Paper 1: **MICRO**

SUMMARY NOTES

Luca F. Righetti (lfr29)

University Of Cambridge | Economics: Part IIA (2018/19)

Table of Contents

Strategic Thinking

Fundamentals

Nash Equilibrium (NE)

- *Definition:* $u_i(a_i,) \geq u_i(a_i', a_{-i})$ for all $a_i' \in A_i$ and $i \in N$
  - There is no gain in utility in playing any alternative a_i', given the other player's action a_{-i}
- *Usefulness*: Can yield counter-intuitive insights:
  - Baress' Paradox: building a road may increase congestions
  - Prisoner's Dilemma: Outcome is not necessarily Pareto Efficient
  - Ketty Genovese Case: Women is openly attacked in front of a crowd. More people present means more agents can intervene but also that the incentive to free ride is larger. Case suggests that the second effect dominates and intervention is less likely!
  - Sun-Tzu: Limiting choices available is a useful commitment device that makes threats credible. E.g. burning a bridge to prevent your own retreat.

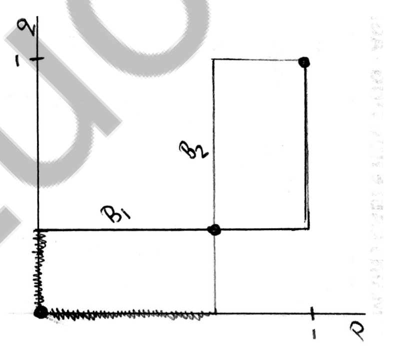
<u>Theoretical Problems</u>

- No NE might exist (e.g. Three-candidate election). Can do better by permitting mixed strategies.
- Multiple NE might exist (e.g. Battle of the Sexes). Can do better by using subgame perfection in extensive form games, restricting weakly dominated strategies, and excluding unstable equilibria.
- MSNE assumes people randomize. But does appear true in cases (e.g. tennis serves, penalties)
- NE entails correct beliefs that are unjustified and involves complex reasoning IRL.
- There is no single algorithm but many rules of thumbs (BR, iteratively deleting strictly dominated strategies, indifference condition for MSNE, backwards induction for SPNE)

- Often the problem is not due to NE itself but due to incorrectly modelling pay-offs due to a simplistic interpretation of agent's preferences. E.g. do not include social pride, altruism, etc.
- We cannot directly compare utility levels between NE since strategic games are based on ordinal, not cardinal, preferences.

Best Responses

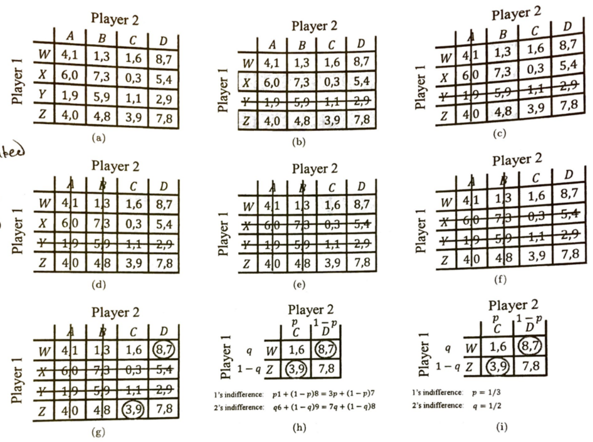
- Called BR Correspondence for discrete and BR function for continuous cases
- $B_i(a_{-i}) = \{a_i \in A_i: u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})\}$ for all $a_i' \in A_i$.
  - a_i gives the highest possible pay-off for i given that a_{-i} has been played.
- a^* is NE iff $a_i^* \in B_i(a_{-i}^*)$ for all $i \in N$ i.e. mutual best responses. Graphically this is intersection

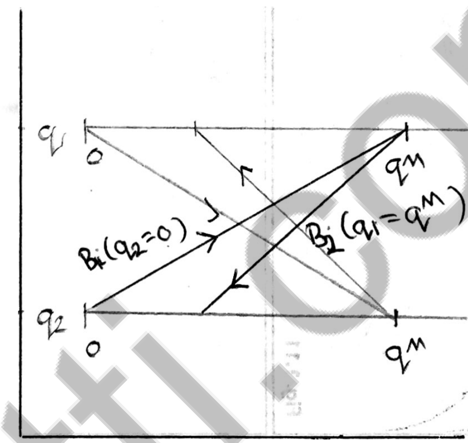


Dominating

- a_i is (strictly) dominant if $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$ for all $a_i' \in A_i \backslash a_i$ and $a_{-i} \in A_i$
- a_i is (strictly) dominated if there exists a_i' s.t. $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$ for all $a_i' \in A_i$
- A <u>strictly</u> dominated strategy is never played in NE. Use this to iteratively delete strategies and simplify the pay-off matrix! A weakly dominated action cannot be played in a <u>strict</u> NE

B weakly dominated by C and D but strictly dominated by a mix of C/D





Preferences

- Note that with v-NM preferences, utilities are cardinal not ordinal.
- If preferences were ordinal the two matrices would represent the same game where $u_1(F,Q) > u_1(Q,Q) > u_1(F,F) > u_1(Q,F)$ but…
- Game 1 P1: $(Q,Q) \sim L\left[\frac{1}{2}\ (\text{F},\text{Q}); \frac{1}{2}\ (\text{F},\text{F})\right]$ | Game 2: $(Q,Q) \succ L\left[\frac{1}{2}\ (\text{F},\text{Q}); \frac{1}{2}\ (\text{F},\text{F})\right]$

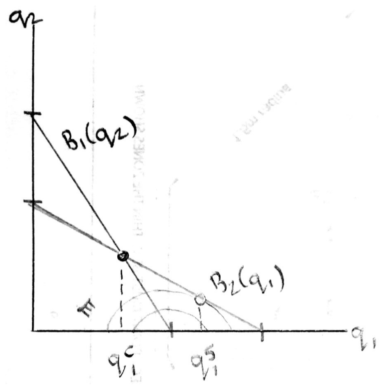
	Q	F
Q	2,2	0,3
F	3,0	1,1

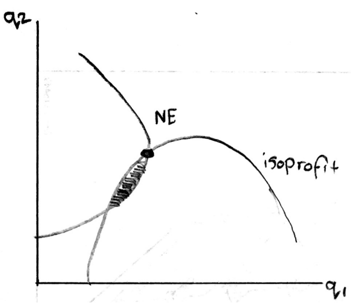
	Q	F
Q	3,3	0,4
F	4,0	1,1

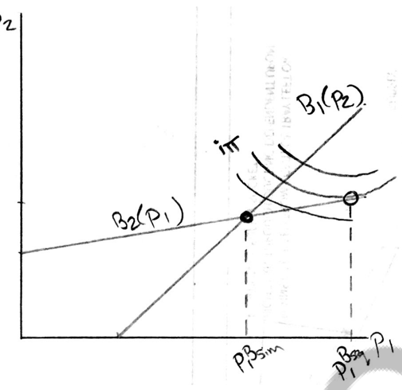
Single-Round Simultaneous Games

Cournot Oligopoly

- *Set-Up:* Firms simultaneously choose $q_i \geq 0$ to $\max \pi_i\,(q_1, \dots, q_n) = q_i P(q_1 + \cdots + q_n) - c(q_i)$
- *Solving:* Take FOC $\left(\frac{\delta \pi_i}{\delta q_i}\right)$ to find $B_i(q_{-i})$. Apply symmetry and solve to find q_i^{NE} [only finds interior!]
  - In simple case: $q_i^{BR} = \frac{a-q_j-c}{2}$; $q_i^{NE} = \frac{a-c}{3}$ [more generally $\frac{a-c}{n+1}$];
  - Where BR_i is a the locus of the highest points of isoprofit lines ($\max \pi_i$ given $\bar{q}_j$)
- *Interpretation:*
  - Strategic substitutes. Best Response to a rise in q_1 is a fall in q_2
  - Both firms make more π if both reduce q_i, hence NE is not PE!
  - More firms leads to lower price (closer to perfect competition)
  - In 3 player game, if A+B merge they make less π and C more! A+B internalize more of ve- externality, hence $\downarrow q_{A+B}, \uparrow p$ (e.g. OPEC cuts back, Russia produces more)
  - Makes more sense for 'big' products (e.g. airplanes).
  - Can reformulate game to common property. Conclude that NE overuses it.







