UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Master Degree in Astrophysics and Cosmology

Exam of Theoretical Cosmology

Fuzzy Dark Matter effect on Small Scale Structure

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Anno Accademico 2022/2023

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Chapter 1

Introduction

It is known that the concordance model of cosmology, the so-called ΛCDM model, is not perfect if it is taken by itself, because it shows some critical issues. To give an example, it can be mentioned the *Missing Satellite Problem.*

Considered true the ΛCDM model, the formation of structure in the Universe follows a *bottom-up* process, in the sense that the first structures formed are galaxies and then, through collisions and merges, cluster and supercluster of galaxies are formed as well.

The hierarchical clustering scenario states that galaxies are surrounded by gigantic regions of dark matter. These regions are called "*halos*" and the first ones formed are relatively small. The halos keep increasing their masses thanks to many merges. For example, the halo from which the Milky Way was born, is thought to be of a mass of $10^{12} M_{\odot}$ as a rough estimate.

These halos attract baryonic matter inside a potential well. This visible matter is then trapped, cooled down and will formed the observable structures in the Universe.

Although a reasonable number of normal-size galaxies can be seen with respect to what the model predicts, there is, however, a lack of small-size galaxies.

In fact, the model predicts way more dwarf galaxies than the observable ones, so the ΛCDM model could be wrong or there could be something missing in the structure formation theory.

This problem of absence of small-size galaxies could be explained by considering the so-called **Fuzzy Dark Matter**.

This particular type of dark matter substitutes the standard CDM (Cold Dark Matter) and in this review it will be found a mathematical equation that shows why, at small scales, structures are less numerous than what ΛCDM model predicts.

In order to have a definition, dark matter is called "wave" if the mass of the main constituent is $< 30\ eV$, and, in particular, it becomes "fuzzy", if this mass is extremely low, of the order of 10^{-22} eV.

Chapter 2

Relativistic Wave Equation

Before starting with the treating, in this first part of the chapter can be found some fundamental information that are used all through the paper.

To begin with, since at large scales the Universe is isotropic and homogeneous at zeroth order approximation, the metric used is the Friedmann–Lemaître–Robertson–Walker (FLRW). In comoving coordinates, the line element is

$$ds^2 = -c^2dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sen^2\theta \, d\varphi^2) \right] \tag{2.1}$$

And since the Universe is almost flat, the parameter k is setted to 0 and the metric becomes

$$\begin{aligned} ds^2 &= -c^2dt^2 + a(t)^2[dr^2 + r^2(d\theta^2 + sen^2\theta \, d\varphi^2)] \\ &= -c^2dt^2 + a(t)^2[dx^2 + dy^2 + dz^2] \end{aligned} \tag{2.2}$$

Another assumption made is that natural units are adopted, so $\hbar$ and c will be considered equal to 1. They will be considered for estimates where correct units are needed. The values of these constants are $h = 4.14 \cdot 10^{-15} eV \cdot s$, $\hbar = h/2\pi$, and $c = 3 \cdot 10^8 \; m/s$

The first step will be prove that the dark matter field can be treated as a classical field. This will be helpful for the next parts.

2.1 Classical Field Justification

Noted with $\bar{r}$ the interparticle separation, it is known that this parameter is dependent on $n^{-1/3}$, with n the number density.

It is given that the dark matter density in the neighborhood of the Solar System is $\rho_{DM,\odot} = 0.4$ $GeV/cm^3 = 4 \cdot 10^{23} \; eV/km^3$. This means that if it is known the mass of dark matter, the number density $n = \rho_{DM,\odot}/m_{DM}$ can be found.

The starting point is to compute the de Broglie wavelength.

$$\begin{aligned}\lambda_{dB} = \frac{2\pi}{mv}\,\hbar c^2 &= 1.49\left(\frac{10^{-6}eV}{m}\right)\left(\frac{250km/s}{v}\right)\,km \\ &= 0.48\left(\frac{10^{-22}eV}{m}\right)\left(\frac{250km/s}{v}\right)\,kpc\end{aligned} \tag{2.3}$$

In this formula, v is the statistical dispersion of velocities about the mean velocity for a group of astronomical objects.

Now, the relation that has to be considered is the case in which $\lambda_{dB} < \bar{r}$. The ratio involving the velocity dispersion is approximated to be of order 1. In this case, since $\bar{r} = (m/\rho)^{1/3}$ the inequality gives the relation

$$m < (1.49 \cdot 10^{-6}\rho^{1/3})^{3/4}eV \sim 34eV$$

This low value of dark matter mass allows to consider dark matter field as a classical field. This can be seen thanks to the following consideration

The number of particles in a de Broglie volume is given by $N = nV = n\lambda_{dB}^3 = \rho\lambda_{dB}^3/m$. Using (2.3), N is given by

$$N = 0.99\left(\frac{34eV}{m}\right)^4\left(\frac{250km/s}{v}\right)^3$$

However, since $m \ll 34eV$ due to the fact that fuzzy DM is considered, the occupancy N is so high that DM must be bosonic, since fermions obey to the Pauli exclusion principle.

