Structure and Interleavings of Relative Interlevel Set Cohomology

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Abstract

The *relative interlevel set cohomology (RISC)* is an invariant of real-valued continuous functions closely related to the *Mayer–Vietoris pyramid* introduced by Carlsson, de Silva, and Morozov. We provide a structure theorem, which applies to the RISC if it is pointwise finite dimensional (pfd) or, equivalently, *q-tame*. Moreover, we provide the notion of an *interleaving* for RISC and we show that it is stable in the sense that any space with two functions that are δ-close induces a δ-interleaving of the corresponding relative interlevel set cohomologies.

1   Introduction

In the present work we study an invariant of real-valued continuous functions closely related to and mostly inspired by the *Mayer–Vietoris pyramid* introduced by [CdM09]. We name this invariant *relative interlevel set cohomology (RISC)*. Roughly speaking, the Mayer–Vietoris pyramid is a graded square shaped diagram and the RISC arises from gluing consecutive layers of the Mayer–Vietoris pyramid in a functorial way to form one large diagram. This procedure has already been suggested by [CdM09] and moreover, the results by [BEMP13] underpin that such a construction with all squares *Mayer–Vietoris diamonds* should be possible. More specifically, [BEMP13, Figures 4 and 6] show that the supports of the indecomposables of each layer align exactly. This raises the question whether a decomposition given by their structure theorem [BEMP13, Theorem 1] is compatible with the connecting homomorphisms gluing consecutive layers as suggested by [CdM09]. As it turns out, even though this cannot be said about an arbitrary choice of a decomposition as in [BEMP13, Theorem 1], we show that such a global decomposition indeed exists. To this end, we provide a structure theorem for the relative interlevel set cohomology itself, which yields the same indecomposables as [BEMP13, Theorem 1] on each of the corresponding layers of the pyramid.

The strong tameness assumptions in [BEMP13, Theorem 1] were weakened by [BLO20, Theorem 10.1] to all layers of the pyramid being pointwise finite-dimensional (pfd). In the present work we show with Proposition 3.12 that the relative interlevel set cohomology is *q-tame* [CdSGO16, Section 1.1] iff it is pfd, which also suffices for our structure theorem to hold. This is very reassuring, as it shows there is no loss of generality when passing from the individual layers of the Mayer–Vietoris pyramid as an invariant to the whole relative interlevel set cohomology. Moreover, following [BdSS15, Section 2.5] we provide a

(super)linear family on the indexing poset $\mathbb{M}$ and we obtain a stability theorem in Section 4. The proof of our structure theorem, which is Theorem 3.4, is inspired by [CB15].

The restriction of RISC to the subposet corresponding to the south face of the pyramid yields essentially the same data as Mayer–Vietoris systems introduced by [BGO19]. In this restricted setting the authors also provide a structure theorem as well as a stability result. While this restriction to the south face retains all information on the level of objects, we lose some information on the level of homomorphisms. In particular, there are different RISC interleavings restricting to identical interleavings of Mayer–Vietoris systems, as shown in Example 4.5.

The tameness assumptions from [CdM09, BEMP13] were also weakened in [CdSKM19] by using measure theory. Roughly speaking, the authors bypass the step involving interleavings of generalized persistence modules [BdSS15] and map functions directly to measures, which they compare with the bottleneck distance of persistence diagrams [CSEH07].

We also note that our construction of the relative interlevel set cohomology, which applies to any cohomology theory, has an analogous homological construction, which is dual in the following sense. For a homology theory valued in graded vector spaces sending weak equivalences to isomorphisms, we may consider the corresponding dual cohomology theory. The resulting RISC is pfd iff this is also the case for the corresponding invariant defined in terms of homology. Moreover, as the duality of vector spaces restricts to an equivalence on finite-dimensional vector spaces, any decomposition of this RISC yields a decomposition of the corresponding "homological" invariant. This way, homological decompositions can be obtained by duality and hence are not treated explicitly in this paper.

In Section 2 we introduce the relative interlevel set cohomology (RISC) as an invariant of real-valued continuous functions. Given a continuous function $f \colon X \to \mathbb{R}$, the study of *interlevel set persistent cohomology* concerns the cohomology (with field coefficients) of preimages $f^{-1}(I)$ of open intervals I. This construction can be extended to the relative cohomology of pairs $f^{-1}(I, C)$, where $I \subseteq \mathbb{R}$ is an open interval and $C \subseteq I$ is the complement of a closed interval. We parametrize the set of all such pairs as well as the cohomological degrees by a lattice $\mathbb{M}$. As it turns out, any continuous function $f \colon X \to \mathbb{R}$ induces a contravariant functor from $\mathbb{M}$ to the category of vector spaces, with some of the internal maps induced by inclusions and the other maps being differentials of a corresponding Mayer–Vietoris sequence. The existence of these differentials is one of our motives to consider preimages of open subsets as opposed to closed subsets of $\mathbb{R}$. We refer to this functor as the *relative interlevel set cohomology (RISC)* of f and we show that it satisfies certain exactness properties. We call such functors *cohomological*; this is Definition C.1. Furthermore, we show in Proposition 2.3 that the RISC is *sequentially continuous* (Definition 2.2), if it is pfd. As a byproduct, any cohomology class from the RISC determines a natural transformation from an indecomposable and vice versa. Dually, natural transformations from a corresponding homological construction to a *sequentially cocontinuous* indecomposable of a certain kind are one-to-one with elements of the dual space of a corresponding homology group. However, this dual space is naturally isomorphic to cohomology. Thus, cohomology even appears in the analogous construction of a decomposition of the corresponding homological invariant. This is the reason why we work with cohomology in place of homology. As noted above, this is no limitation in our context.

In Section 3 we show in Theorem 3.4 that any pfd sequentially continuous cohomological functor decomposes into a direct sum of indecomposables of a certain type. Each indecomposable can be characterized by its support, which is a maximal axis-aligned rectangle as shown in Fig. 3.1a. A posteriori, the upper left vertex of this rectangle gives the corre-

sponding vertex in the *extended persistence diagram* as we define it in Definition 3.1. This close relationship between the indecomposables and the extended persistence diagram as well as the fact that there is just one type of indecomposable was a major motivation for us to glue the layers of the Mayer–Vietoris pyramid to a single diagram. We note that at this point, one may also invoke [BLO20, Theorem 2.11] in place of Theorem 3.4 to obtain a decomposition of the RISC. We are convinced that our proof of Theorem 3.4 is relevant nevertheless, as it is comparatively simple and more elementary than the proof of [BLO20, Theorem 2.11]. We also note that one may obtain *interlevel set cohomology* from RISC by restriction to a subposet of $\mathbb{M}$. Thus, under the assumption that the RISC is pfd, the interlevel set cohomology decomposes as well. Similar results have been shown by [CO20, Section 9.3], [BCB20, Section 5], and [BGO19, Theorem 2.19].

In Section 4 we use the framework provided by [BdSS15, Section 2.5] to define the notion of an interleaving for contravariant functors on $\mathbb{M}$; this is Definition 4.3. Moreover, we show a stability result with Theorem 4.4. In order to prove this theorem, we cannot apply the framework by [BdSS15] directly. The reason for this is that the canonical *(super)linear family* on the indexing poset $\mathbb{M}$ does not preserve the "cohomological degree". As a result, the interleaving homomorphisms will map some cohomology classes to a cohomology class of one degree higher. Resolving these subtleties requires us to study the interplay of the elementary geometry of $\mathbb{M}$ and the relative interlevel set cohomology. Furthermore, we provide Example 4.5, which shows that the induced interleavings of RISC capture more information than the corresponding interleavings of extended persistence or Mayer–Vietoris systems. We end this paper with an elementary form of "quantitative homotopy invariance" for RISC; this is Proposition 4.9.

2 The Relative Interlevel Set Cohomology

We start with specifying the indexing poset $\mathbb{M}$ for the relative interlevel set cohomology. To this end, let $\mathbb{R}$ and $\mathbb{R}^\circ$ denote the posets given by the orders $\leq$ and $\geq$ on $\mathbb{R}$, respectively. Then we may form the product poset $\mathbb{R}^\circ \times \mathbb{R}$, which is a lattice and whose underlying set is the Euclidean plane. Let l_0 and l_1 be two lines of slope -1 in $\mathbb{R}^\circ \times \mathbb{R}$ with l_1 sitting above l_0 as shown in Fig. 2.1. Then the indexing poset is the sublattice $\mathbb{M} \subset \mathbb{R}^\circ \times \mathbb{R}$ given by the convex hull of l_0 and l_1. The degree-shift in cohomology will correspond to the central[1] automorphism $T\colon \mathbb{M} \to \mathbb{M}$ with the following defining property (also see Fig. 2.1):

> Let $u \in \mathbb{M}$, h_0 be the horizontal line through u, let g_0 be the vertical line through u, let h_1 be the horizontal line through $T(u)$, and let g_1 be the vertical line through $T(u)$. Then the lines l_0, h_0, and g_1 intersect in a common point, and the same is true for the lines l_1, g_0, and h_1.

We also note that T is a glide reflection along the bisecting line between l_0 and l_1, and the amount of translation is the distance of l_0 and l_1. Moreover, as a space, $\mathbb{M}/\langle T \rangle$ is a Möbius strip; see also [CdM09].

The region of $\mathbb{M}$ indexing the Mayer–Vietoris pyramid in degree 0 yields a fundamental domain D with respect to the action of $\langle T \rangle \cong \mathbb{Z}$ on $\mathbb{M}$, which we specify now. To this end, suppose l_0 and l_1 intersect the x-axis in $-\pi$ and π, respectively. With this we embed

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[1]By a central automorphism we mean an automorphism that commutes with any other lattice automorphism of $\mathbb{M}$.
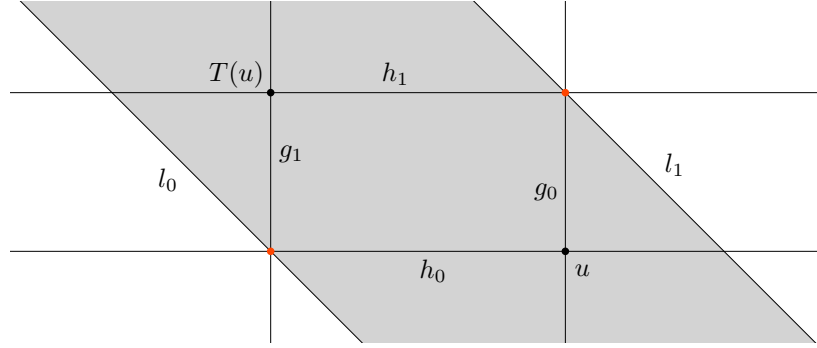
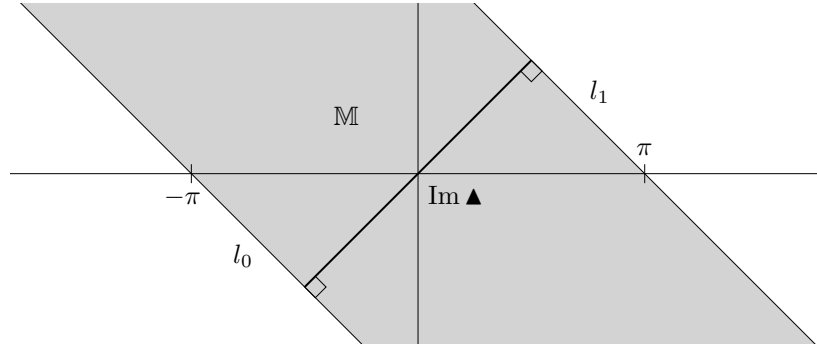Figure 2.1: Incidences defining T. The indexing poset $\mathbb{M}$ is shaded in grey.

Figure 2.2: The strip $\mathbb{M}$ and the image of the embedding $\blacktriangle\colon \overline{\mathbb{R}} \to \mathbb{M}$.

the extended reals $\overline{\mathbb{R}} := [-\infty, \infty]$ into the strip $\mathbb{M}$ by precomposing the diagonal map $\Delta\colon \mathbb{R} \to \mathbb{R}^2, t \mapsto (t,t)$ with the homeomorphism $\arctan\colon \overline{\mathbb{R}} \to [-\pi/2, \pi/2]$, yielding a map

$$\blacktriangle = \Delta \circ \arctan\colon \overline{\mathbb{R}} \to \mathbb{M}, \; t \mapsto (\arctan t, \arctan t)$$

such that $\operatorname{Im}\blacktriangle$ is a perpendicular line segment through the origin joining l_0 and l_1, see Fig. 2.2. We specify the fundamental domain as shown in Fig. 2.3 by

$$D := (\downarrow \operatorname{Im}\blacktriangle) \setminus T^{-1}(\downarrow \operatorname{Im}\blacktriangle),$$

where $\downarrow \operatorname{Im}\blacktriangle$ is the downset of the image of $\blacktriangle$. Fig. 2.4 shows the tessellation of $\mathbb{M}$ induced by T and D.

Now each point in the Mayer–Vietoris pyramid corresponds to a pair of subspaces, which is a preimage of a pair of subspaces of $\mathbb{R}$. To specify such a pair for each point in D the following proposition characterizes a monotone map ρ from $\mathbb{M}$ to the poset of pairs of open subspaces of $\mathbb{R}$, which is locally constant on $\mathbb{M} \setminus D$; a schematic image of ρ is shown in Fig. 2.5.

Proposition 2.1. *Let $\mathcal{P}$ denote the set of pairs of open subspaces of $\mathbb{R}$. Then there is a unique monotone map*

$$\rho = (\rho_1, \rho_0)\colon \mathbb{M} \to \mathcal{P}$$
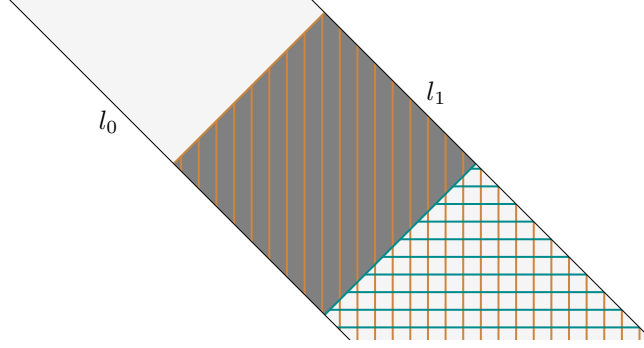
with the following four properties:

Figure 2.3: The fundamental domain $D := \,\downarrow \mathrm{Im}\,\blacktriangle \setminus T(\downarrow \mathrm{Im}\,\blacktriangle)$.
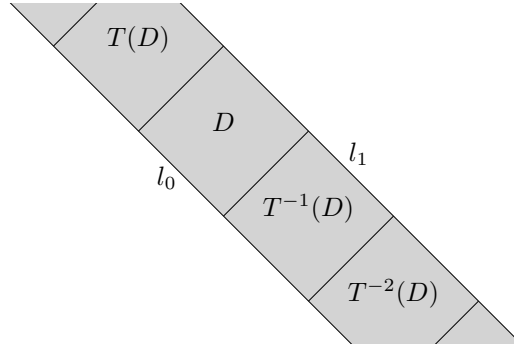

Figure 2.4: The tessellation of $\mathbb{M}$ induced by T and D.

(1) For any $t \in \mathbb{R}$ we have $(\rho \circ \blacktriangle)(t) = (\mathbb{R}, \mathbb{R} \setminus \{t\})$.

(2) For any $u \in \partial \mathbb{M}$ we have $\rho_1(u) = \rho_0(u)$.

(3) For any axis-aligned rectangle contained in $\uparrow D$ the corresponding joins and meets are preserved by ρ_1.

(4) For any axis-aligned rectangle contained in $\downarrow D$ the corresponding joins and meets are preserved by ρ_0.

Moreover, we have the explicit formula

$$\rho(u) = \mathbb{R} \cap \blacktriangle^{-1}\left(\mathrm{int}(\downarrow T(u)), \mathbb{M} \setminus \uparrow u\right),$$

where $\mathrm{int}(\downarrow T(u))$ is the interior of the downset of $T(u)$.

Now let $f \colon X \to \mathbb{R}$ be a continuous function and let $\mathcal{H}^\bullet$ be a cohomology theory with values in the category of graded vector spaces over a fixed field K, which sends weak equivalences to isomorphisms. Then we obtain the contravariant functor

$$F' \colon D \to \left(\mathrm{Vect}_K^{\mathbb{Z}}\right)^\circ, u \mapsto \mathcal{H}^\bullet(f^{-1}(\rho(u))),$$

where $\mathrm{Vect}_K^{\mathbb{Z}}$ is the category of $\mathbb{Z}$-graded vector spaces over K and the circle $\circ$ as an exponent is used to denote the corresponding opposite category. Now the degree-shift yields the
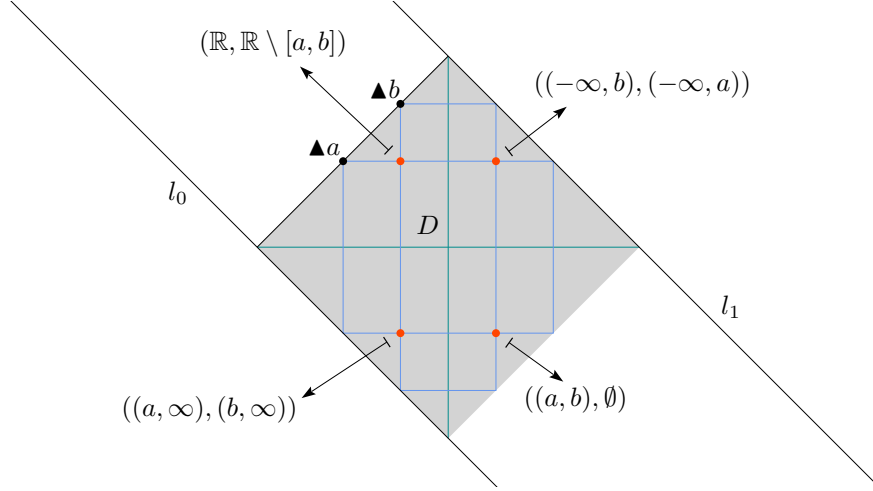
Figure 2.5: A schematic image of ρ.

endofunctor

$$\Sigma \colon \left(\mathrm{Vect}_K^{\mathbb{Z}}\right)^\circ \to \left(\mathrm{Vect}_K^{\mathbb{Z}}\right)^\circ, M^\bullet \mapsto M^{\bullet-1}$$

on graded vector spaces. We intend to replace the contravariant functor F' from D to the category of graded vector spaces with a contravariant functor from $\mathbb{M}$ to mere vector spaces Vect_K carrying the same (and more) information, with precomposition by T taking the place of Σ. As an intermediate step, we extend F' to a functor

$$F \colon \mathbb{M} \to \left(\mathrm{Vect}_K^{\mathbb{Z}}\right)^\circ,$$

which is $\mathbb{Z}$-equivariant or strictly stable in the sense that

$$F \circ T = \Sigma \circ F. \tag{2.1}$$

Now as a map into the objects of $\mathrm{Vect}_K^{\mathbb{Z}}$, such a functor F carries no new information in comparison to F'. Moreover, by (2.1) most of the information carried by F is redundant and we may discard all redundant information by post-composition with the evaluation at 0:

$$\mathrm{ev}^0 \colon \mathrm{Vect}_K^{\mathbb{Z}} \to \mathrm{Vect}_K, \ M^\bullet \mapsto M^0;$$

see also Lemma A.1.