Bertrand Duopoly

- *Set-Up:* Firms simultaneously choose $p_i \geq 0$ to $\max \pi_i$. Consumers go to cheapest seller.
- *Solving:* Cannot take FOC since profits change discontinuously in p_i. Instead apply logic to cases.
  - $BR_i(p_j)\begin{cases} p_i\colon p_i > p_j & \text{if } p_j < c & \pi = 0 \\ p_i\colon p_i \geq p_j & \text{if } p_j = c & \pi = 0 \\ p_j - \epsilon & \text{if } c < p_j \leq p^M & \pi = ve^+ \\ p^M & \text{if } p^M < p_j & \pi = ve^+ \end{cases}$
  - In simple case $p_i^{NE} = p_j^{NE} = c$. Note $p_j < c$ is weakly dominated by $p_j = 0$. This is only NE:
    - $\min\{p_i, p_j\} < c$: Profitable deviation for $< c$ to raise to $= c$ to make zero π.
    - $p_i = c, p_j > c \,|\, p_i > c, p_j = c$: Profitable deviation for $= c$ to raise to $= c + \delta$, increasing π
    - $\min\{p_i, p_j\} > c$: Profitable deviation for max to slightly undercut other to make positive π
  - Where BR_i is a the locus of the lowest points of isoprofit lines ($\max \pi_i$ given $\bar{p}_j$)
- *Interpretation:*
  - Strategic complements: Best Response to a rise in p_1 is a rise in p_2
  - Implies $\pi^{NE} = 0$! Not the case IRL because products are not perfectly homogenous!
  - Makes more sense for everyday businesses (e.g. coffee shops)

<u>Product Differentiation Variation</u>

- *Set-Up:* Consumers' with utility function: $v - (l - x_i)^2 t - p_i$ have preferences l uniformly distributed on the unit interval [0,1]. Firms A,B fixed at location 0,1 respectively
- *Solving:*
  - Find indifferent consumer $[v - (l^*)^2 t - p_1 = v - (1 - l^*)^2 t - p_2 \Rightarrow l^* = \frac{t + p_2 - p_1}{2 - t}$. Assume that v is sufficiently high (i.e. consumer would rather buy either than nothing)
  - Find general demand function $q_1 = \frac{t + p_2 - p_1}{2t}$ and $q_2 = 1 - \frac{t + p_2 - p_1}{2t} = \frac{t - p_2 + p_1}{2t}$ and use to solve $\max \pi = q_i(p_i, p_j)(p_i - c)$ via FOC. Thus conclude $p_1 = p_2 = t + c$
  - By contrast to normal Bertrand, prices aren't driven down to MC but to t+c, with t reflecting how differentiated the products are

Hotelling's Model of Electoral Competition

- *Set-Up:* Politicians choose whether to enter a race and which platform to run on. They have preferences win ≻ tie ≻ stay-out ≻ lose. Voters elect 'least worst' based on some linear scale

<u>Two Players</u>: NE: $x_1 = x_2 = m$ where m is the preference of the median voter. No profitable deviation.

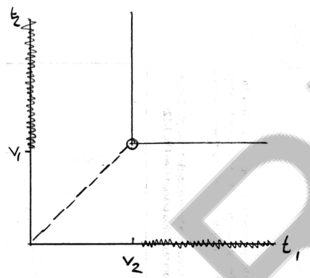
- *Solving:* $BR_1(x_2)\begin{cases} x_1: x_2 < x_1 < 2m - x_2 & \text{if } x_2 < m \\ m & \text{if } x_2 = m \\ x_1: 2m - x_2 < x_1 < x_2 & \text{if } x_2 > m \end{cases}$
- *Interpretation:*
  - Position of median voter explains political waves and should promote centrism (e.g. Tony Blair)
  - In three player scenario there is no NE! Always an incentive for one person to deviate.

<u>Three players:</u> No NE! Always a profitable deviation

- If none enter, profitable deviation to enter and choose m (will win)
- If one enter, profitable deviation to enter and choose m (will at least tie)
- If two enter (and both choose m), profitable deviation to enter just left or right (will win)
- If three enter and at least one loses, profitable deviation for loser to not enter at all
- If three enter and tie, profitable deviation to move optimally (will win)

War of Attrition

- *Set-Up:* Each predator decides how long to fight t_i. The one who is able to hold out longer gets a reward v_i.
- *Solving:*
  - $u_i(t_1, t_2)\begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\ v_i - t_i & \text{if } t_i > t_j \end{cases}$ and $B_i(t_j)\begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\ v_i - t_i & \text{if } t_i > t_j \end{cases}$
  - NE: P1/P2 concedes immediate and P2/P1 at v_1/v_2. There is never an actual fight!
- *Interpretation:* P1, P2 population must be distinct (e.g. conventions has it that the challenger always gives up). If not then Game Theory has no predictive power



Auctions

- 2nd Price Auction: a player's bid equal to true valuation v_i weakly dominates all other bids b_i
- 1st Price Auction: $b_i < v_i$ is not weakly dominated by $b_i = v_i$ or any strategy. Note that one player's strategy will always be weakly dominated
- The single NE in which no player's bid is weakly dominated in a 2nd price auction yields the same outcome as the distinguished equilibrium of a 1st price auction.

Mixing Moves

Basics

Mixed Strategy Nash Equilibrium (MSNE)

Definition and Notations

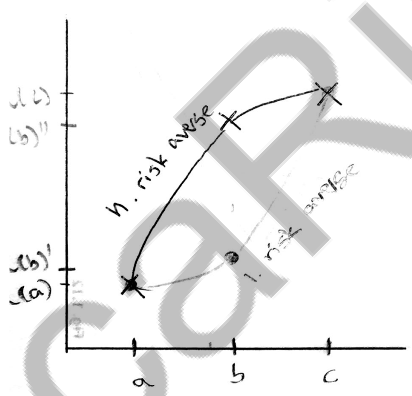
- Player's choice can vary as long as the pattern of choices remains constant
- $a =$ action; $\alpha^* =$ (mixed) strategy
- *Mixed strategy:* A probability distribution over the player's actions
- *Pure strategy:* Only choose a single action with $p = 1$
- *Mixed strategy profile:* a collection of mixed strategies

Nash Equilibrium

- MS is NE iff $U_i(a_i, a_{-i}) \geq U_i(a_i', a_{-i})$ for all $a_i' \in \Delta A_i$
- MSNE has general property that agent is indifferent about everything you mix over with positive probability: $E[U_i(\text{rock})] = E[U_i(\text{paper})] = E[U_i(\text{scissors})]$.
- Any pure NE is also a MSNE. Any game with more than one pure NE will, by definition, also have MSNE, where the agent mixes over these NE. A strictly dominated action is never used in MSNE.
- Every game in which each player has finitely many actions, there is at least one MSNE, assuming vNM preferences.

Exogenous Uncertainty

- vNM preferences can be represented by a Bernoulli payoff function: $U_i(\alpha) = \sum \alpha_i(a_i)E_i(a_i, \alpha_{-i})$. $P \succ Q$ iff $\sum p_a u_i(a) > \sum q_a u_i(a)$
  - Risk taking preferences are incorporated in E not U



- Note that vNM preferences are necessarily cardinal, not ordinal. Do not confuse them!
  - Suppose choice between Bundle A (50% £0 and 50% £1000) and Bundle B (100% £450)
  - Let $u(0) = 0;\ u(1000) = 10;\ u(450) = 4$
  - Note that $U(A) = 0.5 \times 10 + 0.5 \times 0 = 5 > U(B) = 4$
  - If we change to $u(450) = 6$, despite maintaining rank, $U(A) < U(B)$

Bertrand Competition with Sunk Cost

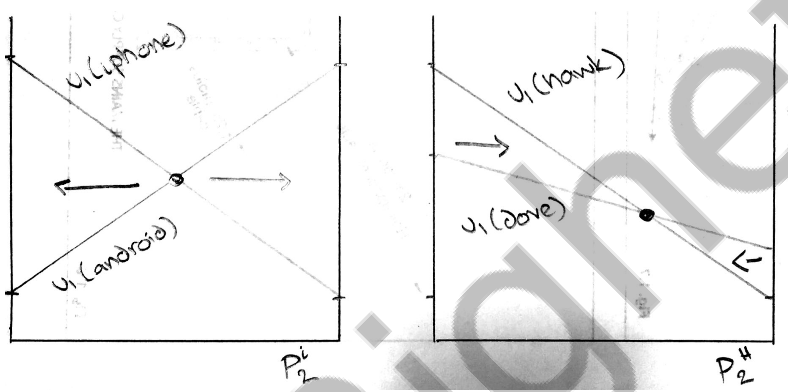
- Let π be prob. of entering and p price drawn at random from CDF F (e.g. $F_j(0.3) = 0.5$ means 50% probability j chooses a price less than 0.3)

- Note pay-off from entering (win+lose): $U_i(p_i) = (p_i - c)\left[(1-\pi_j) + \pi_j\left(1 - F_j(p_i)\right)\right] - k$
- Note pay-off from not entering $= 0$.