With the introduction of a quantum state $\hat{\psi}$, it is possible to split it into $\hat{\psi} = \psi + \delta\hat{\psi}$, the expectation value in a given state and the relative quantum corrections.

For a coherent state, which is a state with similar behavior of a classic harmonic oscillator that describes the oscillating motion of a particle in a quadratic potential well, the occupancy for modes of the typical wavelength obey to

$$\frac{1}{\sqrt{N}} = \frac{\delta\hat{\psi}}{\psi}$$

So if N is too high (boson), then $\delta\hat{\psi} \ll \psi$ and the approximation to consider the field as a classical one is justified.

2.2 Unperturbed Klein-Gordon Equation

Let be ϕ the scalar field with a small mass m given by a spontaneous symmetry breaking. The action of this field is given by

$$\mathcal{S}[\phi] = \int d^4x\sqrt{-g}\;\mathcal{L}[\phi, g_{\mu\nu}] \tag{2.4}$$

$\mathcal{L}$ is the Lagrangian density, which is given by

$$\mathcal{L}[\phi, g_{\mu\nu}] = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.5}$$

The Lagrangian density invloves the potential of the field $V(\phi)$. For the choice of $V(\phi)$ there is a large number of possible potential candidate.

There can be the so-called "Large field models" in which ϕ is bigger than the Planck mass (which is around 10^{19} GeV). This condition arises from the fact that inflation needs a parameter ϵ to be less than 1. This parameter depends on the ratio between ϕ and M_{Pl}. This kind of model shows a potential of the form $V(\phi) \sim \phi^{\alpha}$

There can be also "Small field models" that require ϕ to be smaller than M_{Pl} for the condition $\epsilon < 1$. The potential this time assumes the form

$$V(\phi) = V_0 \left(1 - \left(\frac{\phi}{\mu} \right)^p \right) \tag{2.6}$$

In the last equation μ is a parameter related to ϕ, while p is a natural number greater than 2

It is worth to mention the existence of hybrid models, however they include a second scalar field, and so they are neglected in this work.

Since the mass of the FDM is extremely low, it is natural to choose a potential for a small field, and, in particular, the potential for the axion is

$$V(\phi) = \Lambda^4 \left(1 - cos \left(\frac{\phi}{f} \right) \right) \tag{2.7}$$

The ratio in the cosine is an effective angle, and f is some high energy level. The square of the mass of the field is given by the second derivative of V with respect to ϕ, calculated in $\phi = 0$, so $m^2 = \Lambda^4/f^2$

Since f is very high and ϕ is lower than this energy scale, the cosine can be expanded

$$cos \left(\frac{\phi}{f} \right) \sim 1 - \frac{1}{2!} \left(\frac{\phi}{f} \right)^2 + \frac{1}{4!} \left(\frac{\phi}{f} \right)^4 - \frac{1}{6!} \left(\frac{\phi}{f} \right)^6 + \ \ldots \tag{2.8}$$

Only the free mass term (ϕ^2) is considered, while the self interaction (ϕ^4) is ignored. Higher order terms are useless to considered, since $(\phi/f)^{\alpha}$ becomes extremely small starting from $\alpha = 6$. The expansion then stops at first order in ϕ

In this case

$$V(\phi) = \Lambda^4 \left(1 - \left(1 - \frac{1}{2} \frac{\phi^2}{f^2} \right) \right) = \frac{1}{2} \frac{\Lambda^4}{f^2} \phi^2 = \frac{1}{2} m^2 \phi^2 \tag{2.9}$$

With this potential, it is now possible to derive the Klein Gordon equation for the scalar field ϕ

$$\Box \phi = \frac{\partial V}{\partial \phi} \tag{2.10}$$

From the metric given in (2.1), the matrix needed is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{2.11}$$

The determinant of this matrix is noted with the letter g. The next step is to compute the unperturbed d'Alembertian.

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\; g^{\mu\nu}\partial_\nu) \tag{2.12}$$

In this case, the easy form of $g_{\mu\nu}$ allows a direct computation

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\; g^{\mu\nu}\partial_\nu) \\
&= \frac{1}{\sqrt{-g}}\partial_0(\sqrt{-g}\; g^{00}\partial_0) + \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}\; g^{ii}\partial_i) \\
&= \frac{1}{a^3}\partial_0(-a^3\partial_0) + \frac{1}{a^3}a^3\frac{1}{a^2}\partial_i\partial_i \\
&= -3\frac{a^2}{a^3}\dot{a}\;\partial_0 - \partial_0^2 + \frac{1}{a^2}\nabla^2 \\
&= -\partial_0^2 - 3H\partial_0 + \frac{\nabla^2}{a^2}
\end{aligned} \tag{2.13}$$

Using (2.13) into (2.10) leads to

$$-\ddot{\phi} - 3H\dot{\phi} + \frac{\nabla^2}{a^2}\phi = \partial_\phi V \tag{2.14}$$

For the moment is valid the assumption that ϕ is homogeneous, so the ∇ term in the above equation vanishes.The conclusion is the Klein-Gordon equation that rules the motion of a free particle in a FLRW metric under the influence of a potential V.

