Term Paper

Structural Macroeconometrics

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Topic:

Monte Carlo analysis for the Likelihood-ratio test on structural breaks

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1 INTRODUCTION

Since the rise of dynamic stochastic structural modeling, it became possible to evaluate structural models with the statistical methods of time series analysis. A popular exercise is to test time series on structural breaks. Your task is to evaluate the power and overall goodness of the Likelihood-ratio test on structural breaks. The remainder of the exercise sheet guides you to fulfill the task.

2 THE STRUCTURAL MODEL

The model

A representative household derives utility from consumption C_t with current-period utility $u_t = u(C_t)$. The household produces with capital K_t output Y_t using the production function $exp(Z_t)F(K_t) = Y_t$ where $exp(Z_t)$ is the total factor productivity and follows a Markov process. Output can be used as an investment good I_t and as a consumption good C_t. The accumulation of capital reads as follows: $K_{t+1} = I_t + (1-\delta)K_t$, where δ is the depreciation rate of capital. The household lives forever and chooses in period t C_{t+s} and K_{t+s+1} to optimize its expected lifetimeutility $U_t = E_t \sum_{s=0}^{\infty} \beta^s u_{t+s}$.

Thus, given K_t, the problem of the household reads as follows:

$$max_{C_{t+s},K_{t+s+1}} U_t = E_t \sum_{s=0}^{\infty} \beta^s u_{t+s} \tag{1}$$

$$s.t.\ Y_{t+s} = C_{t+s} + I_{t+s} \tag{2}$$

$$Y_{t+s} = exp(Z_{t+s})F(K_{t+s}) \tag{3}$$

$$K_{t+1+s} = I_{t+s} + (1-\delta)K_{t+s} \tag{4}$$

Exercise: Derive the first-order conditions with respect to C_{t+s} and K_{t+s+1}. Hints: Reduce the constraints (2)-(4) to one equation. Write down the Lagrangian, and take the derivatives with respect to C_t and K_{t+1}.

Solution:
Plugging in (3) and (4) in (2) gives the single constraint
$exp(Z_{t+s})F(K_{t+s}) = C_{t+s} + K_{t+1+s} - (1-\delta)K_{t+s}$.

The Lagrangian reads:

$L_t = E_t \sum_{s=0}^{\infty} \beta^s (u_{t+s} + \lambda_{t+s}(exp(Z_{t+s})F(K_{t+s}) - C_{t+s} - K_{t+1+s} + (1-\delta)K_{t+s}))$

The FOCs are the constraint and:

$$\frac{dL_t}{dC_t} = u'_t - \lambda_t = 0 \tag{5}$$

$$\frac{dL_t}{dK_{t+1}} = E_t(\beta\lambda_{t+1}(exp(Z_{t+1})F'(K_{t+1}) + (1-\delta)) - \lambda_t) = 0 \tag{6}$$

We can exchange expectation and differentiation when taking the derivative with respect to K_{t+1} (and C_t) because K_{t+1} is already known and not stochas-

tic at time t due to the constraint and for a deterministic c and stochastic variable X it holds that
$\frac{d}{dc}E(g(c,X)) = \frac{d}{dc}\int g(c,X)f(X)dX = \int \frac{d}{dc}g(c,X)f(X)dX = E(\frac{d}{dc}g(c,X))$
The expectation operator drops out in (5) because the equation doesn't contain any future values.

Parameterization
Assume the current-utility function is logarithmic ($u_t = ln(C_t)$), the production function reads $F(K_t) = K_t^{\alpha}$, and the Markov process $\{Z_{t+s}\}_{s=0}^{\infty}$ follows $Z_{t+1} = \rho Z_t + \omega\epsilon_{t+1}$, $\epsilon_t \sim N(0,1)$.

Exercise: Cast the model into the canonical form, including the first-order conditions in t. Why is it sufficient to consider only one period? Hint: You may refer to the Bellman principle.

Solution:
$F'(K_t) = \alpha K_t^{\alpha-1}$, $u_t' = \frac{1}{C_t}$
Plug in λ_t and λ_{t+1} from (5) in (6):

$$E_t(\beta\frac{1}{C_{t+1}}(exp(Z_{t+1})\alpha K_{t+1}^{\alpha-1} + (1-\delta)) - \frac{1}{C_t}) = 0$$

The canonical form then includes the first order conditions, constraint and Markov process and reads:

$$\mathbf{0} = \mathbf{E_t}(\mathbf{\Gamma}) = E_t\begin{bmatrix} \beta\frac{1}{C_{t+1}}(exp(Z_{t+1})\alpha K_{t+1}^{\alpha-1} + (1-\delta)) - \frac{1}{C_t} \\ exp(Z_t)K_t^{\alpha} - C_t - K_{t+1} + (1-\delta)K_t \\ Z_{t+1} - \rho Z_t - \omega\epsilon_{t+1} \end{bmatrix}$$

The Bellman principle states that the value of the optimization problem at any point in time depends only on the current state and the optimal policy going forward. Thus, by solving the problem for one period only and establishing optimal policy rules, we can recursively apply this solution to all time periods which will lead to the optimized policy path.