Now in order to obtain such a strictly stable functor F from F' we need to glue consecutive layers using connecting homomorphisms. To this end, let

$$R_D := \{(w, \hat{u}) \in D \times T(D) \mid w \preceq \hat{u} \preceq T(w)\}$$

as in Definition A.3 in Appendix A. As shown in Fig. 2.6, any pair $(w, \hat{u}) \in R_D$ determines an axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ with $T(u) = \hat{u}$. Moreover, as this rectangle is contained in D, the corresponding join $w = v_1 \vee v_2$ and meet $u = v_1 \wedge v_2$ are preserved by ρ by Proposition 2.1.(3-4). Furthermore, since taking preimages is a homomorphism of boolean algebras, f^{-1} also preserves joins and meets, which in this case are the componentwise
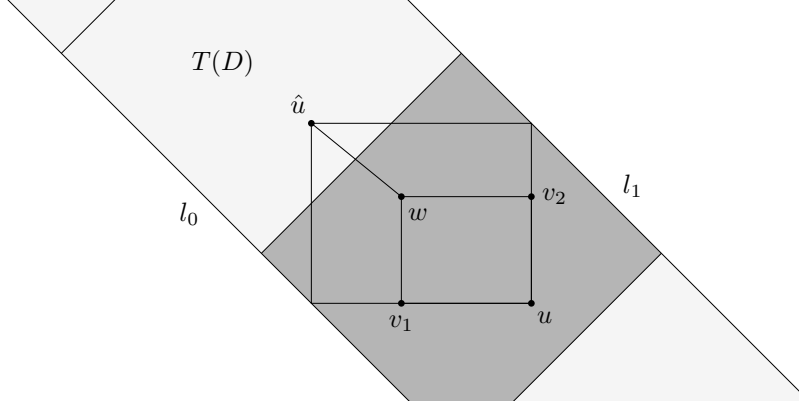
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Figure 2.6: The axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ determined by $(w, \hat{u}) \in R_D$.

unions and intersections. This means that $f^{-1}(\rho(u))$ is the componentwise intersection of $f^{-1}(\rho(v_1))$ and $f^{-1}(\rho(v_2))$, while $f^{-1}(\rho(w))$ is their union. With this we obtain the triad $f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$ of pairs of open subsets of X, which is excisive in each component by excision. Thus, we have the differential

$$\delta'_{(w, \hat{u})} \colon \left(\mathcal{H}^{\bullet - 1} \circ f^{-1} \circ \rho \right)(u) \to \left(\mathcal{H}^{\bullet} \circ f^{-1} \circ \rho \right)(w)$$

of the corresponding Mayer–Vietoris sequence as described in [tom08, Section 17.1.4]. Now let

$$\partial'_{(w, \hat{u})} := \left(\delta'_{(w, \hat{u})} \right)^{\circ} \colon \left(\mathcal{H}^{\bullet} \circ f^{-1} \circ \rho \right)(w) \to \left(\mathcal{H}^{\bullet - 1} \circ f^{-1} \circ \rho \right)(u)$$

be the corresponding homomorphism in the opposite category $\left(\mathrm{Vect}_K^{\mathbb{Z}} \right)^{\circ}$ for all $(w, \hat{u}) \in R_D$. Moreover, let $\mathrm{pr}_1 \colon R_D \to D$ and $\mathrm{pr}_2 \colon R_D \to T(D)$ be the projections to the first and the second component, respectively. Then ∂' is a natural transformation as in the diagram

$$
\begin{array}{ccc}
R_D & \xrightarrow{\mathrm{pr}_1} & D \\
{\scriptstyle \mathrm{pr}_2} \downarrow & \swarrow{\scriptstyle \partial'} & \downarrow {\scriptstyle F'} \\
T(D) & \xrightarrow[\Sigma \circ F' \circ T^{-1}]{} & \left(\mathrm{Vect}_K^{\mathbb{Z}} \right)^{\circ}.
\end{array}
$$

Thus, the functor $F' \colon D \to \left(\mathrm{Vect}_K^{\mathbb{Z}} \right)^{\circ}$ and ∂' determine a unique strictly stable functor $F \colon \mathbb{M} \to \left(\mathrm{Vect}_K^{\mathbb{Z}} \right)^{\circ}$ by Proposition A.6. To obtain a functor of type $\mathbb{M}^{\circ} \to \mathrm{Vect}_K$ from F we post-compose $F^{\circ} \colon \mathbb{M}^{\circ} \to \mathrm{Vect}_K^{\mathbb{Z}}$ with the evaluation at 0 and we define

$$h(f) := \mathrm{ev}^0 \circ F^{\circ}.$$

If $\mathcal{H}^{\bullet}$ is singular cohomology with coefficients in K, then we name $h(f)$ the *relative interlevel set cohomology (RISC) of f with coefficients in K*. Moreover, we note that h extends to a contravariant functor from the category of spaces over the reals $\mathbb{R}$ to the category of

contravariant functors on $\mathbb{M}$ in the following way. For a commutative triangle

$$\begin{array}{ccc} & \overset{\varphi}{\longrightarrow} & \\ X & & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathbb{R} & \end{array}$$

of topological spaces, the map φ yields a continuous map of pairs

$$\left(f^{-1} \circ \rho\right)(u) \to \left(g^{-1} \circ \rho\right)(u),$$

which is natural in $u \in D$. By the functoriality of $\mathcal{H}^{\bullet}$ and the naturality of the Mayer–Vietoris sequence the collection of these maps induces a natural transformation

$$h(\varphi) \colon h(g) \to h(f).$$

By construction any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. 2.6 yields a long exact sequence

$$\cdots \longrightarrow h(f)(T(u)) \longrightarrow$$
$$\longrightarrow h(f)(w) \longrightarrow h(f)(v_1) \oplus h(f)(v_2) \xrightarrow{(1\ -1)} h(f)(u) \longrightarrow$$
$$\longrightarrow h(f)(T^{-1}(w)) \longrightarrow \cdots .$$

By Proposition C.2.1 this is one way of characterizing *cohomological functors* on $\mathbb{M}$, which we define in Definition C.1. We note that the characterization Proposition C.4.4 has been stated by [CdM09] and proven by [CdM09, BEMP13] for any axis-aligned rectangle contained within a tile of the tessellation shown in Fig. 2.4.

Now suppose that $h(f)$ is pointwise finite-dimensional (pfd) or q-tame [CdSGO16, Section 1.1], which is equivalent by Proposition 3.12 below. We end this section by showing that $h(f) \colon \mathbb{M}^{\circ} \to \mathrm{vect}_K$ satisfies the following form of continuity. Here vect_K denotes the category of finite-dimensional vector spaces over K.

Definition 2.2. We say that a contravariant functor $F \colon \mathbb{M}^{\circ} \to \mathrm{Vect}_K$ is *sequentially continuous*, if for any increasing sequence $(u_k)_{k=1}^{\infty}$ in $\mathbb{M}$ converging to u the natural map

$$F(u) \to \varprojlim_k F(u_k) \tag{2.2}$$

is an isomorphism. Dually, a covariant functor $F \colon \mathbb{M} \to \mathrm{Vect}_K$ is *sequentially continuous*, if for any decreasing sequence $(u_k)_{k=1}^{\infty}$ in $\mathbb{M}$ converging to u the natural map (2.2) is an isomorphism, see also Remark 3.15 below.

Proposition 2.3. *The contravariant functor $h(f) \colon \mathbb{M}^{\circ} \to \mathrm{vect}_K$ is sequentially continuous.*

Proof. Let $(u_k)_{k=1}^{\infty}$ be an increasing sequence in $\mathbb{M}$ converging to u. Without loss of generality we assume that $(u_k)_{k=1}^{\infty}$ is contained in a single tile $T^{-n}(D)$ of the tessellation induced by

T and D as shown in Fig. 2.4. We write X_k for the absolute component of $(f^{-1} \circ \rho \circ T^n)(u_k)$ and A_k for the relative component. With this we have the commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{H}^{n-1}(\bigcup_k X_k) & \longrightarrow & \mathcal{H}^{n-1}(\bigcup_k A_k) & \longrightarrow & h(f)(u) & \longrightarrow & \mathcal{H}^n(\bigcup_k X_k) & \longrightarrow & \mathcal{H}^n(\bigcup_k A_k) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\varprojlim_k \mathcal{H}^{n-1}(X_k) & \longrightarrow & \varprojlim_k \mathcal{H}^{n-1}(A_k) & \longrightarrow & \varprojlim_k h(f)(u_k) & \longrightarrow & \varprojlim_k \mathcal{H}^n(X_k) & \longrightarrow & \varprojlim_k \mathcal{H}^n(A_k)
\end{array}
$$

with exact rows. Now for each $k \in \mathbb{N}$ the cohomology of X_k appears as a value of $h(f)$. Similarly the cohomology of A_k is a value of $h(f)$ or it is isomorphic to a direct sum of two values of $h(f)$. So the inverse sequences

$$\left(\mathcal{H}^{n-2}(X_{k+1}) \to \mathcal{H}^{n-2}(X_k)\right)_{k=1}^{\infty},$$
$$\left(\mathcal{H}^{n-2}(A_{k+1}) \to \mathcal{H}^{n-2}(A_k)\right)_{k=1}^{\infty},$$
$$\left(\mathcal{H}^{n-1}(X_{k+1}) \to \mathcal{H}^{n-1}(X_k)\right)_{k=1}^{\infty},$$
$$\text{and} \quad \left(\mathcal{H}^{n-1}(A_{k+1}) \to \mathcal{H}^{n-1}(A_k)\right)_{k=1}^{\infty}$$

are pfd. In particular they satisfy the Mittag-Leffler condition and thus the four vertical maps surrounding $h(f)(u) \to \varprojlim_k h(f)(u_k)$ in the above diagram are isomorphisms by [May99, Section 19.4]. With this it follows from the five lemma that $h(f)(u) \to \varprojlim_k h(f)(u_k)$ is an isomorphism. $\qquad \square$

3 Decomposition

Having defined the relative interlevel set cohomology as a contravariant functor $h(f) \colon \mathbb{M}^\circ \to \mathrm{vect}_K$, we now formalize the notion of an *extended persistence diagram*, originally due to [CSEH09], as a discrete invariant of contravariant functors $F \colon \mathbb{M}^\circ \to \mathrm{vect}_K$ vanishing on $\partial \mathbb{M}$. The *persistence diagram* $\mathrm{Dgm}(F)$ is a multiset $\mu \colon \mathbb{M} \to \mathbb{N}_0$, which counts, for each point $u = (x, y) \in \mathbb{M}$, the maximal number $\mu(u)$ of linearly independent vectors in $F(u)$ born at u; for the functor $h(f)$, these are cohomology classes. Before we provide a more explicit definition, we introduce some notation. We note that a contravariant functor $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ vanishing on $\partial \mathbb{M}$ can equivalently be thought of as a bifunctor $F \colon \mathbb{R} \times \mathbb{R}^\circ \to \mathrm{Vect}_K$ supported on the interior $\mathrm{int}\, \mathbb{M} \subset \mathbb{R} \times \mathbb{R}$, which is covariant in its first argument and contravariant in the second. For axis-parallel internal maps of a functor $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ we use similar notation as with bifunctors. More specifically, if we have $u, v \in \mathbb{M}$ with $u \preceq v$, $u = (x, y)$, and $v = (s, t)$, then we use notation as in the commutative diagram

$$
\begin{array}{ccc}
F(v) = F(s,t) & \xrightarrow{\ F(s \leq x, t)\ } & F(x,t) \\
{\scriptstyle F(s, y \leq t)} \downarrow \quad\quad \searrow {\scriptstyle F(u \preceq v)} & & \downarrow {\scriptstyle F(x, y \leq t)} \\
F(s,y) & \xrightarrow[\ F(s \leq x, y)\ ]{} & F(x,y) = F(u).
\end{array}
$$

Now we define the persistence diagram as

$$\mathrm{Dgm}(F) : \mathrm{int}\, \mathbb{M} \to \mathbb{N}_0, \ u \mapsto \dim_K F(u) - \dim_K \sum_{v \succ u} \mathrm{Im}\, F(u \preceq v) \tag{3.1}$$
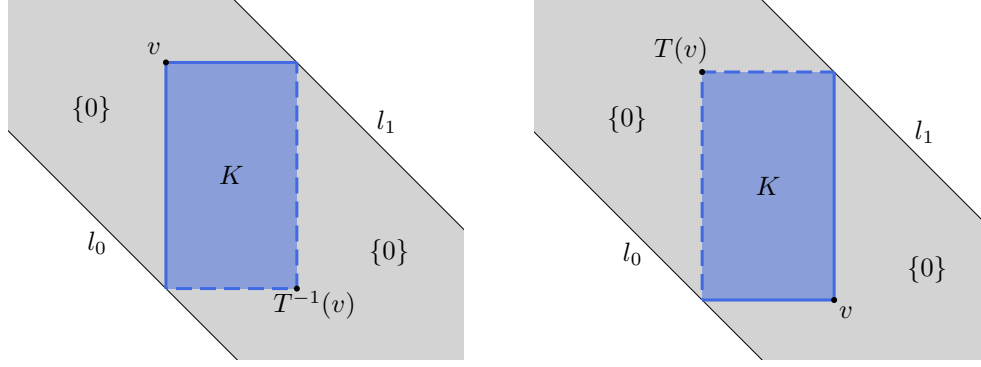
(a) The indecomposable $B_v \colon \mathbb{M}^\circ \to \mathrm{Vect}_K$.   (b) The indecomposable $B^v \colon \mathbb{M} \to \mathrm{Vect}_K$.

Figure 3.1: The contravariant and the covariant indecomposables.

for any functor $F \colon \mathbb{M}^\circ \to \mathrm{vect}_K$ vanishing on $\partial \mathbb{M}$. In the last term, v ranges over all $v \in \mathbb{M}$ with $v \succ u$. Moreover, we note that

$$\sum_{v \succ u} \mathrm{Im}\, F(u \preceq v) = \left(\bigcup_{s < x} \mathrm{Im}\, F(s \leq x, y) \right) + \left(\bigcup_{t > y} \mathrm{Im}\, F(x, y \leq t) \right),$$

for $u = (x, y)$. Now let $f \colon X \to \mathbb{R}$ be a continuous function with pfd RISC $h(f)$.

Definition 3.1 (Extended Persistence Diagram)**.** The *extended persistence diagram of f (over K)* is $\mathrm{Dgm}(f) := \mathrm{Dgm}(f; K) := \mathrm{Dgm}(h(f))$.

Originally the extended persistence diagram was defined in a different way by [CSEH09]; see Section 3.1.2 for details on the connection between these two definitions. Up to isomorphism of ambient sets, the multiset defined by [CSEH09] and the multiset defined here are the same. We note that $\mathrm{Dgm}(f)$ is supported in the downset $\downarrow \mathrm{Im}\, \blacktriangle \subseteq \mathbb{M}$. This definition is consistent with [BBF20, Section 2.2] in the sense that both definitions yield the same multiset for X a finite simplicial complex and f piecewise linear.

Next we show that sequentially continuous pfd cohomological functors decompose into the following type of indecomposables, see also Fig. 3.1a.

Definition 3.2 (Contravariant Block)**.** For $v \in \mathrm{int}\,\mathbb{M}$ we define

$$B_v \colon \mathbb{M}^\circ \to \mathrm{Vect}_K, u \mapsto \begin{cases} K & u \in (\downarrow v) \cap \mathrm{int}\left(\uparrow T^{-1}(v)\right) \\ \{0\} & \text{otherwise,} \end{cases}$$

where $\mathrm{int}\left(\uparrow T^{-1}(v)\right)$ is the interior of the upset of $T^{-1}(v)$ in $\mathbb{M}$ and $\mathrm{int}\,\mathbb{M}$ is the interior of $\mathbb{M}$ in $\mathbb{R}^2$. Whenever it makes sense, the internal maps are identities, otherwise they are zero.

Lemma 3.3 (Yoneda)**.** *Let $G \colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ be a contravariant functor vanishing on $\partial \mathbb{M}$ and let $v \in \mathrm{int}\,\mathbb{M}$. Then the evaluation at $1 \in K = B_v(v)$ yields a linear isomorphism*

$$\mathrm{Nat}(B_v, G) \cong G(v),$$

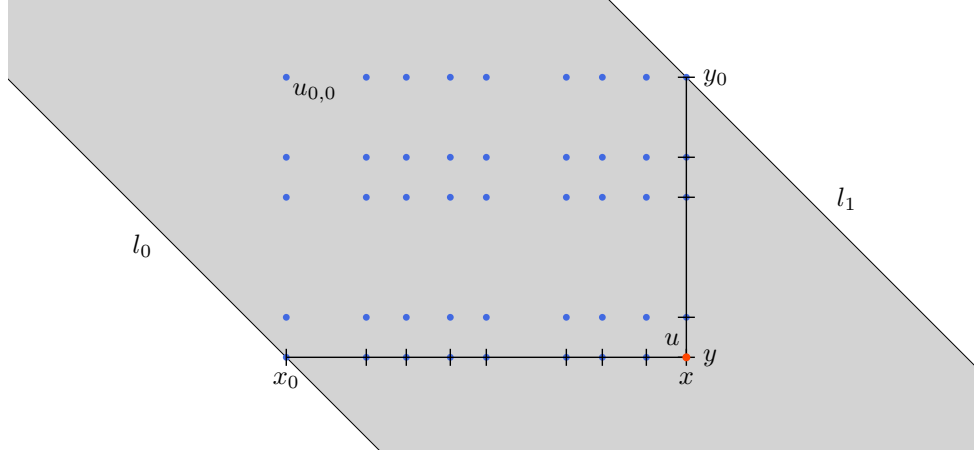*where $\mathrm{Nat}(B_v, G)$ denotes the vector space of natural transformations from B_v to G.*

10

Figure 3.2: The horizontal and vertical filtrations of $F(u)$ by images.

Now let $F\colon \mathbb{M}^\circ \to \text{vect}_K$ be a sequentially continuous cohomological functor, as defined in Definition 2.2 and Definition C.1. For each $v \in \text{int}\,\mathbb{M}$ we choose a basis for a complement of $\sum_{w \succ v} \text{Im}\,F(v \preceq w)$ in $F(v)$. By the Yoneda Lemma 3.3 this yields a natural transformation

$$\varphi\colon \bigoplus_{v \in \text{int}\,\mathbb{M}} (B_v)^{\oplus \mu(v)} \longrightarrow F,$$

where $\mu := \text{Dgm}(F)$.

Theorem 3.4. *The natural transformation*

$$\varphi\colon \bigoplus_{v \in \text{int}\,\mathbb{M}} (B_v)^{\oplus \mu(v)} \longrightarrow F$$

is a natural isomorphism.