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0 \tag{2.15}$$

From this equation it is possible to conclude that dark matter scales as non-relativistic matter. To see this consider energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{2.16}$$

In the last equation, the derivative is the funcional one. The action is given by (2.4) and for the Lagrangian density, the simplest case of a single scalar field, given in (2.5), can be taken.

Since the Lagrangian density is dependent only on $g_{\mu\nu}$ the functional derivative in (2.16) gives

$$\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\left[\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\;\mathcal{L})\right] \\
&= -\frac{2}{\sqrt{-g}}\left[\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\mathcal{L} + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\right] \\
&= -\frac{2}{\sqrt{-g}}\left[-\frac{1}{2}\sqrt{-g}\;g_{\mu\nu}\;\mathcal{L} + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\right] \\
&= -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\;\mathcal{L} \\
&= \partial_\mu\phi\;\partial_\nu\phi + g_{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\;\phi\partial_\beta\phi - V(\phi)\right]
\end{aligned} \tag{2.17}$$

In the third line, the fact $\partial\sqrt{-g}/\partial g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}$ is been used.

It is known that the component $T_0{}^0$ is related to the energy density. As shown in the section before, the field can be written as $\phi = \phi(r,t) = \phi_0(t) + \delta\phi(r,t)$ but the quantum fluctuation can be ignored because $\langle\delta^2\phi\rangle \ll \phi_0^2$ so, for simplicity, the $_0$ index is ignored

In the end

$$T_0^0 = g^{00}T_{00} = -\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = -\rho(t) \tag{2.18}$$

Recalling (2.9) the density is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \tag{2.19}$$

Now, in order to solve (2.15) the JWKB approximation is useful to solve equation of the form

$$\ddot{x} + 2\eta\dot{x} + k^2x = 0 \tag{2.20}$$

In this approximation $\dot{x}/x = -\eta$. The comparison between (2.20) and (2.15) gives $\dot{\phi}/\phi = -3H/2$. Since $H = \dot{a}/a$, this equation can be solved by separating the variables. It turns out

$$ln\ \phi \propto -\frac{3}{2}\ ln\ a \implies \phi \propto\ a^{-3/2} \tag{2.21}$$

Inserting this scale into (2.19) and using the slowroll approximation ($V \gg \dot{\phi}$), the conclusion is that $\rho \propto a^{-3}$ and this is exactly the behavior of non-relativistic matter.

2.3 Klein-Gordon perturbed

Of course, the Universe is full of matter that perturb spacetime. A metric like the one in (2.1) is not perfect to describe the reality of the Universe. The FLRW metric changes when perturbations are included and the Newtonian gauge can be used to describe it. The Newtonian potential is referred with Φ

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^idx^j \tag{2.22}$$

Φ and Ψ are the scalar gravitational potentials. In a universe without anisotropic stresses it is possible to consider $\Phi = \Psi$ since ther is a parameter called "gravitational slip" $\eta = \Phi/\Psi$ which is the ratio between the effective gravitational coupling of light to that of the matter, and is almost equal to 1. A universe without anisotropic stresses shows a stress energy tensor $T_{\mu\nu}$ where the three pressures are identical, so it is invariant under spatial rotations

And so, having $\Phi = \Psi$, the matrix is

$$g_{\mu\nu} = \begin{pmatrix} -1(1+2\Phi) & 0 & 0 & 0 \\ 0 & a^2(t)(1-2\Phi) & 0 & 0 \\ 0 & 0 & a^2(t)(1-2\Phi) & 0 \\ 0 & 0 & 0 & a^2(t)(1-2\Phi) \end{pmatrix} \tag{2.23}$$

The goal is again to compute (2.10), remembering that the d'Alembertian is given by (2.12)

This time the treatment is more complicated than in the unperturbed case, so some approximations are required. In particular, the main assumption is to consider Φ only at first order

$$\begin{aligned} g &= -a^6(1+2\Phi)(1-2\Phi)^3 \\ &= -a^6(1-4\Phi^2)(1-2\Phi)^2 \\ &= -a^6(1-4\Phi^2)(1-4\Phi+4\Phi^2) \\ &= -a^6(1-4\Phi+4\Phi^2-4\Phi^2+16\Phi^3-16\Phi^4) \end{aligned} \tag{2.24}$$