- There is no PSNE: always either a profitable deviation to enter and undercut or to not enter. Instead look to find MSNE, at which i must be indifferent between two pay-offs…
- $(p_i - c)\left[(1-\pi_j) + \pi_j\left(1 - F_j(p_i)\right)\right] - k = 0 \therefore F_j(p_i) = \frac{1}{\pi_j}\left(\frac{p_i - c - k}{p_i - c}\right)$
- Hence, using the fact that $F_j(c+k) = 0$ and $F_j(v) = 1$, solve $\pi_j = \frac{v-c-k}{v-c}$
- Hence, substituting back in, we define $F_j(p_i) = \begin{cases} 0 & \text{if } p < c+k \\ \frac{(v-c)(p-c-k)}{(v-c-k)(p-c)} & \text{if } p \in [c+k, v] \\ 1 & \text{if } p > v \end{cases}$
- *Intuition*: P1 tradeoff in deciding whether to enter (risking loss k but possible profit) and choosing price (↑ p, ↑profit if win but higher risk of loss). P2 strategy balances these forces and vice versa.
  - P1's expected payoff from entering and choosing a price = expected payoff from not

Stability

- A NE is considered stable if, given small deviations, we converge back. E.g. Side blotched lizards give natural examples, with population densities adjusting to maintain balance

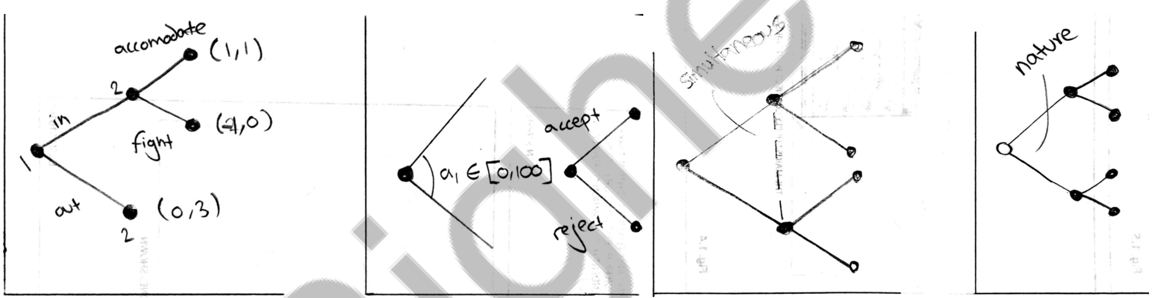


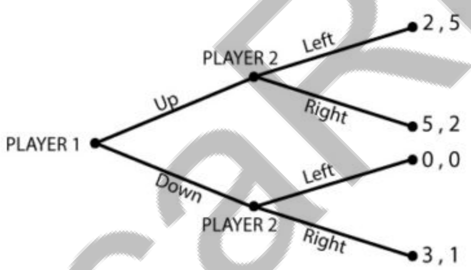
Dynamic Games

Basics

Interpretation

- So far we have only studied simultaneous decisions. Can also model sequential games:
- *Strategy*: now defined as a vector of actions, accounting for every possible situation. A behavioral strategy mixes over actions at each specific node. A mixed strategy mixes over the strategies available.
- *Subgame*: Take an action node to start from and consider only nodes than can be reached from this (and contains only complete information sets)
- There are two ways to represent such games: *Extensive Form* (tree) and Strategic Form (plans)
  - Action Node: Associated with a player choosing an action
  - Branches/Edges: Represent available actions
  - Terminal Node: Assign pay-offs
  - Simultaneous moves: Let players make moves the other does not observe. Creates Info Set:
    - At each node, i has the same actions available
    - No node in the set is a successor to another in the set
    - Each action node belongs to exactly one information set
    - Must take the same action at all nodes in the same information set





		PLAYER 2			
		R , L	L , R	R , R	L , L
PLAYER 1	U	5 , 2	2 , 5	5 , 2	2 , 5
	D	0 , 0	3 , 1	3 , 1	0 , 0

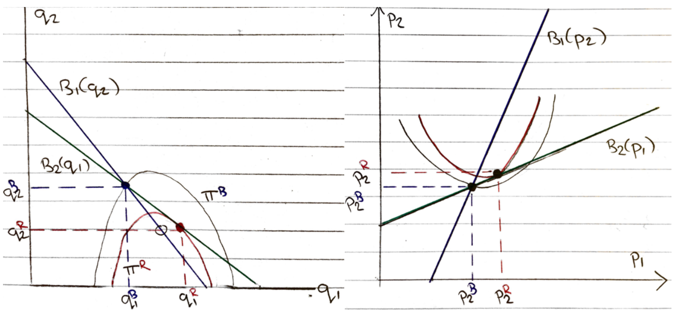
Subgame Perfect Nash Equilibrium (SPNE)

- NE ignores sequential structure of extensive games. Steady state may not be robust (empty threat)
- SPNE is a strategy profile that induces NE in *all* subgames. This includes all those not on-path (i.e. never played irl) and ensures only robust steady states. Hence, not all NE are necessarily SPNE!
- Assuming finite sequence and perfect information, we are guaranteed a SPNE! First find a SPNE for all subgames length 1, then use this to find a SPNE for all subgames length 2, etc.

Single Round Sequential Games

Stackelberg (i.e. Sequential Cournot) Oligopoly

- *Solving:* Use backwards induction: find $q_2(\bar{q}_1)$ [$\max \pi_2 = q_2(A - \bar{q}_1 - q_2 - c)$ gives FOC $B_2(\bar{q}_1) = \frac{A-\bar{q}_1-c}{2}$]; likewise solve for $q_1(\bar{q}_2(q_1))$ [gives unique SPNE $q_1 = \frac{A-C}{2}$ and $q_2 = \frac{A-C}{4}$]
- *Graphically*: P1 takes B_2 as given and chooses point tangent to isoprofit line. Note this is not on B_1
- *Insight*: First mover advantage: P1/P2 produces more/less than Standard Cournot.
    - If order goes P1,P2, P1 then P2 essentially has the First Move advantage!
- *Intuition:* Sequential Cournot ⇒ strategic sub. ⇒ BR to ↑ q_j is ↓ q_i ⇒ First firm can be aggressive and by producing a lot force the other firm to produce less



Sequential Bertrand Duopoly

- *Solving:* Use backwards induction: find $p_2(\bar{p}_1)$ [$\max \pi_2 = q_2(p_2, \bar{p}_1)[p_2 - c]$ gives FOC $B_2(\bar{p}_1) = \frac{t+\bar{p}_1+c}{2}$]; likewise solve for $p_1(\bar{p}_2(p_1))$ [gives unique SPNE $p_1 = \frac{3t+2c}{2}$ and $p_2 = \frac{5t+4c}{4}$]
- *Graphically:* P1 takes B_2 as given and chooses point tangent to isoprofit line. Note this is not on B_1
- *Insight:* Second mover advantage: Both P1 and P2 profit more through higher prices than Standard Bertrand, but with P2 profiting even more so
- *Intuition:* Sequential Bertrand ⇒ strategic comp. ⇒ BR to ↑ p_j is ↑ p_i ⇒ First firm commits to high p so that second firm also sets high p

n-Player ultimatum with uncertainty

- Uncertainty: Add a new player, nature, which randomly chooses which subgame is played. Use an information set to captures the fact that people do not know where they are in the game
- *Set Up:* Nature chooses the disagreement value $d_i \begin{cases} 0 \text{ with prob } p \\ \frac{100}{n} \text{ with prob } 1-p \end{cases}$. P1 proposes split [s.t. $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 100$]. Players observe disagreement vector and iff $a_i \geq d_i$, accept
- *Solving:*
    - If P1 offers $a_j = 0 \ \forall j \neq 1$, probability that all accept p^{n-1}. So expected payoff $100p^{n-1}$
    - If P1 offers $a_j = \frac{100}{n} \ \forall j \neq 1$ probability that all accept 1. So expected payoff $\frac{100}{n}$
    - P1 prefers to offer $\frac{100}{n}$ iff $\frac{1}{n} > p^{n-1}$
- *Intuitively*: If n is very large, uncertainty about what is acceptable removes proposer advantage.

Repetitions

Set-Up

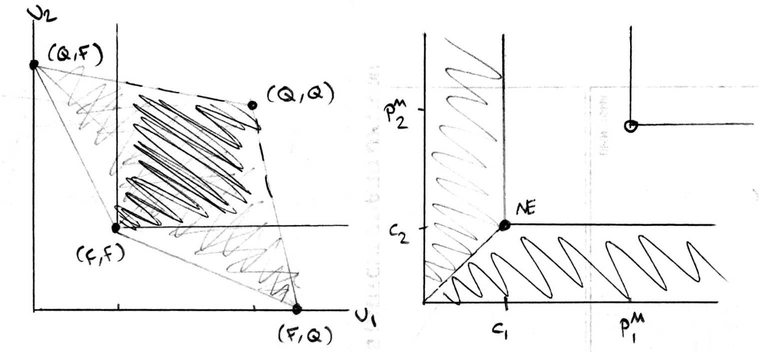
- The game being repeated is called stage-game. $U_i(a_1, a_2 \dots a_T) = \sum_{t=1}^{T} \delta^{t-1} u_i(a_t)$ where $a_t \in A$ and δ is the discount factor, allowing us to incorporate time preferences.
  - Note that this assumes exponential discounting utility function ($U = u_0 + u_0\delta + u_1\delta^2 \dots$), which is a terrible model of behavior IRL since there is no time inconsistency.
  - Any $\sum_{t=1}^{T} \delta^{t-1} u_i(a_t)$ corresponds to some $\sum_{t=1}^{T} c^{t-1}$. We can thus calculate the <u>discounted average stream</u> $c = (1-\delta)V$.
- Sometimes we are interested in $\delta \to 1$ but then payoffs become infinitely large. Hence rescale by multiplying by $1-\delta$
  - $\sum_{t=1}^{\infty} \delta^{t-1} b = \frac{b}{1-\delta}$
  - $\lim_{\delta \to 1} \sum_{t=1}^{\infty} \delta^{t-1} b = \infty$
  - $\sum_{t=1}^{\infty} (1-\delta)\delta^{t-1} b = b$
  - $\lim_{\delta \to 1} \sum_{t=1}^{\infty} (1-\delta)\delta^{t-1} b \ = b$
- If δ close to 1, discounted average payoffs of players is approximate the weighted average of their stage game payoffs
- E.g. if outcome alternates between (C,C) and (C,D): P1: (2+0)/2=1 | P2: (2+3)/2=2.5
- SPNE iff fulfills <u>one-deviation-property</u>: No player can increase their payoff by changing their action at the start of any subgame in which they are the first mover, given the other player's strategies and the rest of their own strategy.
- For finitely repeated games we use backwards induction. There will always be SPNE (Zermelo's T.)
- For infinitely repeated game we know that, given a proposer, all subgames are identical! Hence we cannot have a SPNE with delay (i.e. if optimal to defect in $t = 1$, also optimal to defect in $t = 0$)
  - IRL don't know when games will end, so as long as probability of termination is high we can model games as infinite

