Statistical Models

Course 2024-2025

SEMINAR 2. Multivariate Statistics (Part 2)

1 Basic Problems

PROBLEM 1.- Given the following data sample:

$$\mathbf{X} = \begin{bmatrix} 2 & 3 & 5 & 6 & 9 & 7 & 5 & 4 & 3 & 1 & -1 \\ 1 & 4 & 9 & 6 & 7 & 4 & 3 & 3 & 7 & 6 & 5 \end{bmatrix}$$

a) Generate a qqplot for dimension 1 (see Table 2 for the required quantiles).

b) Based on the plot generated in (a), would you say that dimension 1 is normally distributed?

c) Generate a qqplot for dimension 2.

d) Based on the plot generated in (c), would you say that dimension 2 is normally distributed?

e) Based on the plots generated in (a) and (c), would you say that $\mathbf{X}$ follows a multi-variate Gaussian distribution? Justify.

PROBLEM 2.- Given the data sample from Problem 1,

a) Compute the sample mean and covariance matrix.

b) Calculate the Mahalanobis distances of each data point.

c) Generate a chi-square plot.

d) Based on the plot generated in (c), would you say that $\mathbf{X}$ follows a multi-variate Gaussian distribution? Justify.

PROBLEM 3.- Given the following data sample:

$$\mathbf{X} = \begin{bmatrix} -2 & -3 & -2 & 3 & 0 & 2 & 1 & -1 & -1 & 1 \\ -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 0 & 6 \end{bmatrix}$$

a) Generate a chi-square plot for these data. The chi-square percentiles corresponding to 10 bi-variate samples are: $q_{(1)} \dots q_{(10)} = 0.10,\ 0.33,\ 0.58,\ 0.86,\ 1.20,\ 1.60,\ 2.10,\ 2.77,\ 3.79,\ 5.99$

b) Based on the plot from (a), would you say that dimension 1 is normally distributed? Justify.

c) Based on the plot from (a), would you say that $\mathbf{X}$ follows a multivariate Gaussian distribution? Justify.

PROBLEM 4.- Consider a 3-dimensional population for which we wish to assess multivariate normality at significance level $\alpha = 0.05$. To do so, we obtain a random sample of $n = 25$ and construct the qq-plots displayed in Figure 1. Based on those plots, justify whether each of the following statements are *True* or *False*:

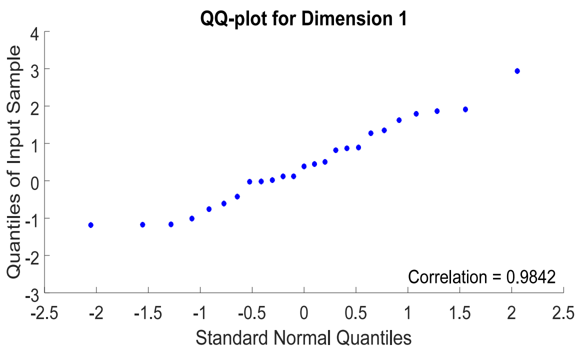
a) We can conclude that each of the 3 dimensions is normal.

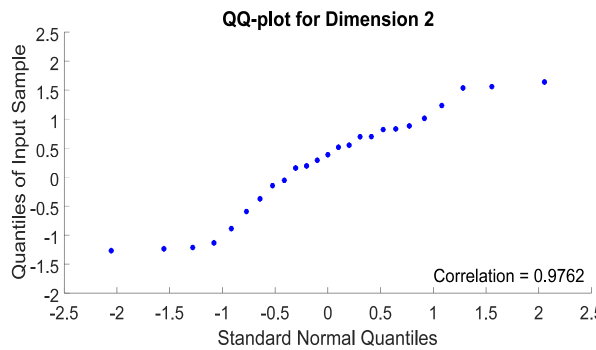
b) We cannot reject that each of the 3 dimensions is normal.

c) We can conclude that at least one of the dimensions is not normal.

d) We can conclude that the population that we are studying is multivariate normal.

e) The provided plots are insufficient to assess multivariate normality.