To show that φ is a natural isomorphism, we have to show pointwise that

$$\varphi_u\colon \bigoplus_{v \in \text{int}\,\mathbb{M}} (B_v(u))^{\oplus \mu(v)} \longrightarrow F(u)$$

is an isomorphism for all $u := (x, y) \in \text{int}\,\mathbb{M}$. To this end, we fix some notation, which we use in the proof of Theorem 3.4 and auxiliary lemmas. As depicted in Fig. 3.2, let x_0 be the x-coordinate of the intersection of l_0 and the horizontal line through u, let $x_1 < x_2 < \cdots < x_{k-1}$ be the points of discontinuity of the function

$$(x_0, x) \to \mathbb{N}_0,\ s \mapsto \text{rank}\,F(s \le x, y), \tag{3.2}$$

and let $x_k := x$. Similarly, let y_0 be the intersection of l_1 and the vertical line through u, let $y_1 > y_2 > \cdots > y_{l-1}$ be the points of discontinuity of the function

$$(y, y_0) \to \mathbb{N}_0,\ t \mapsto \text{rank}\,F(x, y \le t),$$

and let $y_l := y$. Moreover, we set $u_{(i,j)} := (x_i, y_j)$ for $i = 0, \ldots, k$ and $j = 0, \ldots, l$, then we have $u = u_{(k,l)}$ and $T(u) = u_{(0,0)}$. With some abuse of notation, we may also drop the

11

parentheses and write $u_{i,j}$ in place of $u_{(i,j)}$. Furthermore, let $\preceq$ be the colexicographic order on $I := \{0, \ldots, k\} \times \{0, \ldots, l\}$, which is defined by

$$(i, j) \preceq (i', j') \quad :\Leftrightarrow \quad j < j' \ \vee \ (j = j' \ \wedge \ i < i').$$

For any contravariant functor $H \colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ vanishing on $\partial \mathbb{M}$ and $\zeta \in I$ we set

$$H_\zeta := \sum_{\xi \preceq \zeta} \mathrm{Im}\, H(u \preceq u_\xi)$$

to obtain the natural filtration

$$\bigcup_{\zeta \in I} H_\zeta = H(u).$$

For a pair $\zeta := (i, j) \in I$ we will drop the parentheses in the index and write $H_{i,j} = H_\zeta$ in place of $H_{(i,j)}$. With this notation we may write the filtration $\bigcup_{\zeta \in I} H_\zeta$ as

$$\begin{aligned}
\{0\} = H_{0,0} = H_{1,0} &= H_{2,0} = \cdots = H_{k,0} \\
&= H_{0,1} \subseteq H_{1,1} \subseteq H_{2,1} \subseteq \cdots \subseteq H_{k,1} \\
&= H_{0,2} \subseteq H_{1,2} \subseteq H_{2,2} \subseteq \cdots \subseteq H_{k,2} \\
&\ \ \vdots \\
&= H_{0,l} \subseteq H_{1,l} \subseteq H_{2,l} \subseteq \cdots \subseteq H_{k,l} = H(u),
\end{aligned}$$

see also Fig. 3.3. We may describe this filtration more concretely using the equations

$$\begin{aligned}
H_{i,0} &= 0 & &\text{for } i = 0, \ldots, k, & &(3.3) \\
H_{k,j-1} &= H_{0,j} & &\text{for } j = 1, \ldots, l, \text{ and} & &(3.4) \\
H_{i,j} &= \mathrm{Im}\, H(u \preceq u_{i,j}) + \mathrm{Im}\, H(u \preceq u_{k,j-1}) & &\text{for } j = 1, \ldots, l \text{ and } i = 0, \ldots, k. & &(3.5)
\end{aligned}$$

Proof of Theorem 3.4. Now let $H := \bigoplus_{v \in \mathrm{int}\, \mathbb{M}} (B_v)^{\oplus \mu(v)}$. We show that

$$\varphi_\xi \colon H_\xi \to F_\xi$$

is an isomorphism for all $\xi \in I$ by induction on ξ. By (3.3) the map $\varphi_{i,0}$ is an isomorphism for all $i = 0, \ldots, k$. Moreover, $\varphi_{0,j}$ is an isomorphism if $\varphi_{k,j-1}$ is an isomorphism for all $j = 1, \ldots, l$ by (3.4). Thus, in order to complete our proof by induction, it suffices to show that $\varphi_{i,j} \colon H_{i,j} \to F_{i,j}$ is an isomorphism whenever $\varphi_{i-1,j} \colon H_{i-1,j} \to F_{i-1,j}$ is an isomorphism for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$. To this end, we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{i-1,j} & \longrightarrow & H_{i,j} & \longrightarrow & H_{i,j}/H_{i-1,j} & \longrightarrow & 0 \\
& & \Big\downarrow{\varphi_{i-1,j}} & & \Big\downarrow{\varphi_{i,j}} & & \Big\downarrow & & \\
0 & \longrightarrow & F_{i-1,j} & \longrightarrow & F_{i,j} & \longrightarrow & F_{i,j}/F_{i-1,j} & \longrightarrow & 0
\end{array}$$

with exact rows. By the five lemma, it suffices to show that the vertical map on the right hand side is an isomorphism. To this end, we note that H is cohomological, as it is a direct
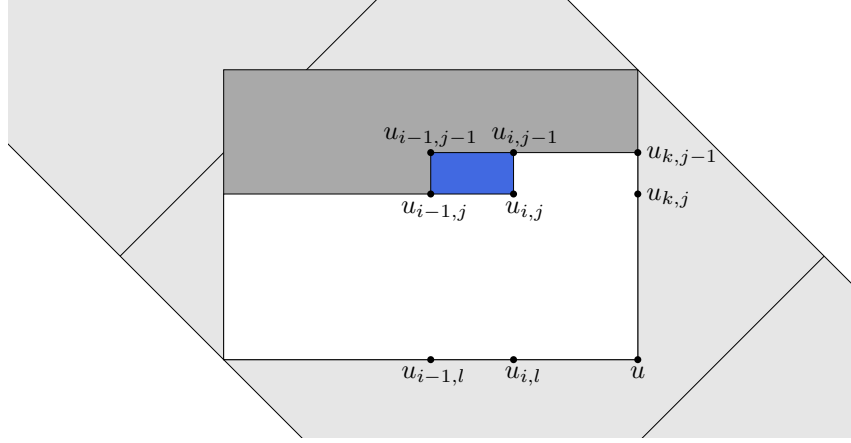
Figure 3.3: The filtration of $H(u)$ in terms of the colexicographic order on I. The large axis-aligned rectangle contains all points such that the corresponding image in $H(u)$ can be non-zero. The subspace $H_{i-1,j} \subseteq H(u)$ is the sum of the images in $H(u)$ corresponding to points in the region shaded in dark grey. If we add the images corresponding to points in the blue rectangle (or just the image corresponding to the lower right vertex $u_{i,j}$), then we obtain $H_{i,j}$ as the next step in the filtration.

sum of cohomological functors. Thus, by Lemma 3.5 below, there is a commutative square

$$
\begin{array}{ccc}
\dfrac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} & \xrightarrow{\ \sim\ } & \dfrac{H_{i,j}}{H_{i-1,j}} \\[2ex]
\Big\downarrow & & \Big\downarrow \\[2ex]
\dfrac{F(u_{i,j})}{\operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})} & \xrightarrow{\ \sim\ } & \dfrac{F_{i,j}}{F_{i-1,j}},
\end{array}
$$

where the two vertical maps are induced by φ. As the two horizontal maps are isomorphisms by Lemma 3.5, it remains to show that the vertical map on the left hand side is an isomorphism. To this end, we consider the commutative square

$$
\begin{array}{ccc}
\dfrac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} & \longrightarrow & \dfrac{H(u_{i,j})}{\sum\limits_{w \succ u_{i,j}} \operatorname{Im} H(u_{i,j} \preceq w)} \\[3ex]
\Big\downarrow & & \Big\downarrow \\[3ex]
\dfrac{F(u_{i,j})}{\operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})} & \longrightarrow & \dfrac{F(u_{i,j})}{\sum\limits_{w \succ u_{i,j}} \operatorname{Im} F(u_{i,j} \preceq w)},
\end{array}
$$

where the vertical maps are induced by φ and the horizontal maps are induced by the internal maps of H and F respectively. We have to show that the vertical map on the left hand side is an isomorphism. The horizontal map at the top and the horizontal map at the bottom are isomorphisms by Corollary 3.10 and Lemma 3.11, respectively. Thus, it suffices

to show that the vertical map on the right hand side is an isomorphism. To this end, we consider the commutative diagram

$$
\begin{array}{ccc}
& K^{\mu(u_{i,j})} & \\
& \downarrow \wr & \\
& B_{u_{i,j}}^{\oplus \mu(u_{i,j})}(u_{i,j}) & \\
& \downarrow \iota & \\
\displaystyle\sum_{w \succ u_{i,j}} \mathrm{Im}\, H(u_{i,j} \preceq w) \longhookrightarrow H(u_{i,j}) \longrightarrow & \dfrac{H(u_{i,j})}{\displaystyle\sum_{w \succ u_{i,j}} \mathrm{Im}\, H(u_{i,j} \preceq w)} \\
\downarrow \qquad\qquad \downarrow{\scriptstyle\varphi_{u_{i,j}}} & \downarrow \\
\displaystyle\sum_{w \succ u_{i,j}} \mathrm{Im}\, F(u_{i,j} \preceq w) \longhookrightarrow F(u_{i,j}) \longrightarrow & \dfrac{F(u_{i,j})}{\displaystyle\sum_{w \succ u_{i,j}} \mathrm{Im}\, F(u_{i,j} \preceq w)}.
\end{array}
$$

Now the image of ι is a complement of $\displaystyle\sum_{w \succ u_{i,j}} \mathrm{Im}\, H(u_{i,j} \preceq w)$ in $H(u_{i,j})$. Moreover, by the construction of φ, the composition of the three vertical maps in the center map the standard basis of $K^{\mu(u_{i,j})}$ to a basis for a complement C of $\displaystyle\sum_{w \succ u_{i,j}} \mathrm{Im}\, F(u_{i,j} \preceq w)$ in $F(u_{i,j})$. As a result, the map $\varphi_{u_{i,j}}$ maps $\mathrm{Im}(\iota)$ isomorphically onto C, hence the vertical map on the right hand side is an isomorphism. As $u \in \mathrm{int}\,\mathbb{M}$ was arbitrary, φ is a natural isomorphism. $\qquad\square$

Lemma 3.5. *For any cohomological functor* $H \colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ *and any pair of indices* $(i,j) \in \{1,\ldots,k\} \times \{1,\ldots,l\}$ *there is an isomorphism*

$$
\frac{H(u_{i,j})}{\mathrm{Im}\, H(u_{i,j} \preceq u_{i-1,j}) + \mathrm{Im}\, H(u_{i,j} \preceq u_{i,j-1})} \xrightarrow{\sim} \frac{H_{i,j}}{H_{i-1,j}}
$$

natural in H.

Proof. We consider the commutative diagram

$$
\begin{array}{ccccc}
H(u_{i-1,j-1}) & \longrightarrow & H(u_{i,j-1}) & \longrightarrow & H(u_{k,j-1}) = H(x, y_{j-1}) \\
\downarrow & & \downarrow & & \downarrow \\
H(u_{i-1,j}) & \longrightarrow & H(u_{i,j}) & \longrightarrow & H(u_{k,j}) = H(x, y_j) \\
\downarrow & & \downarrow & & \downarrow \\
H(u_{i-1,l}) & \longrightarrow & H(u_{i,l}) & \longrightarrow & H(u) = H(x, y).
\end{array}
$$

By Proposition C.2.4 all squares in this diagram are middle exact, see also Fig. 3.3. Thus, by Proposition B.3 the map $H(u \preceq u_{i,j})$ induces an isomorphism

$$
\frac{H(u_{i,j})}{\mathrm{Im}\, H(u_{i,j} \preceq u_{i-1,j}) + \mathrm{Im}\, H(u_{i,j} \preceq u_{i,j-1})} \xrightarrow{\sim} \frac{\mathrm{Im}\, H(u \preceq u_{i,j}) + \mathrm{Im}\, H(u \preceq u_{k,j-1})}{\mathrm{Im}\, H(u \preceq u_{i-1,j}) + \mathrm{Im}\, H(u \preceq u_{k,j-1})}.
$$

Moreover, by (3.5) the codomain of this isomorphism is $H_{i,j}/H_{i-1,j}$ and thus we may write this isomorphism also as

$$\frac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} \xrightarrow{\sim} \frac{H_{i,j}}{H_{i-1,j}}. \qquad \square$$

Before we prove Corollary 3.10 and Lemma 3.11 we need to establish three auxiliary results. To this end, we note that the inclusion

$$\{x_i \mid i = 0, \ldots, k\} \hookrightarrow [x_0, x]$$

has the upper adjoint

$$r_1 \colon [x_0, x] \to \{x_i \mid i = 0, \ldots, k\}, \; s \mapsto \max\{x_i \mid x_i \leq s\}.$$

Similarly

$$r_2 \colon [y, y_0] \to \{y_j \mid j = 0, \ldots, l\}, \; t \mapsto \min\{y_j \mid t \leq y_j\}$$

is the lower adjoint of the inclusion

$$\{y_j \mid j = 0, \ldots, l\} \hookrightarrow [y, y_0].$$

Lemma 3.6. *We have*

$$\begin{aligned}
\operatorname{Im} F(r_1(s) \leq x, y) &= \operatorname{Im} F(s \leq x, y) & \textit{for all } s \in [x_0, x] \textit{ as well as} \\
\operatorname{Im} F(x, y \leq r_2(t)) &= \operatorname{Im} F(x, y \leq t) & \textit{for all } t \in [y, y_0].
\end{aligned}$$

Proof. We prove the first equation, the second can be shown in an analogous manner. To this end, we consider the filtration

$$\bigcup_{s \leq x} \operatorname{Im} F(s \leq x, y) = F(x, y) = F(u).$$

For $s_0 \in [x_0, x)$ the canonical map

$$F(s_0, y) \longrightarrow \varprojlim_{s > s_0} F(s, y) \tag{3.6}$$

is an isomorphism by the sequential continuity of F. As a result, the image of the canonical map

$$\varprojlim_{s > s_0} F(s, y) \longrightarrow F(x, y) = F(u)$$

and the image $\operatorname{Im} F(s_0 \leq x, y)$ are the same. Moreover, the image of (3.6) and the intersection

$$\bigcap_{s > s_0} \operatorname{Im} F(s \leq x, y)$$

are identical, hence

$$\operatorname{Im} F(s_0 \leq x, y) = \bigcap_{s > s_0} \operatorname{Im} F(s \leq x, y).$$

As a result of this equation, the function (3.2) is upper semi-continuous, i.e. the superlevel sets of (3.2) are closed. Moreover, as $x_1, \ldots, x_{k-1}$ are by definition the points of discontinuity of (3.2), we have

$$\operatorname{Im} F(x_i \leq x, y) = \operatorname{Im} F(s \leq x, y)$$

15

for all $i = 0, \ldots, k-1$ and $s \in [x_i, x_{i+1})$. Using the upper adjoint $r_1 \colon [x_0, x] \to \{x_i \mid i = 0, \ldots, k\}$ we can state this last equation without explicit quantification over $\{0, \ldots, k-1\}$ as

$$\operatorname{Im} F(r_1(s) \leq x, y) = \operatorname{Im} F(s \leq x, y) \quad \text{for all } s \in [x_0, x]. \qquad \square$$

Now suppose we have

$$v := (v_1, v_2) \in [u, T(u)] = [x_0, x] \times [y, y_0]$$
$$\text{and} \qquad (s, t) \in [v, T(u)] = [x_0, v_1] \times [v_2, y_0].$$

We consider the commutative square

$$
\begin{array}{ccc}
F(s, t) & \longrightarrow & F(v_1, t) \\
\downarrow & & \downarrow {\scriptstyle F(v_1, v_2 \leq t)} \\
F(s, v_2) & \xrightarrow[\; F(s \leq v_1, v_2) \;]{} & F(v)
\end{array}
$$

and we define

$$F_v(s, t) := \operatorname{Im} F(s \leq v_1, v_2) + \operatorname{Im} F(v_1, v_2 \leq t).$$

Moreover, let

$$r := r_1 \times r_2 \colon [u, T(u)] \to \{u_\xi \mid \xi \in I\},$$

i.e. r is the lower adjoint to the inclusion $\{u_\xi \mid \xi \in I\} \hookrightarrow [u, T(u)]$. Then we have

$$(F_v \circ r)(s, t) \subseteq F_v(s, t).$$

Furthermore, in the special case that $v = u$ we have equality:

$$(F_u \circ r)(s, t) = F_u(s, t);$$

as a result of Lemma 3.6. By the following lemma this is true even when $v \neq u$.

Lemma 3.7. *We have $(F_v \circ r)(s, t) = F_v(s, t)$.*

Remark 3.8. In general it may very well happen that

$$\operatorname{Im} F(r_1(s) \leq v_1, v_2) \neq \operatorname{Im} F(s \leq v_1, v_2)$$
$$\text{or} \quad \operatorname{Im} F(v_1, v_2 \leq r_2(t)) \neq \operatorname{Im} F(v_1, v_2 \leq t).$$

Thus, it is crucial to consider the two summands of $F_v(s, t)$ in conjunction.

Proof. It suffices to show that

$$F_v(r_1(s), t) = F_v(s, t) \tag{3.7}$$
$$\text{and} \quad F_v(s, r_2(t)) = F_v(s, t), \tag{3.8}$$

independent of s and t, since this implies that

$$(F_v \circ r)(s, t) = F_v(r_1(s), r_2(t)) = F_v(s, r_2(t)) = F_v(s, t).$$

16

We show (3.8), our proof of (3.7) is similar. Now

$$F_v(s, r_2(t)) = \operatorname{Im} F(s \le v_1, v_2) + \operatorname{Im} F(v_1, v_2 \le r_2(t)),$$

which is a subspace of

$$\operatorname{Im} F(s \le v_1, v_2) + \operatorname{Im} F(v_1, v_2 \le t) = F_v(s, t) \subseteq F(v).$$

Moreover, this inclusion

$$F_v(s, r_2(t)) \hookrightarrow F_v(s, t)$$

induces a canonical map

$$\pi_v \colon \frac{F(v)}{F_v(s, r_2(t))} \longrightarrow \frac{F(v)}{F_v(s, t)}.$$

Now in order to prove (3.8), it suffices to show that π_v is an isomorphism. In the special case that $v = u$, we already have $F_u(s, r_2(t)) = F_u(s, t)$ by the second equation from the previous lemma, hence

$$\pi_u = \operatorname{id} \colon \frac{F(u)}{F_u(s, r_2(t))} \xrightarrow{=} \frac{F(u)}{F_u(s, t)}.$$

Our approach is to reduce the general case for π_v to this special case of $\pi_u = \operatorname{id}$. To this end, we consider the commutative diagram

$$(3.9)$$

By Proposition C.2.4 all axis-aligned squares and rectangles in this diagram are middle exact. Now $F(u \preceq v)$ maps $\operatorname{Im} F(v_1, v_2 \le r_2(t))$ to a subspace of $\operatorname{Im} F(x, y \le r_2(t))$, hence $F_v(s, r_2(t))$ is mapped to a subspace of $F_u(s, r_2(t))$. Similarly $F(u \preceq v)$ maps $F_v(s, t)$ to a subspace of $F_u(s, t)$. As a result we obtain the commutative diagram

17

from which we obtain the induced commutative square

$$
\begin{array}{ccc}
\dfrac{F(v)}{F_v(s, r_2(t))} & \longrightarrow & \dfrac{F(u)}{F_u(s, r_2(t))} \\[2ex]
\pi_v \downarrow & & \| \pi_u = \mathrm{id} \\[2ex]
\dfrac{F(v)}{F_v(s,t)} & \longrightarrow & \dfrac{F(u)}{F_u(s,t)}.
\end{array}
$$

As all axis-aligned squares and rectangles of (3.9) are middle exact, the two horizontal maps of this square are isomorphisms by Proposition B.3, hence π_v is an isomorphism as well. $\quad\square$

Lemma 3.9. *The restriction of* $\mathrm{Dgm}(F)$ *to* $(\uparrow u) \cap \mathrm{int}(\downarrow T(u))$ *is supported on the grid* $\{u_\xi \mid \xi \in I\}$.

