IGERT Data Science : Graph Algorithms

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Why study Graphs and Graph algorithms

Graphs are useful in modeling complex systems with interacting components

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Graphs are useful in modeling complex systems with interacting components

We look at, understand, and interact with graphs locally: Graph algorithms compute more global properties

Graphs - Mathematics and Computer Science

Graphs have a deep Mathematical theory

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“Intrinsic” to computer science - represent hard problems, program states and execution, search structures, etc.

Network Science :: Graphs

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Network Science

Network Science :: Graphs

$$=$$

Statistical Mechanics :: Mechanics

Examples of “real networks”

- **The internet**
- **The WWW**
- **The Power Grid**
- **Collaboration network**
- **Citations**

An indispensable tool : Visualization

Courtesy : The OPTE project

Goals of Network Science

Understand and infer "macro" properties of networks

- **Network vulnerability**
- **How things spread : information, epidemics etc.**
- **Identifying "important" nodes in networks**
- **...**

Page rank : Google's secret formula

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Depends on computing over the WWW network - EFFICIENTLY

Graph theory and Algorithms

Page rank : Google's secret formula

Depends on computing over the WWW network - EFFICIENTLY

Graph algorithms : indispensable to compute statistics on large networks

Fundamental graph problems

- **Traversals**
- **Connectivity**
- **Shortest paths**
- **Identifying subgraphs with specific structure**
- **Characterizing graphs**
- **Optimization problems**
- **...**

The Shortest Path problem

Given graph and a vertex s find shortest paths from s to all other vertices.

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- **Map routing, robot navigation, urban traffic planning**
- **Network routing protocols (OSPF, BGP, RIP)**
- **...**

Traversals : Depth First Search (DFS)

Visit all vertices of the graph

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Visit all vertices of the graph

Procedure DFS(G, v)
begin
  label v as discovered
  forall the *edges from* v *to* w *in* $G.adjacentEdges(v)$ **do**
    if w *is not labeled as discovered* **then**
      recursively call DFS(G, w)
    end
  end
end

Non recursive DFS

```
Procedure DFS-iterative(G, s)
begin
    S ← empty stack
    S.push(s)
    while S is not empty do
        v ← S.pop()
        if v is not labeled as discovered then
            label v as discovered
            forall the edges from v to w in G.adjacentEdges(v) do
                S.push(w)
            end
        end
    end
end
```

Breadth First Search

Can compute shortest paths in unweighted graphs!

Breadth First Search

```
Procedure BFS(G, s)
begin
    s.dist ← 0
    for currDist ← 0 to |V| do
        for all vertices v do
            if v not labeled as discovered and v.dist == currDist then
                Label v as discovered
                forall the edges from v to w in G.adjacentEdges(v) do
                    Handle w
                end
            end
        end
    end
end
```

Breadth First Search

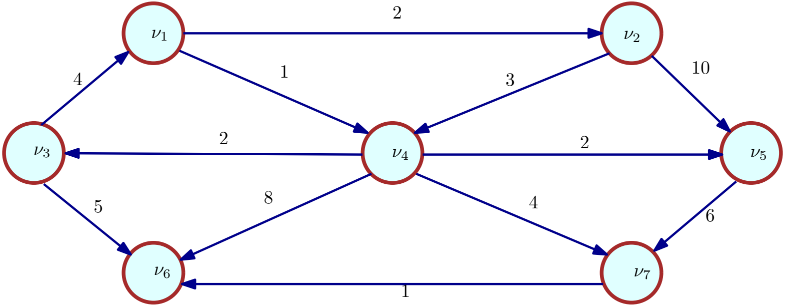
Runtime : $O(|V|^2)$

Breadth First Search

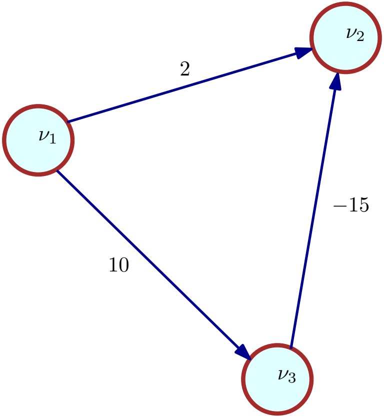
Runtime : $O(|V|^2)$

Can achieve $O(|V| + |E|)$ **using a smarter implementation**

Edge weighted graphs



Negative edge weights



Main structural properties of Shortest Paths

- **Prefixes of shortest paths are shortest paths**

- **Shortest paths from s - make up a tree**

Advancing the frontier

- **At any time we know the distance of some vertices**

Advancing the frontier

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- **How do we discover further?**