Nash Reversion Strategy

- Can also use Nash reversion strategy: Use stage game NE as threat/punishment for deviation to sustain cooperative behavior.
  - Finite games never have a credible Nash reversion strategy since people will always have an incentive to cheat in the ultimate period, hence the pen-ultimate period.
- Can use this to break out of bad NE. As $\delta \to 1$ (i.e. discount factor increases), agents become more patient and more Nash reversion strategies become SPNE. E.g.
  - Grim Trigger: Cooperate until a first defection by the other. Then always Defect. Play minmax punishments forever
    $$s_i(a_i^1, \dots, a_i^t) = \begin{cases} C & \text{if } (a_j^1, \dots, a_j^t) = (c, \dots, c) \\ D & \quad\quad \text{otherwise} \end{cases}$$
  - Finite Punishment: Cooperate until a first defection by the other. Then Defect for k periods
  - Tit-for-Tat: Cooperate in t if other player cooperated in $t-1$. Else Defect.

Possible Payoffs

- Can get any weights by varying frequency of a_t. But to find payoffs that are feasible comes down to finding the harshest possible punishment: $\min_{a_1 \in A_1} \max_{a_2 \in A_2} u_2(a_1, a_2)$
- *Folk Theorem:* In the limit, as $\delta \to 1$ a payoff vector can be obtained in NE iff it is feasible and all players receive more than their min-max



Rubinstein Bargaining Model

- *Set Up*: Pi offers division of pie a_i; Pj accepts or rejects; if rejects roles switch; repeat for infinite rounds. Players discount future by factor δ. Note we cannot have SPNE with delay!
- *Solving:* Let V_1 = expected payoff in round 1 for P1 and V_2 = expected payoff in round 2 for P2
  - In round 1, P1 must offer P2 δV_2 (expected payoff to P2 in round 2) thus $V_1 = 100 - \delta V_2$
  - In round 2 P2 will offer P1 $\delta \hat{V}_1$ (expected payoff to P1 in round 3) thus $V_2 = 100 - \delta \hat{V}_1$
  - Since infinite rounds, $\hat{V}_1 = V_1$. Thus have simultaneous equations to solve
  - Hence know P1 will offer $\frac{100\delta}{1+\delta}$ in round 1 and P2 accepts. End of game.
- *Intuitively*: higher δ, more patient, need to give larger amount now as players value future more

Finitely (T) Repeated Prisoner's Dilemma

- Unique SPNE where players (D, D) in all periods. Every NE generates outcome path (D, D) (strategy may specify other actions for histories with different outcomes, but never occur in equ)
- *Backwards Induction*: In T, players play (D, D) as there is no future to impact. In T-1, players know whatever happens, (D, D) in T, hence actions have no impact on future, thus (D, D). Repeat.

Infinitely Repeated Prisoner's Dilemma

- Many NE: Simplest (always D, always D): No incentive to deviate since if C will get lower pay-off
- Can also use (Grim Trigger, Grim Trigger) for sufficiently high δ. Intuitively, $\uparrow \delta$, more patient, increase in immediate utility from defecting is less than the future utility due to punishment phase.
  - If follow: $2 + 2\delta + 2\delta^2 \ldots = 2 + 2\delta + 2\delta^2 \ldots + \delta^{t-1}\left(\frac{2}{1-\delta}\right)$
  - If defect: $2 + 2\delta + 2\delta^2 \ldots + 3\delta^{t-1} + \delta^t + \delta^{t+1} + \cdots = 2 + 2\delta + 2\delta^2 \ldots + \delta^{t-1}\left(3 + \frac{\delta}{1-\delta}\right)$
  - Thus $\frac{2}{1-\delta} \geq 3 + \frac{\delta}{1-\delta}$ so $\delta \geq \frac{1}{2}$

Silly Errors Corrected:

- For strategies in dynamic game, remember to include all contingencies
- SPNE of repeated game: look at on-path and off-path!
- In repeated game punishment phase, deviations cannot be profitable since players choose actions that form a NE in the stage game

Social Choice & Welfare

- Social Choice Rules [SCR]: aggregate individual inputs (e.g. votes, preferences) into collective outputs (e.g. collective decisions)

Arrow's Impossibility Theorem

Argument

- There exists no SCR for aggregating the preferences of two or more individuals over three or more alternatives into collective preferences, where this method satisfies:
  1. Universal Domain: SCR cope with any level of pluralism in its inputs.
  2. Ordering: SCR produce 'rational' social preferences, avoiding Condorcet cycles.
  3. Weak Pareto Principle: when all strictly prefer alternative x to y, so does society [if $x \succ_i y \; \forall \; i$ then $x \succ y$].
  4. Independence of Irrelevant Alternatives: social preference between any two alternatives x and y depend only on the individual preferences between x and y, not other alternatives such as z.
  5. Non-dictatorship: no one can determine the social preference, regardless of other individuals
- Implies choice of SCR is arbitrary, being based on subjective view of which axioms to sacrifice.
  - Riker: mathematical proof of the impossibility of populist democracy and thus entertained the idea of abandoning democracy all together.

Criticism

- In an ideal world and general application axioms attractive but in reality we can often relax them…
  - [1.]: IRL SCR does not need to deal with every kind of profile but only types that satisfy certain cohesion conditions.
    - E.g. Black: if we just have single-peaked preferences (e.g. tax or quantity of public good) the most preferred alternative of the median individual relative is always a Condorcet winner
  - [3.]: IRL may relax this to ensure minimal liberalism.
    - 'Lady Chatterley's Lover' thought-experiment: if we wish to respect individual rights, may sacrifice Paretian efficiency or limit our domain to suitably 'tolerant' preference profiles

- … but we cannot just freely relax any axiom for any case.
  - [2.]: Gibbard: if we replace with quasi-transitivity, resulting aggregation possibilities are still limited
  - [5.]: Ethically may not tolerate any relaxation for state elections or referendums

Interpersonal Comparisons

- If welfare is hedonic utility, which can be experienced only from a first-person perspective, interpersonal comparisons are hard to justify (implicitly assumed by Arrow's Impossibility Theorem).
- If welfare is objective satisfaction of subjective preferences or an objective state, then we can make such comparisons and have richer set of inputs with which to judge SCRs (can satisfy!)
  - Sen's reinterpretation of the Impossibility Theorem: relying on ordinal preferences is insufficient for social choices. Hence imperative to use interpersonally comparable welfare measurements

- 'Ordinal measurability with No interpersonal Comparability' [ONC]
  - Arrow and Robbins: We may be able know that $x >_1 y$ and $y >_2 x$ but we can never know how much more 1 prefers x to y relative to 2
- 'Ordinal measurability with interpersonal Level Comparability' [OLC].
  - Only the rank of outcomes matters, we can meaningfully compare ranks between agents.

- Allows for Rawls's "veil of ignorance" (or max-min principle), where social alternatives are ranked according to the welfare of the worst-off individual:
  - For any profile $< W_1, W_2, \ldots, W_n >$ and choices $x, y \in X$, xRy iff $min_i \in N(W_i(x)) \geq min_i \in N(W_i(y))$
- 'Cardinal measurability with interpersonal Unit Comparability' [CUC].
  - Preferences not only have rank but also a value
  - Allows for Classical utilitarianism: arithmetic and unweighted sum of total individual utilities:
    - For any profile $< W_1, W_2, \ldots, W_n >$ and choices $x, y \in X$, xRy iff $\sum W_i(x) \geq \sum W_i(y)$

Plurality Rule and Borda Count

Decisions under Uncertainty

Basics

Preferences

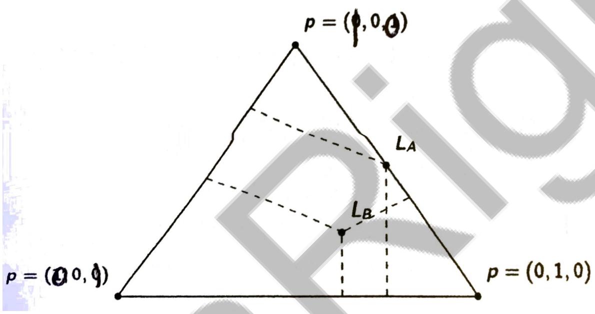
- Preference to choice: Rational preference relations imply coherent choice (and can infer vice versa)
- Preference to utility: Rational preference relations represented by utility. Choice is opt. problem
- Utility Maximizer has preferences such that there exists a utility function $U: \mathcal{L} \to \mathbb{R}$

