II Periodic Homogenization

Take $\Omega \subset \mathbb{R}^d$ a bounded open set and for $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ consider the functionals

$$\mathcal{E}_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, Du\right)dx$$

where $f : \mathbb{R}^d \times \mathbb{R}^{m\times d} \to \mathbb{R}_+$ is a function which is

- periodic in the first variable $f(y+k, A) = f(y, A)$ $\forall y \in \mathbb{R}^d$ $\forall A \in \mathbb{R}^{m\times d}$ and $\forall k \in \mathbb{Z}$
- satisfies a growth condition
$$\alpha(|A|^p - 1) \le f(y, A) \le \beta(|A|^p + 1)$$
$\forall y, A$ and $p > 1$ is given.

Thm The Γ-limit $\mathcal{E}$ in $L^p(\Omega;\mathbb{R}^m)$ is given for $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$\mathcal{E}(u) = \int f_{hom}(Du)\,dx$$

where

$$f_{hom}(A) = \lim_{T\to+\infty} \frac{1}{T^d} \inf\left\{ \int_{(0,T)^d} f(x, A + D\phi)\,dx \;\middle|\; \phi \in W_0^{1,p}((0,T)^d;\mathbb{R}^m) \right\}$$

Rmk 1 One must show that the limit exists (see Braides)

Lemma f_{hom} is continuous.

Lemma f_{hom} is cvx along the directions $A + \mathbb{R}(a \otimes e_i)$ where (e_i) is the canonical basis of $\mathbb{R}^d$

Corollary $|f_{hom}(A) - f_{hom}(B)| \leq C(|A|^{p-1} + |B|^{p-1})|A-B|$

Proof of Γ-limit

Step 1 the (UP)

Given $u \in W^{1,p}$ $\exists$ a sequence of piecewise affine functions u_n which converges to u in $W^{1,p}$ (P_1 finite element appx) [Assume Ω is Lip]

thanks to the above corollary one can show that

$$\lim \int f_{hom}(Du_j) = \int f_{hom}(Du)$$

We then show that when u is piece-wise affine then $\exists$ $u_\varepsilon \to u$ such that $\limsup \mathcal{E}_\varepsilon(u) \leq \mathcal{E}(u)$ and then a diagonal trick allows to conclude

to build u_ε one proceeds independently in each subdomains when u is affine. Hence one considers an open domain $O \subset \mathbb{R}^d$ and $u = Ax$ on O

Then given $\delta > 0$ one lets

$$O^\delta = \bigcup_{z \in \delta \mathbb{Z}^d,\ z + (0,\delta)^d \subset O} z + (0,\delta)^d$$

and one observes that $|O \setminus O^\delta| \to 0$ as $\delta \downarrow 0$.

Given $\underset{\overset{\vee}{e}}{\varepsilon}$ small $(\varepsilon \ll \delta)$ we let $T = \lfloor \frac{\delta}{\varepsilon} \rfloor - 1$

If $\varepsilon > 0$ is small enough we can find $\phi_\varepsilon \in W_0^{1,p}$ such that

$$\frac{1}{T^d} \int f(y, A + D\phi_\varepsilon(y)) \leq f_{hom}(A) + \varepsilon$$

We then let

$$u_\varepsilon(x) := \begin{cases} Ax & \text{if } x \in O \setminus O^\delta \\ Ax + \varepsilon \phi_\varepsilon\left(\frac{x}{\varepsilon} - \lfloor \frac{z}{\varepsilon} \rfloor\right) & \text{if } x \in z + (0,\delta)^d \subset \mathbb{R}, \; z \in \delta \mathbb{Z}^d \end{cases}$$

In each cube $z + (0,\delta)^d$, using the periodicity of f,

$$\int_{z+(0,\delta)^d} f\left(\frac{x}{\varepsilon}, Du_\varepsilon(x)\right) = \int_{z+(0,\delta)^d} f\left(\frac{x}{\varepsilon} - \lfloor \frac{z}{\varepsilon} \rfloor, A + D\phi_\varepsilon\left(\frac{x}{\varepsilon} - \lfloor \frac{z}{\varepsilon} \rfloor\right)\right) dx$$

$$= \varepsilon^d \int_{(0,\frac{\delta}{\varepsilon})^d} f(y + \theta_\varepsilon, A + D\phi_\varepsilon(y + \theta_\varepsilon)) dy$$

where we have done the change of variables

$$y = \frac{(x-z)}{\varepsilon} \qquad \theta_\varepsilon = \frac{z}{\varepsilon} - \lfloor \frac{z}{\varepsilon} \rfloor \in [0,1]$$