Proof. Let $v := (v_1, v_2) \in (\uparrow u) \cap \mathrm{int}(\downarrow T(u)) \setminus \{u_\xi \mid \xi \in I\}$. We have to show that $\mathrm{Dgm}(F)(v) = 0$. As $v \notin \{u_\xi \mid \xi \in I\}$ we have $v \neq r(v)$, which implies $v \prec r(v)$. Thus, we have $v_1 > r_1(v_1)$ or $v_2 < r_2(v_2)$. Without loss of generality we assume that $v_2 < r_2(v_2)$. Now let $j = 0, \ldots, l-1$ be such that $y_j = r_2(v_2) > v_2$. Considering the commutative diagram

$$
\begin{array}{ccc}
F(x_0, y_j) & \longrightarrow & F(v_1, y_j) \\
\downarrow & & \downarrow \\
F(x_0, v_2) & \longrightarrow & F(v)
\end{array}
$$

we see that

$$
\begin{aligned}
(F_v \circ r)(x_0, v_2) &= F_v(x_0, y_j) \\
&= \mathrm{Im}\, F(x_0 \leq v_1, v_2) + \mathrm{Im}\, F(v_1, v_2 \leq y_j) \\
&\subseteq \sum_{w \succ v} \mathrm{Im}\, F(v \preceq w).
\end{aligned}
$$

Moreover, the previous lemma implies that

$$
\begin{aligned}
(F_v \circ r)(x_0, v_2) &= F_v(x_0, v_2) \\
&= \mathrm{Im}\, F(x_0 \leq v_1, v_2) + \mathrm{Im}\, F(v_1, v_2 \leq v_2) \\
&= \mathrm{Im}\, F(x_0 \leq v_1, v_2) + F(v) \\
&= F(v).
\end{aligned}
$$

The previous two chains of equations (and an inclusion) taken together we obtain

$$
F(v) \subseteq \sum_{w \succ v} \mathrm{Im}\, F(v \preceq w) \subseteq F(v),
$$

hence

$$
\begin{aligned}
\mathrm{Dgm}(F)(v) &= \dim_K F(v) - \dim_K \sum_{w \succ v} \mathrm{Im}\, F(v \preceq w) \\
&= \dim_K F(v) - \dim_K F(v) = 0. \qquad\qquad \square
\end{aligned}
$$

Corollary 3.10. *For* $H = \bigoplus_{v \in \operatorname{int} \mathbb{M}} (B^v)^{\oplus \mu(v)}$, *the canonical map*

$$\frac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} \longrightarrow \frac{H(u_{i,j})}{\sum\limits_{w \succ u_{i,j}} \operatorname{Im} H(u_{i,j} \preceq w)} \tag{3.10}$$

is an isomorphism.

Proof. We consider the restriction of $\operatorname{Dgm}(F)$ to the blue rectangle in Fig. 3.3. By the previous lemma, this restriction can be non-zero only at the vertices $u_{i-1,j-1}$, $u_{i,j-1}$, $u_{i-1,j}$, or $u_{i,j}$. Thus, any indecomposable summand of H, which is not born at $u_{i,j}$ and yet alive at $u_{i,j}$, is alive at $u_{i-1,j}$ or $u_{i,j-1}$, hence

$$\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1}) = \sum\limits_{w \succ u_{i,j}} \operatorname{Im} H(u_{i,j} \preceq w).$$

As a result, the canonical map (3.10) is an identity. $\qquad\square$

Lemma 3.11. *The canonical map*

$$\frac{F(u_{i,j})}{\operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})} \longrightarrow \frac{F(u_{i,j})}{\sum\limits_{w \succ u_{i,j}} \operatorname{Im} F(u_{i,j} \preceq w)} \tag{3.11}$$

is an isomorphism.

Proof. Let $v := u_{i,j}$ and let $R := [v, u_{i-1,j-1}] \setminus \{v\}$, i.e. R is the blue rectangle in Fig. 3.3 except for the vertex $v = u_{i,j}$. Then we have the inclusion

$$\operatorname{Im} F(v \preceq w) \subseteq F_v(w)$$

for any $w \in R$, and thus

$$\sum\limits_{w \succ u_{i,j}} \operatorname{Im} F(u_{i,j} \preceq w) = \sum\limits_{w \in R} \operatorname{Im} F(v \preceq w) = \sum\limits_{w \in R} F_v(w).$$

Moreover, by Lemma 3.7

$$\begin{aligned}
F_v(w) = (F_v \circ r)(w) &= F_v(u_{i-1,j-1}) \\
&= \operatorname{Im} F(v \preceq u_{i-1,j}) + \operatorname{Im} F(v \preceq u_{i,j-1}) \\
&= \operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})
\end{aligned}$$

for any $w \in R$ and as a result the denominators of the domain and the codomain of (3.11) are identical, hence (3.11) is the identity. $\qquad\square$

Now we generalize Theorem 3.4 from pfd cohomological functors to q-tame [CdSGO16, Section 1.1] cohomological functors.

Proposition 3.12. *Let* $F \colon \mathbb{M}^\circ \to \operatorname{Vect}_K$ *be a cohomological functor which is q-tame in the sense that* $F(u \preceq v)$ *has finite rank for all* $u \prec v \in \mathbb{M}$. *Then F is pfd.*

Proof. Let $(x, y) \in \operatorname{int} \mathbb{M}$. We show that $F(x, y)$ is finite-dimensional. To this end, let $\delta > 0$ be such that $(x - \delta, y), (x + \delta, y + \delta) \in \mathbb{M}$. Now let

$$x_0 := x - \delta, \qquad\qquad x_1 := x, \qquad\qquad x_2 := x + \delta,$$
$$y_1 := y, \quad \text{and} \qquad\qquad y_2 := y + \delta.$$

We consider the commutative diagram

$$
\begin{array}{ccccc}
F(x_0, y_2) & \longrightarrow & F(x_1, y_2) & \longrightarrow & F(x_2, y_2) \\
\downarrow & & \downarrow & & \downarrow \\
F(x_0, y_1) & \longrightarrow & F(x_1, y_1) & \longrightarrow & F(x_2, y_1) \\
\| & & \| & & \| \\
F(x_0, y_1) & \longrightarrow & F(x_1, y_1) & \longrightarrow & F(x_2, y_1).
\end{array}
$$

By Proposition C.2.4 all squares in this diagram are middle exact, hence

$$
\frac{F(x_1, y_1)}{\operatorname{Im} F(x_0 \le x_1, y_1) + \operatorname{Im} F(x_1, y_1 \le y_2)} \cong \frac{\operatorname{Im} F(x_1 \le x_2, y_1) + \operatorname{Im} F(x_2, y_1 \le y_2)}{\operatorname{Im} F(x_0 \le x_2, y_1) + \operatorname{Im} F(x_2, y_1 \le y_2)}
$$

by Proposition B.3. As F is q-tame, the numerator on the right hand side and both denominators are finite-dimensional. Thus, the numerator $F(x_1, y_1) = F(x, y)$ on the left has to be finite-dimensional as well. $\qquad\square$

Corollary 3.13. *Let $F \colon \mathbb{M}^\circ \to \operatorname{Vect}_K$ be a q-tame cohomological functor. Then F is pfd and*

$$
F \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}} (B_v)^{\oplus \mu(v)},
$$

where $\mu := \operatorname{Dgm}(F)$.

3.1 Connections to Level Set and Extended Persistence

We now use Theorem 3.4 to connect RISC to two other variants of persistence, namely level set persistence [CdM09] and extended persistence [CSEH09]. A posteriori this also implies that our Definition 3.1 of the extended persistence diagram is consistent with the original definition by [CSEH09]. We make use of the connection to extended persistence in our discussion of Example 4.5. Other than this, we describe these connections for the reader who is already familiar with either of these two notions.

3.1.1 The Level Set Barcode

Here we connect the extended persistence diagram, as defined in Definition 3.1, to the *level set barcode*, as introduced by [CdM09] as *levelset zigzag persistence*; see also [BEMP13, CdSKM19]. The extended persistence diagram is a multiset of points in $\mathbb{M}$ while the level set barcode is a multiset of pairs of an integer degree and an interval in $\mathbb{R}$. To connect the two notions we describe a bijection between $\operatorname{int} \mathbb{M}$ and $\mathbb{Z} \times \mathcal{I}$, where $\mathcal{I} \subset 2^{\mathbb{R}}$ is the set of non-empty intervals in $\mathbb{R}$. To this end, let $u \in \operatorname{int} \mathbb{M}$ and let $n \in \mathbb{Z}$ be the unique integer with $T^n(u) \in D$. We define

$$
\nu(u) := n \quad \text{and} \quad I(u) := (\rho_1 \circ T^n)(u) \setminus (\rho_0 \circ T^n)(u)
$$

and we think of $\nu(u)$ as the degree associated to u and of $I(u)$ as the associated interval. With this we obtain the bijection

$$\beta \colon \operatorname{int} \mathbb{M} \to \mathbb{Z} \times \mathcal{I}, \ u \mapsto (\nu(u), I(u)) \,.$$

Now let $f \colon X \to \mathbb{R}$ be a piecewise linear function with X a finite simplicial complex and let $\mu := \operatorname{Dgm}(f) \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ be the associated extended persistence diagram. We argue that

$$\mu \circ \beta^{-1} \colon \mathbb{Z} \times \mathcal{I} \to \mathbb{N}_0$$

is the level set barcode of f in the following way. We choose a *representation* $p \colon S \to \operatorname{int} \mathbb{M}$ of the multiset $\mu = \operatorname{Dgm}(f)$. This means that $p \colon S \to \operatorname{int} \mathbb{M}$ is some map of sets with

$$\mu(v) = \# p^{-1}(v) \quad \text{for all } v \in \operatorname{int} \mathbb{M}.$$

We now describe how $\beta \circ p \colon S \to \mathbb{Z} \times \mathcal{I}$ is a representation of the levelset barcode of $f \colon X \to \mathbb{R}$. For each $v \in \operatorname{int} \mathbb{M}$ we choose a basis $\{\omega_s\}_{s \in p^{-1}(v)}$ for a complement of

$$\sum_{w \succ v} \operatorname{Im} h(f)(v \preceq w)$$

in $h(f)(v)$. Moreover, for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ we set

$$S_{n,t} := \{s \in S \mid n = (\nu \circ p)(s) \text{ and } t \in (I \circ p)(s)\} \,.$$

As $t \in (\rho_1 \circ T^n \circ p)(s)$ for any $s \in S_{n,t}$, the corresponding cohomology class $\omega_s \in (\mathcal{H}^n \circ \rho \circ T^n \circ p)(s)$ has a pullback

$$\omega_s|_t \in \mathcal{H}^n(f^{-1}(t)).$$

Now suppose we can show that $\{\omega_s|_t\}_{s \in S_{n,t}}$ is a basis of $\mathcal{H}^n(f^{-1}(t))$ for any $t \in \mathbb{R}$, then this suggests that $\beta \circ p$ is a representation of the levelset barcode for the following reason. For each $s \in S$ we obtain a degree $n := (\nu \circ p)(s)$ as well as an interval $(I \circ p)(s)$. Moreover, for each $t \in (I \circ p)(s)$ we have a basis element $\omega_s|_t \in \mathcal{H}^n(f^{-1}(t))$ in the n-th cohomology of the fiber of t. Thus, the entire family $\{\omega_s\}_{s \in S}$ induces a simultaneous decomposition of the cohomology spaces of all fibers of f in such a way that any two basis elements associated to the same $s \in S$ arise as pullbacks of the same cohomology class ω_s. While this is not the original definition of the level set barcode from [CdM09], it comes very close and we omit the remaining details. We conclude with the following proposition providing the missing ingredient to the above argument.

Proposition 3.14. *Let $t \in \mathbb{R}$. Then the family $\{\omega_s|_t\}_{s \in S_{n,t}}$ is a basis of $\mathcal{H}^n(f^{-1}(t))$.*

Proof. Without loss of generality we assume that f is a simplicial map to some finite geometric simplicial complex with $\mathbb{R}$ as its ambient space and vertex set $V \subset \mathbb{R}$. Now let

$$U_t := \{u \in D \mid t \in I(u)\} = (\downarrow \blacktriangle t) \cap \operatorname{int} \left(\uparrow \left(T^{-1} \circ \blacktriangle\right)(t)\right),$$

see also Fig. 3.4. Then we have

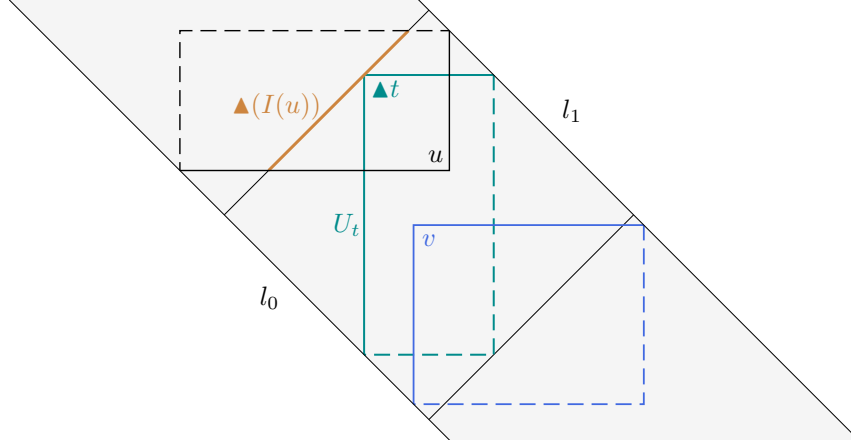$$S_{n,t} = (T^n \circ p)^{-1}(U_t). \tag{3.12}$$

21

Figure 3.4: The axis-aligned rectangle U_t and the support of the indecomposable B_v.

Moreover, let $\varepsilon > 0$ be such that $(t - \varepsilon, t + \varepsilon) \cap V \setminus \{t\} = \emptyset$. Then the homotopy invariance of $\mathcal{H}^n$ implies that the induced map

$$\mathcal{H}^n(f^{-1}(t - \delta, t + \delta)) \to \mathcal{H}^n(f^{-1}(t))$$

is an isomorphism for any $0 < \delta \leq \varepsilon$. Now

$$\{t\} = \bigcap_{u \in U_t} \rho_1(u)$$

and furthermore, $\{(t - \delta, t + \delta) \mid \delta > 0\}$ is a final subset of $\{\rho_1(u) \mid u \in U_t\}$. Thus, we have

$$\begin{aligned}
\mathcal{H}^n(f^{-1}(t)) &\cong \varinjlim_{u \in U_t} \left(\mathcal{H}^n \circ f^{-1} \circ \rho_1\right)(u) \\
&\cong \varinjlim_{u \in U_t} \left(\mathcal{H}^n \circ f^{-1} \circ \rho\right)(u) \\
&= \varinjlim_{u \in U_t} \left(h(f) \circ T^{-n}\right)(u).
\end{aligned} \tag{3.13}$$

Here the second isomorphism follows from the fact that U_t has a final subset on which ρ_0 is empty, hence $(\rho_1, \emptyset)$ and ρ agree on this subset. By the Yoneda Lemma 3.3 there is a unique natural transformation

$$\varphi_s \colon B_{(T^n \circ p)(s)} \to h(f) \circ T^{-n}$$

sending $1 \in K = B_{(T^n \circ p)(s)}((T^n \circ p)(s))$ to $\omega_s \in (h(f) \circ T^{-n} \circ p)(s)$ for any $s \in S$. By Theorem 3.4 the family $\{\varphi_s\}_{s \in S}$ yields a natural isomorphism

$$\varphi \colon \bigoplus_{s \in S} B_{(T^n \circ p)(s)} \to h(f) \circ T^{-n}.$$

Now for $v \in \operatorname{int} \mathbb{M}$ we have

$$\varinjlim_{u \in U_t} B_v(u) \cong \begin{cases} K & v \in U_t \\ \{0\} & v \notin U_t, \end{cases}$$

see also Fig. 3.4. In conjunction with (3.12) and (3.13) we obtain that $\{\omega_s|_t\}_{s \in S_{n,t}}$ is a basis for $\mathcal{H}^n(f^{-1}(t))$. $\qquad \square$
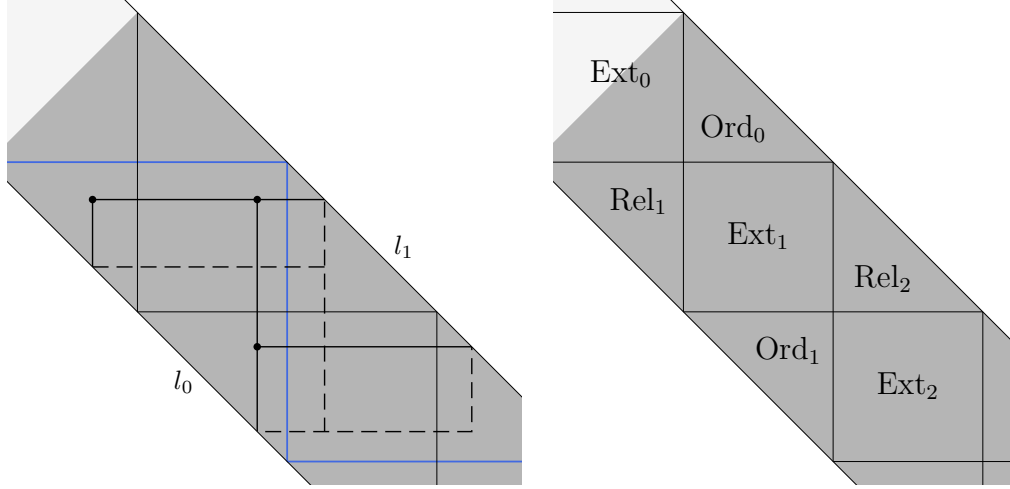
Figure 3.5: In the graphic on the left we see the subposet of $\mathbb{M}$ corresponding to extended persistent cohomology shaded in blue as well as three vertices contained in the domains corresponding to 1-dimensional relative, extended, and ordinary persistent cohomology. On the right we see the regions in the strip $\mathbb{M}$ corresponding to the ordinary, relative, and extended subdiagrams [CSEH09]. In both figures the support $\downarrow \mathrm{Im} \blacktriangle$ of RISC and of extended persistence diagrams is shaded in grey.

3.1.2 Extended Persistence

Here we describe how the extended persistence diagram as defined by [CSEH09] corresponds to our Definition 3.1. We note that the connection between the levelset barcode and extended persistence is well-understood [CdM09, BEMP13]. Here we provide a correspondence between RISC and extended persistence without requiring any finiteness assumptions other than the RISC being pfd.