Critical points for the QQ-plot correlation coefficient test for normality

Sample size n	Significance levels α .01	.05	.10
5	.8299	.8788	.9032
10	.8801	.9198	.9351
15	.9126	.9389	.9503
20	.9269	.9508	.9604
25	.9410	.9591	.9665
30	.9479	.9652	.9715
35	.9538	.9682	.9740

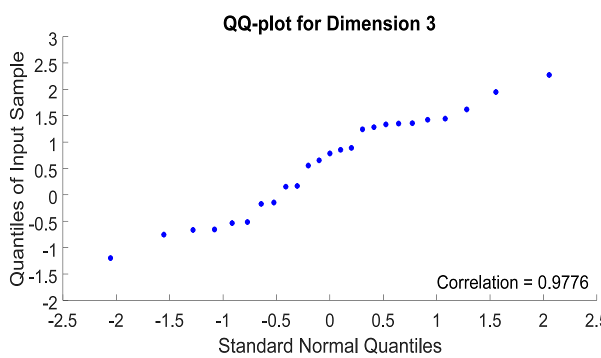


Figure 1: Information for problem 4

PROBLEM 5.- Given the following two bi-dimensional random sample from Gaussian distributions with similar covariance matrices:

$$\mathbf{X_1} : \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \mathbf{X_2} : \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

a) Calculate the pooled covariance $\mathbf{S_{pool}}$

b) Test the null hypothesis that both samples come from populations with equal means, at significance level $\alpha = 0.1$.

c) Construct 90% confidence intervals for the difference between the means along each dimension.

PROBLEM 6.- Two random samples have the following summary statistics:

$$n_1 = 45 \qquad \overline{\mathbf{x}}_\mathbf{1} = \begin{bmatrix} 20 \\ 55 \end{bmatrix} \qquad \mathbf{S_1} = \begin{bmatrix} 140 & 230 \\ 230 & 730 \end{bmatrix}$$

$$n_2 = 55 \qquad \overline{\mathbf{x}}_\mathbf{2} = \begin{bmatrix} 13 \\ 36 \end{bmatrix} \qquad \mathbf{S_2} = \begin{bmatrix} 86 & 190 \\ 190 & 560 \end{bmatrix}$$

a) Compute the appropriate T^2 to test the hypothesis $\mu_\mathbf{1} - \mu_\mathbf{2} = \mathbf{0}$ at significance level $\alpha = 0.05$.

b) Compute the 95% confidence region for $\mu_\mathbf{1} - \mu_\mathbf{2}$. Use it to confirm the result in (a).

PROBLEM 7.- The table below shows the marks for $n = 5$ students from 2 different Universities (A and B), on two aspects: science proficiency score and communication proficiency score. We wish to answer the following research question: *on average, do students from University A perform differently than students from University B, considering jointly both of the provided scores?*

Students from University A					
Science score	10	9	8	10	8
Communication score	8	6	6	4	6
Students from University B					
Science score	9	6	5	5	5
Communication score	8	8	10	8	6

Table 1: Data for Problem 7

a) State the null and alternative hypotheses required for the comparison indicated above. Include the mathematical formulation.

b) Indicate what method would you use to test the hypotheses stated in (a). Indicate what assumptions would be needed in order for your method to be valid. For this problem, you do not need to test whether the assumptions hold; we will assume that they do.

c) Test the hypotheses stated in (a) at significance level $\alpha = 0.05$. Hint: use $F_{2,7}(0.95) \simeq 4.74$

d) Clearly state your conclusion and interpret it to answer the above research question.

PROBLEM 8.- Given the following two independent random samples and assuming that they come from normal populations:

$$\mathbf{X}_1 = \begin{bmatrix} 6 & 2 & 7 & 0 \\ 3 & -4 & 1 & 2 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 & -3 & 3 & -2 \\ 1 & -4 & 0 & 1 \end{bmatrix}$$

we want to assess whether there is a statistically significant difference in their population mean.

a) State the null and alternative hypotheses.

b) Compute the required Hotelling's T^2

c) Compare the value calculated in (b) with the critical value for $\alpha = 0.10$. What can you conclude about the hypotheses stated in (a)?

PROBLEM 9.- Now we consider the same data from Problem 8, but under the assumption that these are not two independent samples but, instead, correspond to samples of the same experimental units measured two different times. Therefore, the samples are *paired*.

We want to assess whether there is a statistically significant difference in their population mean.

a) State the null and alternative hypotheses.

b) Compute the required Hotelling's T^2

c) Compare the value calculated in (b) with the critical value for $\alpha = 0.10$. What can you conclude about the hypotheses stated in (a)?

d) Does your result agree with the one in Problem 8? If not, which one is the *correct* answer?