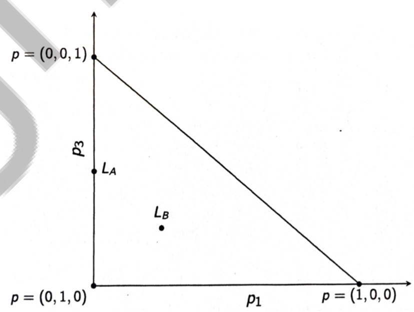
- Houthakker's Axiom of Revealed Preferences: if x and y both contained in both A and B (subsets of X), and if $x \in C_R(A)$ and $y \in C_R(B)$ then it is also true that $x \in C_R(B)$ and $y \in C_R(A)$
  - WARP is like HARP but only with respect to choice sets actually observed $C_R(\{x, y\})$ not $C_R(B)$
- If $C_R(.)$ satisfies non-emptiness and HARP, then $\succcurlyeq c_R$ is rational (complete and transitive)
  - It does not matter whether we start with a complete and transitive preference relation or with a choice function that is non-empty valued and satisfies HARP. One gives the other
  - Rational preference relations can be represented by a utility function (i.e. coherent choice)

Lotteries

Simple Lottery

- L is a list $L = (p_1, \dots, p_n)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$
- Can be represented geometrically either as [i] a point in the (N-1) dimensional simplex, where
- p_n= perpendicular length from point to side opposite vertex n, or [ii] Marschak-Machina triangle





Compound Lottery

- Given K simple lotteries, $L_k = (p_1^k, \dots, p_n^k)$ and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, compound lottery $(\alpha_1, L_1; \dots; \alpha_K, L_K)$ is risky alternative that yields simple lottery L_k with probability α_k
- Note the DM only cares about reduced lottery over final outcomes

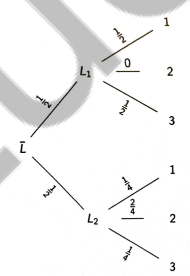


Figure: A compound lottery

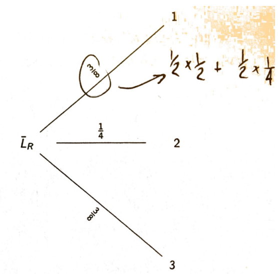
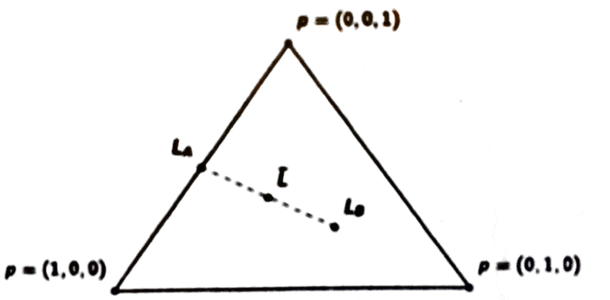


Figure: The reduced lottery



Expected Utility Theory

Assumptions

Rational and Continuous Preferences

- Rational: Preferences are complete, transitive, and reflexive
- Continuous: For any L, there is some probability $\alpha \in [0,1]$ such that $L \sim L' = \{b, \alpha; w, 1-\alpha\}$
  - Where b is the best and w is the worst possible outcome of X

(!) Independence Axiom

- Mixing each of two lotteries with a third one does not affect the original preference ordering: $L \succcurlyeq L'$ iff $\alpha L + (1-\alpha)L'' \geq \alpha L' + (1-\alpha)L''$ where $\alpha \in [0,1]$
- Intuitively, if one prefers L to L' then one must also prefer the chance of getting L to the same chance of getting L' given that the alternative in both cases is the same L''

Independence Axiom - Another Graphical Representation

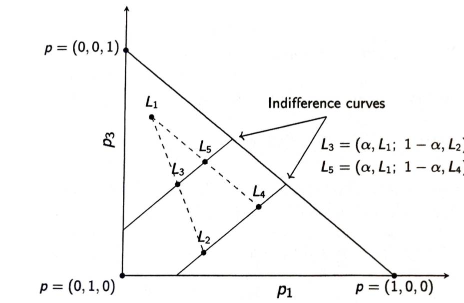


Figure: Lotteries in a 3-outcome world - the Marschak-Machina triangle

? if $L_2 \sim L_4$ adding L_1 to mix should not make difference ∴ $L_5 \sim L_3$

Linked to utility maximizer:

Let $L \succcurlyeq L'$ thus $U[L] \geq U[L']$. If $\alpha L + (1-\alpha)L'' \succcurlyeq \alpha L' + (1-\alpha)L''$

Then $U[\alpha L + (1-\alpha)L''] \geq U[\alpha L' + (1-\alpha)L'']$

Then $\alpha U[L] + (1-\alpha)U[L''] \geq \alpha U[L'] + (1-\alpha)U[L'']$

Then $U[L] \geq U[L']$, which is necessarily true

- Machina triangle representation: indifference curves are linear and parallel, which is a result we can prove from the independence axiom also

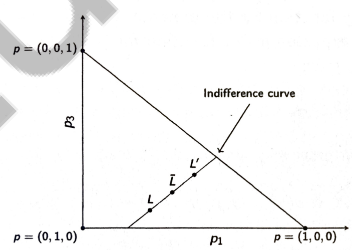
Consequentialist Approach

- DM are able to make all necessary calculations
- Indifferent among all compound lotteries that reduce to the same simple lottery

Modelling Risk

- Expected Utility Function (von-Neumann Morgenstern)
  - Discrete Case: $E[u(X)] = \sum_{n=1}^{N} p_n u(x_n)$: ($u$ is the felicity function; p is subjective prob.)
  - Continuous Case: $E[u(X)] = \int p(x)u(x)$
- Expected Utility Theorem: $L \succcurlyeq L'$ iff $\sum p_i u(x_i) \geq \sum p_i' u(x_i)$ where $\tilde{U}(L) = \alpha U(L) + \beta$ if $\alpha > 0$
- Note this is equivalent to the assumption that IC over lotteries are parallel, straight lines

Expected Utility - Linearity



$L \sim L'$
$E[U(L)] = E[U(L')]$

Let $E[U(L)] = E[U(L')] = 10$

$\bar{L} = 0.5L + 0.5L'$

$E[U(\bar{L})] = 0.5E[U(L)] + 0.5E[U(L')]$
$E[U(\bar{L})] = 0.5 \times 10 + 0.5 \times 10 = 10$

$\bar{L} \sim L \sim L'$

Figure: Indifference curves in a 3-outcome world - the Marschak-Machina triangle

Additional

Intensity of Preferences

- No longer strictly ordinal (can show preference of x over y is more intense than z over w)
  - If $L = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right) \succcurlyeq L' = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$
  - Then $\frac{1}{2}u(x) - \frac{1}{2}u(y) > \frac{1}{2}u(z) - \frac{1}{2}u(w)$ then $u(x) - u(y) > u(z) - u(w)$

States of Nature

- State-Dependent Utility means $u_i(.)$ itself depends on which state of the world actually obtains
  - e.g. if sunny: $u_i(x) = 2\sqrt{x}$; if rainy: $u_i(x) = \sqrt{x}$

Revealed Likelihood

- Suppose don't know preference but observe $L\{£1, s = 1;\ £0, s = 2\} \succcurlyeq L'\{£0, s = 1;\ £1, s = 2\}$
- We can thus infer that DM thinks $s = 1$ is more likely than $s = 2$ (i.e. subjective probability)

Criticisms

Corner Cases

- "Discontinuity in the neighborhood of certainty"
- Allais Paradox: disproves EUT IRL using large payoffs and probabilities close to 0,1
  - Prospect Theory: Modify EUT by giving 'objective' probabilities a subjective weighting function
  - Regret Theory: Allow DM to account for what they may have received if they chose differently
- Andreoni and Sprenger: Find that 'linear in probabilities' implications fails near certainty
  - Note Uncertainty Equivalent (value q such that DM is indifferent between L_A and L_q i.e. $pu(y) + (1-p)u(z) = qu(z) + (1-q)u(0)$
  - Thus $q = \frac{pu(y)+(1-p)u(z)-u(0)}{u(z)-u(0)}$ and EUT implies $\frac{dq}{dp} = \frac{u(y)-u(z)}{u(z)-u(0)} < 0$

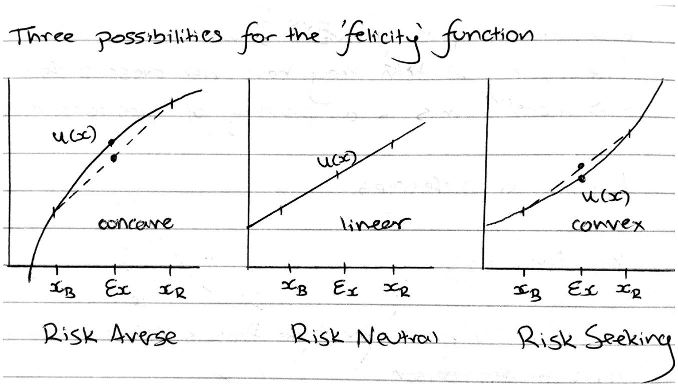
Ambiguity Aversion

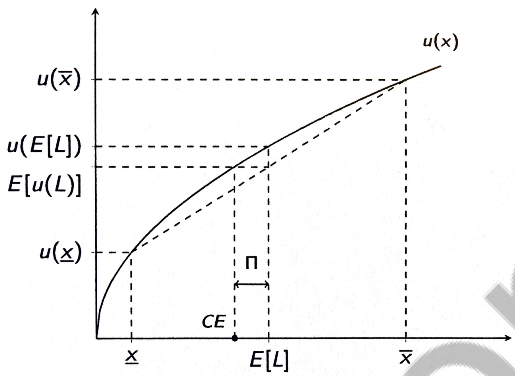
- Ellsberg Paradox: Urn I contains 100 red and black balls in unknown ratio and Urn 2 in 50:50. Suppose you win if red is picked. Suppose you win if black is picked. Contradicting implications!