By the choice of ϕ_ε and using that
$\theta_\varepsilon + (0,T)^d \subset (0, \frac{\delta}{\varepsilon})^d$

$$\int_{(0,\frac{\delta}{\varepsilon})^d} f(y+\theta_\varepsilon, A + D\phi_\varepsilon(y+\theta_\varepsilon)) \le$$

$$\le C \frac{\delta^{d-1}}{\varepsilon^{d-1}} |A|^p + T^d (f_{hom}(A + \varepsilon))$$

Hence

$$\int_{z+(0,\delta)^d} f(\tfrac{x}{\varepsilon}, Du_\varepsilon(x))\, dx \le$$

$$\le C \delta^{d-1} \varepsilon |A|^p + \delta^d \left(\frac{T}{\delta/\varepsilon}\right)^d (f_{hom}(A) + \varepsilon)$$

and using $|O^\delta| = \delta^d \# \{ z \in \mathbb{Z}^d \mid z + (0,\delta) \subset O \}$

$$\int_O f(\tfrac{x}{\varepsilon}, Du_\varepsilon) \le C |O \setminus O^\delta| |A|^p +$$

$$+ C |O^\delta| \frac{\varepsilon}{\delta} |A|^p$$

$$+ |O^\delta| \left(\frac{T}{\delta/\varepsilon}\right)^d (f_{hom}(A) + \varepsilon)$$

So that

$$\limsup_{\varepsilon \to 0} \int_O f(\tfrac{x}{\varepsilon}, Du_\varepsilon) \le |O^\delta| f_{hom}(A) + C |O \setminus O^\delta| |A|^p$$

$\delta \to 0$

$\to |O| f_{hom}(A)$

step 2 (LB)

let's start with an easier case where we consider functions u_ε s.t. $u_\varepsilon \to u = Ax$ in $L^p(B;\mathbb{R}^m)$ where B is a ball centered at 0

and we wish to show that

$$|B| f_{hom}(A) \leq \liminf_{\varepsilon \to 0} \int_B f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) dx$$

the RHS after a change of variable is

$$\varepsilon^d \int_{B/\varepsilon} f(y, Dv_\varepsilon)\, dy$$

$$v_\varepsilon(y) = u_\varepsilon(\varepsilon y)/\varepsilon .$$

If the liminf is finite there is a subsequence s.t. $\int f(x/\varepsilon_k, Du_{\varepsilon_k})$ is finite and goes to the liminf.

In particular Du_{ε_k} is globally bounded in $L^p(B)$ and it converges weakly to A

$$\int A\varphi\,dx = -\int \operatorname{div}(\varphi)\,A x - \lim_h \int \operatorname{div}\varphi\, u_{\varepsilon_h} = \lim_h \int Du_{\varepsilon_h}\varphi$$

We denote u_ε the subsequence.

In a first stage we assume that we work in the unit cube $Q = (0,1)^d$ rather than in the ball and we observe that if $T = 1/\varepsilon$

$$\varepsilon^d \int_{Q/\varepsilon} f(y, Dv_\varepsilon)\,dy = \frac{1}{T^d}\int_{(0,T)^d} f(y, A + (Dv_\varepsilon - A))\,dy$$

$$\gtrapprox f_{hom}(A)$$

the inequality would be true if we had

$v_\varepsilon - Ax = 0$ on $\partial(0,T)^d$.

To make this work we consider $\delta > 0$ and a cut-off $\eta \in C_c^\infty(Q;[0,1])$ with $|\nabla\eta| \le \frac{c}{\delta}$ and $\eta = 1$ on $Q^\delta = (\delta, 1-\delta)^d$

then we consider

$$u_\varepsilon^\delta = Ax + \eta(u_\varepsilon - Ax).$$

If we replace u_ε with u_ε^δ the inequality above becomes true and we can assert that

$$f_{hom}(A) \le \int_Q f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta(x)\right)dx$$

Now we need to understand the error between this and the energy of u_ε.

$$Du_\varepsilon^\delta = A + (u_\varepsilon - Ax) \otimes \nabla\eta + \eta\,(Du_\varepsilon - A)$$

$$\int_Q f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) \le \int_Q f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) \quad \overbrace{Du_\varepsilon + (1-\eta)(Du_\varepsilon - A)}$$

$$+ \boxed{C\,|Q \setminus Q^\delta|\,(1+|A|^p)} + \boxed{\frac{C}{\delta^p}\int_{Q\setminus Q^\delta} |u_\varepsilon - Ax|^p\,dx} + C\int_{Q\setminus Q^\delta} |Du_\varepsilon|^p\,dx$$

the first term can be made arbitrarily small choosing δ small

the second term goes to 0 as $\varepsilon \to 0$ since we have assumed $u_\varepsilon \to Ax$ strongly in L^p

However there is no clear way to bound the last term.