Advancing the frontier

- **At any time we know the distance of some vertices**
- **How do we discover further?**
- **Relaxation of an edge** $(v, w) : d(w) = \min(d(w), d(v) + c_{vw})$

Djikstra algorithm : Generalize BFS

- **Store d_v, v.isKnown, v.path**

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Djikstra algorithm : Generalize BFS

- **Store d_v, v.isKnown, v.path**
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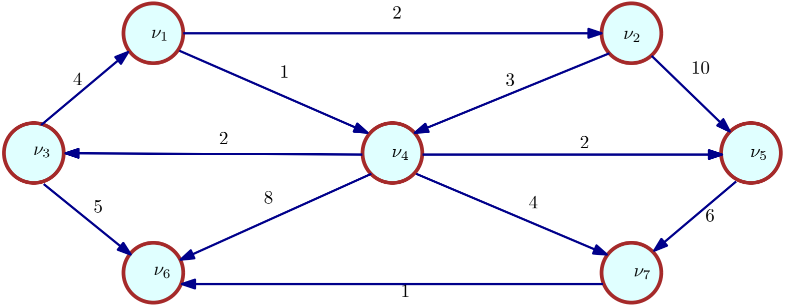
- **Relax all edges outgoing from it**
- **Repeat until all vertices are known**

Example of Djikstra in action



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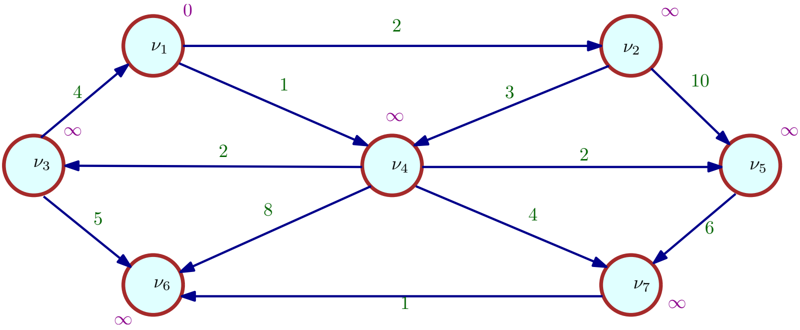

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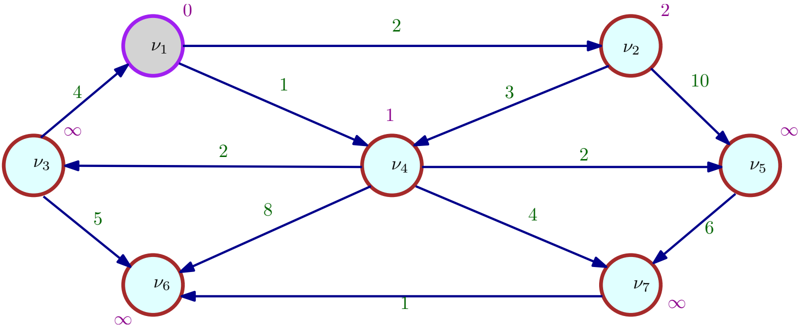

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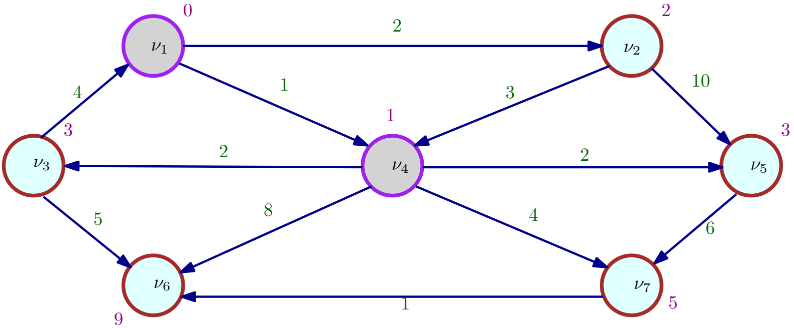