Considering only terms at first order in Φ, the determinant in (2.24) becomes

$$g = -a^6(1-4\Phi) \tag{2.25}$$

For the next step remember the expansion $(1+x)^\alpha \sim 1+\alpha x$ for small x. This expansion is justified because the order of magnitude of Φ is 10^{-4}

$$\sqrt{-g} = a^3(1-4\Phi)^{1/2} \sim a^3(1-2\Phi) \tag{2.26}$$

Before the computation of the d'Alembertian, the terms of $g^{\mu\nu}$ can also be expanded

$$g^{00} = -\frac{1}{1+2\Phi} \sim -1+2\Phi \qquad\qquad g^{ii} = \frac{1}{a^2(1-2\Phi)} \sim \frac{1+2\Phi}{a^2} \tag{2.27}$$

The computation of the d'Alembertian will be splitted into three parts

$$\begin{aligned} \Box &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\; g^{\mu\nu}\partial_\nu) \\ &= \underbrace{\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})\; g^{\mu\nu}\partial_\nu}_{\mathcal{A}} + \underbrace{\partial_\mu(g^{\mu\nu})\partial_\nu}_{\mathcal{B}} + \underbrace{g^{\mu\nu}\partial_\mu\partial_\nu}_{\mathcal{C}} \end{aligned} \tag{2.28}$$

The first computation will be $\mathcal{A}$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})\; g^{\mu\nu}\partial_\nu = \frac{1}{\sqrt{-g}}\partial_0(\sqrt{-g})\; g^{00}\partial_0 + \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g})\; g^{ii}\partial_i \tag{2.29}$$

And since

$$\begin{aligned} \partial_0\sqrt{-g} &= 3a^2\dot{a}(1-2\Phi) - 2a^3\dot{\Phi} \\ \partial_i\sqrt{-g} &= -2a^3\partial_i\Phi \end{aligned}$$

The first partial result is

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})\; g^{\mu\nu}\partial_\nu &= \frac{-3a^2\dot{a}(1-2\Phi)^2}{a^3(1-2\Phi)}\partial_0 + \frac{2a^3\dot{\Phi}(1-2\Phi)}{a^3(1-2\Phi)}\partial_0 - \frac{2a^3\partial_i\Phi}{a^3(1-2\Phi)}\frac{1}{a^2}(1+2\Phi)\partial_i \\ &= -(1-2\Phi)3H\partial_0 + 2\dot{\Phi}\partial_0 - \frac{2}{a^2}\partial_i A\partial_i \end{aligned} \tag{2.30}$$

For the last term it is been used the fact that $(1+2\Phi)/(1-2\Phi) \sim (1+2\Phi)^2$ and since only first order terms in Φ are considered, that ratio can be put similar to 1 since it will be multiply by $\partial_i \Phi$ that is already a first order term.

The $\mathcal{B}$ term is

$$\begin{aligned}\partial_\mu g^{\mu\nu}\partial_\nu &= \partial_0 g^{00}\partial_0 = +\partial_i g^{ii}\partial_i \\ &= 2\dot{\Phi}\partial_0 + \frac{2}{a^2}\partial_i A \partial_i \end{aligned} \tag{2.31}$$

While $\mathcal{C}$ is the most immediate

$$\begin{aligned}g^{\mu\nu}\partial_\mu\partial_\nu &= g^{00}\partial_0^2 + g^{ii}\partial_i\partial_i \\ &= -(1-2\Phi)\partial_0^2 + \frac{1+2\Phi}{a^2}\nabla^2 \end{aligned} \tag{2.32}$$

Now the only thing to do is to sum (2.30), (2.31) and (2.32)

$$\Box = -(1-2\Phi)3H\partial_0 + 2\dot{\Phi}\partial_0 - \frac{2}{a^2}\partial_i A\partial_i + 2\dot{\Phi}\partial_0 + \frac{2}{a^2}\partial_i A\partial_i - (1-2\Phi)\partial_0^2 + \frac{1+2\Phi}{a^2}\nabla^2$$

And then

$$\Box = -(1-2\Phi)(\partial_0^2 + 3H\partial_0) + a^{-2}(1+2\Phi)\nabla^2 + 4\dot{\Phi}\partial_0 \tag{2.33}$$

Rather than the whole ϕ it is more useful to work with a complex ψ and so the field can be defined in another way

$$\phi(r,t) = \frac{1}{\sqrt{2}\,m}\left[\psi(r,t)e^{-imt} + \psi^*(r,t)e^{imt}\right] \tag{2.34}$$

The only oscillations considered are the slow ones. The WKB approximation works to solve linear differential equations with spatially variating coefficients. Since we are considering a field in slow roll, the condition is $|\ddot{\psi}| \ll m|\dot{\psi}|$. In other words the condition is $\partial_t \ll m$, and since for Schrodinger $\partial_t \sim \nabla^2/m \sim k^2/m$, it leads to $k^2/m \ll m$. The first term id the momentum term, while the latter is the rest mass. This means that the field is slowly variating.