Now let $f \colon X \to \mathbb{R}$ be a continuous function such that its RISC $h(f) \colon \mathbb{M} \to \mathrm{Vect}_K$ is pfd. We consider the left hand side of Fig. 3.5 and the restriction of $h(f)$ to the subposet of $\mathbb{M}$, which is shaded in blue in this figure. Here each point on the horizontal blue line segment to the upper left is assigned the cohomology space in degree 0 of an open sublevel set of f, of X, or of a pair with X as the first component and an open superlevel set as the second component. Up to isomorphism of posets, this is the extended persistent cohomology of f in degree 0. (Strictly speaking, in the original definition, which is for piecewise linear functions, closed sublevel sets and closed superlevel sets are used. When considering continuous functions with fewer restrictions, it is not uncommon to consider preimages of open subsets in place of closed sets.) Similarly, any point on the vertical blue line segment in the center is assigned the cohomology space of some pair of preimages in degree 1 and any point on the horizontal blue line at the lower right is assigned the cohomology of some pair in degree 2. By Theorem 3.4 the RISC $h(f)$ decomposes into contravariant blocks as in Definition 3.2. Now the support of each such contravariant block intersects exactly one of these blue line segments. We focus on the vertical blue line segment in the center of the graphic on the left in Fig. 3.5, which carries the extended persistent cohomology of f in degree 1. Any choice of decomposition of $h(f)$ yields a decomposition of its restriction to this line segment and thus of persistent cohomology in degree 1. Moreover,

the support of the contravariant block assigned to any of the black dots in the graphic on the left of Fig. 3.5 intersects the vertical line segment.

First assume that the black dot on the lower right appears in $\mathrm{Dgm}(f)$. Then the restriction of the associated contravariant block to the vertical blue line segment is a direct summand of the persistent cohomology of f in degree 1 and the intersection of its support with the blue line segment is the life span of the corresponding feature in the sense that the point of intersection of the upper edge marks the birth of a cohomology class that dies as soon as it is pulled back to the open sublevel set corresponding to the point of intersection of the lower edge. Moreover, this life span is encoded by the position of this black dot. Now this particular black dot on the lower right of the left graphic in Fig. 3.5 is contained in the triangular region labeled as Ord_1 in the graphic on the right hand side of Fig. 3.5. Furthermore, any vertex of the extended persistence diagram $\mathrm{Dgm}(f)$ contained in the triangular region labeled Ord_1 describes a feature of f, which is born at some open sublevel set and also dies at some open sublevel set. Thus, up to reparametrization the restriction of $\mathrm{Dgm}(f)$ to the region labeled Ord_1 is the ordinary persistence diagram of f in degree 1.

Now suppose that the black dot on the upper left of the left hand image in Fig. 3.5 appears in $\mathrm{Dgm}(f)$. Then the intersection of the support of the associated contravariant block with the vertical blue line segment describes the life span of a feature which is born at the cohomology of X relative to some open superlevel set in degree 1 and also dies at some relative cohomology space. Moreover, this is true for any vertex of $\mathrm{Dgm}(f)$ contained in the triangular region labeled Rel_1 in the graphic on the right in Fig. 3.5. Thus, up to reparametrization the restriction of $\mathrm{Dgm}(f)$ to the region labeled Rel_1 is the relative subdiagram of f in degree 1.

Finally, the black dot to the upper right in the left hand image of Fig. 3.5 (if in $\mathrm{Dgm}(f)$), or any other vertex of $\mathrm{Dgm}(f)$ in the square region labeled Ext_1 in the graphic on the right in Fig. 3.5, describes a feature, which is born at the cohomology of X relative to some open superlevel set and dies at the cohomology of some sublevel set of X. Thus, the restriction of $\mathrm{Dgm}(f)$ to the square region labeled Ext_1 is the extended subdiagram of f in degree 1.

As we have analogous correspondences for each line segment of the subposet of $\mathbb{M}$ shaded in blue in the left image of Fig. 3.5, we obtain a partition of the lower right part of the strip $\mathbb{M}$ into regions, corresponding to ordinary, relative, and extended subdiagrams of $\mathrm{Dgm}(f)$ analogous to [CSEH09].

3.2   Decomposition of Homological Functors

We note that $\mathbb{M}$ is self-dual as a lattice. Thus, there is an obvious dual version of Theorem 3.4. Now we state this result for the sake of completeness.

Remark 3.15. The reflection at the diagonal

$$\mathbb{M} \to \mathbb{M}, \ (x, y) \mapsto (y, x)$$

is a self-duality of the lattice $\mathbb{M}$ in the sense that it is order-reversing and interchanges joins and meets. In particular, any covariant functor on $\mathbb{M}$ can be made into a contravariant functor and vice versa by precomposition with this reflection.

The following is dual to Definition 3.2 in the sense of this remark, see also Fig. 3.1b.

Definition 3.16 (Block). For $v \in \mathrm{int}\,\mathbb{M}$ we define

$$B^v \colon \mathbb{M} \to \mathrm{Vect}_K, \ w \mapsto \begin{cases} K & w \in (\uparrow v) \cap \mathrm{int}(\downarrow T(v)) \\ \{0\} & \text{otherwise,} \end{cases}$$

where $\text{int}(\downarrow T(v))$ is the interior of the downset of $T(v)$ in $\mathbb{M}$. Whenever it makes sense, the internal maps are identities, otherwise they are zero.

Similarly we have the dual of Lemma 3.3.

Lemma 3.17 (Yoneda)**.** *Let* $G\colon \mathbb{M} \to \text{Vect}_K$ *be a functor vanishing on* $\partial\mathbb{M}$ *and let* $v \in \text{int}\,\mathbb{M}$. *Then the evaluation at* $1 \in K = B^v(v)$ *yields a linear isomorphism*

$$\text{Nat}(B^v, G) \cong G(v),$$

where $\text{Nat}(B^v, G)$ *denotes the vector space of natural transformations from* B^v *to* G.

Now let $F\colon \mathbb{M} \to \text{vect}_K$ be a sequentially continuous *homological functor*, as defined in Definition 2.2 and Definition C.3. For each $v \in \text{int}\,\mathbb{M}$ we choose a basis for a complement of $\sum_{u \prec v} \text{Im}\, F(u \preceq v)$ in $F(v)$. By Lemma 3.17 this yields a natural transformation

$$\varphi\colon \bigoplus_{v \in \text{int}\,\mathbb{M}} (B^v)^{\oplus\mu(v)} \to F,$$

where $\mu := \text{Dgm}(F)$ is defined dually to (3.1) as

$$\text{Dgm}(F) : \text{int}\,\mathbb{M} \to \mathbb{N}_0, \ v \mapsto \dim_K F(v) - \dim_K \sum_{u \prec v} \text{Im}\, F(u \preceq v). \tag{3.14}$$

With this we may state the dual of Theorem 3.4 in the sense of Remark 3.15.

Theorem 3.18. *The natural transformation*

$$\varphi\colon \bigoplus_{v \in \text{int}\,\mathbb{M}} (B^v)^{\oplus\mu(v)} \to F$$

is a natural isomorphism.

4 Interleavings

Let X be a non-empty topological space. For two functions $f, g\colon X \to \mathbb{R}$ we define

$$\mathbf{d}(f, g) := (\inf(g - f), \sup(g - f)),$$

then $\mathbf{d}$ can be thought of as a type of distance assigning two values as it satisfies a form of triangle inequality: If $\mathbf{d}(f_1, f_2), \mathbf{d}(f_2, f_3) \in \mathbb{R}^\circ \times \mathbb{R}$, then also $\mathbf{d}(f_1, f_3) \in \mathbb{R}^\circ \times \mathbb{R}$ and

$$\mathbf{d}(f_1, f_3) \preceq \mathbf{d}(f_1, f_2) + \mathbf{d}(f_2, f_3). \tag{4.1}$$

Now let $(x, y) = \mathbf{d}(f, g) \in \mathbb{R}^2$ for two functions $f, g\colon X \to \mathbb{R}$, then $\frac{1}{2}(y - x)$ is the minimal infinity distance between f and any shift of g. Moreover, $-\frac{1}{2}(x + y)$ is the shift, for which this minimum is attained. This is the information we obtain from $\mathbf{d}$ about f and g and (4.1) captures in a single inequality how shifts and the infinity distance interact for three functions. In this section we describe how this type of distance $\mathbf{d}$ and the triangle inequality (4.1) resurface in a different form when considering relative interlevel set cohomology.
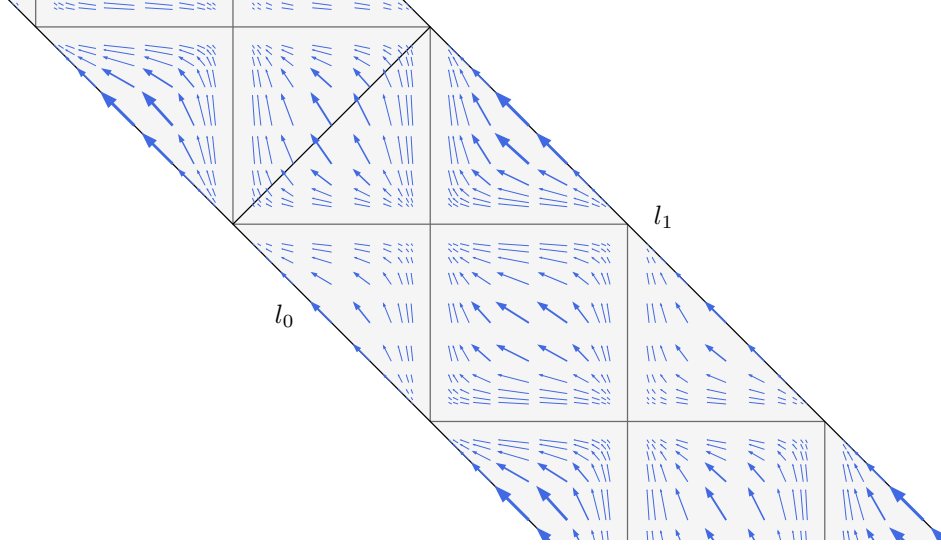
Figure 4.1: The lattice automorphism $\alpha_a\colon \mathbb{M} \to \mathbb{M}$ for $a = (-0.315, 0.525)$.

As in several instances before, suppose that l_0 and l_1 intersect the x-axis in $-\pi$ and π, respectively. Moreover, let $\mathrm{Aut}(\mathbb{M})$ be the automorphism group of $\mathbb{M}$ in the category of lattices. Then there is a unique group homomorphism

$$\alpha\colon \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{M}) \quad \text{with} \quad \mathrm{ev}_0 \circ \alpha = \arctan \times \arctan,$$

where $\mathrm{ev}_0\colon \mathrm{Aut}(\mathbb{M}) \to \mathbb{M}$, $\varphi \mapsto \varphi(0)$ is the evaluation at the origin, see also Fig. 4.1. We provide an explicit description of α in Remark 4.1 below. We also note that $\mathbb{R}^\circ \times \mathbb{R}$ is an abelian group object in the category of lattices and the uncurrying

$$\alpha^b\colon (\mathbb{R}^\circ \times \mathbb{R}) \times \mathbb{M} \to \mathbb{M}$$

of α exhibits $\mathbb{M}$ as a module over $\mathbb{R}^\circ \times \mathbb{R}$. In particular, $\alpha\colon \mathbb{R}^\circ \times \mathbb{R} \to \mathrm{Aut}(\mathbb{M})$ is monotone. For better readability we will often write the argument of α as an index by writing α_a in place of $\alpha(a)$ for $a \in \mathbb{R}^\circ \times \mathbb{R}$.

Remark 4.1 (An Explicit Description of α). Let $a := (a_1, a_2) \in \mathbb{R}^\circ \times \mathbb{R}$. We construct $\alpha_a\colon \mathbb{M} \to \mathbb{M}$ as follows. First we have the inverse radial projection or S^1-valued inverse tangent function

$$\phi\colon \mathbb{R} \to S^1, \ y \mapsto \frac{1}{\sqrt{1 + y^2}}(1, y) = \exp(i \arctan(y)).$$

Then we define

$$g_a\colon S^1 \to S^1, (x, y) \mapsto \begin{cases} (0, y), & x = 0 \\ \phi\left(\frac{y}{x} + a_2\right), & x > 0 \\ -\phi\left(\frac{y}{x} - a_1\right), & x < 0. \end{cases}$$

As $g_a(i) = i$, where $i = (0, 1)$ is the imaginary unit, there is a unique continuous map $\tilde{g}_a\colon \mathbb{R} \to \mathbb{R}$ with

$$\tilde{g}_a\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \text{and} \quad (g_a \circ \exp)(it) = \exp(i\tilde{g}_a(t)) \quad \text{for all } t \in \mathbb{R}.$$

26

With $\sigma\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto \pi - t$ being the reflection at $\pi/2$ we have

$$\alpha_a = (\sigma \circ \tilde{g}_a \circ \sigma) \times \tilde{g}_a.$$

Now suppose we have $a := \mathbf{d}(f,g) \in \mathbb{R}^\circ \times \mathbb{R}$. In the following we describe how this induces a natural transformation

$$\vec{h}(f,g)\colon h(g) \circ \alpha_a \to h(f).$$

To this end, let $F\colon \mathbb{M} \to \left(\mathrm{Vect}_K^{\mathbb{Z}}\right)^\circ$ be defined as in the construction of $h(f)$ in Section 2. Or, in other words, we define F to be the transform of $(h(f))^\circ$ under the 2-adjunction from Lemma A.1. Completely analogously we have a functor $G\colon \mathbb{M} \to \left(\mathrm{Vect}_K^{\mathbb{Z}}\right)^\circ$ as we would use it in the construction of $h(g)$. We use Lemma A.7 to construct a natural transformation $\varphi\colon F \to G \circ \alpha_a$ and then we obtain $\vec{h}(f,g)\colon h(g) \circ \alpha_a \to h(f)$ as $\vec{h}(f,g) := \mathrm{ev}^0 \circ \varphi^\circ$ by whiskering with the evaluation at 0 analogously to the construction of $h(f)$ from F. Now suppose $a = (x,y)$ and $(r,s) \subseteq \mathbb{R}$ is an open interval, then we have

$$f^{-1}(r,s) \subseteq g^{-1}(r+x, s+y)$$

and thus

$$(f^{-1} \circ \rho)(u) \subseteq (g^{-1} \circ \rho \circ \alpha_a)(u)$$

for all $u \in \mathbb{M}$. As in [BdSS15, Section 3], this induces a linear map

$$\mathcal{H}^\bullet(f^{-1} \circ \rho)(u) \leftarrow \mathcal{H}^\bullet(g^{-1} \circ \rho \circ \alpha_a)(u),$$

which is natural in $u \in \mathbb{M}$. Now we have

$$
\begin{aligned}
F(u) &= \left(\mathcal{H}^\bullet \circ f^{-1} \circ \rho\right)(u) &&\text{for all } u \in D \text{ and}\\
(G \circ \alpha_a)(u) &= \left(\mathcal{H}^\bullet \circ g^{-1} \circ \rho \circ \alpha_a\right)(u) &&\text{for all } u \in E := \alpha_{(-a)}(D),
\end{aligned}
$$

see also Fig. 4.2. Thus, we may restrict F and $G \circ \alpha_a$ to the intersection $D \cap E$ to obtain the natural transformation

$$\eta\colon F|_{D\cap E} \to (G \circ \alpha_a)|_{D\cap E}.$$

As shown in Fig. 4.2, the intersection $D \cap E$ is no fundamental domain in general, so this does not describe a natural transformation from F to $G \circ \alpha_a$. In order to extend η to a natural transformation $\varphi\colon F \to G \circ \alpha_a$ we use Lemma A.7. So we need to supply the missing ingredient, which is a natural transformation

$$\nu\colon F|_{D\cap T(E)} \to (G \circ \alpha_a)|_{D\cap T(E)}.$$

To this end, we consider the monotone map

$$\xi\colon u \mapsto ((g^{-1} \circ \rho_1 \circ \alpha_a)(u), (f^{-1} \circ \rho_0)(u))$$

from $\mathbb{M}$ to the set of pairs of open subspaces of X. In some sense ξ interpolates between $f^{-1} \circ \rho$ and $g^{-1} \circ \rho \circ \alpha_a$, since we have the chain of inclusions

$$f^{-1} \circ \rho \subseteq \xi \subseteq g^{-1} \circ \rho \circ \alpha_a$$

pointwise in $\mathbb{M}$ and moreover, ξ agrees with $f^{-1} \circ \rho$ when restricted to the region shaded in red in Fig. 4.3. In particular we have
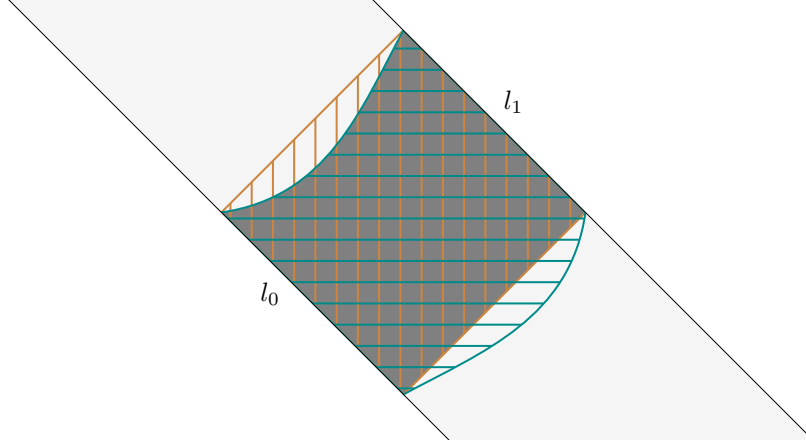
27

Figure 4.2: The fundamental domain D, its shift E by $\alpha(-a)$, and their intersection shaded in grey.

$$\left(\mathcal{H}^{\bullet} \circ \xi\right)(u) = \left(\mathcal{H}^{\bullet} \circ f^{-1} \circ \rho\right)(u) = F(u) \tag{4.2}$$

for any point u contained in the red region. Furthermore, ξ and $g^{-1} \circ \rho \circ \alpha_a$ agree, when restricted to the region shaded in blue. Thus, if $T^{-1}(w)$ is contained in the blue region for some $w \in \mathbb{M}$, then

$$\begin{aligned}
\left(\mathcal{H}^{\bullet-1} \circ \xi \circ T^{-1}\right)(w) &= \left(\mathcal{H}^{\bullet-1} \circ g^{-1} \circ \rho \circ \alpha_a \circ T^{-1}\right)(w) \\
&= (\Sigma \circ G \circ T^{-1} \circ \alpha_a)(w) \\
&= (G \circ \alpha_a)(w).
\end{aligned} \tag{4.3}$$

Here the second equality follows from the fact that T is a central automorphism and the third from G being strictly stable. By Proposition 2.1.(3-4) the map ξ preserves the joins and meets of any axis-aligned rectangle contained in $D \cup E$. Thus, any axis-aligned rectangle in $D \cup E$ gives rise to some relative Mayer–Vietoris sequence of subspaces of X. In particular, if $v \in D \cap T(E)$, then the axis-aligned rectangle shown in Fig. 4.3 gives rise to a Mayer–Vietoris sequence with the differential

$$\delta \colon \left(\mathcal{H}^{\bullet-1} \circ \xi \circ T^{-1}\right)(v) \to \left(\mathcal{H}^{\bullet} \circ \xi\right)(v).$$

By combining this with the equations (4.2) and (4.3) and passing to the opposite category we define

$$\nu_v := \delta^{\circ} \colon F(v) \to (G \circ \alpha_a)(v).$$

The naturality of $\nu \colon F|_{D \cap T(E)} \to (G \circ \alpha_a)|_{D \cap T(E)}$ follows from the naturality of the Mayer–Vietoris sequence.