PROBLEM 10.- Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be two random samples of the same variable before and after a given treatment:

$$\mathbf{X_1} = \begin{bmatrix} 31 & 10 & 14 & 22 & 16 & 33 & 49 & 12 & -16 & 59 \\ 21 & -17 & -10 & -37 & 13 & 6 & -25 & 24 & -15 & -20 \\ 31 & 39 & 37 & 39 & 47 & 30 & 43 & 75 & 10 & 24 \end{bmatrix}$$

$$\mathbf{X_2} = \begin{bmatrix} 46 & 17 & 26 & 52 & 15 & 29 & 40 & 44 & 37 & 38 \\ 18 & 12 & 10 & -16 & -1 & -7 & 24 & 24 & -1 & 16 \\ 27 & 29 & 1 & 14 & 30 & 11 & 8 & 44 & 48 & 1 \end{bmatrix}$$

a) Evaluate whether there is a difference between the samples at $\alpha = 0.95$ significance level.

b) Construct 95% simultaneous confidence intervals for the mean difference in each of the 3 dimensions.

c) Construct 95% simultaneous confidence intervals for the mean of each variable in each of the 3 dimensions. Compare these results with those from (b).

d) Idem (c) but using Bonferroni-corrected 95% confidence intervals.

PROBLEM 11.- We are testing a device that is part of a VAR[1] system. The device aims to determine the (x, y) position of the soccer ball in the field and the first thing that we want to check is whether on average it is accurate. To this end, we gather a random sample of size 10 with the following information:

$\mathbf{x}_{device}$ = the (x, y) position estimated by the device.

$\mathbf{x}_{truth}$ = the correct (x, y) position of the ball that should have been measured.

The samples are:

$$\mathbf{X}_{device} = \begin{bmatrix} 1 & 13 & 22 & 30 & 36 & 51 & 60 & 70 & 79 & 88 \\ -2 & 9 & 20 & 32 & 37 & 48 & 63 & 70 & 81 & 92 \end{bmatrix}$$

$$\mathbf{X}_{truth} = \begin{bmatrix} 0 & 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 \\ 0 & 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 \end{bmatrix}$$

Since n is small, it is important to know whether the error of the device follows a multivariate normal distribution. The error, in this case, should be considered with sign, and calculated as:

$$\mathbf{e} = \mathbf{x}_{device} - \mathbf{x}_{truth}$$

[1] Video Assistant Referee

a) Perform all the required steps to assess multivariate normality of the above error. Clearly indicate your conclusion. To assist you in this Problem, Table 2 provides some values of the inverse normal and chi-square distributions.

b) Indicate how would you proceed to assess if the device is accurate enough. Justify your choice

c) Perform the calculations required to implement the procedure suggested in (b) and indicate your conclusion.

PROBLEM 12.- We wish to assess if the results from the Pisa report published at the end of 2023 had an impact on the overall opinion of parents about the Catalan education system. To this end, researchers gathered the opinion of 4 parents one week **B**efore and one week **A**fter the Pisa results were made public (the same 4 parents were interviewer before and after). Each parent was asked to rate how good was the current education in 2 aspects: Math and Reading (first and second row, respectively, of the matrices below):

$$\mathbf{X_B} = \begin{bmatrix} 7 & 8 & 9 & 10 \\ 6 & 9 & 6 & 9 \end{bmatrix} \qquad \mathbf{X_A} = \begin{bmatrix} 7 & 7 & 8 & 8 \\ 5 & 7 & 6 & 8 \end{bmatrix}$$

We then consider the following research question: "has the overall opinion of parents changed due to the publication of the Pisa report?", which we wish to assess at significance level $\alpha = 0.10$

(a) State the null and alternative hypotheses for this problem. Include the mathematical formulation, defining any variable that you use without ambiguities.

(b) Indicate what method would you use to answer the research questions of this Problem.

(c) Indicate the assumptions needed so that the results provided by the selected method are valid.

(d) Test the hypotheses stated in (a) with the method selected in (b), considering that all assumptions stated in (c) are fulfilled. You will need one of the values below from the F distribution (make sure to use the correct one for this problem):

$$F_{2,2}(0,90) = 9.00 \qquad F_{2,3}(0,90) \simeq 5.46 \qquad F_{2,4}(0,90) \simeq 4.32 \qquad F_{2,5}(0,90) \simeq 3.78$$
$$F_{3,2}(0,90) \simeq 9.16 \qquad F_{3,3}(0,90) \simeq 5.39 \qquad F_{3,4}(0,90) \simeq 4.19 \qquad F_{4,3}(0,90) \simeq 5.34$$

(e) Interpret the obtained result in terms of the research question.

