$\boxed{\text{GBS}}$

$U^n \qquad\qquad\qquad U^{n+1}$

$t_n \qquad\qquad\qquad t_{n+1}$

$\ell = 1$: $N_1 \longrightarrow U_1, h_1$

$\ell = 8$: $N_2 \longrightarrow U_2, h_2$

ℓ : $N_\ell \longrightarrow U_\ell, h_\ell$

$h = \Delta t / 2N$ ~ damos 2 pasos

$$U(h) = a_0 + a_1 \cdot h^3 + \dots + a_q \cdot h^{2q}$$

$(a_0, \dots, a_q) \rightarrow$ Incógnita $\longrightarrow q+1$ level

$$\begin{pmatrix} 1 & h_1^2 & \dots & h_1^{2q} \\ & & \ddots & \\ 1 & h_{q+1}^2 & \dots & h_{q+1}^{2q} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_q \end{pmatrix} = \begin{pmatrix} U_1 \\ \vdots \\ U_3 \end{pmatrix}$$

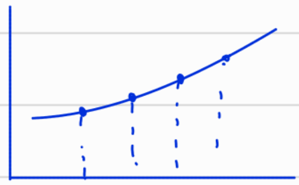
Para resolver utilizamos Lagrange:

$$U(h) = \sum_{j=0}^{q+1} \ell_j(x)\, U_j$$

$\ell \rightarrow$ level

$N \rightarrow$ ley que rige para un ℓ $\rightarrow$ $h = \dfrac{\Delta t}{2N}$ ↪ Esta N se calcula

$h \rightarrow$ intervalo en que divides un level



Aplicar Leap Frog a el problema del oscilador:

$\ddot{x} + x = 0$

$x(0) = 1 \qquad \dot{x}(0) = 0$

Solución $x(t) = \cos(t)$

$\boxed{\text{LEAP FROG}}$

$$U^{n+1} = U^n + 2\Delta t\, F^n$$

→ El problema lo vamos a convertir en 1er orden con ayuda de la variable compleja

$z = x + i\dot{x} \qquad z(0) = x(0) + i\,\dot{x}(0) = 0$

$\dot{z} = \dot{x} + i\ddot{x} = \dot{x} + i(-x) = -i\,\overbrace{(x + i\dot{x})}^{z} \longrightarrow \dot{z} = -i \cdot z, \quad z(0) = 1$

↳ $\ddot{x} + x = 0$

$\ddot{x} = -x$

Si $z(t) = e^{-it}$

ERROR LOCAL DE TRUNCACIÓN DE LEAP FROG

$$T^{n+1} = U(t_{n+1}) - U(t_{n-1}) - 2\Delta t \cdot F(U(t_n), t_n)$$

$$= \cancel{U(t_n)} + \cancel{\dot{U}(t_n) \cdot \Delta t} + \cancel{\ddot{U}(t_n) \cdot \frac{\Delta t^2}{2}} + \dddot{U}(t_n) \cdot \frac{\Delta t^3}{3!} + \cancel{U^{iv}(t_n) \cdot \frac{\Delta t^4}{4!}} + U^{v}(t_n) \cdot \frac{\Delta t^5}{5!} + \dots$$

$$-\cancel{\vec{v}}(t_n) + \dot{\vec{v}}(t_n)\cdot\Delta t - \ddot{\vec{v}}(\Delta t)\cdot\cancel{\frac{\Delta t^2}{2!}} + \dddot{\vec{v}}(\Delta t)\cdot\frac{\Delta t^3}{3!} - \overset{iv}{\vec{v}}\cdot\cancel{\frac{\Delta t^{IV}}{4!}} + \overset{v}{\vec{v}}\cdot\frac{\Delta t^5}{5!} + \dots$$

$$-2\Delta t\cdot F(\cancel{\vec{v}}(t_n),t_n) = \boxed{2\dddot{\vec{v}}(t_n)\cdot\frac{\Delta t^3}{3!} + 2\overset{v}{\vec{v}}\cdot\frac{\Delta t^5}{5!}}$$

$$\frac{dv}{dt} = f(v,t)$$

Volviendo al problema

$z(t) = e^{-it}$

$z_{n+1} = z_{n-1} + 2\Delta t\, F_n = z_{n-1} - 2i\Delta t\cdot z_n$

$z_0 = 1$

1^{er} paso Aplicar Euler

$z_1 = z_0 + \Delta t\cdot F_0 = 1 - \underbrace{i}_{}\cdot\Delta t$ $\quad \dot{z} = -ie^{-it} \Rightarrow z_0 = -ie^{0t} = -i$

Por tanto, la ecuación a resolver es:

$$\begin{array}{l|l}
z_{n+1} = z_{n-1} - 2i\cdot\Delta t\cdot z_n & z_n = C\cdot x^n \\
z_0 = 1 & \cancel{c}\cdot r^{n+1} = \cancel{c}\cdot r^{n-1} - 2i\cdot\Delta t\cdot\cancel{c}\cdot r^n \\
z_1 = 1 - i\Delta t & r^{n+1} = r^{n-1} - 2i\Delta t\cdot r^n \rightarrow \frac{r^{n+1}}{r^{n-1}} = 1 - \frac{2i\Delta t\, r^n}{r^{n-1}}
\end{array}$$