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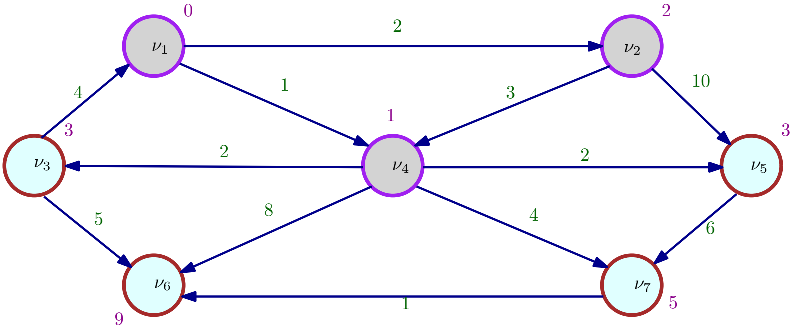

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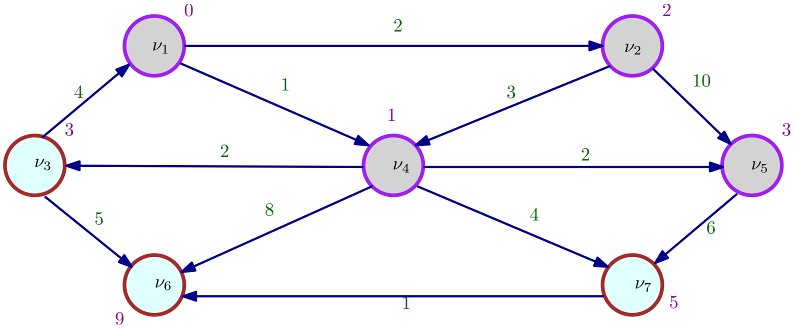

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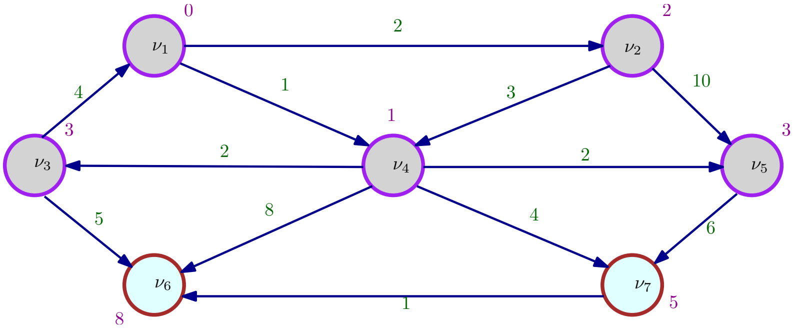

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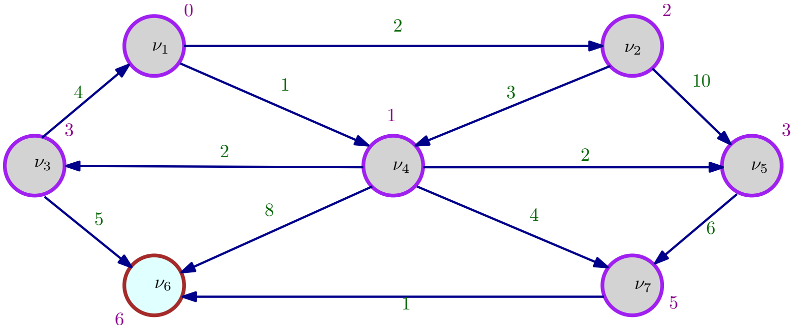

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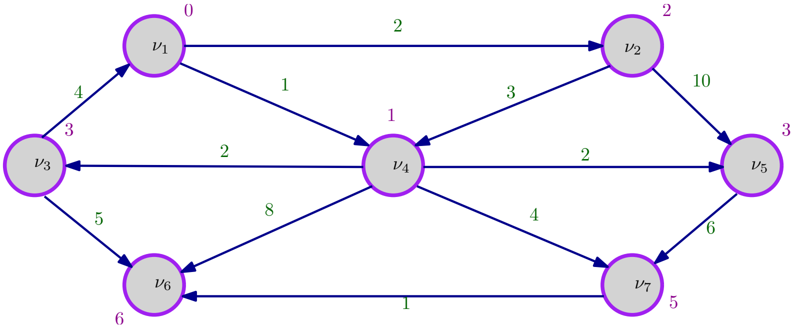