(f) Draw a confidence region for the mean difference in Math and Reading scores before and after the Pisa report. Make the plot as detailed as you can and faithful to your results in the previous points.

2 Lab Practice Problems

PROBLEM 13.- File `SM22_Seminar_2_Head_Brothers.xlsx` contains measurements of the length and breadth of the heads of pairs of adult brothers in 25 randomly sampled families. We wish to analyze this dataset with techniques that assume multivariate Gaussianity:

a) Assess whether each of the 4 dimensions of these data is normally distributed.

b) Assess whether each pair of dimensions is normally distributed.

c) Compute a chi-squared plot for all 4 dimensions considered together.

d) Identify possible outliers. If one or more outliers are found, concisely discuss what shall be done with it/them.

e) Based on all the above information, indicate whether you would find it acceptable to test this dataset using statistical techniques that assume the data following an (approximately) multivariate normal distribution. Make a clear and concise justification of your answer.

PROBLEM 14.- File `SM22_Seminar_2_Alcohol_and_Tobacco.xlsx` contains data from 200 cases of oesophageal cancer and 775 controls. For each subject, the table provides his/her alcohol and tobacco consumption, indicated in grams per day. We want to test whether the average alcohol and tobacco consumption (combined as bivariate samples) are significantly higher in subjects that developed oesophageal cancer than they are in controls.

a) State the null and alternative hypotheses to be tested (make sure to include the mathematical formulation).

b) Inspect your data for multivariate normality. Clearly indicate whether you would advise in favor or against using statistical methods that assume multivariate normality in these data.

c) Based on your conclusions in (b), test the hypotheses stated in (a) at significance level $\alpha = 0.05$. Indicate what conclusion you reach.

d) Calculate and plot a 95% confidence region for the difference of (alcohol, tobacco) consumption between cancer cases and controls. Show that it conforms with the conclusion in (c).

PROBLEM 15.- File `SM22_Seminar_2_Height_and_Weight.xlsx` contains data from men and women engaged in regular exercise. Each subject was asked to self-report his/her height and weight. Actual values of height and weight were also measured. We are interested in the following question: on average, do men and women differ in the bias with which they perceive their body height and weight (considered jointly)?

a) Inspect the data and discuss what dimensions shall be used and in what manner (e.g. as provided, added, multiplied, ...). State the appropriate null and alternative hypotheses.

b) Assess whether your data (or the relevant subsets) can be analyzed under the assumption of multivariate normality. If you find that this is not the case, propose and implement some solution. Carefully justify your choice.

c) Test the null hypothesis at significance level $\alpha = 0.05$. Clearly indicate and justify the statistic you choose. Indicate your conclusion.

d) Compute 95% confidence intervals for the bias in perceived height and for the bias in perceived weight. Since we are analyzing multivariate data, the intervals should hold simultaneously at the specified confidence level.

PROBLEM 16.- Given the 4-dimensional random sample from file `SM22_Seminar_2_4d.xlsx`, inspect the data to determine if it0 follows a multivariate normal distribution.

a) Indicate graphically if each of the dimensions is normally distributed.

b) Indicate graphically if each pair of dimensions is normally distributed.

c) Compute Q-Q plots for each dimension.

d) Compute a chi-squared plot for all dimensions considered together.

e) Use the above plots to look for outliers.

f) Based on all the above information, indicate whether you would find acceptable to test this dataset using statistical techniques that assume the data following an (approximately) multi-variate normal distribution.

PROBLEM 17.- File `SM22_Seminar_2_Plants.xlsx` contains data samples from 2 types of plants. Each row of the file contains one sample, for which there are 5 columns, as follows: i) type of plant (1 or 2); ii) sepal length; iii) sepal width; iv) petal length; v) petal width. We want to test if the two types of plants have different mean in terms of these 4 variables that have been measured, using Hotelling's T^2.

a) State the null and alternative hypotheses to be tested.

b) Inspect the data and indicate whether it fulfills the assumptions necessary to use Hotelling's T^2 (independence of the random samples can be assumed).

  b1. Test the assumptions of multivariate normality.

  b2. Inspect (roughly) the assumption of equality of covariances.

c) Based on your conclusions in (b), determine if any of the T^2 statistics discussed in class could be employed to test the hypotheses in (a).[2]

d) Test the hypothesis indicated in (a) at $\alpha = 0.05$ significance level and indicate what is the conclusion.