Next we show that the natural transformations μ and ν satisfy property (1) from Lemma A.7. Here it suffices to show that the solid square in

$$
\begin{array}{ccc}
F(v) & \xrightarrow{\nu_v = \delta^{\circ}} & (G \circ \alpha_a)(v) \\
{\scriptstyle F(u \preceq v)} \big\uparrow & {\scriptstyle (\delta')^{\circ}} \nearrow & \big\uparrow {\scriptstyle (G \circ \alpha_a)(u \preceq v)} \\
F(u) & \xrightarrow[\eta_u]{} & (G \circ \alpha_a)(u)
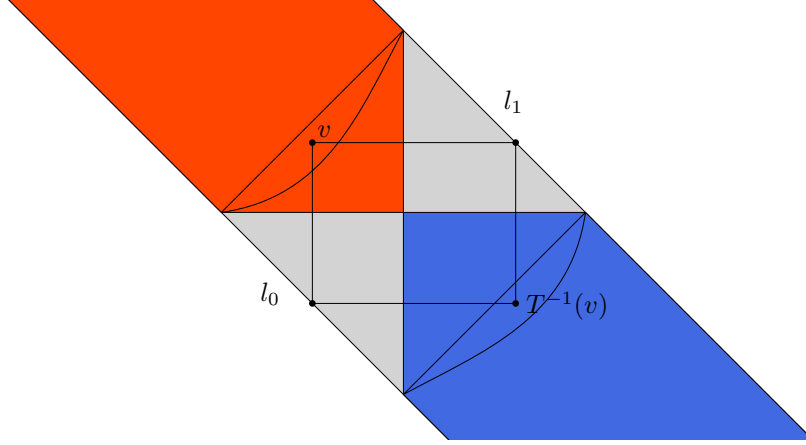\end{array}
\tag{4.4}
$$

Figure 4.3: The maps ξ and $f^{-1} \circ \rho$ coincide on the red region, whereas ξ and $g^{-1} \circ \rho \circ \alpha_a$ agree on the blue region.

commutes for all u contained in the region shaded in red in Fig. 4.4. For such a point u, we construct a linear map $(\delta')^\circ$ as indicated with the dashed arrow in (4.4) and then we show that both triangles commute. To this end, we consider the axis-aligned rectangle in Fig. 4.4 with u a vertex. As ξ preserves the corresponding join u and the meet $T^{-1}(v)$, this gives rise to a Mayer–Vietoris sequence with differential

$$\delta' \colon \left(\mathcal{H}^{\bullet - 1} \circ \xi \circ T^{-1} \right)(v) \to \left(\mathcal{H}^\bullet \circ \xi \right)(u).$$

By equations (4.2) and (4.3) the domain and codomain of $(\delta')^\circ$ match up with the dashed arrow in (4.4). Now the upper triangle commutes by the naturality of the Mayer–Vietoris sequence. Moreover, we have pointwise $\xi \subseteq g^{-1} \circ \rho \circ \alpha_a$ and thus, using the naturality of the Mayer–Vietoris sequence once again, we obtain the commutativity of the lower triangle in (4.4).

Next we show that μ and ν satisfy property (2) from Lemma A.7. Here it suffices to show that the solid square in

$$\begin{array}{ccc}
F(w) & \xrightarrow{\ (\Sigma \circ \eta \circ T^{-1})_w\ } & (G \circ \alpha_a)(w) \\
{\scriptstyle F(v \preceq w)} \Big\uparrow & {\scriptstyle (\delta'')^\circ} \nearrow & \Big\uparrow {\scriptstyle (G \circ \alpha_a)(v \preceq w)} \\
F(v) & \xrightarrow[\ \nu_v = \delta^\circ\ ]{} & (G \circ \alpha_a)(v)
\end{array} \qquad (4.5)$$

commutes for all w contained in the region shaded in green in Fig. 4.5. As with property (1) we provide the dashed arrow and then we show that both triangles commute. To this end, we consider the axis-aligned rectangle shown in Fig. 4.5 with v and $T^{-1}(w)$ vertices. As ξ preserver the join v and the meet $T^{-1}(w)$, this gives rise to a Mayer–Vietoris sequence with differential

$$\delta'' \colon \left(\mathcal{H}^{\bullet - 1} \circ \xi \circ T^{-1} \right)(w) \to \left(\mathcal{H}^\bullet \circ \xi \right)(v).$$

By equations (4.2) and (4.3) the domain and the codomain of $(\delta'')^\circ$ match up with the dashed arrow in (4.5). The lower triangle commutes by the naturality of the Mayer–Vietoris
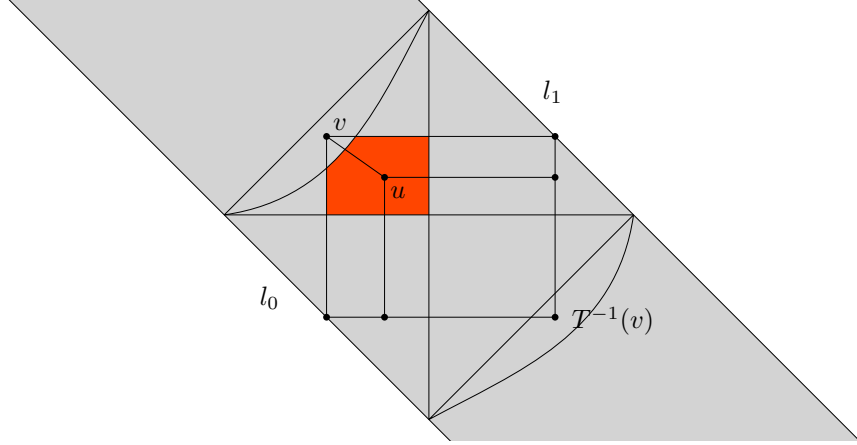
Figure 4.4: The points u and v with corresponding axis-aligned rectangles in $D \cup E$.
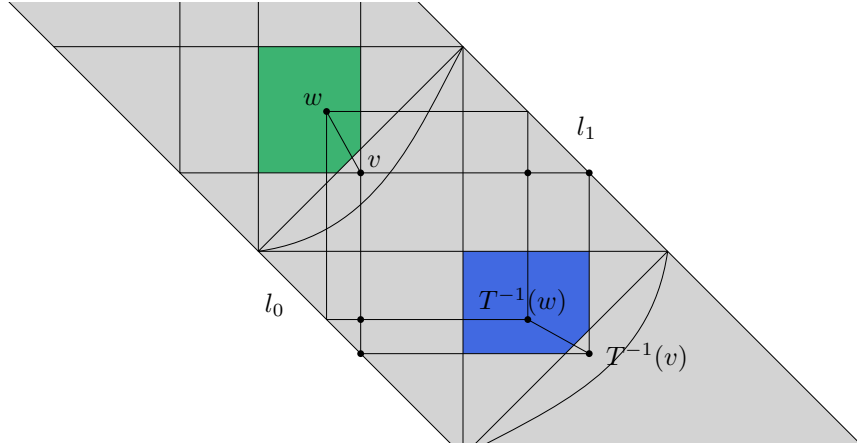

Figure 4.5: The points v and w with corresponding axis-aligned rectangles in $D \cup E$.

sequence. Moreover, we have pointwise $f^{-1} \circ \rho \subseteq \xi$ and thus, using the naturality of the Mayer–Vietoris sequence once again, we obtain the commutativity of the upper triangle in (4.5).

With the assumptions of Lemma A.7 satisfied we may extend η and ν to a unique strictly stable natural transformation $\varphi \colon F \to G \circ \alpha_a$. We define $\vec{h}(f,g) \colon h(g) \circ \Omega_\delta \to h(f)$ as $\vec{h}(f,g) := \mathrm{ev}^0 \circ \varphi^\circ$ by whiskering with the evaluation at 0.

With the next proposition we show that $\vec{h}$ and h preserves triangles in some sense. To this end, let $f_1, f_2, f_3 \colon X \to \mathbb{R}$ be functions with $\mathbf{d}(f_1, f_2), \mathbf{d}(f_2, f_3) \in \mathbb{R}^\circ \times \mathbb{R}$. Moreover, let

$$a := \mathbf{d}(f_1, f_2), \quad b := \mathbf{d}(f_2, f_3), \text{ and } c := \mathbf{d}(f_1, f_3),$$

then we have $c \preceq a + b$ by the triangle inequality (4.1).

Proposition 4.2 (Compatibility with Composition)**.** *The following square of functors and*

natural transformations

$$
\begin{array}{ccc}
h(f_3) \circ \alpha_{a+b} & \xrightarrow{\ \vec{h}(f_2,f_3)\circ\alpha_a\ } & h(f_2) \circ \alpha_a \\
{\scriptstyle h(f_3)\circ\alpha_{c\preceq a+b}}\Big\downarrow & & \Big\downarrow{\scriptstyle \vec{h}(f_1,f_2)} \\
h(f_3) \circ \alpha_c & \xrightarrow[\ \vec{h}(f_1,f_3)\ ]{} & h(f_1)
\end{array}
\qquad (4.6)
$$

commutes.

Proof. Let $h^{\#}(f_1)$ be the transform of $h(f_1)$ under the 2-adjunction from Lemma A.1. We define $h^{\#}(f_2)$, $h^{\#}(f_3)$, $\vec{h}^{\#}(f_1,f_2)$, $\vec{h}^{\#}(f_2,f_3)$, and $\vec{h}^{\#}(f_1,f_3)$ analogously. Then the commutativity of (4.6) is equivalent to the commutativity of

$$
\begin{array}{ccc}
h^{\#}(f_3) \circ \alpha_{a+b} & \xrightarrow{\ \vec{h}^{\#}(f_2,f_3)\circ\alpha_a\ } & h^{\#}(f_2) \circ \alpha_a \\
{\scriptstyle h^{\#}(f_3)\circ\alpha_{c\preceq a+b}}\Big\downarrow & & \Big\downarrow{\scriptstyle \vec{h}^{\#}(f_1,f_2)} \\
h^{\#}(f_3) \circ \alpha_c & \xrightarrow[\ \vec{h}^{\#}(f_1,f_3)\ ]{} & h^{\#}(f_1)
\end{array}
$$

by Lemma A.1. As all functors and natural transformations in this square are strictly stable, it suffices to check the commutativity for any point $v \in D$. To this end, we partition D into the three regions

$$
\begin{aligned}
D_1 &:= D \cap \alpha_{(-a-b)}(D), \\
D_2 &:= \alpha_{(-a)}(D) \setminus \alpha_{(-a-b)}(D), \quad \text{and} \\
D_3 &:= D \setminus \alpha_{(-a)}(D).
\end{aligned}
$$

For $v \in D_1$ the commutativity follows from the functoriality of the cohomology theory $\mathcal{H}^{\bullet}$. For $v \in D_2 \cup D_3$ we consider the monotone map

$$
\xi_0 \colon u \mapsto ((f_3^{-1} \circ \rho_1 \circ \alpha_{a+b})(u), (f_1^{-1} \circ \rho_0)(u)),
$$

where $\alpha_{a+b} := \alpha(a+b)$. Then the axis-aligned rectangle shown in Fig. 4.3 gives rise to a Mayer–Vietoris sequence with differential

$$
\delta \colon \left(\mathcal{H}^{\bullet-1} \circ \xi_0 \circ T^{-1}\right)(v) \to \left(\mathcal{H}^{\bullet} \circ \xi_0\right)(v).
$$

By equations (4.2) and (4.3) we have

$$
\left(\mathcal{H}^{\bullet} \circ \xi_0\right)(v) = h^{\#}(f_1)(v) \quad \text{and} \quad \left(\mathcal{H}^{\bullet-1} \circ \xi_0 \circ T^{-1}\right)(v) = \left(h^{\#}(f_3) \circ \alpha_{a+b}\right)(v). \qquad (4.7)
$$

Moreover, by the naturality of the Mayer–Vietoris sequence, the triangle

$$
\begin{array}{ccc}
\left(h^{\#}(f_3) \circ \alpha_{a+b}\right)(v) & & \\
{\scriptstyle \left(h^{\#}(f_3)\circ\alpha_{c\preceq a+b}\right)_v}\Big\downarrow & \searrow{\scriptstyle \delta} & \\
\left(h^{\#}(f_3) \circ \alpha_c\right)(v) & \xrightarrow[\ \vec{h}^{\#}(f_1,f_3)_v\ ]{} & h^{\#}(f_1)(v)
\end{array}
$$

commutes. It remains to show that the triangle

$$\left(h^\#(f_3) \circ \alpha_{a+b}\right)(v) \xrightarrow{\left(\vec{h}^\#(f_2,f_3)\circ\alpha_a\right)_v} \left(h^\#(f_2) \circ \alpha_a\right)(v)$$

with a diagonal map δ to $h^\#(f_1)(v)$ and a vertical map $\vec{h}^\#(f_1,f_2)_v$ (4.8)

commutes. For $v \in D_3$ we consider the monotone map

$$\xi_1 \colon u \mapsto ((f_2^{-1} \circ \rho_1 \circ \alpha_a)(u), (f_1^{-1} \circ \rho_0)(u)),$$

which we used for the construction of $\vec{h}^\#(f_1, f_2)$. By equations (4.2) and (4.3) we have

$$\left(\mathcal{H}^\bullet \circ \xi_1\right)(v) = h^\#(f_1)(v) \quad \text{and} \quad \left(\mathcal{H}^{\bullet-1} \circ \xi_1 \circ T^{-1}\right)(v) = \left(h^\#(f_2) \circ \alpha_a\right)(v).$$

In conjunction with (4.7) this allows us to rewrite (4.8) as

$$\begin{array}{ccc}
\left(\mathcal{H}^{\bullet-1} \circ \xi_0 \circ T^{-1}\right)(v) & \xrightarrow{\left(\vec{h}^\#(f_2,f_3)\circ\alpha_a\right)_v} & \left(\mathcal{H}^{\bullet-1} \circ \xi_1 \circ T^{-1}\right)(v) \\
\Big\downarrow{\delta} & & \Big\downarrow{\vec{h}^\#(f_1,f_2)_v} \\
\left(\mathcal{H}^\bullet \circ \xi_0\right)(v) & =\!=\!=\!=\!=\!= & \left(\mathcal{H}^\bullet \circ \xi_1\right)(v).
\end{array}$$

Moreover, we have pointwise $\xi_1 \subseteq \xi_0$, and thus the commutativity of this square follows from the naturality of the Mayer–Vietoris sequence. For $v \in D_2$ we consider the monotone map

$$\xi_2 \colon u \mapsto ((f_3^{-1} \circ \rho_1 \circ \alpha_b)(u), (f_2^{-1} \circ \rho_0)(u)),$$

which we used for the construction of $\vec{h}^\#(f_2, f_3)$. By equations (4.2) and (4.3) we have

$$\left(\mathcal{H}^\bullet \circ \xi_2 \circ \alpha_a\right)(v) = \left(h^\#(f_2) \circ \alpha_a\right)(v)$$

$$\text{and} \quad \left(\mathcal{H}^{\bullet-1} \circ \xi_2 \circ \alpha_a \circ T^{-1}\right)(v) = \left(\mathcal{H}^{\bullet-1} \circ \xi_2 \circ T^{-1} \circ \alpha_a\right)(v) = \left(h^\#(f_3) \circ \alpha_{a+b}\right)(v).$$

Here the first equality on the second line follows from the fact that T is a central automorphism. In conjunction with (4.7) this allows us to rewrite (4.8) as

$$\begin{array}{ccc}
\left(\mathcal{H}^{\bullet-1} \circ \xi_2 \circ \alpha_a \circ T^{-1}\right)(v) & \xrightarrow{\left(\vec{h}^\#(f_2,f_3)\circ\alpha_a\right)_v} & \left(\mathcal{H}^\bullet \circ \xi_2 \circ \alpha_a\right)(v) \\
\Big\| & & \Big\downarrow{\vec{h}^\#(f_1,f_2)_v} \\
\left(\mathcal{H}^{\bullet-1} \circ \xi_0 \circ T^{-1}\right)(v) & \xrightarrow{\quad\delta\quad} & \left(\mathcal{H}^\bullet \circ \xi_0\right)(v).
\end{array}$$

Moreover, we have pointwise $\xi_0 \subseteq \xi_2 \circ \alpha_a$, and thus the commutativity of this square follows from the naturality of the Mayer–Vietoris sequence. $\qquad\square$

Next we show how this proposition implies *stability* for RISC in the sense of [BdSS15]. The group homomorphism

$$\Omega \colon \mathbb{R} \to \mathrm{Aut}(\mathbb{M}), \ \delta \mapsto \Omega_\delta := \alpha_{(-\delta,\delta)},$$

describes a *(super)linear family* on $\mathbb{M}$ in the sense of [BdSS15, Section 2.5].

Definition 4.3 (δ-Interleaving, [BdSS15]). Let $\delta \geq 0$ and let $F, G\colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ be contravariant functors on $\mathbb{M}$. Then a δ-*interleaving* of F and G is a pair of natural transformations

$$\varphi\colon G \circ \Omega_\delta \to F \quad \text{and} \quad \psi\colon F \circ \Omega_\delta \to G$$

such that both triangles in the diagram

$$(4.9)$$

commute. Moreover, we say that F and G are δ-*interleaved* if there is a δ-interleaving.

Theorem 4.4 (Stability). *Let $\delta \geq 0$ and let $f, g\colon X \to \mathbb{R}$ be continuous functions with $|f(x) - g(x)| \leq \delta$ for all $x \in X$. Then $h(f)$ and $h(g)$ are δ-interleaved. More specifically, the interleaving natural transformations can be given as the compositions*

$$\varphi\colon h(g) \circ \Omega_\delta \xrightarrow{h(g) \circ \alpha_{a \preceq (-\delta, \delta)}} h(g) \circ \alpha_a \xrightarrow{\vec{h}(f,g)} h(f)$$

$$and \quad \psi\colon h(f) \circ \Omega_\delta \xrightarrow{h(f) \circ \alpha_{a' \preceq (-\delta, \delta)}} h(f) \circ \alpha_{a'} \xrightarrow{\vec{h}(g,f)} h(g),$$

where $(x,y) := a := \mathbf{d}(f,g)$ and $a' := (-y, -x) = \mathbf{d}(g,f)$.

Proof. By Proposition 4.2 we have the commutative diagram

which yields the commutativity of the triangle on the right hand side in (4.9). By symmetry, the proof that the left triangle commutes is completely analogous. $\qquad\square$
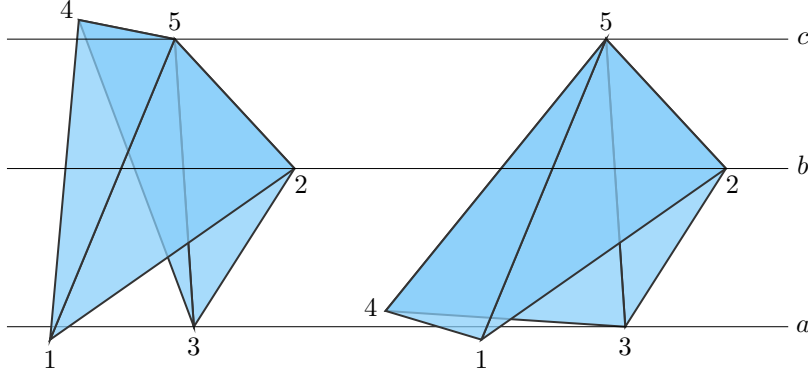
Figure 4.6: The function f depicted as a height function on the left hand side and the height function g' on the right, see Example 4.5.