$r^2 + 2i\Delta t r - 1 = 0 \leftarrow r^2 = 1 - 2i\Delta t\, r$

$r_1 = -i\Delta t + \sqrt{-\Delta t^2 + 1}$

$r_2 = -i\Delta t - \sqrt{-\Delta t^2 + 1}$

Si $\boxed{\Delta t < 1}$

- $|x_1| = \sqrt{\underbrace{1-\Delta t^2}_{Re^2} + \underbrace{\Delta t^2}_{Im^2}} = 1$
- $|r_2| = \sqrt{1-\Delta t^2 + \Delta t^2} = 1$

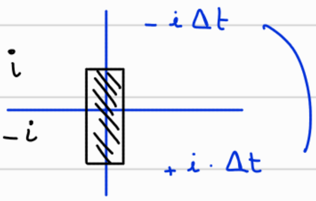
Si $\boxed{\Delta t > 1}$

- $r_1 = (-\Delta t + \sqrt{|\Delta t^2 - 1|})\cdot i$
- $r_2 = (-\Delta t - \sqrt{\Delta t^2 - 1})\cdot i$

Si $\lambda = \pm i$

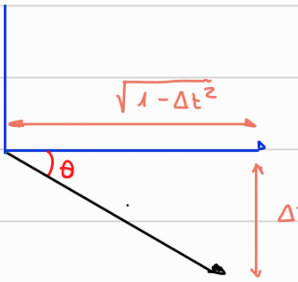
$\lambda_1\cdot\Delta t = +i\Delta t$

$\lambda_2\cdot\Delta t = -i\Delta t$



Sale de la región de estabilidad si $\Delta t > 1$.

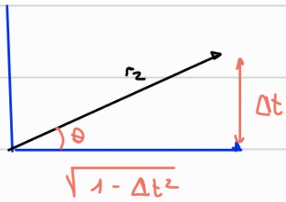
$r_1 = e^{-i\theta}$



$$\operatorname{tg}\theta = \frac{\Delta t}{\sqrt{1-\Delta t^2}}$$

$r_2 = -(\sqrt{1-\Delta t^2} + i\,\Delta t$

$r_2 = -e^{i\theta}$
↳ El conjugado



$$\longrightarrow z_n = C_1 \cdot r_1^n + C_2 \cdot r_2^n = C_1 \cdot e^{-i\cdot n\cdot\theta} + C_2\,(-e^{i\theta})^n =$$

$$= C_1\, e^{-in\theta} + \boxed{C_2\,(-1)^n \cdot e^{i\cdot n\theta}}$$

Solución espúrea $(-1)^n \rightarrow$ ∿∿ → No tiene sentido físico.

$z_0 = 1 = C_1 \cdot e^0 + C_2\,(-1)^0 \cdot e^0$

$z_1 = 1 - i\,\Delta t = C_1 \cdot e^{-i\theta} + C_2\,(-1)^1 \cdot e^{i\theta}$

$= C_1 e^{-i\theta} - C_2 e^{i\theta}$

$$\begin{pmatrix} 1 & 1 \\ e^{-i\theta} & e^{i\theta} \end{pmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{pmatrix} 1 \\ 1-\Delta t\cdot i \end{pmatrix} \longrightarrow \text{Se obtiene } C_1 \text{ y } C_2$$

$$z_n = \underbrace{\left(1 - \frac{\Delta t^2}{4}\right)}_{\text{error cometido}} \underbrace{e^{-in\theta}}_{z(t)=e^{it}} - \underbrace{\frac{\Delta t^2}{4}}_{\text{Error}} \underbrace{(-1)^n}_{\text{oscilante}} e^{in\theta} + O(\Delta t^4)$$

$\theta \simeq \Delta t$ | tiende a 0

$z(t_n) = e^{-i\cdot n\,\Delta t}$

Para que hubiera solución en ≠ levels, debe ser de orden menor que la solución válida.

Para hacer el error más pequeño → hacemos un filtrado (z_n^*)

$$z_n^* = \frac{1}{4}\,(z_{n+1} + 2z_n + z_{n-1}) = \frac{1}{2}\,(1+\cos\theta)\cdot\left(1 - \frac{\Delta t^2}{2}\right) e^{-in\theta} - \underbrace{\frac{\Delta t^2}{16}\cdot(-1)^n e^{in\theta}\cdot \overbrace{\operatorname{sen}^2\left(\frac{\theta}{2}\right)}^{\Delta t^2/2}}_{\frac{\Delta t^2}{8}}$$

$\frac{\Delta t^2}{8}$ → más pequeña que la solución de la izquierda.