[2] If none is suitable, try transforming one or more variables. If you cannot find a satisfactory transformation, you can remove the dimension(s) that prevent your data from meeting the assumptions required to apply Hotelling's T^2.

3 Answers to Selected Problems

PROBLEM 2.-

a)

$$\overline{\mathbf{x}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 8 & 1.5 \\ 1.5 & 5.2 \end{bmatrix}$$

b) 3.1715; 0.2592; 3.0801; 0.5794; 3.3545; 1.6213; 1.0978; 0.8132; 1.0978; 1.6213; 3.3037

c) See Figure 2

d) Most likely not. It is also interesting to confirm this by a scatter plot or an estimate of the bi-variate density (Figure 3).

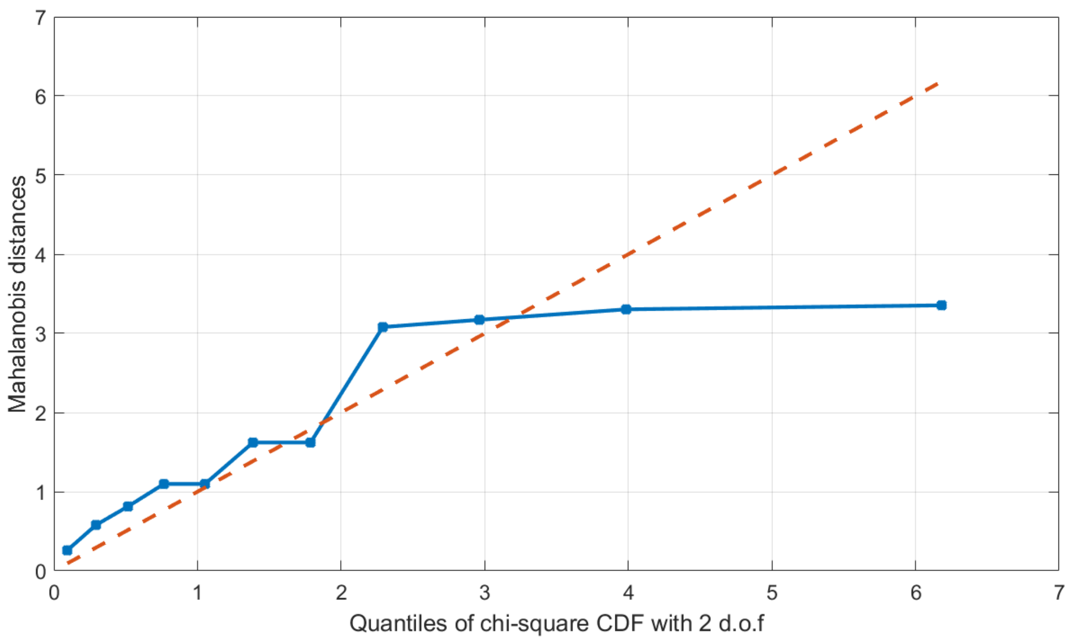


Figure 2: Chi-square plot for problem 2

PROBLEM 3.-

a) See Figure 4

b) The plot is not informative about the normality of dimension 1.

c) There is some deviation from the 45-degree line, although given the small sample size it is hard to conclude how important it is. Thus, the plot is rather inconclusive in this case. Any strong statement in favor or against normality would be wrong.

PROBLEM 6.- (a) $\hat{\theta}_{T^2} = 14.41 > T^2_{critic} = 6.24$. Reject the null hypothesis (means differ significantly); (b) See Figure 5