Attitudes Towards Risk

Basics

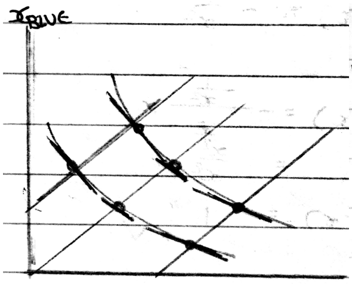
- Certainty Equivalent: CE s.t. $u(CE) = E[U(L)]$: Amount of money for which an individual is indifferent between choosing the certain amount CE and gamble L
- Risk Premium: $\Pi = E[U(L)] - CE_L$. Max amount of income that the risk averse person would sacrifice in order to eliminate the risk associated with a particular prospect
- Risk-Neutral: $CE_L = E[U(L)]$. Risk-Averse: $CE_L < E[U(L)]$. Risk-Seeking: $CE_L > E[U(L)]$
  - Properties e.g. Risk-Aversion:
    - Always prefer a mixture of prospect P_2 with its certainty equivalent E_2 (i.e. $P_2' \succ P_2$)
    - Has a concave felicity function if v-NM [or more generally quasi-concave U-f()]

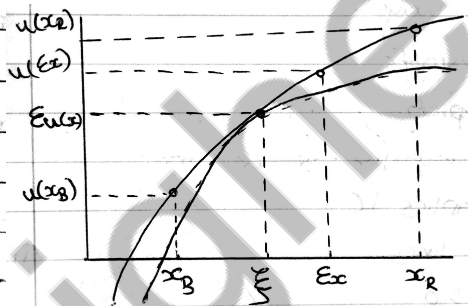


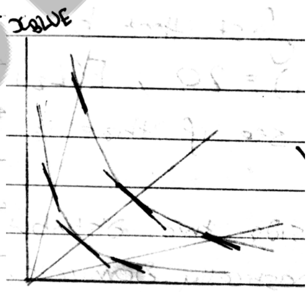


Measures of Risk-Aversion

- Risk aversion is reflected in the concavity of u-f. Let $A(w)$ be set of acceptable gambles. Starting from $(0,0)$, if x_2 falls by δ, need x_1 to rise more than δ to compensate if risk averse.
- Need to normalize by first derivative so that measure is invariant to scaling…
- Arrow Pratt Absolute-Risk-Aversion: $r(x) = -\frac{u''(x)}{u'(x)}$ [i.e. curvature of u]. Risk averse if $r(x) > 0$
  - If $r_u(w)$ decreases in w, DARA. Likewise CARA and IARA.
  - CARA: $u(x) = -\frac{1}{\alpha}e^{-\alpha x}$ gives $r(x) = \alpha$
    - Graphically, always the same distance from CE-line. Thus lines are parallel.



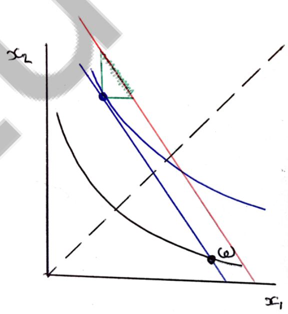


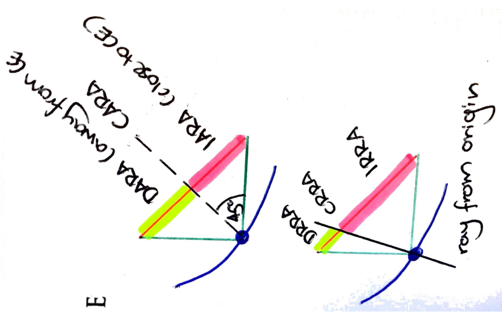


- Arrow Pratt Relative-Risk-Aversion: $\rho(x) = -x\frac{u''(x)}{u'(x)}$ [i.e. elasticity of marginal felicity]
  - If $\rho_u(w)$ decreases in w, DRRA. Likewise CRRA and IRRA.
  - CRRA: $u(x) = \frac{1}{1-\beta}x^{1-\beta}$ gives $\rho(x) = \beta$
    - MRS is the same (prop. risk $\frac{x_2}{x_1}$). Moving budget constraint moves optimal bundle along ray
- Note since $\rho(w) = r(w)w$, we know $\uparrow r(x)$ implies $\uparrow \rho(x)$ [but not vice versa!]

<u>Graphical Narrative</u>

- Start on endowment [black]. Impose budget constraint and reoptimize [blue]. Observe wealth shock [red] and since we assume a normal good can narrow down possible points [green].



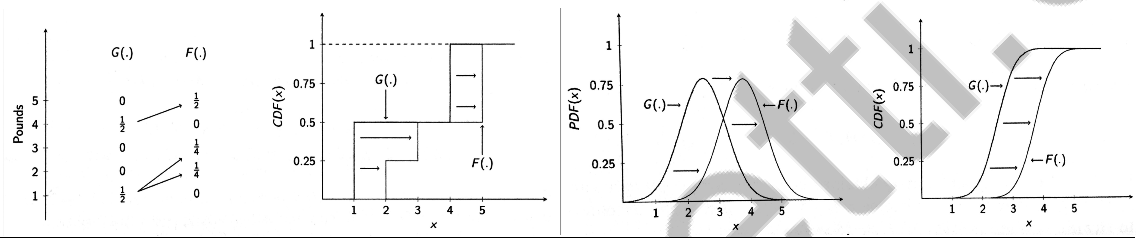


Comparison of Outcome Distributions

- Depends on expected outcome and degree of risk
- Consider two cumulative distributions of monetary payoffs $F(.)$ And $G(.)$
- Every DM who values more over less prefers $F(.)$ To $G(.)$
- For every amount of money, x, the probability of getting at least x is higher under $F(.)$ Than $G(.)$

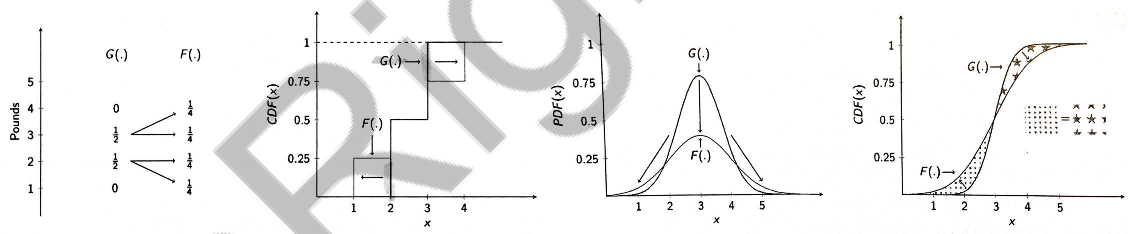
First-Order Stochastic Dominance

- L_1 FOSD L_2 iff $\forall\, \tilde{x}, p_1(x \geq \tilde{x}) \geq p_2(x \geq \tilde{x})$
- $F(.)$ FODS $G(.)$ iff for every non-decreasing function u, $\int u(x)dF(x) \geq \int u(x)dG(x)$
  (i.e. every expected utility maximizer prefers $F(.)$ to $G(.)$)
  - Does not imply every possible return of $F(.)$ is larger than every possible return of $G(.)$
  - Does imply mean $F(.)$ is larger than mean $G(.)$ [$\int xdF(x) \geq \int xdG(x)$] but not vice versa
  - Intuitively, exclusively shifting probability from lower outcomes to higher ones. Requires an increase in the mean value of the lottery



Second-Order Stochastic Dominance

- $G(.)$ SOSD $F(.)$ iff $F(.)$ is a mean-preserving spread of $G(.)$. In other words iff have same mean and for every non-decreasing, concave function u, $\int u(x)dG(x) \geq \int u(x)dF(x)$
  (i.e. $G(.)$ is less risky than $F(.)$ and thus every risk-averse utility maximizer prefers $G(.)$ to $F(.)$)



Information

States and Messages

- Objective probability p_i is replaced subjective probability belief π_s so DM $\max_{\alpha} U(x) = \sum \pi_s u(x_{\alpha s})$
- Signals will lead to a revision of π_s and thus imply a different optimal action. Learning the signal will not change the behavior but simply defines which behavior pattern they are actually going to take.

