det\left(\mathbf{J}^T\right) d\xi d\eta d\zeta \tag{9}$$

Coordonnées cylindriques

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{10}$$

$$\begin{cases} \frac{\partial x}{\partial r} &= \cos\theta \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial x}{\partial \tilde{z}} &= 0 \end{cases} \qquad \begin{cases} \frac{\partial y}{\partial r} &= \sin\theta \\ \frac{\partial y}{\partial \theta} &= r\cos\theta \\ \frac{\partial y}{\partial \tilde{z}} &= 0 \end{cases} \qquad \begin{cases} \frac{\partial z}{\partial r} &= 0 \\ \frac{\partial z}{\partial \theta} &= 0 \\ \frac{\partial z}{\partial \tilde{z}} &= 1 \end{cases} \tag{11}$$

$$\begin{aligned} \mathbf{i}_r &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial r}||}\frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2\theta+\cos^2\theta}}\begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} = \begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} \\ \mathbf{j}_\theta &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \theta}||}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2\sin^2\theta+r^2\cos^2\theta}}\begin{bmatrix}-r\sin\theta \\ r\cos\theta \\ 0\end{bmatrix} = \begin{bmatrix}-\sin\theta \\ \cos\theta \\ 0\end{bmatrix} \\ \mathbf{k}_{\tilde{z}} &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \tilde{z}}||}\frac{\partial \mathbf{r}}{\partial \tilde{z}} = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} \end{aligned} \tag{12}$$

$$\mathbf{J} = \begin{bmatrix}\cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{13}$$

$$d\mathbf{r} = \begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} dr + \begin{bmatrix}-r\sin\theta \\ r\cos\theta \\ 0\end{bmatrix} d\theta + \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} d\tilde{z} = dr\mathbf{i}_r + rd\theta\mathbf{i}_\theta + d\tilde{z}\mathbf{i}_{\tilde{z}} = \begin{bmatrix}dr \\ rd\theta \\ d\tilde{z}\end{bmatrix} \tag{14}$$

$$\mathbf{g} = \begin{bmatrix}1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{15}$$

$$dl^2 = dr^2 + r^2d\theta^2 + d\tilde{z}^2 \tag{16}$$

$$dV = det\left(\mathbf{J}^T\right) drd\theta d\tilde{z} = rdrd\theta d\tilde{z} \tag{17}$$

Coordonnées spheriques

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{18}$$

$$\begin{cases} \frac{\partial x}{\partial r} & = \sin\theta\cos\varphi \\ \frac{\partial x}{\partial \theta} & = r\cos\theta\cos\varphi \\ \frac{\partial x}{\partial \varphi} & = -r\sin\theta\sin\varphi \end{cases} \quad \begin{cases} \frac{\partial y}{\partial r} & = \sin\theta\sin\varphi \\ \frac{\partial y}{\partial \theta} & = r\cos\theta\sin\varphi \\ \frac{\partial y}{\partial \varphi} & = r\sin\theta\cos\varphi \end{cases} \quad \begin{cases} \frac{\partial z}{\partial r} & = \cos\theta \\ \frac{\partial z}{\partial \theta} & = -r\sin\theta \\ \frac{\partial z}{\partial \varphi} & = 0 \end{cases} \tag{19}$$

$$\begin{aligned}
\mathbf{i}_r &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial r}||}\frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}}\begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix} \\
\mathbf{j}_\varphi &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \varphi}||}\frac{\partial \mathbf{r}}{\partial \varphi} = \frac{1}{\sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi}}\begin{bmatrix}-r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0\end{bmatrix} = \begin{bmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{bmatrix} \\
\mathbf{k}_\theta &= \frac{1}{||\frac{\partial \mathbf{r}}{\partial \theta}||}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}}\begin{bmatrix}r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta\end{bmatrix}
\end{aligned} \tag{20}$$

$$\mathbf{J} = \begin{bmatrix}\sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta\end{bmatrix} \tag{21}$$

$$\begin{aligned} d\mathbf{r} &= \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} dr + r\sin\theta \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} d\varphi + r \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix} d\theta = \\ &= dr\mathbf{i}_r + r\sin\theta d\varphi\mathbf{j}_\varphi + rd\theta\mathbf{k}_\theta = \begin{bmatrix} dr \\ r\sin\theta d\varphi \\ rd\theta \end{bmatrix} \end{aligned} \tag{22}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2 \end{bmatrix} \tag{23}$$

$$dl^2 = dr^2 + r^2\sin^2\theta d\varphi^2 + r^2 d\theta^2 \tag{24}$$

$$dV = det\left(\mathbf{J}^T\right) drd\theta d\tilde{z} = r^2\sin\theta drd\varphi d\theta \tag{25}$$