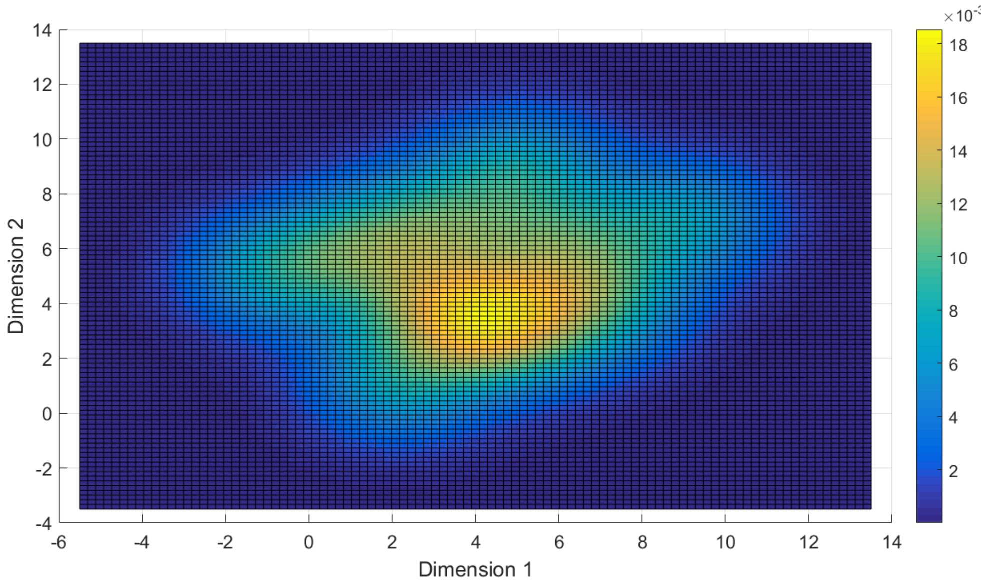


Figure 3: Kernel density estimate for problem 2 using Matlab's function 'ksdensity'

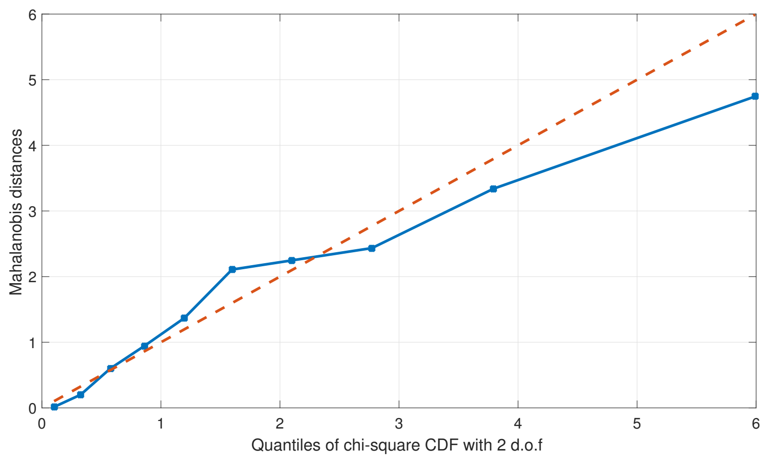


Figure 4: Chi-square plot for problem 3

PROBLEM 7.-

a) H_0: $\mu_A = \mu_B$, $\quad H_{ALT}$: $\mu_A \neq \mu_B$
where μ_A is the population mean of the bivariate (science, communication) scores of University A, and μ_B the same for University B.

b) Hotelling's T^2; Assumptions: multivariate normality of each population; both populations independent from each other and with equal co-variances.

c) $\hat{\theta}_{T^2} = 16.25 > T^2_{critic} = 10.828 \Rightarrow$ Reject H_0.

d) The average performance of students from Universities A and B are different at the specified significance level.