Equation (2.10) with the potential expressed in (2.9) and the d'Alembertian in equation (2.33) becomes

$$-(1-2\Phi)(\partial_0^2 + 3H\partial_0)\phi + a^{-2}(1+2\Phi)\nabla^2\phi + 4\dot{\Phi}\partial_0\phi = m^2\phi \tag{2.35}$$

In order to solve this equation, the first and second derivatives of ϕ are needed

$$\begin{aligned}\partial_0\phi &= \frac{1}{\sqrt{2}\,m}\dot{\psi}e^{-imt} - \frac{i}{\sqrt{2}}\psi e^{-imt} \\ &= \left(\frac{1}{\sqrt{2}\,m}\dot{\psi} - \frac{i}{\sqrt{2}}\psi\right)e^{-imt}\end{aligned} \tag{2.36}$$

$$\partial_0^2\phi = \frac{1}{\sqrt{2}\,m}\ddot{\psi}e^{-imt} - \frac{i}{\sqrt{2}}\dot{\psi}e^{-imt} - \frac{i}{\sqrt{2}}\dot{\psi}e^{-imt} - \frac{m}{\sqrt{2}}\psi e^{-imt}$$
$$= \left(i\sqrt{2}\dot{\phi} - \frac{m}{\sqrt{2}}\psi\right)e^{-imt} \tag{2.37}$$

With the assumption that ψ is slowly variating, and adding the condition for slow roll ($\epsilon = -\dot{H}/H^2$) and the WKB approximation, it results that $\Phi \sim \epsilon^2, H/m \sim \epsilon$ and $\dot{\psi} \sim \epsilon$, some term that appear in (2.35) can be neglected since only quadratic term on ϵ are considered. For simplicity, e^{-imt} is omitted in every term and simplified later.

$$\begin{aligned}
-\partial_0^2\phi &= i\sqrt{2}\dot{\psi} + \frac{m}{\sqrt{2}}\psi \\
-3H\partial_0\phi &= -\frac{3H}{\sqrt{2}m}\dot{\psi} + \frac{3Hi}{\sqrt{2}}\psi \\
2\Psi\partial_0^2\phi &= -2\Psi i\sqrt{2}\dot{\psi} - \sqrt{2}\Psi m\psi \\
6\Psi H\partial_0\phi &= \frac{6\Psi H}{\sqrt{2}m}\dot{\psi} - \frac{6\Psi Hi}{\sqrt{2}}\psi \\
\frac{1}{a^2}\nabla^2\phi &= \frac{1}{\sqrt{2}ma^2}\nabla^2\psi \\
\frac{2\Psi}{a^2}\nabla^2\phi &= \frac{2\Psi}{\sqrt{2}ma^2}\nabla^2\psi \\
4\dot{\Psi}\partial_0\phi &= \frac{4\dot{\Psi}}{\sqrt{2}m}\dot{\psi} - \frac{4\dot{\Psi}i}{\sqrt{2}}\psi \\
m^2\phi &= \frac{m}{\sqrt{2}}\psi
\end{aligned}$$

Once only the term quadratic in ϵ are involved, the result is

$$i\sqrt{2}\dot{\psi} + \frac{m}{\sqrt{2}}\psi + \frac{3Hi}{\sqrt{2}}\psi - \sqrt{2}\Psi m\psi + \frac{1}{\sqrt{2}ma^2}\nabla^2\psi = \frac{m}{\sqrt{2}}\psi \tag{2.38}$$

Dividing by $\sqrt{2}$, the non-linear Shrödinger equation is then obtained

$$i\left[\dot{\psi} + \frac{3}{2}H\psi\right] + \frac{1}{2ma^2}\nabla^2\psi = \Psi m\psi \tag{2.39}$$

It is confirmed the non-relativistic behavior for unperturbed background, since $\nabla^2\psi$ and Ψ would be zero and equation (2.39) would be

$$\dot{\psi} + \frac{3}{2}H\psi = 0 \tag{2.40}$$

This is already been solved in equation (2.21)

Chapter 3

Fluid Dynamics

3.1 From Klein-Gordon to fluid treatment

Dark matter can be seen as a superfluid, which is a fluid without viscosity. In this way the motion of the fluid does not dissipate kinetic energy. Recalling equation (2.39), it is possible to define the field ψ in the following way

$$\psi(r,t) = \sqrt{\frac{\rho(r,t)}{m}}\, e^{i\theta(r,t)} \tag{3.1}$$

So that $\rho(r,t) = m|\psi(r,t)|^2$. If short timescale are considered, than H tends to 0 since for small t it is true that $H = \dot{a}/a \to 0$ and the equation (2.39) become

$$i\dot{\psi} = -\frac{1}{2m}\nabla^2\psi + \Psi m\psi \tag{3.2}$$

In this equation in formulated in physical coordinates, since $x_p = a \cdot x_c$ and so $\nabla_c = a \cdot \nabla_p$