0.2.2 Problème B

Rappels théoriques

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = y\left(\xi, \eta, \zeta\right) \\ z = z\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{26}$$

$$f\left(x, y, z\right) = f\left(x\left(\xi, \eta, \zeta\right), y\left(\xi, \eta, \zeta\right), z\left(\xi, \eta, \zeta\right)\right) \tag{27}$$

$$f\left(\xi, \eta, \zeta\right) = f\left(\xi\left(x, y, z\right), \eta\left(x, y, z\right), \zeta\left(x, y, z\right)\right) \tag{28}$$

$$\nabla^2_{xyz} f\left(x, y, z\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{29}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{cases} \tag{30}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) = \\ & = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right) = \\ & = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2} \\ \frac{\partial^2 f}{\partial z^2} & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) = \\ & = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2} \end{cases} \tag{31}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = \left(\dfrac{\partial \xi}{\partial x} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x} \dfrac{\partial}{\partial \zeta} \right) \\ \dfrac{\partial}{\partial y} = \left(\dfrac{\partial \xi}{\partial y} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y} \dfrac{\partial}{\partial \zeta} \right) \\ \dfrac{\partial}{\partial z} = \left(\dfrac{\partial \xi}{\partial z} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial z} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial z} \dfrac{\partial}{\partial \zeta} \right) \end{cases} \tag{32}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x} \dfrac{\partial}{\partial \zeta} \right)\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)^2 \dfrac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\dfrac{\partial^2 f}{\partial \eta \partial \xi} + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \xi} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x} \dfrac{\partial}{\partial \zeta} \right)\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial x} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \dfrac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \eta} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} & = \left(\dfrac{\partial \xi}{\partial x} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial x} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial x} \dfrac{\partial}{\partial \zeta} \right)\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial x} = \\ & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\dfrac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\frac{\partial \zeta}{\partial x}\right)^2 \dfrac{\partial^2 f}{\partial \zeta^2} \end{cases} \tag{33}$$

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial y} & = \left(\dfrac{\partial \xi}{\partial y} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y} \dfrac{\partial}{\partial \zeta} \right)\left(\dfrac{\partial f}{\partial \xi}\right)\dfrac{\partial \xi}{\partial y} = \\ & = \left(\dfrac{\partial \xi}{\partial y}\right)^2 \dfrac{\partial^2 f}{\partial \xi^2} + \left(\dfrac{\partial \eta}{\partial y}\right)\left(\dfrac{\partial \xi}{\partial y}\right)\dfrac{\partial^2 f}{\partial \eta \partial \xi} + \left(\dfrac{\partial \zeta}{\partial y}\right)\left(\dfrac{\partial \xi}{\partial y}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \xi} \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial y} & = \left(\dfrac{\partial \xi}{\partial y} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y} \dfrac{\partial}{\partial \zeta} \right)\left(\dfrac{\partial f}{\partial \eta}\right)\dfrac{\partial \eta}{\partial y} = \\ & = \left(\dfrac{\partial \xi}{\partial y}\right)\left(\dfrac{\partial \eta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \xi \partial \eta} + \left(\dfrac{\partial \eta}{\partial y}\right)^2 \dfrac{\partial^2 f}{\partial \eta^2} + \left(\dfrac{\partial \zeta}{\partial y}\right)\left(\dfrac{\partial \eta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \zeta \partial \eta} \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial y} & = \left(\dfrac{\partial \xi}{\partial y} \dfrac{\partial}{\partial \xi} + \dfrac{\partial \eta}{\partial y} \dfrac{\partial}{\partial \eta} + \dfrac{\partial \zeta}{\partial y} \dfrac{\partial}{\partial \zeta} \right)\left(\dfrac{\partial f}{\partial \zeta}\right)\dfrac{\partial \zeta}{\partial y} = \\ & = \left(\dfrac{\partial \xi}{\partial y}\right)\left(\dfrac{\partial \zeta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \xi \partial \zeta} + \left(\dfrac{\partial \eta}{\partial y}\right)\left(\dfrac{\partial \zeta}{\partial y}\right)\dfrac{\partial^2 f}{\partial \eta \partial \zeta} + \left(\dfrac{\partial \zeta}{\partial y}\right)^2 \dfrac{\partial^2 f}{\partial \zeta^2} \end{cases} \tag{34}$$

$$
\begin{cases}
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \xi}\right)\frac{\partial \xi}{\partial z} = \\
 & = \left(\frac{\partial \xi}{\partial z}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \eta\partial \xi}+\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta\partial \xi} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial z} = \\
 & = \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi\partial \eta}+\left(\frac{\partial \eta}{\partial z}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \zeta\partial \eta} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} & = \left(\frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\right)\left(\frac{\partial f}{\partial \zeta}\right)\frac{\partial \zeta}{\partial z} = \\
 & = \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi\partial \zeta}+\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta\partial \zeta}+\left(\frac{\partial \zeta}{\partial z}\right)^2\frac{\partial^2 f}{\partial \zeta^2}
\end{cases}
\tag{35}
$$