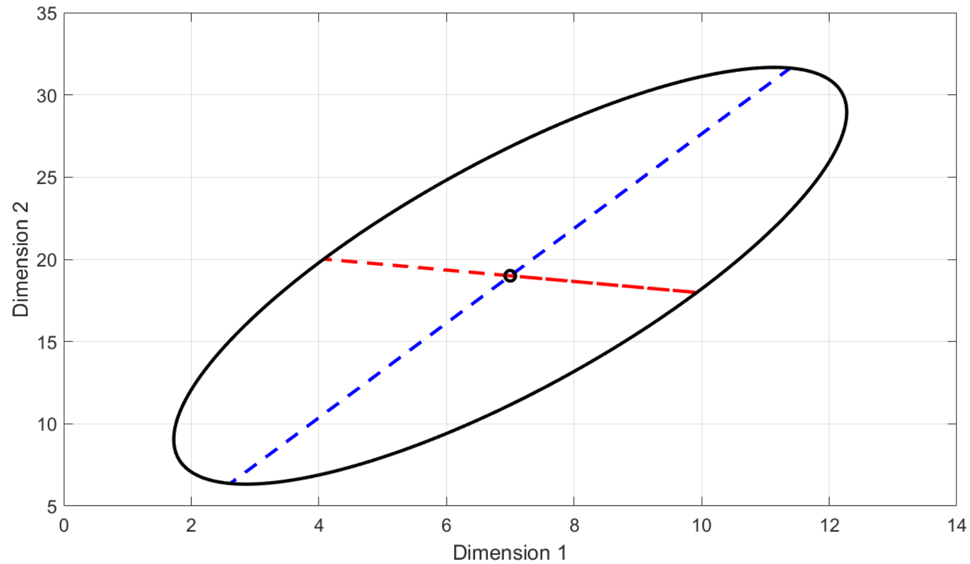


Figure 5: Confidence region for Problem 6.

PROBLEM 9.-

a) H_0 : The population means are equal, $\mu_1 - \mu_2 = \mathbf{0}$. H_A : The population means are different, $\mu_1 - \mu_2 \neq \mathbf{0}$.

b) $\hat{\theta}_{T^2} = 38$

c) $T^2_{critic} = 27 \Rightarrow$ we reject H_0 at the specified significance level and conclude that the means of the populations from which we obtained the random samples are different at the specified significance level.

PROBLEM 10.-

a) $\hat{\theta}_{T^2} = 23.5$; $F_{equival} = 6.09 > F_{critic} = 4.347 \rightarrow H_0$ is rejected.

b) $(-39.9989;\ 17.1989)$, $(-41.2554;\ 13.4554)$, and $(-11.8438;\ 44.2438)$

c) μ_1 : $(-4.5627;\ 50.5627)$, $(-32.8506;\ 20.8506)$, and $(15.6255;\ 59.3745)$
μ_2 : $(18.4454;\ 50.3546)$, $(-9.6796;\ 25.4796)$, and $(-0.5216;\ 43.1216)$

d) μ_1 : $(3.2548;\ 42.7452)$, $(-25.2351;\ 13.2351)$, and $(21.8297;\ 53.1703)$
μ_2 : $(22.9705;\ 45.8295)$, $(-4.6936;\ 20.4936)$, and $(5.6675;\ 36.9325)$