A solution (due to Marcellini) consists

in considering not 1 but N cut offs

$\eta_1, \dots, \eta_N$ for a given integer s.t.

$$\eta_i \in C_c^\infty\left(Q^{\frac{(i-1)\delta}{N}}\right) \quad \eta_i = 1 \text{ on } Q^{\frac{i\delta}{N}}$$

$|\nabla \eta_i| \leq \frac{CN}{\delta}$ We consider for ε the functions $u_\varepsilon^i = Ax + \eta_i (u_\varepsilon - Ax)$ and

$\forall i$ $$Du_\varepsilon^i = Du_\varepsilon + (1-\eta_i)(Du_\varepsilon - A) + (u_\varepsilon - Ax) \otimes \nabla \eta_i$$

$$f_{hom}(A) \leq \frac{1}{\left|Q^{\frac{(i-1)\delta}{N}}\right|} \int_{Q^{\frac{(i-1)\delta}{N}}} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^i\right)$$

$$\leq \frac{1}{\left(1 - 2\frac{(i-1)\delta}{N}\right)^d} \int_{Q^{\frac{(i-1)\delta}{N}}} f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)$$

$$+ \frac{CN^p}{\delta^p} \int_{Q^{\frac{i\delta}{N}} \setminus Q^{\frac{(i-1)\delta}{N}}} |u_\varepsilon - Ax|^p$$

$$+ \int_{Q^{\frac{i\delta}{N}} \setminus Q^{\frac{(i-1)\delta}{N}}} |Du_\varepsilon - A|^p dx$$

Now since Du_ε is bounded in L^p one has for a constant C

$$\sum_i \int_{Q^{\frac{i\delta}{N}} \setminus Q^{\frac{(i-1)\delta}{N}}} |Du_\varepsilon - A|^p \leq \int_Q |Du_\varepsilon - A|^p \leq C$$

so that there exists for each $\varepsilon > 0$ an index $i(\varepsilon)$ for which

$$\int_{Q^{\frac{i\delta}{N}} \setminus Q^{\frac{(i-1)\delta}{N}}} |Du_\varepsilon - A|^p \leq \frac{C}{N}$$

Choosing for each ε this good index i we find that

$$f_{hom}(A) \leq \frac{1}{\left(1 - 2\frac{\delta}{N}\right)^d} \int_Q f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + \frac{CN^p}{\delta^p} \int_Q |u_\varepsilon - Ax|^p$$

$$+ \frac{C}{N}$$

so that using $u_\varepsilon \to Ax$

$$f_{hom}(A) \leq \frac{1}{\left(1 - 2\frac{\delta}{N}\right)^d} \liminf_{\varepsilon \to 0} \int_Q f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + \frac{C}{N}$$

It remains to send first $N \to +\infty$ and then $\delta \to 0$ to conclude.

FROM Q TO B

We cover B with smaller and smaller disjoint cubes and use the previous result

if $(Q_i)_{i=1}^k$ are disjoint cubes in B one has

$$|B| f_{hom}(A) = |B \setminus \bigcup^k Q_i| f_{hom}(A) + $$
$$+ \sum |Q_i| f_{hom}(A)$$
$$\leq |B \setminus \bigcup^k Q_i| f_{hom}(A) + \sum^k \liminf \int_{Q_i} f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)$$
$$\leq |B \setminus \bigcup^k Q_i| f_{hom}(A) + \liminf \int_B f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)$$

Taking more and more cubes the first term ~~on~~ can be made arbitrarily small!

Localization

Now we want to consider an arbitrary $u_\varepsilon \to u \in L^p$ and show that

$$\int_\Omega f_{hom}(Du(x))\,dx \leq \liminf \int_\Omega f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)$$

We assume that the RHS is not $+\infty$

and we consider a subsequence s.t.

$$\lim_{k\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_k}, Du_{\varepsilon_k}\right) = \liminf_{\varepsilon\to 0} \int_\Omega f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)$$

The idea is to "localize" the problem

near each point, use $Du_\varepsilon(x) \simeq Du(x)$

near each point and then use the

previous result.

The classical method is due to Fonseca

and Müller.