Parameter values

If not otherwise stated, t represents one-quarter year and the parameter values follow Table 1.

Table 1: Calibration of the model

Parameter	Description	Value
α	Capital share	**0.3**
β	Discount factor	0.99
δ	Rate of capital depreciation	0.02
ρ	AR(1) coefficient	$\mathbf{\{0.85, 0.95\}}$
ω	Conditional standard deviation of Z_t	0.01

Note: Bold numbers differ from the lecture's default.

Exercise: Cast the canonical form in a recursive form ('the model's solution') by approximating the policy function with a first-order perturbation. **Hint:** You may use the lecture's code repository, especially `CreatModel_bench.m`, `CreatModel_est.m`, `SolveD.m`, and `SolveS.m`. Solve for $\rho \in \{0.85, 0.95\}$.

Solution:
Introduce a perturbation parameter σ: $Z_{t+1} = \rho Z_t + \sigma\omega\epsilon_{t+1}$. The first-order perturbation solution reads:

$$C_t = C^* + H_y((K_t - K^*) \;\; Z_t \;\; \sigma)'$$

$$K_{t+1} = K^* + H_x((K_t - K^*) \;\; Z_t \;\; \sigma)'$$

A * denotes the deterministic steady-state value which can be solved for by plugging in $\epsilon_{t+1} = \epsilon^* = 0$, $K_t = K_{t+1} = K^*$, $C_t = C_{t+1} = C^*$, $Z_t = Z_{t+1} = Z^* = 0$ into the canonical form. Then, given the recursive form, you plug in the solution and Markov process for C_{t+1}, C_t, K_{t+1}, Z_{t+1} into the canonical form, which is performed by egamma.m . Then we differentiate the first two entries of Γ with respect to K_t, Z_t and σ around the steady-state. I do this numerically which is done by scores.m . Then one solves for H_y and H_x so that the derivatives are all 0 which is done by Solve-FOP.m. Further, stability conditions have to hold to rule out the unstable out of 2 possible solutions: the roots of the state-VAR representation must be smaller than 1 in absolute which holds as long as $abs(\frac{dK_{t+1}}{dK_t}) < 1$.
As this is only a first-order perturbation the certainty equivalence holds and the last entry of the policy functions is 0. The steady state values do not depend on ρ and are for this parameterization $K^* = ((1/\beta - 1 + \delta)/\alpha)^{1/(\alpha-1)} = 26.6984$ and $C^* = K^{*\alpha} + (1-\delta)K^* - K^* = 2.1449$
For $\rho = 0.95$ the policy functions are:

$H_y = (0.0456 \;\; 0.6771 \;\; 0)$ and $H_x = (0.9645 \;\; 2.0017 \;\; 0)$

and for $\rho = 0.85$ the policy functions are:

$H_y = (0.0456 \;\; 0.3662 \;\; 0)$ and $H_x = (0.9645 \;\; 2.3126 \;\; 0)$

3 MONTE CARLO SETUP

Exercise: Given the linear model solution, simulate 2×500 times series for output, investment, and consumption. For the first subset of simulations, simulate 550 periods with $\rho = 0.95$ and after 50 periods with $\rho = 0.85$. Do the same, for the second subset, just with 600 periods and 100 for $\rho = 0.95$ and $\rho = 0.85$, respectively. Burn the first 500 for each subset of simulations.

Hint: For each subset of simulations, you may iterate the first periods over a state space model with a loop. Use the state (forecast) to start to iterate over the periods after the structural break. Use a loop to repeat this 500 times.

Solution:
See Script.m , section Monte Carlo. Done jointly with the next task.

4 FILTER EVALUATION

Exercise: Use the Likelihood Ratio to verify $H_0 : \Theta_{1:T/2} = \Theta_{T/2+1:T}$, with $\Theta \in \{\rho, \omega\}$, where $T \in \{100, 200\}$, given the other parameters are calibrated correctly. Hints: Calculate the value of the Maximum Likelihood of both subsamples (before and after the structural break) and over the whole sample. Note that the initialization of the Kalman filter for the likelihood evaluation of the second era should be more profound than the unconditional first and second moments. Afterward, use the test from p. 49, Ch 5 of the lecture's slides. For maximization, you may use mleRBC.m from the lecture's code. You may restrict maximization to the use the local optimizer fminsearch and as a guess the actual parameter values or their mean, respectively.