How efficient is Djikstra

Naive implementation : $O(|E| + |V^2|) = O(|V|^2)$

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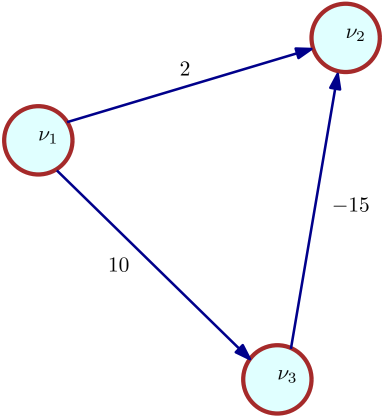
Using a heap : $O(|E| \log |V|)$

Using Fibonacci heap : $O(|E| + |V| \log |V|)$

Negative edge weights!



The Bellman Ford algorithm

Procedure Bellman-Ford(G, s) **begin**
    s.dist $\leftarrow 0$;
    for $i \leftarrow 1$ **to** $|V|$ **do**
        Relax each edge
    end
end

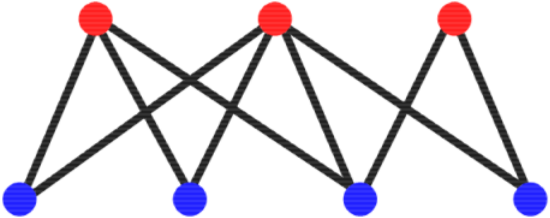
Runtime : $O(|E||V|)$

All-Pairs Shortest Path

- **Can run $|V|$ Djikstra's -** $O(|E||V|\log|V|)$
- **Floyd Warshall -** $O(|V|^3)$

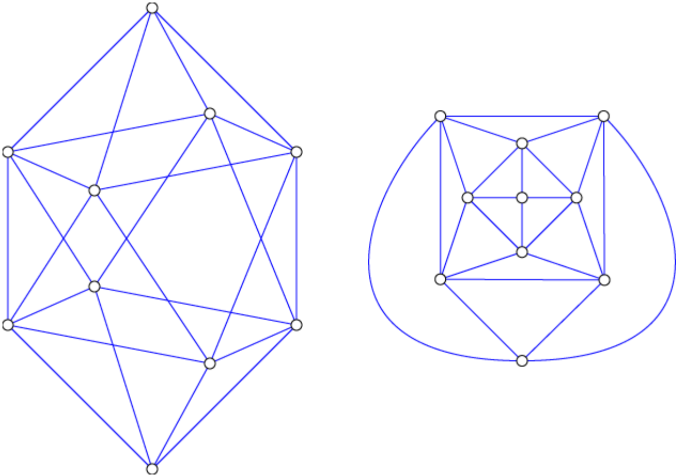
Graphs with special structure

Bipartite : $V = V_1 \cup V_2$, **edges only between** V_1, V_2



Graphs with special structure

Planar: Can be drawn in the plane without crossings - sparse



Back to networks

How can we model real networks?

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Doing mathematical analysis on the abstracted models helps us understand networks

Models should fit

Models should be guided by observed properties

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Test their computed properties against observed ones

Some common network models

- **Random graph models - why?**
- **Random (n, m) graphs**
- **Erdös-Rényi model**
- **Albert-Barbási model - preferential attachment**
- **Watts Strogatz model - small world phenomena**

Summary statistics and Performance metrics

- **Diameter and mean path length**
- **Clustering, average clustering, global clustering**
- **Degree distribution**
- **Centrality and its distribution**

The definitions

Clustering coefficient: $C_i = 2L_i/k_i(k_i - 1)$

Global clustering coefficient : $\frac{3\times\textbf{Number of triangles}}{\textbf{Number of connected triples}}$

Betweenness Centrality

$$g(v) = \textbf{Normalizing factor} \times \sum_{s\neq v\neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Why so many network models are needed?

- **Erdös-Rényi model does not capture gloabl clustering**

- **Watts-Strogatz model captures small world properties**

- **Networks *grow* - how can we accomodate that?**

Graphs algorithms - the message

Network Science is important

Graphs algorithms - the message

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Understanding graphs and computing with them is the key to understanding networks better

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Understanding graphs and computing with them is the key to understanding networks better

Need approximate but very fast algorithms for basic tasks