First $\mu_k = f\left(\frac{x}{\varepsilon_k}, Du_{\varepsilon_k}(x)\right)dx$ is a

sequence of bounded measure which

are uniformly bounded such that

we may assume $\mu_k \overset{*}{\rightharpoonup} \mu$ where μ

is a nonnegative measure

$$\lim_{k\to+\infty} \int_\Omega \varphi \, d\mu_k = \int_\Omega \varphi \, d\mu \qquad \varphi \in C_0^0(\Omega)$$

We know that

$$\mu(\Omega) \le \liminf \mu_k(\Omega)$$

so that (LB) inequality holds if we can show $\mu \geq f_{hom}(Du(x))\,dx$

By Radon - Nikodym's thm we know

$$\mu = g(x)\,dx + \nu \quad \text{with} \quad \nu \perp dx \quad \nu \geq 0$$

and $$g(x) = \lim_{\rho \to 0} \frac{\mu(B_\rho(x))}{|B_\rho(x)|} \quad \mathcal{L}\text{-a.e.}$$

Hence we need to show that

$$g(x) \geq f_{hom}(x, Du(x))$$

Lemma Given $x \in \Omega$ for a.e. $\rho > 0$ one has

$$\lim_{k \to +\infty} \mu_k(B_\rho(x)) = \mu(B_\rho(x))$$

Now, for each $n \geq 1$ one can find

$$\rho_n \leq \rho_{n-1} \quad \text{s.t.}$$

$$\left| \frac{\mu(B_{\rho_n}(x))}{|B_{\rho_n}(x)|} - g(x) \right| \leq \frac{1}{n}$$

and then one can find $\varepsilon_{k_n} \leq \varepsilon_{k_{n-1}}$ s.t.

$$\left| \frac{\mu_{k_n}(B_{\rho_n}(x))}{|B_{\rho_n}|} - \frac{\mu(B_{\rho_n}(x))}{|B_{\rho_n}|} \right| \leq \frac{1}{n}$$

we then denote ε_{k_n} with ε_n and

$\varepsilon'_n = \varepsilon_n / \rho_n$

Since ε_n is chosen "after ρ_n" (for a fixed given value of ρ_n) we should have (and we can assume that) $\varepsilon'_n \to 0$ as $n \to 0$

Then the change of variable $z = x + \rho_n y$ yields

$$\frac{\mu_{k_n}(B_{\rho_n}(x))}{|B_{\rho_n}|} = \frac{1}{|B_{\rho_n}|} \int_{B_{\rho_n}(x)} f\left(\frac{z}{\varepsilon_n}, Du_\varepsilon(z)\right) dz$$

$$= \frac{1}{|B_1|} \int_{B_1} f\left(\frac{x}{\varepsilon_n} + \frac{y}{\varepsilon'_n}, Du_\varepsilon(x + \rho_n y)\right) dy$$

$$= \frac{1}{|B_1|} \int_{B_1} f\left(\theta_n + \frac{y}{\varepsilon'_n}, Du_\varepsilon(x + \rho_n y)\right) dy$$

$$= \frac{1}{|B_1|} \int_{B_1} f\left(\frac{y + \varepsilon'_n \theta_n}{\varepsilon'_n}, D\tilde{u}_\varepsilon(y)\right) dy$$

where $\theta_n = \frac{x}{\varepsilon_n} - \left\lfloor \frac{x}{\varepsilon_n} \right\rfloor \in [0,1]^d$

and $\tilde{u}_n(y) = \dfrac{u_\varepsilon(x + \rho_n y) - u_\varepsilon(x)}{\rho_n}$

Assume we can show $\hat{u}_n \to Du(x)\cdot y$ in L^p $n \to +\infty$. Then back to the previous case of a sequence of fcts converging to an affine fct in a ball (modulo the varying translations $\varepsilon_n' \theta_n$) it follows that

$$f_{hom}(Du(x)) \leq \liminf_n \frac{1}{|B_1|}\int_{B_1} f\left(\frac{y+\varepsilon_n'\theta_n}{\varepsilon_n'}, D\tilde{u}_n(y)\right)dy$$

Putting everything together we will deduce

$$f_{hom}(Du(x)) \leq g(x)$$

It remains to show that $\hat{u}_n \to Du(x)\cdot y$ (it will true for almost all x)

First observe that for any $\varepsilon, \rho > 0$

$$\left\| \frac{u_\varepsilon(x+\rho\,\cdot) - u_\varepsilon(x)}{\rho} - \frac{u(x+\rho\,\cdot) - u(x)}{\rho} \right\|_{L^p(B_1)}$$

$$\leq \frac{1}{\rho}\left(\int_{B_1} |u_\varepsilon(x+\rho y) - u(x+\rho y)|^p dy\right)^{1/p}$$

$$+ \frac{1}{\rho}|B_1|^{\frac{1}{p}}|u_\varepsilon(x) - u(x)|$$

Since we could have assumed $u_\varepsilon(x) \to u(x)$ a.e. (up to a subsequence), the RHS goes to 0 as $\varepsilon \to 0$: therefore when we have chosen ε_{k_n} for a fixed ρ_n we could have required in addition that

$$\left\| \tilde{u}_n - \frac{u(x+\rho\,\cdot) - u(x)}{\rho_n} \right\|_{L^p(B_1)} \leq \frac{1}{n}$$