Table: Consequences of terminal choices

		States (s)			
		1	2	...	S
Choices (α)	1	x_{11}	x_{12}	...	x_{1S}
	2	x_{21}	x_{22}	...	x_{2S}
	...	...	...	...	...
	A	x_{A1}	x_{A2}	...	x_{AS}
Beliefs (π)		π_1	π_2	...	π_S

- $\alpha \in \{1, ..., A\}$: *Terminal* action
- $s \in \{1, ..., S\}$: State of the world
- $x_{\alpha s}$: Outcome in state s resulting from action α

Table: Joint probability matrix, $J \equiv [j_{sm}]$

		Messages (m)				Probabilities for states (π)
	J	1	2	...	M	
States (s)	1	j_{11}	j_{12}	...	j_{1M}	π_1
	2	j_{21}	j_{22}	...	j_{2M}	π_2
	...	...	...	...	...	...
	S	j_{S1}	j_{S2}	...	j_{SM}	π_S
Probabilities for messages (q)		q_1	q_2	...	q_M	1

$$\sum_{m=1}^{M} j_{sm} \equiv \pi_s \ , \ \sum_{s=1}^{S} j_{sm} \equiv q_m$$

Table: Likelihood matrix, $L \equiv [l_{sm}] \equiv [q_{m|s}]$

		Messages (m)				
	L	1	2	...	M	
States (s)	1	$q_{1\|1}$	$q_{2\|1}$	...	$q_{M\|1}$	1
	2	$q_{1\|2}$	$q_{2\|2}$	...	$q_{M\|2}$	1
	...	...	...	...	...	...
	S	$q_{1\|S}$	$q_{2\|S}$	...	$q_{M\|S}$	1

$$l_{sm} \equiv q_{m|s} \equiv \frac{j_{sm}}{\pi_s}$$

$$\sum_{m=1}^{M} q_{m|s} \equiv \frac{1}{\pi_s} \sum_{m=1}^{M} j_{sm} \equiv \frac{1}{\pi_s} \pi_s = 1$$

Table: Potential posterior matrix, $\Pi \equiv [\pi_{s|m}]$

		Messages (m)			
	Π	1	2	...	M
States (s)	1	$q_{1\|1}$	$q_{2\|1}$	...	$q_{M\|1}$
	2	$q_{1\|2}$	$q_{2\|2}$	...	$q_{M\|2}$
	...	...	...	...	...
	S	$q_{1\|S}$	$q_{2\|S}$	...	$q_{M\|S}$
		1	1	...	1

$$\pi_{s|m} \equiv \frac{j_{sm}}{q_m}$$

$$\sum_{s=1}^{S} \pi_{s|m} \equiv \frac{1}{q_m} \sum_{s=1}^{S} j_{sm} \equiv \frac{1}{q_m} q_m = 1$$

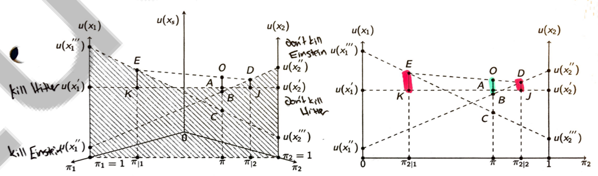
Bayesian Updating

- $\pi_{s|m} = \frac{j_{sm}}{q_m} = \pi_s \frac{q_{m|s}}{q_m} = \pi_s \frac{q_{m|s}}{\sum^S \pi_{s'} q_{m|s'}}$ or $P(S|M) = \frac{P(M|S)P(S)}{P(M)} = \frac{\text{Likelihood} * \text{Prior prob. of S occurring}}{\text{Prob. of M}}$
- Note that from this we see that what matters is
  - Confidence $P(S)$: higher the prior confidence; reflected more in the final conclusion
  - Quantity of Evidence $P(M|S)$: greater the mass of new evidence, higher impact on final conclusion
  - Extremity of Evidence $P(M)^{-1}$: more surprising the evidence; higher impact on the final conclusion

Value of Information

- If terminal action chosen immediately, the DM $\max_{\alpha} U(\alpha; \pi) = \sum_{s=1}^{S} \pi_s u(x_{\alpha s})$ (with solution α_0)
- If prior message m is received, the DM $\max_{\alpha} U(\alpha; \pi_{|m}) = \sum_{s=1}^{S} \pi_{s|m} u(x_{\alpha s})$ (with solution a_m)
- Value of message m: $\omega_m \equiv U(\alpha_m; \pi_{|m}) - U(\alpha_0; \pi_{|m})$
  - i.e. expected utility gain from revision of optimal action in terms of revised probabilities
  - Note this is necessarily non-negative, even if it is unreliable, but potentially zero
- Value of message service μ: $\Omega(\mu) = E_m[\omega_m] = \sum_{m=1}^{M} q_m \left[U(\alpha_m; \pi_{|m}) - U(\alpha_0; \pi_{|m})\right] = \sum_{m=1}^{M} q_m \sum_{s=1}^{S} \pi_{s|m} u(x^*_{sm}) - \sum_{m=1}^{M} q_m \sum_{s=1}^{S} \pi_{s|m} u(x^*_{s0}) = \sum_{m=1}^{M} q_m \sum_{s=1}^{S} \pi_{s|m} q_m u(x^*_{sm}) - \sum_{s=1}^{S} \pi_{s|m} u(x^*_{s0})$ [where x^*_{sm} is state s associated with optimal action α^*_m (and likewise x^*_{s0} for α_0)]
  - i.e. difference between expected utility with and without the service
  - Note that μ is valuable only if it results in a revision of the optimal action

The Value of Information - A Graphical Representation



→ $A = U(x_1|\pi)$, $B = U(x_2|\pi)$, $C = U(x_3|\pi)$

→ $D = U(x_2|\pi_{|2})$, $J = U(x_1|\pi_{|2})$

→ $E = U(x_3|\pi_{|1})$, $K = U(x_1|\pi_{|1})$

→ O: Ex-ante expected utility from information

→ Red line: Value of information (in utility units)

expected gain from getting message:
| - Prob weighted avg
=

Principal Agent Problem

Moral Hazzard

- A form of post-contractual opportunism that arises due to information asymmetry, with action that has efficiency consequences not freely observable. Tackle this via implementing incentive contracts.

Set Up

- P wants to induce A to take some action b (versus a) via an incentive payment $w(\quad)$, which is costs c to A. P cannot observe action but instead output $x = e + \epsilon$ where $\epsilon \sim N(0, \sigma^2)$
  - Assume risk-neutral P (i.e. corporation) and risk-averse A (i.e. individual) [show this in exam]
  - Generally $U_P = x - w(x)$ and $U_A = w(x) - c(a)$
- P faces two obstacles:
  - [PC] participation constraint (A has some reservation utility): $w[x(b)] - c(b) \geq \bar{u}$
  - [ICC] incentive compatibility (A picks best a): $w[x(b)] - c(b) \geq w[x(a)] - c(a) \ \forall \ a \in A$
- We will use linear incentive scheme $w = \alpha + \beta(e + \epsilon + \gamma y)$ [see signal for γy]
  - α for PC; governs division of surplus. $\beta(e + x + \gamma y)$ for ICC; required for prod. efficiency

Guide to Solving

- Specifically, Principal $EU_P = E[S - w] = (1 - \beta)e - \alpha$
- Specifically, Agent is CARA: $EU_A = E[w] - \frac{1}{2}\rho_A Var(w) - c(e) = \alpha + \beta e - \frac{1}{2}\rho_A \beta^2 \sigma^2 - \frac{1}{2}Ce^2$

  [Using $CE = E(L) - \frac{1}{2}rE(L)Var(L)$, where $\frac{1}{2}rVar(w)$ is risk premium for CARA $u_A(w) = -e^{-rw}$]
- P maximizes $EU_P + EU_A$ since P's opt e is also the PE outcome. i.e. P seeks to find e that maximizes surplus, then, via wage setting process, get as large a share of this as possible

Use Backwards Induction to solve:

1. Use U_A to find optimal quantity of effort for any given parameters. Here $e = \beta / C$.
2. Substitute back into original equation to find PC holding: $\alpha + \frac{\beta^2}{2}\left(\frac{1}{C} - \rho_A \sigma^2\right) \geq \bar{u}$
3. Max profits subject to PC $\max \pi = e - \alpha - \beta e$ s.t. $\alpha + \frac{\beta^2}{2}\left(\frac{1}{C} - \rho_A \sigma^2\right) \geq \bar{u}$, giving us $\beta = \frac{1}{\rho_A C \sigma^2 + 1}$
4. Choose the optimal value α, and check P wishes to participate also.

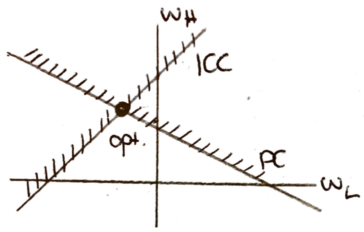
Note: $\beta^* = \frac{R'p(e^*)}{1 + \rho_A c''(e^*)\sigma^2}$, thus $\frac{R'p(e^*) - \beta^*}{c''(e)} = r\sigma^2\beta^*$ i.e. P's M Net Benefit = M Transaction Cost
  - $\uparrow R'p(e^*), \uparrow MR, \uparrow P$'s willingness to incentivize effort, $\uparrow \beta^*$
  - $\uparrow r, \uparrow A$ risk aversion, $\uparrow$ compensation needed, $\uparrow \beta^*$
  - $\uparrow \sigma^2, \uparrow A$ risk exposed to, futile to use wage incentives, $\downarrow \beta^*$
  - $\uparrow c''(e), \downarrow$ responsiveness of e to $\beta, \downarrow$ responsiveness of A to incentives, $\downarrow \beta^*$

Full Information Benchmark

- P: $\max x(b) - w[x(b)]$ s.t. $w[x(b)] - c(b) > \bar{u}$ and $w[x(b)] - c(b) \geq w[x(a)] - c(a) \ \forall \ a \in A$
- Many solutions: $w(x) = \begin{cases} \bar{u} + c(b^*) \text{ if } b = b^* \\ -\infty \text{ otherwise} \end{cases}$ or $w(x) = x - F$ s.t. $x(b^*) - F - c(b^*) = \bar{u}$
  - Note under competition (as opposed to monopoly) $\pi \to 0$ so $x - (x - F) = 0$ so $F = 0$
- P absorbs all risk (graphically, P and $A's$ ICs are tangent along 45-degree line). Makes intuitive sense since P is risk neutral whilst A is risk averse.