Probability	Inverse Normal	Inverse Chi-square 2 d.o.f.	3 d.o.f.	4 d.o.f	Probability	Inverse Normal	Inverse Chi-square 2 d.o.f.	3 d.o.f.	4 d.o.f
0.01	−2.3263	0.0201	0.1148	0.2971	0.50	0.0000	1.3863	2.3660	3.3567
0.02	−2.0537	0.0404	0.1848	0.4294	0.51	0.0251	1.4267	2.4196	3.4209
0.03	−1.8808	0.0609	0.2451	0.5351	0.52	0.0502	1.4679	2.4740	3.4861
0.04	−1.7507	0.0816	0.3002	0.6271	0.53	0.0753	1.5100	2.5294	3.5521
0.05	−1.6449	0.1026	0.3518	0.7107	0.54	0.1004	1.5531	2.5857	3.6191
0.06	−1.5548	0.1238	0.4012	0.7884	0.55	0.1257	1.5970	2.6430	3.6871
0.07	−1.4758	0.1451	0.4487	0.8616	0.56	0.1510	1.6420	2.7013	3.7562
0.08	−1.4051	0.1668	0.4949	0.9315	0.57	0.1764	1.6879	2.7608	3.8265
0.09	−1.3408	0.1886	0.5401	0.9987	0.58	0.2019	1.7350	2.8213	3.8979
0.10	−1.2816	0.2107	0.5844	1.0636	0.59	0.2275	1.7832	2.8831	3.9706
0.11	−1.2265	0.2331	0.6280	1.1268	0.60	0.2533	1.8326	2.9462	4.0446
0.12	−1.1750	0.2557	0.6710	1.1884	0.61	0.2793	1.8832	3.0106	4.1201
0.13	−1.1264	0.2785	0.7136	1.2488	0.62	0.3055	1.9352	3.0764	4.1970
0.14	−1.0803	0.3016	0.7558	1.3081	0.63	0.3319	1.9885	3.1437	4.2755
0.15	−1.0364	0.3250	0.7978	1.3665	0.64	0.3585	2.0433	3.2126	4.3557
0.16	−0.9945	0.3487	0.8395	1.4241	0.65	0.3853	2.0996	3.2831	4.4377
0.17	−0.9542	0.3727	0.8810	1.4810	0.66	0.4125	2.1576	3.3554	4.5215
0.18	−0.9154	0.3969	0.9225	1.5374	0.67	0.4399	2.2173	3.4297	4.6074
0.19	−0.8779	0.4214	0.9638	1.5933	0.68	0.4677	2.2789	3.5059	4.6954
0.20	−0.8416	0.4463	1.0052	1.6488	0.69	0.4959	2.3424	3.5842	4.7857
0.21	−0.8064	0.4714	1.0465	1.7039	0.70	0.5244	2.4079	3.6649	4.8784
0.22	−0.7722	0.4969	1.0879	1.7589	0.71	0.5534	2.4757	3.7479	4.9738
0.23	−0.7388	0.5227	1.1293	1.8136	0.72	0.5828	2.5459	3.8336	5.0719
0.24	−0.7063	0.5489	1.1709	1.8681	0.73	0.6128	2.6187	3.9221	5.1730
0.25	−0.6745	0.5754	1.2125	1.9226	0.74	0.6433	2.6941	4.0136	5.2774
0.26	−0.6433	0.6022	1.2544	1.9769	0.75	0.6745	2.7726	4.1083	5.3853
0.27	−0.6128	0.6294	1.2963	2.0313	0.76	0.7063	2.8542	4.2066	5.4969
0.28	−0.5828	0.6570	1.3385	2.0857	0.77	0.7388	2.9394	4.3087	5.6127
0.29	−0.5534	0.6850	1.3810	2.1402	0.78	0.7722	3.0283	4.4150	5.7329
0.30	−0.5244	0.7133	1.4237	2.1947	0.79	0.8064	3.1213	4.5258	5.8581
0.31	−0.4959	0.7421	1.4666	2.2494	0.80	0.8416	3.2189	4.6416	5.9886
0.32	−0.4677	0.7713	1.5098	2.3042	0.81	0.8779	3.3215	4.7630	6.1251
0.33	−0.4399	0.8010	1.5534	2.3593	0.82	0.9154	3.4296	4.8904	6.2681
0.34	−0.4125	0.8310	1.5973	2.4145	0.83	0.9542	3.5439	5.0247	6.4185
0.35	−0.3853	0.8616	1.6416	2.4701	0.84	0.9945	3.6652	5.1665	6.5770
0.36	−0.3585	0.8926	1.6862	2.5259	0.85	1.0364	3.7942	5.3170	6.7449
0.37	−0.3319	0.9241	1.7313	2.5821	0.86	1.0803	3.9322	5.4773	6.9233
0.38	−0.3055	0.9561	1.7768	2.6386	0.87	1.1264	4.0804	5.6489	7.1137
0.39	−0.2793	0.9886	1.8227	2.6955	0.88	1.1750	4.2405	5.8335	7.3182
0.40	−0.2533	1.0217	1.8692	2.7528	0.89	1.2265	4.4145	6.0333	7.5390
0.41	−0.2275	1.0553	1.9161	2.8106	0.90	1.2816	4.6052	6.2514	7.7794
0.42	−0.2019	1.0895	1.9636	2.8689	0.91	1.3408	4.8159	6.4915	8.0434
0.43	−0.1764	1.1242	2.0116	2.9277	0.92	1.4051	5.0515	6.7587	8.3365
0.44	−0.1510	1.1596	2.0602	2.9870	0.93	1.4758	5.3185	7.0603	8.6664
0.45	−0.1257	1.1957	2.1095	3.0469	0.94	1.5548	5.6268	7.4069	9.0444
0.46	−0.1004	1.2324	2.1593	3.1075	0.95	1.6449	5.9915	7.8147	9.4877
0.47	−0.0753	1.2698	2.2099	3.1687	0.96	1.7507	6.4378	8.3112	10.0255
0.48	−0.0502	1.3079	2.2612	3.2306	0.97	1.8808	7.0131	8.9473	10.7119
0.49	−0.0251	1.3467	2.3132	3.2933	0.98	2.0537	7.8240	9.8374	11.6678
0.50	0.0000	1.3863	2.3660	3.3567	0.99	2.3263	9.2103	11.3449	13.2767

Table 2: Values of inverse normal and Chi-square distributions with 2, 3 and 4 degrees of freedom (d.o.f.).