$$
\frac{\partial^2 f}{\partial \eta\partial \xi}=\frac{\partial^2 f}{\partial \xi\partial \eta}\qquad \frac{\partial^2 f}{\partial \eta\partial \zeta}=\frac{\partial^2 f}{\partial \zeta\partial \eta}\qquad \frac{\partial^2 f}{\partial \xi\partial \zeta}=\frac{\partial^2 f}{\partial \zeta\partial \xi}
\tag{36}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = & \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial x}\right)^2\frac{\partial^2 f}{\partial \zeta^2}+ \\
& + 2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi\partial \eta}+2\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \xi\partial \zeta}+2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial^2 f}{\partial \eta\partial \zeta}+ \\
& + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} = & \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \eta^2}+\left(\frac{\partial \zeta}{\partial y}\right)^2\frac{\partial^2 f}{\partial \zeta^2}+ \\
& + 2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi\partial \eta}+2\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \xi\partial \zeta}+2\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial^2 f}{\partial \eta\partial \zeta}+ \\
& + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}+\frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2}
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial z^2} = &\left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial \zeta}{\partial z}\right)^2 \frac{\partial^2 f}{\partial \zeta^2} + \\
&+ 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \eta} + 2\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
&+ \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial f}{\partial \zeta}\frac{\partial^2 \zeta}{\partial z^2}
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\nabla^2_{\xi\eta\zeta} f(\xi,\eta,\zeta) = &\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \xi^2} + \\
&+ \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} + \\
&+ \left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2\right]\frac{\partial^2 f}{\partial \zeta^2} + \\
&+ 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \eta} + \\
&+ 2\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \eta \partial \zeta} + \\
&+ 2\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \zeta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \zeta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right]\frac{\partial^2 f}{\partial \xi \partial \zeta} + \\
&+ \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right)\frac{\partial f}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\frac{\partial f}{\partial \eta} + \\
&+ \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right)\frac{\partial f}{\partial \zeta}
\end{aligned}
\tag{40}
$$

Coordonnées cylindriques

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{41}$$

$$\begin{cases} \frac{\partial r}{\partial x} & = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\ & = \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} & = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\ & = \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} & = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} & = -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ & = -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} & = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ & = \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\ \frac{\partial \theta}{\partial z} & = 0 \end{cases} \quad \begin{cases} \frac{\partial \tilde{z}}{\partial x} & = 0 \\ \frac{\partial \tilde{z}}{\partial y} & = 0 \\ \frac{\partial \tilde{z}}{\partial z} & = 1 \end{cases} \tag{42}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} & = \frac{\partial}{\partial x}\left(\cos\theta\right) = \\ & = -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\ \frac{\partial^2 r}{\partial y^2} & = \frac{\partial}{\partial y}\left(\sin\theta\right) = \\ & = \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\ \frac{\partial^2 r}{\partial z^2} & = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \theta}{\partial x^2} & = \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\ & = -\frac{1}{r}\frac{\partial}{\partial x}\left(\sin\theta\right) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\ & = -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\ & = \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\ & = \frac{1}{r}\frac{\partial}{\partial y}\left(\cos\theta\right) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\ & = -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\ & = -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} & = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \tilde{z}}{\partial x^2} & = 0 \\ \frac{\partial^2 \tilde{z}}{\partial y^2} & = 0 \\ \frac{\partial^2 \tilde{z}}{\partial z^2} & = 0 \end{cases} \tag{43}$$

$$\begin{aligned} \nabla^2_{r\theta\tilde{z}} f\left(r,\theta,\tilde{z}\right) = & \left[\cos^2\theta + \sin\theta^2\right]\frac{\partial^2 f}{\partial r^2} + \left[\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right]\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \tilde{z}^2} + \\ & + 2\left[-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right]\frac{\partial^2 f}{\partial r\partial\theta} + \\ & + \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial f}{\partial r} + \left(\frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2}\right)\frac{\partial f}{\partial \theta} = \\ = & \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \tilde{z}^2} \end{aligned} \tag{44}$$