Solution:
See Script.m , section Monte Carlo. I use C_t as the observation variable and the state variables are $S_t = (1 \quad K_t \quad Z_t)'$. The linear state-space model is derived from the recursive form and Markov process and reads as:

$$S_t = FS_{t-1} + \eta_t, \eta_t \sim N(0, Q) \tag{7}$$

$$C_t = HS_t \tag{8}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ (1 - H_x(1))K^* & H_x(1) & H_x(2) \\ 0 & 0 & \rho \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}$$

$$H = \begin{bmatrix} (C^* - H_y(1)K^*) & H_y(1) & H_y(2) \end{bmatrix}$$

$H_{x,y}(i)$ denotes the ith entry of $H_{x,y}$.
The test statistics is $LR = 2(L(\hat{\Theta}_{1:T/2}, \hat{\Theta}_{T/2+1:T}) - L(\hat{\Theta}_{1:T})) \overset{asy}{\sim} \chi^2(dim(\Theta))$ where $L(\hat{\Theta})$ is the maximized log-likelihood.

To estimate the parameters under the null I maximize the likelihood from the output of a Kalman filter over the full sample. The filter is initialized, for given parameters, with the unconditional mean and variance of the states: $E(S_t) = (1 \quad K^* \quad 0)'$ can be simply derived applying the expectation operator to (7) and setting $E(S_t) = E(S_{t-1})$ assuming stationarity. By the same procedure we set $Var(S_t) = FVar(S_t)F' + Q$ (due to independence between η_t and S_{t-1}). By vectorizing both sides we get
$vec(Var(S_t)) = (F \otimes F)vec(Var(S_t)) + vec(Q)$
$\iff vec(Var(S_t)) = (I - (F \otimes F))^{-1}vec(Q)$.

To maximize the likelihood under the alternative I initialize the Kalman filter only once and switch the parameters of the state space model after the break and maximize with respect to the parameters before and after the break simultaneously. This implies that the LR statistic is always positive because at least the same likelihood as under the null could be achieved, if the parameters after and before the break are the same as the MLE without break.

5 DISCUSSION

Exercise What do the results tell us on the power of the test? What does this imply for cases where we cannot – and can – reject H_0 in an empirical exercise?

Solution:
Even for only 50 observations before and after the break the test has estimated power of 0.83 even at the 0.001 level, implying that the test is quite powerful at detecting a break of this magnitude. For a larger sample size of 100 observations before and after the break the test has estimated power of 1. Therefore non-rejection of this test can in practice be interpreted as strong evidence that there is no structural break of this magnitude.
Rejection at a small significance level of course implies that there is a structural break for one of the parameters in Θ. Simulations under the true null with $\rho = 0.95$ at the 0.05 level show that there are no severe size distortions of the test with rejection rates of 0.052 and 0.062 for the smaller and larger sample size respectively. Note that the sample mean of M i.i.d Bernoulli variables with p=0.05 follows asymptotically $\sim N(0.05, 0.0475/M)$ so for 500 Monte Carlo simulations and $\pm$ 2 SE regions rejection rates in simulations between 0.03 and 0.07 are expected even if the size is correct at 0.05.

Some comments are in order however that make the test less appealing in practice than it seems in this simulation: First, the power of the test depends of course on the magnitude of the break and weaker breaks are less likely to be detected.
Second, we're assuming a correctly specified likelihood based on normally

distributed shocks. A violation of this assumption would also violate the test statistic and exact distributional assumptions of the underlying shocks are quite strong. The test size at the 0.05 level is 0.18 and 0.188 for the smaller and larger sample respectively, if the shocks in the simulations are generated from a t(5) distribution (normalized to have variance 1). This is despite parameter estimates appearing to still be consistent due to QMLE. A workaround may be to bootstrap the QLR test statistic to find its asymptotic distribution but this is computationally demanding.

Lastly, we assumed the break date to be known which is also unrealistic in practice. One workaround is to go through a set of possible break dates and reject the null if the maximum LR statistic exceeds some critical value. The maximum of several LR statistics however does no longer follow a standard chi-square distribution under the null of no break. This likely leads to a loss of power and to higher computational demand as the null distribution of such a statistic requires simulation.