The next step is to insert equation (3.1) into equation (3.2). The terms are computed separately

$$\begin{aligned}
i\dot{\psi} &= \frac{i}{\sqrt{m}}\,\partial_0\sqrt{\rho}\, e^{i\theta} - \sqrt{\frac{\rho}{m}}\,\partial_0\theta\, e^{i\theta} \\
\nabla\psi &= \frac{1}{\sqrt{m}}\nabla\sqrt{\rho}\, e^{i\theta} + i\sqrt{\frac{\rho}{m}}\nabla\theta\, e^{i\theta} \\
\nabla^2\psi &= \frac{1}{\sqrt{m}}\nabla^2\sqrt{\rho}\, e^{i\theta} + \frac{i}{\sqrt{m}}\,\nabla\sqrt{\rho}\,\nabla\theta\, e^{i\theta} + \frac{i}{\sqrt{m}}\,\nabla\sqrt{\rho}\,\nabla\theta\, e^{i\theta} + \\
&\quad + i\sqrt{\frac{\rho}{m}}\,\nabla^2\theta\, e^{i\theta} - \sqrt{\frac{\rho}{m}}\,(\nabla\theta)^2 e^{i\theta}
\end{aligned}$$

Inserting everything into equation(3.2), the terms $e^{i\theta}$ and $-\sqrt{\rho/m}$ can be simplified. Now the Schrödinger equation is splitted into its real and imaginary part

The real part leads to

$$\frac{\partial\theta}{\partial t} = \frac{1}{2m}\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho} - \frac{1}{2m}(\nabla\theta)^2 - \Psi m \tag{3.3}$$

And the imaginary part equation is

$$\frac{\partial\sqrt{\rho}}{\partial t} = -\frac{1}{2m}(2\nabla\sqrt{\rho}\;\nabla\theta + \sqrt{\rho}\;\nabla^2\theta) \tag{3.4}$$

Instead of treating $\partial\sqrt{\rho}$ it is more useful to work with $\partial\rho$

$$\begin{aligned}\frac{1}{2}\;\rho^{-1/2}\;\frac{\partial\rho}{\partial t} &= -\frac{1}{2m}\left(2\;\frac{1}{2}\;\rho^{-1/2}\;\nabla\rho\;\nabla\theta + \rho^{1/2}\;\nabla^2\theta\right)\\ &= -\frac{\rho^{-1/2}}{2m}(\nabla\rho\;\nabla\theta + \rho\nabla^2\theta)\end{aligned} \tag{3.5}$$

Now deleting $\rho^{-1/2}/2$, the result is then

$$\frac{\partial\rho}{\partial t} = -\frac{1}{m}(\nabla\rho\nabla\theta + \rho\;\nabla^2\theta) \tag{3.6}$$

This final equation can be changed a little. If $A = \rho$ and $B = \nabla\theta$, in (3.6) it is possible to recognize the form $\nabla(AB) = \nabla A\;B + A\;\nabla B$

In the end

$$\frac{\partial\rho}{\partial t} = -\frac{1}{m}\nabla(\rho\;\nabla\theta) \tag{3.7}$$

3.2 Probability Current

Equation (3.7) can be written in a even more simpler form by considering the following computation. It is known that the probability density related to a field is given by

$$P(r,t) = \psi^*(r,t)\psi(r,t) \tag{3.8}$$

and if in a position the probability to find a particle decreases, it means that it increases in another place for the conservation of the total probability. There is therefore a flow, or, in other words, a **current**. The goal is to relate the probability density to this current that is called $\boldsymbol{J}$

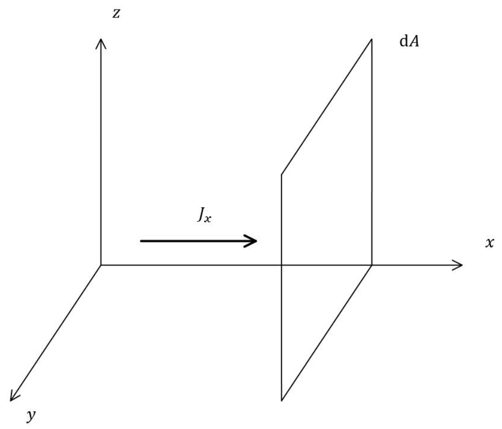

Figure 3.1: Geometrical representation of the probability current

Let be J_x the probability per unit time and unit area that passes through dA parallel to yz plane. Now it is possible to compute the rate of change in the probability density in a more general way without specifying the direction of $\boldsymbol{J}$. Remember that the goal is to link P to $\boldsymbol{J}$, so it can be written

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\nabla \cdot \boldsymbol{J} \\ &= \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\end{aligned} \tag{3.9}$$

For the derivative of ψ and ψ^*, equation (3.2) is used in a slight different form

$$\dot{\psi} = -i\left[\frac{1}{2m}\left(\frac{\nabla}{i}\right)\left(\frac{\nabla}{i}\right)\psi + \Psi m \psi\right] \tag{3.10}$$