Coordonnées spheriques

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{45}$$

$$\begin{cases} \frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\sin\theta\cos\varphi}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\sin\theta\sin\varphi}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \end{cases} \quad \begin{cases} \frac{\partial\varphi}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{\sin\varphi}{r\sin\theta} \\ \frac{\partial\varphi}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{\cos\varphi}{r\sin\theta} \\ \frac{\partial\varphi}{\partial z} &= 0 \end{cases} \quad \begin{cases} \frac{\partial\theta}{\partial x} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}x}{\cancel{2}z\sqrt{x^2+y^2}} = \\ &= \frac{1}{r}\cos\theta\cos\varphi \\ \frac{\partial\theta}{\partial y} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}y}{\cancel{2}z\sqrt{x^2+y^2}} = \\ &= \frac{1}{r}\cos\theta\sin\varphi \\ \frac{\partial\theta}{\partial z} &= -\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\sqrt{x^2+y^2}}{z^2} = \\ &= -\frac{1}{r}\sin\theta \end{cases} \tag{46}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} &= \cos\theta\cos\varphi\frac{\partial\theta}{\partial x} - \sin\theta\sin\varphi\frac{\partial\varphi}{\partial x} = \\ &= \frac{1}{r}\left(\cos^2\varphi\cos^2\theta + \sin^2\varphi\right) \\ \frac{\partial^2 r}{\partial y^2} &= \cos\theta\sin\varphi\frac{\partial\theta}{\partial y} + \sin\theta\cos\varphi\frac{\partial\varphi}{\partial y} = \\ &= \frac{1}{r}\left(\sin^2\varphi\cos^2\theta + \cos^2\varphi\right) \\ \frac{\partial^2 r}{\partial z^2} &= -\sin\theta\frac{\partial\theta}{\partial z} = \\ &= \frac{1}{r}\sin^2\theta \end{cases} \tag{47}$$

$$\begin{cases} \frac{\partial^2\varphi}{\partial x^2} &= \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial r}{\partial x} - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\varphi}{\partial x} + \frac{1}{r}\frac{\sin\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial x} = \\ = & \frac{1}{r^2}\sin\varphi\cos\varphi + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\ \frac{\partial^2\varphi}{\partial y^2} &= -\frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial r}{\partial y} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\varphi}{\partial y} - \frac{1}{r}\frac{\cos\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial y} = \\ = & -\frac{1}{r^2}\sin\varphi\cos\varphi - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\ \frac{\partial^2\varphi}{\partial z^2} &= 0 \end{cases} \tag{48}$$

$$\begin{cases} \frac{\partial^2\theta}{\partial x^2} & = -\frac{1}{r^2}\cos\varphi\cos\theta\frac{\partial r}{\partial x} - \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\varphi}{\partial x} - \frac{1}{r}\cos\varphi\sin\theta\frac{\partial\theta}{\partial x} = \\ & = -\frac{2}{r^2}\cos^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\sin^2\varphi}{\tan\theta} \\ \frac{\partial^2\theta}{\partial y^2} & = -\frac{1}{r^2}\sin\varphi\cos\theta\frac{\partial r}{\partial y} - \frac{1}{r}\sin\varphi\sin\theta\frac{\partial\varphi}{\partial y} + \frac{1}{r}\cos\varphi\cos\theta\frac{\partial\theta}{\partial y} = \\ & = -\frac{2}{r^2}\sin^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\cos^2\varphi}{\tan\theta} \\ \frac{\partial^2\theta}{\partial z^2} & = \frac{1}{r^2}\sin\theta\frac{\partial r}{\partial z} - \frac{1}{r}\cos\theta\frac{\partial\theta}{\partial z} = \\ & = \frac{2}{r^2}\cos\theta\sin\theta \end{cases} \tag{49}$$

$$\nabla^2_{r\varphi\theta} f\left(\xi,\eta,\zeta\right) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{2}{r^2\tan\theta}\frac{\partial f}{\partial\theta} \tag{50}$$

0.2.3 Problème C

Rappels théoriques

$$\begin{cases} x = x\left(\xi, \eta, \zeta\right) \\ y = x\left(\xi, \eta, \zeta\right) \\ z = x\left(\xi, \eta, \zeta\right) \end{cases} \longleftrightarrow \begin{cases} \xi = \xi\left(x, y, z\right) \\ \eta = \eta\left(x, y, z\right) \\ \zeta = \zeta\left(x, y, z\right) \end{cases} \tag{51}$$

$$f\left(x, y, z\right) = f\left(x\left(\xi, \eta, \zeta\right), y\left(\xi, \eta, \zeta\right), z\left(\xi, \eta, \zeta\right)\right) \tag{52}$$

$$f\left(\xi, \eta, \zeta\right) = f\left(\xi\left(x, y, z\right), \eta\left(x, y, z\right), \zeta\left(x, y, z\right)\right) \tag{53}$$

$$\nabla f_{xyz} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial y}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\partial f}{\partial z}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{i}^x + \frac{\partial f}{\partial y}\mathbf{j}^y + \frac{\partial f}{\partial z}\mathbf{k}^z \tag{54}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial \xi}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \eta}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} + \frac{\partial f}{\partial \zeta}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{55}$$

$$\begin{aligned} \mathbf{i}^\xi &= \frac{1}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2}}\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \\ \frac{\partial \xi}{\partial z} \end{bmatrix} \\ \mathbf{j}^\eta &= \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2}}\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial z} \end{bmatrix} \\ \mathbf{k}^\zeta &= \frac{1}{\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2}}\begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{bmatrix} \end{aligned} \tag{56}$$