Example 4.5. Let C_4 be the cyclic graph on four vertices $\{1, 2, 3, 4\}$. Moreover, let $5 * C_4$ be the abstract simplicial cone over C_4 with 5 as the tip of the cone and let $X := |5 * C_4|$ be the geometric realization of $5 * C_4$. Furthermore, let $a < b < c$ and let $f, g' \colon X \to \mathbb{R}$ be the unique simplexwise linear functions with

$$f(1) = f(3) = a = g'(1) = g'(3) = g'(4)$$
$$f(2) = b = g'(2),$$
$$\text{and} \quad f(4) = f(5) = c = g'(5),$$

which are also depicted as height functions in Fig. 4.6. Now let $\delta := \frac{1}{2}(c - a)$ and let $g := \delta + g'$. Then we have $\|f - g\|_\infty = \delta$ and thus there is a δ-interleaving with $\varphi := \vec{h}(f, g)$ as one of the two interleaving natural transformations by the previous theorem. Moreover, we have $h(g) \circ \alpha_{(\delta, \delta)} = h(g')$ and thus

$$h(g) \circ \Omega_\delta = h(g) \circ \alpha_{(\delta, \delta)} \circ \alpha_{(-2\delta, 0)} = h(g') \circ \alpha_{(-2\delta, 0)}$$

and $\varphi = \vec{h}(f, g) = \vec{h}(f, g')$. Now let

$$\bar{a} := \arctan a,$$
$$\bar{b} := \arctan b,$$
$$\bar{c} := \arctan c,$$
$$\text{and} \quad \tilde{c} := \arctan(2c - a).$$

Then we have

$$\mathrm{Dgm}(f) = \mathbf{1}_{\left\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{a})\right\}} \quad \text{and} \quad \mathrm{Dgm}(g') = \mathbf{1}_{\left\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{c} - 2\pi)\right\}},$$

where $\mathbf{1}_{\left\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{a})\right\}} \colon \mathrm{int}\,\mathbb{M} \to \mathbb{N}_0$ is the indicator function of the subset $\left\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{a})\right\} \subset \mathrm{int}\,\mathbb{M}$; see also Fig. 4.7. As

$$\alpha_{(-2\delta, 0)}(\tilde{c}, \bar{a}) = (\bar{c}, \bar{a}) \quad \text{and} \quad \alpha_{(-2\delta, 0)}\left(\pi - \bar{b}, \bar{c} - 2\pi\right) = \left(\pi - \bar{b}, \bar{c} - 2\pi\right),$$
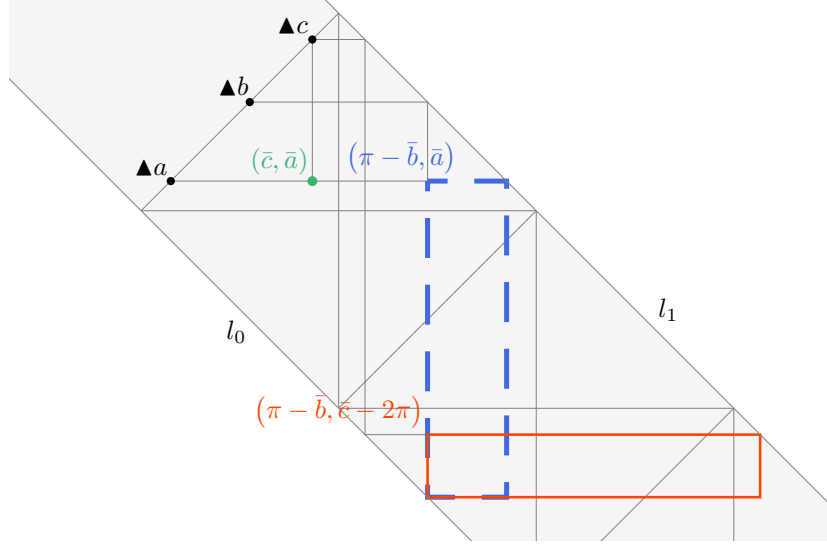
34

Figure 4.7: The indecomposables $B_{\left(\pi-\bar{b},\bar{a}\right)}$ and $B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)}$.

we have $h(g') \circ \alpha_{(-2\delta,0)} \cong B_{(\tilde{c},\bar{a})} \oplus B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)}$ and thus there is a commutative diagram

$$
\begin{array}{ccccc}
B_{(\tilde{c},\bar{a})} \oplus B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)} & \overset{\sim}{\longrightarrow} & h(g') \circ \alpha_{(-2\delta,0)} & = & h(g) \circ \Omega_\delta \\
\Big\downarrow & & \Big\downarrow \vec{h}(f,g') & & \Big\downarrow \varphi \\
\scriptstyle B_{(\tilde{c},\bar{a})\preceq(\bar{c},\bar{a})} \oplus B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)\preceq\left(\pi-\bar{b},\bar{a}\right)} & & & & \\
B_{(\bar{c},\bar{a})} \oplus B_{\left(\pi-\bar{b},\bar{a}\right)} & \overset{\sim}{\longrightarrow} & h(f) & = & h(f).
\end{array}
$$

We consider the natural transformation

$$
\eta := B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)\preceq\left(\pi-\bar{b},\bar{a}\right)} \colon B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)} \to B_{\left(\pi-\bar{b},\bar{a}\right)}.
$$

As is shown in Fig. 4.7 the supports of $B_{\left(\pi-\bar{b},\bar{a}\right)}$ and $B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)}$ have a non-empty intersection. For any u in this intersection, η_u maps $B_{\left(\pi-\bar{b},\bar{c}-2\pi\right)}(u) = K$ identically onto $B_{\left(\pi-\bar{b},\bar{a}\right)}(u) = K$. Moreover, Fig. 4.7 also shows that the intersection of supports is disjoint from the regions of $\mathbb{M}$ covered by extended persistence or the south face of the Mayer–Vietoris pyramid, see also Section 3.1.2. In particular, the corresponding interleaving homomorphisms of extended persistence [BS14, Section 6] and of Mayer–Vietoris systems [BGO19, Section 2.3] do not detect the non-trivial homomorphism between the corresponding summands. Nevertheless, derived level sets persistence and derived extended (or even derived sublevel sets) persistence will recognize the corresponding homomorphism as well. This is consistent with [BGO19, Remark 4.9], where the corresponding homomorphism of derived level sets persistence is provided as a counter-example to the canonical functor to Mayer–Vietoris systems being faithful.

Next we describe how $\vec{h}$ is compatible with precomposition. To this end, let $f, g \colon Y \to \mathbb{R}$

be continuous functions, let $\varphi\colon X \to Y$ be a continuous map, let $a := \mathbf{d}(f,g)$, and let $b := \mathbf{d}(f \circ \varphi, g \circ \varphi)$, then we have $b \preceq a$.

Proposition 4.6 (Compatibility with Precomposition)**.** *We have the commutative diagram*

$$
\begin{array}{ccc}
h(g) \circ \alpha_a & \xrightarrow{\;h(\varphi) \circ \alpha_a\;} & h(g \circ \varphi) \circ \alpha_a \\
\Big\downarrow{\scriptstyle \vec{h}(f,g)} & & \Big\downarrow{\scriptstyle h(g \circ \varphi) \circ \alpha_{b \preceq a}} \\
& & h(g \circ \varphi) \circ \alpha_b \\
& & \Big\downarrow{\scriptstyle \vec{h}(f \circ \varphi, g \circ \varphi)} \\
h(f) & \xrightarrow[\;h(\varphi)\;]{} & h(f \circ \varphi).
\end{array}
$$

Proof. Here the horizontal natural transformations are induced by inclusions. The vertical transformations are induced by inclusions for some points, and they are given as differentials of Mayer–Vietoris sequences for other points. In the former case the commutativity follows from the functoriality of $\mathcal{H}^\bullet$. In the latter case the commutativity follows from the naturality of the Mayer–Vietoris sequence. $\qquad\square$

Finally we provide a form of homotopy invariance for h and $\vec{h}$. To this end, we consider a commutative square

$$
\begin{array}{ccc}
X \times [0,1] & \xrightarrow{\;\Gamma\;} & Y \\
\Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle g} \\
X & \xrightarrow[\;f\;]{} & \mathbb{R}
\end{array}
$$

of spaces as well as the continuous maps $\varphi := \Gamma(-,0)$ and $\psi := \Gamma(-,1)$ from X to Y. Then we may think of φ and ψ as homomorphisms in the category of spaces over the reals $\mathbb{R}$ with domain f and codomain g. Moreover, we may think of Γ as a *homotopy* from φ to ψ. The following lemma states that *homotopic* homomorphisms of spaces over the reals are identified under h.

Lemma 4.7. *We have $h(\varphi) = h(\psi)\colon h(f) \to h(g)$.*

As usual, we say that two homomorphisms of spaces over the reals are *homotopy inverses* of one another if they are composable both ways and both compositions are homotopic to the corresponding identities. Moreover, we say that a homomorphism is a *homotopy equivalence*, if it has a homotopy inverse.

Corollary 4.8. *The functor h maps homotopy equivalences to natural isomorphisms.*

Now from this corollary to Lemma 4.7 we may in turn deduce a generalization of Lemma 4.7. Suppose we have functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ as well as a map $\Gamma\colon X \times [0,1] \to Y$ with $\mathbf{d}(f \circ \pi_1, g \circ \Gamma) \in \mathbb{R}^\circ \times \mathbb{R}$. Morally the following proposition says that h and $\vec{h}$ also "identify approximately homotopic maps".

Proposition 4.9 (Quantitative Homotopy Invariance)**.** *For $\varphi := \Gamma(-,0)$ and $\psi := \Gamma(-,1)$ as well as*

$$a := \mathbf{d}(f, g \circ \varphi),$$
$$b := \mathbf{d}(f, g \circ \psi),$$
$$and \quad c := \mathbf{d}(f \circ \pi_1, g \circ \Gamma),$$

we have the commutative square

$$
\begin{array}{ccc}
h(g) & \xrightarrow{\;\;h(\psi)\circ\alpha_c\;\;} & h(g \circ \psi) \circ \alpha_c \\
{\scriptstyle h(\varphi)\circ\alpha_c}\Big\downarrow & & \Big\downarrow{\scriptstyle h(g\circ\psi)\circ\alpha_{b \preceq c}} \\
& & h(g \circ \psi) \circ \alpha_b \\
& & \Big\downarrow{\scriptstyle \vec{h}(f,g\circ\psi)} \\
h(g \circ \varphi) \circ \alpha_c \xrightarrow{\;h(g\circ\varphi)\circ\alpha_{a \preceq c}\;} & h(g \circ \varphi) \circ \alpha_a \xrightarrow{\;\vec{h}(f,g\circ\varphi)\;} & h(f).
\end{array}
\tag{4.10}
$$

Proof. We consider the trapezium

$$
\begin{array}{c}
h(g) \circ \alpha_c \\
{\scriptstyle h(\Gamma)}\Big\downarrow \qquad\qquad {\scriptstyle h(\varphi)\circ\alpha_c} \\
h(g \circ \Gamma) \circ \alpha_c \xrightarrow{\;h(i_0)\circ\alpha_c\;} h(g \circ \varphi) \circ \alpha_c \\
{\scriptstyle \vec{h}(f\circ\pi_1,g\circ\Gamma)}\Big\downarrow \qquad\qquad \Big\downarrow{\scriptstyle h(g\circ\varphi)\circ\alpha_{a \preceq c}} \\
\qquad\qquad h(g \circ \varphi) \circ \alpha_a \\
\qquad\qquad \Big\downarrow{\scriptstyle \vec{h}(f,g\circ\varphi)} \\
h(f \circ \pi_1) \xrightarrow[\;h(i_0)\;]{\;\sim\;} h(f) \\
{\scriptstyle \wr}\Big\uparrow{\scriptstyle h(\pi_1)} \\
h(f)
\end{array}
$$

of functors and natural transformations. Here both triangles commute by the functoriality of h. The square in the center commutes by Proposition 4.6. Moreover, π_1 and i_0 are homotopy inverses of one another, hence both are homotopy equivalences. By the previous corollary the natural transformations $h(\pi_1)$ and $h(i_0)$ are natural isomorphisms. Thus, the three natural transformations along the left edge of this trapezium provide a natural transformation

$$\eta \colon h(g) \circ \alpha_c \to h(f)$$

from $h(g) \circ \alpha_c$ to $h(f)$, which is equal to the composition of the natural transformations along the other edges of the trapezium. These natural transformations along the other edges

are the same as the transformations on the left and at the bottom of (4.10). Completely analogously we see that η is identical to the composition of the other two sides of the square (4.10). □

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A   Stable Functors on the Strip $\mathbb{M}$

Definition. A *strictly stable category* is a pair of a category $\mathcal{C}$ and an automorphism of categories $\Sigma\colon \mathcal{C} \to \mathcal{C}$.

Example. The poset $\mathbb{M} \subset \mathbb{R}^\circ \times \mathbb{R}$, seen as a thin category, is a strictly stable category when endowed with the automorphism $T\colon \mathbb{M} \to \mathbb{M}$.

Now let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strictly stable categories endowed with automorphisms Σ_1 and Σ_2, respectively.

Definition. A functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is *strictly stable* if $F \circ \Sigma_1 = \Sigma_2 \circ F$. *Strictly stable* natural transformations are defined analogously.

With these definitions strictly stable categories form a strict 2-category. Moreover, by dropping the associated automorphism, we obtain a strict *forgetful* 2-functor from strictly stable categories to categories. We will now construct a right adjoint to this forgetful functor. To this end, let $\mathcal{D}$ be an ordinary category, then we may associate a strictly stable category $\mathcal{D}^{\mathbb{Z}}$ to $\mathcal{D}$.

Definition. The objects of $\mathcal{D}^{\mathbb{Z}}$ are maps $M_\bullet\colon n \mapsto M_n$ from $\mathbb{Z}$ to the class of objects in $\mathcal{D}$. Homomorphisms, composition, and identities are defined pointwise in $\mathcal{D}^{\mathbb{Z}}$. As the associated automorphism we choose

$$\Sigma\colon \mathcal{D}^{\mathbb{Z}} \to \mathcal{D}^{\mathbb{Z}}, \quad M_\bullet \mapsto M_{\bullet-1}.$$

This construction yields a strict 2-functor from categories to strictly stable categories.

Example. The category of $\mathbb{Z}$-graded vector spaces $\mathrm{Vect}_K^{\mathbb{Z}}$ over K is the strictly stable category associated to Vect_K.

Lemma A.1. *The forgetful functor and $(-)^{\mathbb{Z}}$ form a strict 2-adjunction with the evaluation at 0 as the counit*

$$\mathrm{ev}_0\colon \mathcal{D}^{\mathbb{Z}} \to \mathcal{D}, \quad M_\bullet \mapsto M_0.$$

Depending on the context, we may also write $M^\bullet$ in place of $M_\bullet$ and ev^0 in place of ev_0. Now let $\mathcal{A}$ be an additive (or pointed) category, let $C \subset \mathbb{M}$ be a convex subposet, and let $F\colon C \to \mathcal{A}$ be a functor vanishing on $C \cap \partial\mathbb{M}$.

Lemma A.2. *Let $u, v \in C$ with $u \preceq v \not\preceq T(u)$. Then $F(u \preceq v) = 0$.*

Now let Σ be an automorphism of $\mathcal{A}$ and let $F \colon \mathbb{M} \to \mathcal{A}$ be a strictly stable functor vanishing on $\partial \mathbb{M}$, let D be a convex subposet of $\mathbb{M}$ that is a fundamental domain with respect to the action of $\langle T \rangle$, and let $F' := F|_D$.

Definition A.3. We set $R_D := \big\{ (v, w) \in D \times T(D) \mid v \preceq w \preceq T(v) \big\}$.

If we view R_D as a subposet of $D \times T(D)$ with the product order, we obtain the two functors $F' \circ \mathrm{pr}_1 = F \circ \mathrm{pr}_1$ and $\Sigma \circ F' \circ T^{-1} \circ \mathrm{pr}_2 = F \circ \mathrm{pr}_2$, where $\mathrm{pr}_1 \colon R_D \to D$ and $\mathrm{pr}_2 \colon R_D \to T(D)$ are the projections to the first and the second component, respectively. The following definition provides a natural transformation $\partial(F, D)$ as in the diagram

$$
\begin{array}{ccc}
R_D & \xrightarrow{\ \mathrm{pr}_1\ } & D \\
{\scriptstyle \mathrm{pr}_2}\big\downarrow & \ \ \ {}^{\partial(F,D)} & \big\downarrow {\scriptstyle F} \\
T(D) & \xrightarrow[\ F\ ]{} & \mathcal{A}.
\end{array}
$$

Definition. We set $\ \partial(F, D) \colon F \circ \mathrm{pr}_1 \Rightarrow F \circ \mathrm{pr}_2, (v, w) \mapsto F(v \preceq w)$.

In the following statement we will use $w \preceq T(v) \in T^{n+1}(D)$ as a shorthand for $w, T(v) \in T^{n+1}(D)$, $n \in \mathbb{Z}$, and $w \preceq T(v)$.

Lemma A.4. *Suppose $\partial' := \partial(F, D)$, then*

$$
F(v \preceq w) = \begin{cases}
(\Sigma^n \circ F' \circ T^{-n})(v \preceq w) & v, w \in T^n(D) \\
(\Sigma^n \circ \partial')_{(T^{-n}(v), T^{-n}(w))} & w \preceq T(v) \in T^{n+1}(D) \\
0 & \textit{otherwise}
\end{cases}
\tag{A.1}
$$

for all $v \preceq w \in \mathbb{M}$.

The previous lemma shows that F is determined by its restriction $F|_D$ and the natural transformation $\partial(F, D)$.

Now let $F' \colon D \to \mathcal{A}$ be an arbitrary functor vanishing on $D \cap \partial \mathbb{M}$, let ∂' be a natural transformation as in the diagram

$$
\begin{array}{ccc}
R_D & \xrightarrow{\ \mathrm{pr}_1\ } & D \\
{\scriptstyle \mathrm{pr}_2}\big\downarrow & \ \ \ {}^{\partial'} & \big\downarrow {\scriptstyle F'} \\
T(D) & \xrightarrow[\ \Sigma \circ F' \circ T^{-1}\ ]{} & \mathcal{A},
\end{array}
$$

and let $F \colon \mathbb{M} \to \mathcal{A}$ be defined by equation (A.1). We aim to show that F is a functor.