So, in this way

$$\begin{aligned}\frac{\partial P}{\partial t} &= \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi \\ &= -i\left[\psi^* \frac{1}{2m}\left(\frac{\nabla}{i}\right)\left(\frac{\nabla}{i}\right)\psi + \Psi m \psi^* \psi - \psi\, \frac{1}{2m}\left(-\frac{\nabla}{i}\right)\left(-\frac{\nabla}{i}\right)\psi^* - \Psi m \psi \psi^*\right] \\ &= -i\left[\psi^* \frac{1}{2m}\left(\frac{\nabla}{i}\right)\left(\frac{\nabla}{i}\right)\psi - \psi\, \frac{1}{2m}\left(-\frac{\nabla}{i}\right)\left(-\frac{\nabla}{i}\right)\psi^*\right] \\ &= -\frac{1}{2mi}\Big(\psi^* \nabla(\nabla\psi) - \psi \nabla(\nabla \psi^*)\Big)\end{aligned} \tag{3.11}$$

The terms inside the parenthesis are in the form $A\nabla(\nabla B) = \nabla(A\nabla B) - \nabla A \nabla B$

$$\begin{aligned}\Big(\psi^* \nabla(\nabla\psi) - \psi \nabla(\nabla \psi^*)\Big) &= \nabla(\psi^* \nabla \psi) - \nabla\psi^* \nabla \psi - \Big(\nabla(\psi \nabla \psi^*) - \nabla\psi \nabla\psi^*\Big) \\ &= \nabla(\psi^* \nabla\psi - \psi \nabla \psi^*)\end{aligned}$$

Inserting it into equation (3.11)

$$\frac{\partial P}{\partial t} = -\nabla \underbrace{\left(\frac{1}{2m} \psi^* \left(\frac{\nabla}{i} \right) \psi + \frac{1}{2m} \psi \left(-\frac{\nabla}{i} \right) \psi^* \right)}_{\boldsymbol{J}} \tag{3.12}$$

In this way the form $\partial_t P = -\nabla \cdot \boldsymbol{J}$ was found. For completeness, inserting $\hbar$ in order to have the correct units allows to highlight the momentum operator $-i\hbar\nabla$

$$\begin{aligned} \boldsymbol{J} &= \frac{1}{2m} \psi^* \left(\frac{\hbar}{i} \nabla \right) \psi + \frac{1}{2m} \psi \left(-\frac{\hbar}{i} \nabla \right) \psi^* \\ &= \frac{1}{2m} [\psi^* \hat{P} \psi + \psi \hat{P}^* \psi^*] \end{aligned}$$

The next step is to put equation (3.1) into (3.12)

$$\begin{aligned} \boldsymbol{J} &= \frac{1}{2m} \psi^* \frac{\nabla}{i} \psi - \frac{1}{2m} \psi \frac{\nabla}{i} \psi^* \\ &= \frac{1}{2m} \frac{(\sqrt{\rho})^2}{i} \; e^{-i\theta} \; i\nabla\theta \; e^{i\theta} - \frac{1}{2m} \frac{(\sqrt{\rho})^2}{i} \; e^{i\theta} \; (-i)\nabla\theta \; e^{-i\theta} \\ &= \frac{\rho}{m} \; \nabla\theta \end{aligned} \tag{3.13}$$

It is known that the current density vector $\boldsymbol{J}$ is the amount of charge per unit time that flows through a unit area perpendicular to the direction of motion $\boldsymbol{J} = d\boldsymbol{I}/dA$

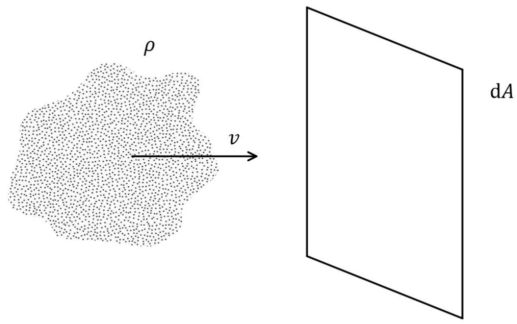

Figure 3.2: Representation of the motion of the charges that generate the current

Consider $\boldsymbol{v}$ the velocity of the charge at time t perpendicular to dA. During dt, the charges that will go through dA will be the ones contained in the volume formed by dA and I

$$I = \frac{dq}{dt} = \frac{\rho \, dV}{dt} = \frac{\rho \; v \; dt \; dA}{dt} = \rho \; v \; dA$$

So

$$\boldsymbol{J} = \frac{d\boldsymbol{I}}{dA} = \frac{\rho \; \boldsymbol{v} \; dA}{dA} = \rho \; \boldsymbol{v} \tag{3.14}$$

By equating equations (3.13) and (3.14) an advantageous relation is then obtained

$$\begin{aligned}\rho\,\boldsymbol{v} &= \frac{\rho}{m}\nabla\theta \\ \boldsymbol{v} &= \frac{1}{m}\nabla\theta\end{aligned} \tag{3.15}$$