$$
\begin{aligned}
\nabla_{\xi\eta\zeta} f =& \frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2+\left(\frac{\partial \xi}{\partial z}\right)^2}\cdot \mathbf{i}^{\xi}+\\
&+\frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2+\left(\frac{\partial \eta}{\partial z}\right)^2}\cdot \mathbf{j}^{\eta}+\\
&+\frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2+\left(\frac{\partial \zeta}{\partial y}\right)^2+\left(\frac{\partial \zeta}{\partial z}\right)^2}\cdot \mathbf{k}^{\zeta} =\\
=&\begin{bmatrix}
\frac{\partial f}{\partial \xi}\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2+\left(\frac{\partial \xi}{\partial z}\right)^2}\\
\frac{\partial f}{\partial \eta}\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2+\left(\frac{\partial \eta}{\partial z}\right)^2}\\
\frac{\partial f}{\partial \zeta}\sqrt{\left(\frac{\partial \zeta}{\partial x}\right)^2+\left(\frac{\partial \zeta}{\partial y}\right)^2+\left(\frac{\partial \zeta}{\partial z}\right)^2}
\end{bmatrix}
\end{aligned}
\tag{57}
$$

Coordonnées cylindriques

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \tilde{z} \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \tilde{z} = z \end{cases} \tag{58}$$

$$\begin{cases} \frac{\partial r}{\partial x} & = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \\ & = \frac{\cancel{r}\cos\theta}{\cancel{r}} \\ \frac{\partial r}{\partial y} & = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \\ & = \frac{\cancel{r}\sin\theta}{\cancel{r}} \\ \frac{\partial r}{\partial z} & = 0 \end{cases} \quad \begin{cases} \frac{\partial \theta}{\partial x} & = -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ & = -\frac{1}{1+\tan^2\theta}\frac{r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} & = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ & = \frac{1}{1+\tan^2\theta}\frac{1}{r\cos\theta} = \frac{\cos\theta}{r} \\ \frac{\partial \theta}{\partial z} & = 0 \end{cases} \quad \begin{cases} \frac{\partial \tilde{z}}{\partial x} & = 0 \\ \frac{\partial \tilde{z}}{\partial y} & = 0 \\ \frac{\partial \tilde{z}}{\partial z} & = 1 \end{cases} \tag{59}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} & = \frac{\partial}{\partial x}(\cos\theta) = \\ & = -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r} \\ \frac{\partial^2 r}{\partial y^2} & = \frac{\partial}{\partial y}(\sin\theta) = \\ & = \cos\theta\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r} \\ \frac{\partial^2 r}{\partial z^2} & = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \theta}{\partial x^2} & = \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = \\ & = -\frac{1}{r}\frac{\partial}{\partial x}(\sin\theta) - \sin\theta\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \\ & = -\frac{\cos\theta}{r}\frac{\partial\theta}{\partial x} + \frac{\sin\theta}{r^2}\frac{\partial r}{\partial x} = \\ & = \frac{2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial y^2} & = \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \\ & = \frac{1}{r}\frac{\partial}{\partial y}(\cos\theta) + \cos\theta\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \\ & = -\frac{\sin\theta}{r}\frac{\partial\theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} = \\ & = -\frac{2\sin\theta\cos\theta}{r^2} = -\frac{\sin 2\theta}{r^2} \\ \frac{\partial^2 \theta}{\partial z^2} & = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 \tilde{z}}{\partial x^2} & = 0 \\ \frac{\partial^2 \tilde{z}}{\partial y^2} & = 0 \\ \frac{\partial^2 \tilde{z}}{\partial z^2} & = 0 \end{cases} \tag{60}$$

$$\begin{cases} \sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1 \\ \sqrt{\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2} = \frac{1}{r} \\ \sqrt{\left(\frac{\partial \tilde{z}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial y}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial z}\right)^2} = 1 \end{cases} \tag{61}$$

$$\begin{aligned}
\mathbf{i}^r &= \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \\
\mathbf{j}^\theta &= \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \\
\mathbf{k}^\zeta &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned} \tag{62}$$

$$\nabla_{r\theta\tilde{z}} f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \tilde{z}} \end{bmatrix} \tag{63}$$

Coordonnées spheriques

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \tag{64}$$

$$\begin{cases} \frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\sin\theta\cos\varphi}{\cancel{r}} \\ \frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\sin\theta\sin\varphi}{\cancel{r}} \\ \frac{\partial r}{\partial z} &= \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \\ &= \frac{\cancel{r}\cos\theta}{\cancel{r}} \end{cases} \qquad \begin{cases} \frac{\partial \varphi}{\partial x} &= -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{y}{x^2} = \\ &= -\frac{\sin\varphi}{r\sin\theta} \\ \frac{\partial \varphi}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \\ &= \frac{\cos\varphi}{r\sin\theta} \\ \frac{\partial \varphi}{\partial z} &= 0 \end{cases} \qquad \begin{cases} \frac{\partial \theta}{\partial x} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}x}{\cancel{2}z\sqrt{x^2+y^2}} = \\ &= \frac{1}{r}\cos\theta\cos\varphi \\ \frac{\partial \theta}{\partial y} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\cancel{2}y}{\cancel{2}z\sqrt{x^2+y^2}} = \\ &= \frac{1}{r}\cos\theta\sin\varphi \\ \frac{\partial \theta}{\partial z} &= -\frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{\sqrt{x^2+y^2}}{z^2} = \\ &= -\frac{1}{r}\sin\theta \end{cases} \tag{65}$$