It remains to see that we could also have chosen ρ_n so that

$$\left\| \frac{u(x+\rho_n y) - u(x)}{\rho_n} - Du(x)\cdot y \right\|_{L^p_y(B_1)} \leq \frac{1}{n}$$

For fixed $\rho > 0$ (and assuming first $u \in C^1$) one has

$$\int_{B_1} \left| \frac{u(x+\rho y) - u(x)}{\rho} - Du(x)\cdot y \right|^p dy$$

$$= \int_{B_1} \left| \frac{1}{\rho} \int_0^\rho \left((Du(x+sy) - Du(x))\cdot y \right) ds \right|^p dy$$

$$\leq \int_{B_1} \frac{1}{\rho} \int_0^\rho \left| \left(Du(x+sy) - Du(x) \right) \cdot y \right|^p ds\, dy \qquad y = \frac{z-x}{s}$$

$$\leq \frac{1}{\rho} \int_0^\rho \left[\frac{1}{s^d} \int_{B_s(x)} | Du(z) - Du(x) |^p dz \right] ds$$

this quantity goes a.e. to 0 as $\rho \to 0$ since we have the following thm.

Eventually the (LB) is now proved since we have all the elements to see that $g(x) \geq f_{hom}(Du(x))$ a.e.

thm let $h \in L^p(\Omega)$. Then for a.e. $x \in \Omega$

$$\lim_{\delta \to 0} \frac{1}{|B_\delta|} \int_{B_\delta(x)} |h(z) - h(x)|^p dz = 0$$

- Convex Case

We assume that $f(y, \cdot)$ is convex. In this case, f_{hom} can be represented by a "cell formula" (see [3]). Let us define

$$\hat{f}_{hom}(A) = \inf \left\{ \int_{(0,1)^d} f(y, A + Du(y))dy \, : \, u \in W^{1,p}_\sharp((0,1)^d; \mathbb{R}^m) \right\}$$

where $W^{1,p}_\sharp((0,1)^d)$ is the set of periodic functions in $W^{1,p}$ ($W^{1,p}(\mathbb{R}^d/\mathbb{Z}^d)$). Observe that in the convex case, the infimum defining $\hat{f}_{hom}$ is in fact a min and we can denote u_A a solution. The function u_A is called a *corrector*.

- Ex $\quad m = d = 1 \quad f(y,p) = a(y)|p|^2$

$a : \mathbb{R} \to \mathbb{R}$ mes. 1-periodic s.t.

$$0 < \lambda \leq a(y) \leq \Lambda < +\infty \quad \forall y$$

Let u_k a solution to

$$\min_{H^1_0(0,k)} \int_0^k a(y)|p + v'(y)|^2 dy$$

EL $\Rightarrow$ $[a(p + u_k')]' = 0$. In particular

$\exists\, c \in \mathbb{R}$ s.t. $\quad a(p + u_k') = c$ a.e. in $(0,k)$

Since $a \neq 0$ and $u_k(0) = 0$

$$u_k(x) = \int_0^x \left(\frac{c}{a(y)} - p \right) dy$$

we use now $u_k(k) = 0$

$$\Rightarrow \quad c = p \left(\int_0^1 \frac{dy}{a(y)} \right)^{-1}$$

$$f_{hom}(p) = \lim_{k \to +\infty} \frac{1}{k} \int_0^k a(y) [p + u_k'(y)]^2 dy$$

$$= \lim_{k \to +\infty} \frac{c}{k} \int_0^k [p + u_k'(y)] dy$$

$$= cp = p^2 \left(\int_0^1 \frac{dy}{a(y)} \right)^{-1}$$

- <u>Comments</u> Γ-cv is different from pointwise cv

$$f_n(x) = \begin{cases} 0 & |x| \geq \frac{1}{n} \\ 1 - |2nx + 1| & -\frac{1}{n} \leq x \leq 0 \\ |2nx - 1| - 1 & 0 \leq x \leq \frac{1}{n} \end{cases}$$

$f_n \to 0$ pointwise

$$f_n \xrightarrow{\Gamma} \begin{cases} 0 & x \neq 0 \\ -1 & x = 0 \end{cases}$$