3.3 Madelung equations

Equation (3.15) is a fundamental relation, since it simplifies equation (3.7)

$$\frac{\partial\rho}{\partial t} = -\frac{1}{m}\nabla(\rho\nabla\theta) = -\nabla(\rho\boldsymbol{v}) \tag{3.16}$$

And also equation (3.3)

$$\begin{aligned}\frac{\partial\theta}{\partial t} &= \frac{1}{2m}\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho} - \frac{1}{2m}(\nabla\theta)^2 - \Psi m \\ &= \frac{1}{2m}\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho} - \frac{m}{2}v^2 - \Psi m\end{aligned} \tag{3.17}$$

The equations (3.16) and (3.17) are known as *Madelung equations*. It is possible to identify θ as the velocity potential (possible thanks to equation (3.15)) and this can be seen as an equation of motion for an electrically charged fluid, but with is a strange term that involve $\nabla^2\sqrt{\rho}$

$$\frac{\partial\theta}{\partial t} = \underbrace{-\frac{1}{2}mv^2}_{K} - \underbrace{\Psi m}_{P} + \underbrace{\frac{1}{2m}\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}}_{Q}$$

K is the kinetic term, P is the potential term and Q is known as the $quantummechanicalenergy$. Rewriting the last equation by taking the gradient of equation (3.17) and dividing by m, the result is the more familiar Euler equation

$$\frac{1}{m}\nabla\frac{\partial\theta}{\partial t} = -\frac{1}{2}\nabla v^2 - \nabla\Psi + \frac{1}{2m^2}\nabla\left(\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right) \tag{3.18}$$

In the left hand side it is possible to switch the derivatives and apply equation (3.15) to obtain $\partial_t\boldsymbol{v}$. Noting that $\nabla(A\cdot B) = (A\cdot\nabla)B + (B\cdot\nabla)A + A\times(\nabla\times B) + B\times(\nabla\times A)$ the kinetic term ∇v^2 can be also changed. Keep attention, since $\nabla\times\boldsymbol{v}\propto\nabla\times\nabla\theta = 0$ since the rotor of the gradient is zero

$$\nabla(v^2) = \nabla(\boldsymbol{v}\boldsymbol{v}) = 2(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$$

So equation (3.18) became the quantum Euler equation, which is the usual Euler equation with the extra term given by the quantum potential

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla\Psi + \frac{1}{2m^2}\nabla\left(\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right) \tag{3.19}$$

3.4 Jeans scale

Suppose the density fluctuation is

$$\rho = \bar{\rho}(1+\delta) \qquad |\delta| \ll 1 \tag{3.20}$$

Consider equation (3.19) and pick the two terms on the right hand side. The computation to do is to compare them to verify when gravity dominates and when quantum pressure does. Taking the divergence

$$\nabla^2\Psi = \frac{1}{2m^2}\nabla^2\underbrace{\left(\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right)}_{\nabla^2 ln\sqrt{\rho}}$$

So

$$\nabla^2\Psi = \frac{1}{4m^2}\nabla^4 ln\rho$$

It is known that the Newtonian potential obey to $\nabla^2\Psi = 4\pi G\rho$ and inserting equation(3.20) instead of ρ

$$\begin{aligned}
4\pi G\bar{\rho} + 4\pi G\delta\bar{\rho} &= \frac{1}{4m^2}\nabla^4 ln\Big(\bar{\rho}(1+\delta)\Big) \\
&= \frac{1}{4m^2}\nabla^4\Big(ln\bar{\rho} + ln(1+\delta)\Big) \\
&= \frac{1}{4m^2}\nabla^4 ln\bar{\rho} + \frac{1}{4m^2}\nabla^4 ln(1+\delta)
\end{aligned}$$

Consider only first order perturbation terms in which $ln(1+\delta) \sim \delta$. The result is

$$4\pi G\delta\bar{\rho} = \frac{1}{4m^2}\nabla^4\delta \tag{3.21}$$

It is important to work in the Fourier space, where $\nabla \longrightarrow i\boldsymbol{k}$

$$4\pi G\delta\bar{\rho} = \frac{1}{4m^2}k^4\delta$$

Finally, the Jeans scale is found

$$k_J = (16\pi G\bar{\rho})^{1/4}m^{1/2} \tag{3.22}$$

Chapter 4

Conclusion

From equation (3.22), it is immediate to see that if k is larger than the Jeans scale quantum pressure dominates. In the opposite regime, on course, gravity dominates.

Since the k parameter is inversely proportional to the length λ, large k means that small scales are considered. In this scales quantum pressure dominates, so this could explain why small-scales structure are less numerous than what the model predicts.

Gravity is not sufficient to win the force exercised by the quantum pressure and this prevents the accretion of ordinary matter in the potential well of the halo and, consequentely, the formation of the galaxy itself.

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