$$\begin{cases} \frac{\partial^2 r}{\partial x^2} &= \cos\theta\cos\varphi\frac{\partial\theta}{\partial x} - \sin\theta\sin\varphi\frac{\partial\varphi}{\partial x} = \\ &= \frac{1}{r}\left(\cos^2\varphi\cos^2\theta + \sin^2\varphi\right) \\ \frac{\partial^2 r}{\partial y^2} &= \cos\theta\sin\varphi\frac{\partial\theta}{\partial y} + \sin\theta\cos\varphi\frac{\partial\varphi}{\partial y} = \\ &= \frac{1}{r}\left(\sin^2\varphi\cos^2\theta + \cos^2\varphi\right) \\ \frac{\partial^2 r}{\partial z^2} &= -\sin\theta\frac{\partial\theta}{\partial z} = \\ &= \frac{1}{r}\sin^2\theta \end{cases} \tag{66}$$

$$\begin{cases} \frac{\partial^2 \varphi}{\partial x^2} &= \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial r}{\partial x} - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\varphi}{\partial x} + \frac{1}{r}\frac{\sin\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial x} = \\ = & \frac{1}{r^2}\sin\varphi\cos\varphi + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\ \frac{\partial^2 \varphi}{\partial y^2} &= -\frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial r}{\partial y} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\varphi}{\partial y} - \frac{1}{r}\frac{\cos\varphi}{\sin^2\theta}\cos\theta\frac{\partial\theta}{\partial y} = \\ = & -\frac{1}{r^2}\sin\varphi\cos\varphi - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\tan^2\theta} \\ \frac{\partial^2 \varphi}{\partial z^2} &= 0 \end{cases} \tag{67}$$

$$\begin{cases} \frac{\partial^2\theta}{\partial x^2} & = -\frac{1}{r^2}\cos\varphi\cos\theta\frac{\partial r}{\partial x} - \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\varphi}{\partial x} - \frac{1}{r}\cos\varphi\sin\theta\frac{\partial\theta}{\partial x} = \\ & = -\frac{2}{r^2}\cos^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\sin^2\varphi}{\tan\theta} \\ \frac{\partial^2\theta}{\partial y^2} & = -\frac{1}{r^2}\sin\varphi\cos\theta\frac{\partial r}{\partial y} - \frac{1}{r}\sin\varphi\sin\theta\frac{\partial\varphi}{\partial y} + \frac{1}{r}\cos\varphi\cos\theta\frac{\partial\theta}{\partial y} = \\ & = -\frac{2}{r^2}\sin^2\varphi\cos\theta\sin\theta + \frac{1}{r^2}\frac{\cos^2\varphi}{\tan\theta} \\ \frac{\partial^2\theta}{\partial z^2} & = \frac{1}{r^2}\sin\theta\frac{\partial r}{\partial z} - \frac{1}{r}\cos\theta\frac{\partial\theta}{\partial z} = \\ & = \frac{2}{r^2}\cos\theta\sin\theta \end{cases} \tag{68}$$

$$\begin{cases} \sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1 \\ \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2} = \frac{1}{r\sin\theta} \\ \sqrt{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2} = \frac{1}{r} \end{cases} \tag{69}$$

$$\begin{aligned} \mathbf{i}^r &= \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} \\ \mathbf{j}^\varphi &= r\sin\theta \begin{bmatrix} -\frac{\sin\varphi}{r\sin\theta} \\ \frac{\cos\varphi}{r\sin\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} \\ \mathbf{k}^\theta &= r \begin{bmatrix} \frac{1}{r}\cos\theta\cos\varphi \\ \frac{1}{r}\cos\theta\sin\varphi \\ -\frac{1}{r}\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix} \end{aligned} \tag{70}$$

$$\nabla_{r\varphi\theta} f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi} \\ \frac{1}{r}\frac{\partial f}{\partial\theta} \end{bmatrix} \tag{71}$$