Lemma A.5. *Let $v \preceq w \not\preceq T(v)$, then $F(v \preceq w) = 0$.*

Proof. If $v, w \in T^n(D)$ for some n, the statement follows from Lemma A.2 and the defining equation (A.1). In any other case the result follows directly from the construction (A.1). $\square$

Lemma. *Let $u \preceq v \preceq w \preceq T(u)$, then*

$$
F(u \preceq w) = F(v \preceq w) \circ F(u \preceq v).
$$

Proof. Without loss of generality we assume $u \in D$. Since $D \cup T(D)$ is convex we have $v, w \in D \cup T(D)$. If $w \in D$ we are done, since D is convex and F' is a functor. Suppose $w \in T(D)$ and $v \in D$, then

$$\begin{aligned} F(u \preceq w) &= \partial'_{(u,w)} \\ &= \partial'_{(v,w)} \circ F'(u \preceq v) \\ &= F(v \preceq w) \circ F(u \preceq v) \end{aligned}$$

by the naturality of ∂' in the first argument. Similarly if $v, w \in T(D)$, then

$$\begin{aligned} F(u \preceq w) &= \partial'_{(u,w)} \\ &= (\Sigma \circ F' \circ T^{-1})(v \preceq w) \circ \partial'_{(u,v)} \\ &= F(v \preceq w) \circ F(u \preceq v) \end{aligned}$$

follows from the naturality of ∂' in its second argument. $\qquad\square$

Lemma. *The data for F yields a functor.*

Proof. For all $u \preceq v \preceq w \in \mathbb{M}$ we have to show the equation

$$F(u \preceq w) = F(v \preceq w) \circ F(u \preceq v).$$

If $w \preceq T(u)$, then we are done by the previous lemma. Otherwise Lemma A.5 implies that $F(u \preceq w) = 0$ and thus we have to show

$$0 = F(v \preceq w) \circ F(u \preceq v).$$

In case $v \npreceq T(u)$ or $w \npreceq T(v)$, Lemma A.5 applies to the right hand side of this equation as well.

Now suppose $v \preceq T(u)$ and $w \preceq T(v)$. Since $\partial\,(\downarrow T(u))$ divides $\mathbb{M}$ into two connected components there is some point $v' \in [v \preceq w] \cap \partial\,(\downarrow T(u))$. Two applications of the previous lemma yield

$$\begin{aligned} F(v \preceq w) \circ F(u \preceq v) &= F(v' \preceq w) \circ F(v \preceq v') \circ F(u \preceq v) \\ &= F(v' \preceq w) \circ F(u \preceq v'). \end{aligned}$$

We are done if we can show that $F(u \preceq v') = 0$.

Now $F|_{[u, T(u)]}$ is a functor by the previous lemma. Moreover, $u \preceq v'$ factors through a point in $\partial\mathbb{M}$ by our choice of v'. And since $F|_{\partial\mathbb{M}} = 0$ we obtain $F(u \preceq v') = 0$ and thus the desired result. $\qquad\square$

Altogether the previous lemma and Lemma A.4 imply the following.

Proposition A.6. *For any functor $F' \colon D \to \mathcal{A}$ vanishing on $D \cap \partial\mathbb{M}$ together with a natural transformation*

$$\begin{array}{ccc} R_D & \xrightarrow{\ \mathrm{pr}_1\ } & D \\ {\scriptstyle \mathrm{pr}_2}\Big\downarrow & \Swarrow{\scriptstyle \partial'} & \Big\downarrow{\scriptstyle F'} \\ T(D) & \xrightarrow[\ \Sigma \circ F' \circ T^{-1}\ ]{} & \mathcal{A}, \end{array}$$

there is a unique strictly stable functor $F \colon \mathbb{M} \to \mathcal{A}$ with

$$F|_D = F', \quad F|_{\partial\mathbb{M}} = 0, \quad and \quad \partial(F, D) = \partial'.$$

41

We end this appendix by discussing how one may extend partially defined natural transformations. To this end, let $F, G\colon \mathbb{M} \to \mathcal{A}$ be strictly stable functors vanishing on $\partial\mathbb{M}$ and let $D, E \subset \mathbb{M}$ be convex fundamental domains with $D \subset E \cup T(E)$.

Lemma A.7. *Let $\eta\colon F|_{D\cap E} \to G|_{D\cap E}$ and $\nu\colon F|_{D\cap T(E)} \to G|_{D\cap T(E)}$ be natural transformations with the following two properties:*

(1) For any $u \in D \cap E$ and $v \in D \cap T(E)$ with $u \preceq v \preceq T(u)$ the diagram

$$
\begin{array}{ccc}
F(v) & \xrightarrow{\;\nu_v\;} & G(v) \\
{\scriptstyle F(u \preceq v)}\big\uparrow & & \big\uparrow{\scriptstyle G(u \preceq v)} \\
F(u) & \xrightarrow[\;\eta_u\;]{} & G(u)
\end{array}
$$

commutes.

(2) For any $v \in D \cap T(E)$ and $w \in T(D \cap E)$ with $v \preceq w \preceq T(v)$ the diagram

$$
\begin{array}{ccc}
F(w) & \xrightarrow{\;(\Sigma \circ \eta \circ T^{-1})_w\;} & G(w) \\
{\scriptstyle F(v \preceq w)}\big\uparrow & & \big\uparrow{\scriptstyle G(v \preceq w)} \\
F(v) & \xrightarrow[\;\nu_v\;]{} & G(v)
\end{array}
$$

commutes.

Then the natural transformations η and ν extend uniquely to a single strictly stable natural transformation from F to G.

B   Middle Exact Squares

Definition B.1. We say that a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\;f_{AB}\;} & B \\
{\scriptstyle f_{AC}}\big\downarrow & & \big\downarrow{\scriptstyle f_{BD}} \\
C & \xrightarrow[\;f_{CD}\;]{} & D
\end{array}
$$

of vector spaces (or modules) is *middle exact* if the sequence

$$
A \xrightarrow{\;\begin{pmatrix} f_{AB} \\ f_{AC} \end{pmatrix}\;} B \oplus C \xrightarrow{\;\begin{pmatrix} f_{BD} & -f_{CD} \end{pmatrix}\;} D
$$

is exact (at the middle term $B \oplus C$).

Now suppose we have two adjacent middle exact squares

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
D & \longrightarrow & E & \longrightarrow & F,
\end{array}
$$

with the maps denoted by f_{AB}, $f_{AC} = f_{BC} \circ f_{AB}$, and so forth.

Lemma B.2. *We have $f_{EF}^{-1}\left(\operatorname{Im} f_{DF} + \operatorname{Im} f_{BF}\right) = \operatorname{Im} f_{DE} + \operatorname{Im} f_{BE}$.*

Proof. It is clear that the right hand side is a subspace of the left hand side. The other inclusion follows from the following diagram chase. Suppose we have

$$u \in f_{EF}^{-1}\left(\operatorname{Im} f_{DF} + \operatorname{Im} f_{BF}\right),$$

then there are vectors $v \in B$ and $w \in D$ with

$$f_{EF}(u) = f_{DF}(w) + f_{BF}(v).$$

Moreover, we have that

$$f_{EF}(u - f_{DE}(w) - f_{BE}(v)) = 0,$$

hence the term

$$\begin{pmatrix} 0 \\ u - f_{DE}(w) - f_{BE}(v) \end{pmatrix} \in C \oplus E$$

is in the kernel of $\begin{pmatrix} f_{CF} & -f_{EF} \end{pmatrix}$. By the exactness of the sequence

$$B \xrightarrow{\begin{pmatrix} f_{BC} \\ f_{BE} \end{pmatrix}} C \oplus E \xrightarrow{\begin{pmatrix} f_{CF} & -f_{EF} \end{pmatrix}} F$$

there is a $v' \in B$ with

$$f_{BE}(v') = u - f_{DE}(w) - f_{BE}(v),$$

which is equivalent to $f_{DE}(w) + f_{BE}(v + v') = u$. $\square$

Now we consider a diagram of four adjacent middle exact squares

$$\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
D & \longrightarrow & E & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
G & \longrightarrow & H & \longrightarrow & I.
\end{array} \tag{B.1}$$

We note that middle exact squares "compose" to middle exact squares. If we compose the two squares in the first row of (B.1) as well as the two squares in the second row and then transpose[2] the diagram, we obtain this diagram

$$\begin{array}{ccccc}
A & \longrightarrow & D & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & F & \longrightarrow & I
\end{array} \tag{B.2}$$

of two adjacent middle exact squares, which we will use at the end of the proof of the following proposition.

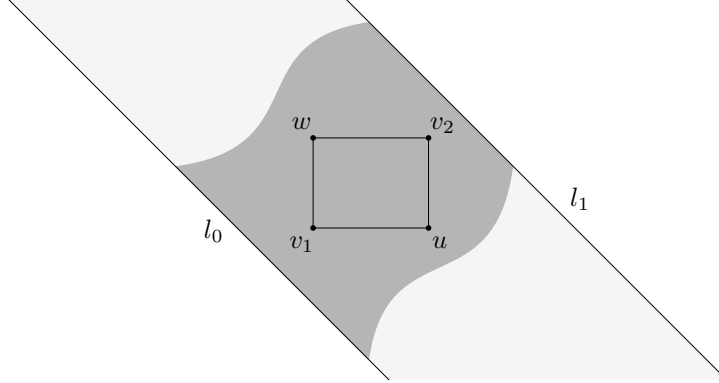[2] i.e. we reflect the diagram along the antidiagonal

Figure C.1: An axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ contained in the fundamental domain D, which is shaded in grey.

Proposition B.3. *The map f_{EI} induces a natural isomorphism*

$$\frac{E}{\operatorname{Im} f_{DE} + \operatorname{Im} f_{BE}} \cong \frac{\operatorname{Im} f_{EI} + \operatorname{Im} f_{CI}}{\operatorname{Im} f_{DI} + \operatorname{Im} f_{CI}}.$$

Proof. As the upper right square of (B.1) is middle exact we have

$$\operatorname{Im} f_{BF} = \operatorname{Im} f_{EF} \cap \operatorname{Im} f_{CF}. \tag{B.3}$$

With this we obtain the chain of three isomorphisms and two equalities

$$\begin{aligned}
\frac{E}{\operatorname{Im} f_{DE} + \operatorname{Im} f_{BE}} &\cong \frac{\operatorname{Im} f_{EF}}{\operatorname{Im} f_{DF} + \operatorname{Im} f_{BF}} \\
&= \frac{\operatorname{Im} f_{EF}}{\operatorname{Im} f_{DF} + \operatorname{Im} f_{EF} \cap \operatorname{Im} f_{CF}} \\
&= \frac{\operatorname{Im} f_{EF}}{\operatorname{Im} f_{EF} \cap (\operatorname{Im} f_{DF} + \operatorname{Im} f_{CF})} \\
&\cong \frac{\operatorname{Im} f_{EF} + \operatorname{Im} f_{CF}}{\operatorname{Im} f_{DF} + \operatorname{Im} f_{CF}} \\
&\cong \frac{\operatorname{Im} f_{EI} + \operatorname{Im} f_{CI}}{\operatorname{Im} f_{DI} + \operatorname{Im} f_{CI}}.
\end{aligned}$$

Here the first isomorphism follows from the previous lemma applied to the two squares at the top of (B.1) and the first isomorphism theorem. The first equality follows from (B.3). The second equality follows from the modular law for the lattice of subspaces. The second isomorphism follows from the second isomorphism theorem and the last isomorphism again from the previous lemma applied to (B.2) and the first isomorphism theorem. $\qquad \square$

C   Cohomological Functors on $\mathbb{M}$

Let $F \colon \mathbb{M}^{\circ} \to \operatorname{Vect}_K$ be a contravariant functor vanishing on $\partial \mathbb{M}$. Moreover, suppose there is a convex fundamental domain $D \subset \mathbb{M}$ with respect to $\langle T \rangle$, such that for any axis-aligned
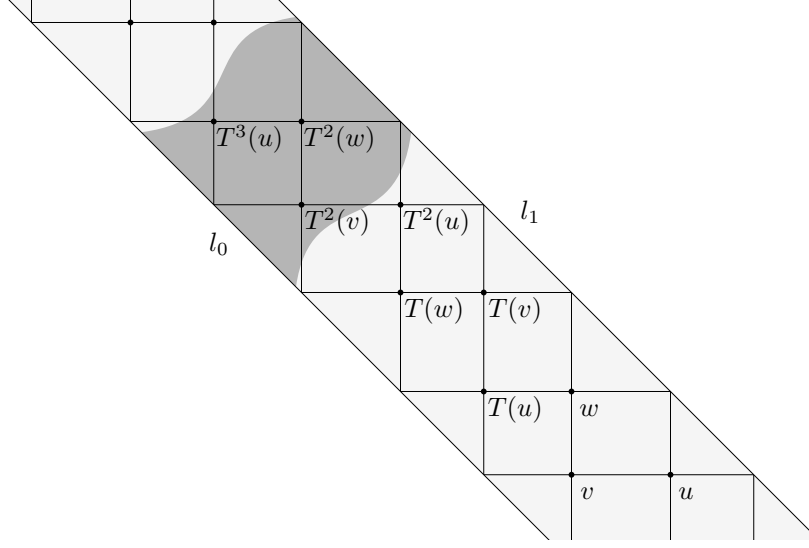
Figure C.2: The linear subposet given by the orbits of u, v, and w. The region shaded in dark grey is our fundamental domain D.

rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. C.1, the long sequence

$$\cdots \longrightarrow F(T(u)) \longrightarrow$$
$$\longrightarrow F(w) \longrightarrow F(v_1) \oplus F(v_2) \xrightarrow{(1\ -1)} F(u) \longrightarrow \tag{C.1}$$
$$\longrightarrow F(T^{-1}(w)) \longrightarrow \cdots$$

is exact. We show that (C.1) is exact for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$. We begin with the special case that $v_2 \in \partial\mathbb{M}$; then we have $F(v_2) \cong \{0\}$ and we set $v := v_1$. As shown in Fig. C.2 the union of the orbits of u, v, and w form a subposet, which is isomorphic to $\mathbb{Z}$. As D is convex, the intersection of D and this subposet consists of three consecutive points; in Fig. C.2 these are $T^2(v)$, $T^2(w)$, and $T^3(u)$. Moreover, these three consecutive points describe an axis-aligned rectangle contained in D. Thus, the restriction of F to this subposet yields the long exact sequence

$$\cdots \longrightarrow F(T(u)) \longrightarrow$$
$$\longrightarrow F(w) \longrightarrow F(v) \longrightarrow F(u) \longrightarrow \tag{C.2}$$
$$\longrightarrow F(T^{-1}(w)) \longrightarrow \cdots .$$
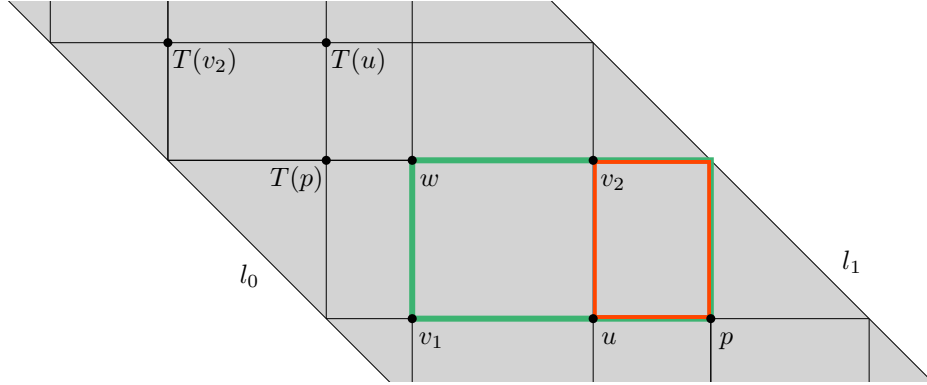
45

Figure C.3: Axis-aligned rectangles determined by u, v_1, v_2, and w.

Definition C.1. We say that a contravariant functor $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ vanishing on $\partial\mathbb{M}$ is *cohomological*, if for any axis-aligned rectangle with one corner lying on l_1 and the other corners $u \preceq v \preceq w \in \mathbb{M}$, the long sequence (C.2) is exact.

This notion of a cohomological functor is inspired by the theory of *triangulated categories*. A *cohomological* functor on a triangulated category yields a long exact sequence for any *distinguished triangle*, see for example [KS90, Definitions 1.5.2 and 1.5.1]. Here the defining property of a cohomological functor is that it yields long exact sequences for certain "triangles" in $\mathbb{M}$ or certain geodesic triangles on the Möbius strip $\mathbb{M}/\langle T \rangle$.

Now suppose we have an arbitrary axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ as shown in Fig. C.3. Together with the additional point $p \in \mathbb{M}$ we obtain two axis-aligned rectangles with one corner on l_1. As F is cohomological, the commutative diagram

$$\ldots \longrightarrow F(T(v_1)) \longrightarrow F(T(p)) \longrightarrow F(w) \longrightarrow F(v_1) \longrightarrow F(p) \longrightarrow \ldots$$
$$\downarrow \qquad\qquad \| \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \|$$
$$\ldots \longrightarrow F(T(u)) \longrightarrow F(T(p)) \longrightarrow F(v_2) \longrightarrow F(u) \longrightarrow F(p) \longrightarrow \ldots$$

has exact rows. Thus, by [tom08, Lemma 10.7.4] and [BW56, Lemma 7.4] the long sequence (C.1) is exact. From this we obtain the following.

Proposition C.2. *For a contravariant functor $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_K$ vanishing on $\partial\mathbb{M}$ the following are equivalent.*

(1) *There is a convex fundamental domain $D \subset \mathbb{M}$ such that for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ the long sequence (C.1) is exact.*

(2) *The contravariant functor F is cohomological.*

(3) *For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the long sequence (C.1) is exact.*

(4) *For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the square*

$$\begin{array}{ccc} F(w) & \longrightarrow & F(v_2) \\ \downarrow & & \downarrow \\ F(v_1) & \longrightarrow & F(u) \end{array}$$

is middle exact.

Proof. Above we have shown that (1) implies (2) and that (2) implies (3). It is clear that (3) implies both (1) and (4). Moreover, if we consider Fig. C.2, then we see that any three consecutive points of the sub-$\langle T \rangle$-set generated by u, v, and w describe an axis-aligned rectangle with one vertex on the boundary $\partial \mathbb{M}$ and thus (4) implies (2). □

We may dualize Definition C.1 and Proposition C.2 in the sense of Remark 3.15 as follows.

Definition C.3. We say that a functor $F\colon \mathbb{M} \to \mathrm{Vect}_K$ vanishing on $\partial \mathbb{M}$ is *homological*, if for any axis-aligned rectangle with one corner lying on l_1 and the other corners $u \preceq v \preceq w \in \mathbb{M}$, the long sequence

$$\cdots \longrightarrow F(T^{-1}(w)) \longrightarrow$$
$$\longrightarrow F(u) \longrightarrow F(v) \longrightarrow F(w) \longrightarrow$$
$$\longrightarrow F(T(u)) \longrightarrow \cdots .$$

is exact.

Proposition C.4. *For a functor $F\colon \mathbb{M} \to \mathrm{Vect}_K$ vanishing on $\partial \mathbb{M}$ the following are equivalent.*

(1) *There is a convex fundamental domain $D \subset \mathbb{M}$ such that for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ the long sequence*

$$\cdots \longrightarrow F(T^{-1}(w)) \longrightarrow$$
$$\longrightarrow F(u) \longrightarrow F(v_1) \oplus F(v_2) \xrightarrow{(1 \ \ -1)} F(w) \longrightarrow \tag{C.3}$$
$$\longrightarrow F(T(u)) \longrightarrow \cdots$$

is exact.

(2) *The functor F is homological.*

(3) *For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the long sequence (C.3) is exact.*

(4) *For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the square*

$$\begin{array}{ccc} F(u) & \longrightarrow & F(v_1) \\ \downarrow & & \downarrow \\ F(v_2) & \longrightarrow & F(w) \end{array}$$

is middle exact.