1 Déplacement et mesures de déformation

1.1 Énoncé

1.2 Corrigé

1.2.1 Rappels théoriques

$$\mathbf{y} = \mathbf{y}\left(\mathbf{x}\right) = \mathbf{x} + \mathbf{u}\left(\mathbf{x}\right) \tag{72}$$

$$\mathbf{y} = \mathbf{x} + \mathbf{u}\left(\mathbf{x}\right) \longleftrightarrow \mathbf{u}\left(\mathbf{x}\right) = \mathbf{y} - \mathbf{x} \tag{73}$$

$$\underline{\underline{\mathbf{F}}} = \nabla \mathbf{y} \longleftrightarrow F_{ij} = \frac{\partial y_i}{\partial x_j} \tag{74}$$

$$d\mathbf{y} = \underline{\underline{\mathbf{F}}} d\mathbf{x} \tag{75}$$

$$d\mathbf{u} = d\mathbf{y} - d\mathbf{x} = \underline{\underline{\mathbf{F}}} d\mathbf{x} - d\mathbf{x} = \left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}}\right) d\mathbf{x} \tag{76}$$

$$d\mathbf{u} = \nabla_{\mathbf{x}} \mathbf{u} d\mathbf{x} \tag{77}$$

$$\nabla_{\mathbf{x}} \mathbf{u} = \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} \tag{78}$$

$$dS^2 = d\mathbf{x}^T d\mathbf{x} \tag{79}$$

$$ds^2 = d\mathbf{y}^T d\mathbf{y} = \left(\underline{\underline{\mathbf{F}}} d\mathbf{x}\right)^T \underline{\underline{\mathbf{F}}} d\mathbf{x} = d\mathbf{x}^T \underbrace{\underline{\underline{\mathbf{F}}}^T \underline{\underline{\mathbf{F}}}}_{\underline{\underline{\mathbf{C}}}} d\mathbf{x} = d\mathbf{x}^T \underline{\underline{\mathbf{C}}} d\mathbf{x} \tag{80}$$

$$
\begin{aligned}
ds^2 &= d\mathbf{y}^T d\mathbf{y} = \\
&= \left(d\mathbf{x} + d\mathbf{u}\left(\mathbf{x}\right)\right)^T \left(d\mathbf{x} + d\mathbf{u}\left(\mathbf{x}\right)\right) = \\
&= d\mathbf{x}^T d\mathbf{x} + d\mathbf{x}^T d\mathbf{u}\left(\mathbf{x}\right) + d\mathbf{u}\left(\mathbf{x}\right)^T d\mathbf{x} + d\mathbf{u}\left(\mathbf{x}\right)^T d\mathbf{u}\left(\mathbf{x}\right) = \\
&= d\mathbf{x}^T d\mathbf{x} + d\mathbf{x}^T \nabla_{\mathbf{x}}\mathbf{u} d\mathbf{x} + d\mathbf{x}^T \nabla_{\mathbf{x}}\mathbf{u}^T d\mathbf{x} + d\mathbf{x}^T \nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u} d\mathbf{x} = \\
&= d\mathbf{x}^T \left(\underline{\underline{\mathbf{I}}} + \underbrace{\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u}}_{2\underline{\underline{\mathbf{E}}}}\right) d\mathbf{x} = \\
&= d\mathbf{x}^T \left(\underline{\underline{\mathbf{I}}} + 2\underline{\underline{\mathbf{E}}}\right) d\mathbf{x}
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
\underline{\underline{\mathbf{E}}} &= \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u}\right) = \\
&= \frac{1}{2}\left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{I}}} + \left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}}\right)^T \left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}}\right)\right) = \\
&= \frac{1}{2}\left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{I}}}\right) = \\
&= \frac{1}{2}\left(\underline{\underline{\mathbf{F}}}^T \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}}\right) = \frac{1}{2}\left(\underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}}\right)
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
ds^2 - dS^2 &= d\mathbf{y}^T d\mathbf{y} - d\mathbf{x}^T d\mathbf{x} = \\
&= d\mathbf{x}^T \left(\underline{\underline{\mathbf{I}}} + 2\underline{\underline{\mathbf{E}}}\right) d\mathbf{x} - d\mathbf{x}^T d\mathbf{x} = \\
&= d\mathbf{x}^T \left(\underline{\underline{\mathbf{I}}} + 2\underline{\underline{\mathbf{E}}} - \underline{\underline{\mathbf{I}}}\right) d\mathbf{x} = \\
&= d\mathbf{x}^T 2\underline{\underline{\mathbf{E}}} d\mathbf{x} = \\
&= d\mathbf{x}^T \left(\underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}}\right) d\mathbf{x}
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
\nabla_{\mathbf{x}}\mathbf{u} \ll 1 \longleftrightarrow \underline{\underline{\mathbf{E}}} &= \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \cancelto{\sim 0}{\nabla_{\mathbf{x}}\mathbf{u}^T \nabla_{\mathbf{x}}\mathbf{u}}\right) \sim \\
&\sim \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T\right) = \underline{\underline{\epsilon}}
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
\underline{\underline{\epsilon}} &= \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T\right) = \\
&= \frac{1}{2}\left(\underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} + \underline{\underline{\mathbf{F}}}^T - \underline{\underline{\mathbf{I}}}\right) = \\
&= \frac{1}{2}\left(\underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{F}}}^T - 2\underline{\underline{\mathbf{I}}}\right) = \\
&= \frac{1}{2}\left(\underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{F}}}^T\right) - \underline{\underline{\mathbf{I}}}
\end{aligned}
\tag{85}
$$

1.2.2 Problème A

$$\begin{cases} y_1 = kx_1 \\ y_2 = kx_2 \\ y_3 = kx_3 \end{cases} \longleftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad k \in \mathbb{R} \tag{86}$$

$$\begin{cases} u_1 = y_1 - x_1 = kx_1 - x_1 = (k-1)\,x_1 \\ u_2 = y_2 - x_2 = kx_2 - x_2 = (k-1)\,x_2 \\ u_3 = y_3 - x_3 = kx_3 - x_3 = (k-1)\,x_3 \end{cases} \longleftrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (k-1) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{87}$$

$$\underline{\underline{\mathbf{F}}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \tag{88}$$

$$\begin{aligned} \nabla_{\mathbf{x}} \mathbf{u} &= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \\ &= \underline{\underline{\mathbf{F}}} - \underline{\underline{\mathbf{I}}} = \\ &= \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \end{aligned} \tag{89}$$

$$
\begin{aligned}
\underline{\underline{\mathbf{E}}} &= \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T + \nabla_{\mathbf{x}}\mathbf{u}^T\nabla_{\mathbf{x}}\mathbf{u}\right) = \\
&= \frac{1}{2}\left(\begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} + \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}\right) + \\
&+ \frac{1}{2}\left(\begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}\begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}\right) = \\
&= \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (k-1)^2 & 0 & 0 \\ 0 & (k-1)^2 & 0 \\ 0 & 0 & (k-1)^2 \end{bmatrix}
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
\underline{\underline{\epsilon}} &= \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^T\right) = \\
&= \frac{1}{2}\left(\begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} + \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}\right) = \\
&= \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}
\end{aligned}
\tag{91}
$$

$$
\underline{\underline{\mathbf{E}}} \sim \underline{\underline{\epsilon}} \longleftrightarrow \frac{\frac{1}{2}(k-1)^2}{(k-1) + \frac{1}{2}(k-1)^2} \leq e \tag{92}
$$

$$
(k-1)\left(\frac{1}{2}(1-e)(k-1) - e\right) \leq 0 \tag{93}
$$

$$
1 \leq k \leq \frac{1+e}{1-e} \tag{94}
$$

$$
\begin{aligned}
dV &= dy_1 dy_2 dy_3 - dx_1 dx_2 dx_3 = \\
&= k^3 dx_1 dx_2 dx_3 - dx_1 dx_2 dx_3 = \\
&= \left(k^3 - 1\right) dx_1 dx_2 dx_3
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
\frac{dV}{V} &= \frac{dy_1 dy_2 dy_3 - dx_1 dx_2 dx_3}{dx_1 dx_2 dx_3} = \\
&= \frac{(k^3 - 1)\, dx_1 dx_2 dx_3}{dx_1 dx_2 dx_3} = k^3 - 1
\end{aligned} \tag{96}
$$

e	$\frac{1+e}{1-e}$	$\varepsilon_{xx}\big\rvert_{k=\frac{1+e}{1-e}}$	$\varepsilon_{yy}\big\rvert_{k=\frac{1+e}{1-e}}$	$\varepsilon_{zz}\big\rvert_{k=\frac{1+e}{1-e}}$	$\frac{dV}{V}$
0.001 (0.1%)	1.002	0.002 (0.2%)	0.002 (0.2%)	0.002 (0.2%)	0.006 (0.6%)
0.005 (0.5%)	1.01	0.01 (1%)	0.01 (1%)	0.01 (1%)	0.03 (3%)
0.01 (1%)	1.02	0.02 (2%)	0.02 (2%)	0.02 (2%)	0.06 (6%)
0.05 (5%)	1.11	0.11 (11%)	0.11 (11%)	0.11 (11%)	0.35 (35%)

Une éprouvette métallique à section rectangulaire est soumise à une charge de traction selon sa longueur. En appelant x la direction selon laquelle la charge est appliquée, y et z les coordonnées dans la section de l'éprouvette, le tenseur de petites déformations dans l'éprouvette est égal à

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & -\nu\varepsilon_{xx} & 0 \\ 0 & 0 & -\nu\varepsilon_{xx} \end{bmatrix} \qquad 0 < \nu < \frac{1}{2} \tag{97}$$

1. Déterminer l'expression du tenseur dans sections orientée à 30°, 45°, 60°, 90° par rapport à l'axe x. Utilisez la loi de rotation de tenseurs et aussi les cercles de Mohr.
2. On observe que la fracture de l'éprouvette se produit dans une section à 45°. Sur la base des résultats au point précédent, qu'est-ce qu'on peut conclure sur la mécanique de la fracture de l'éprouvette ?