Hidden Action

- $\max E(\pi|b) = \sum(x_i - w_i)\pi_{ib}$ s.t. $\sum u(w_i)\pi_{ib} - c_b \geq \sum u(w_i)\pi_{ia} - c_a$ and $\sum u(w_i)\pi_{ib} - c_b \geq \bar{u}$
- There is risk as rewards are backed on an observable outcome that is imperfectly linked to effort due to random, unobservable element ϵ.
- *Intuition*: In order to incentive A to exert high effort they must be exposed to sufficient risk so that they have an incentive to do so. But if too much, P must raise the average payoff to compensate.
  - In theory if A is not risk averse then we could just impose all risk on A
  - Results in some inefficiency. We can never get Pareto Efficiency under these conditions
- In both cases $EU_A = 0$ (PC binds in both cases) but EU_P is now lower since the expected wage of A must increases to compensate them for the income risk.



	$R = 10$	$R = 30$	
$e = 1$	$p = \frac{2}{3}$	$p = \frac{1}{3}$	$E(U_P \mid e = 1) = 50/3$
$e = 2$	$p = \frac{1}{3}$	$p = \frac{2}{3}$	$E(U_P \mid e = 2) = \frac{70}{3}$

Exam-Example

- P is risk neutral $U_P = R$ and A is risk averse $U_A(w,e) = \sqrt{w} - (e-1)$ with reservation $\bar{u} = 1$

- ICC: Thus know that P wants to induce $e = 2$
  - $E(U_A|e = 2) = \frac{1}{3}(\sqrt{w_{30}} - 1) + \frac{2}{3}(\sqrt{w_{10}} - 1) > E(U_A|e = 1) = \frac{2}{3}(\sqrt{w_{30}} - 0) + \frac{1}{3}(\sqrt{w_{10}} - 0)$
  - Simplifies to $\frac{1}{3}\sqrt{w_{30}} - 1 \geq \frac{1}{3}\sqrt{w_{10}}$
- PC: A must also still be willing to participate $[\sqrt{w} - (e-1) \geq 1]$ at all under $e = 2$
  - $E(U_A|e = 2) = \frac{1}{3}(\sqrt{w_{30}} - 1) + \frac{2}{3}(\sqrt{w_{10}} - 1) \geq 1$
- Solving together we get $w_{30} = 9$ and $w_{10} = 0$

Extensions

Signal

- *Informativeness Principle*: It is optimal to base A's pay on signal y iff doing so allows A's effort e and random shock ϵ to be estimated with lower error
- Now have observable signal y (info about ϵ unaffected by e) so now $w = \alpha + \beta(e + \epsilon + \gamma y)$
  - $EU_P = CE_P = R_P(e) - (\alpha + \beta e)$
  - $EU_A = CE_A = \alpha + \beta e - c(e) - \frac{1}{2} r\, \beta^2 Var(\epsilon + \gamma y)$
- First find γ^*, which minimizes Var, and then solve as normal. Generally, $\gamma^* = -\frac{Cov(\epsilon,y)}{Var(y)}$
  - $\uparrow Var(y), \uparrow$ signal noise, $\uparrow$ attach less weight to signal, $\downarrow |y^*|$
  - $\uparrow Cov(\epsilon, y)$. Now if y is high ϵ is likely high, so high x may be more due to ϵ, so worker should get less, so this is reflected in negative γ

Comparative Performance Evaluation

- Assume there are two agents A and B
  - Absolute: $x_A = e_A + \epsilon_A + \epsilon_C$. A max $\alpha + \beta e_A - \frac{1}{2} r\beta^2 Var(\epsilon_A + \epsilon_C) - c(e_A)$
  - Relative: $x_A = e_A - e_B + \epsilon_A - \epsilon_B$. A max $\alpha + \beta(e_A - e_B) - \frac{1}{2} r\beta^2 Var(\epsilon_A - \epsilon_B) - c(e_A)$

- In both cases $\frac{dc(e_A)}{de_A}\Big|_{e_A^*} = \beta$ i.e. induce same level of effort. Now ask, which generates the lower risk premium (lower risk premium → less cost bore by A due to risk → need to compensate A by less)
- Relative is better iff $Var(\epsilon_A + \epsilon_C) > Var(\epsilon_A - \epsilon_B)$. If independent, that is $Var[\epsilon_C] > Var[\epsilon_B]$

<u>Multi-tasking</u>

- *Equal-Compensation Principle*: If P wants A to allocate effort optimally across tasks but cannot monitor each separately then employee's MR-of-return on each activity must be equal (interior solution) or the activity with lower MR-of-return receives no effort (corner solution)
- Now two tasks so $EU_P: R_P(e_1, e_2) - w \mid EU_A: \alpha + \beta_1 e_1 + \beta_2 e_2 - c(e_1, e_2) - \frac{1}{2} r(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2)$
- FOCs give $\beta_1 = \beta_2 = c'(e_1 + e_2)$. Thus $\beta_1 = \beta_2$ is necessary condition for A to choose $e_1, e_2 > 0$

<u>Empirical Evidence</u>

- Lazear (2000): 1994-95 Safelite Glass Corporation: A switch to piece rate leads to 44% increase in the average level of output per worker. 1/2 due to incentive effects and 1/3 due to selection effects
- Foster and Rosenzweig (1994): farming households: workers deplete body mass 10% more under piece-rate/self-employed scheme. Workers consume 23%/16% more calories on these days
- But… Gneezy and Rustichinni (2000): Children charity collection: 0% cut > 10% cut > 1% cut
- Gibbons and Murphy (1990): Executives are penalized when competitor's outperform but best predictor is entire stock market. Would expect higher correlation among shock within same industry!

Adverse Selection

Basic Set-Up

- Assume risk neutral P monopsony and two types of agents: low- A_1 and high-cost A_2 [i.e. $c_2(q) > c_1(q)\ \forall\ q$]. P is able to observe each A's q but not how costly it is to each.
- Assume single crossing property $c'_2(q) > c'_1(q)\ \forall\ q$
  - Thus $c_2(q_2) - c_2(q_1) > c_1(q_2) - c_1(q_1)$ if $q_1 < q_2$. A_1 produces at least as much as A_2
- Generally, $EU_P(q_1, q_2, w_1, w_2) = \pi_1(q_1 - w_1) + \pi_2(q_2 - w_2)$

Full Information Benchmark

- P: $\max_{q_1, q_2} q_1 + q_2 - w_1 - w_2$ s.t. $w_1 - c_1(q_1) \geq 0$ and $w_2 - c_2(q_2) \geq 0$ [i.e PC binds]
- Taking FOC $c_1'(q_1^*) = c_2'(q_2^*) = 1$. Hence P offers each A_t $w_t(q) = \begin{cases} c_t(q_t^*) \text{ if } q = q_t^* \\ < c_t(q_t^*) \text{if } q \neq q_t^* \end{cases}$
- A_1 produces q_1^* and A_2 produces q_2^*. P extracts the entire surplus. Outcome is Pareto Efficient.
- Note, A_1 prefers (w_2, q_2^*) to (w_1, q_1^*) [i.e. gets surplus D] so has incentive to pretend to be A_2 Now P needs to ensure A_1 chooses $q_1$and A_2 chooses q_2.

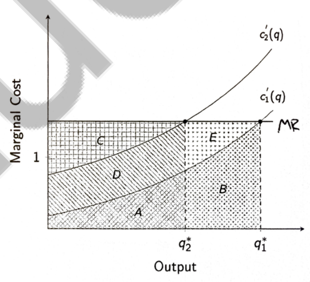


Figure: Production under full information

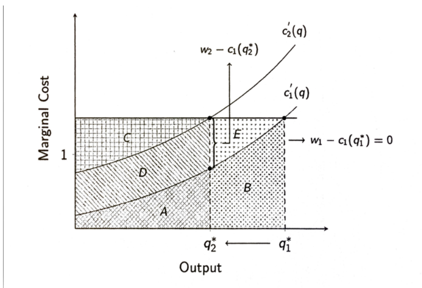


Figure: Incentive for agent 1 to deviate under asymmetric information

Monopoly Asymmetry

- $P\text{: max}\,\pi_1(q_1 - w_1) + \pi_2(q_2 - w_2)$ s.t. …
  - $[A_1]$: [PC] $w_1 - c_1(q_1) \geq 0$ and [ICC] $w_1 - c_1(q_1) \geq w_2 - c_1(q_2)$
  - $[A_2]$: [PC] $w_2 - c_2(q_2) \geq 0$ and [ICC] $w_2 - c_2(q_2) \geq w_1 - c_2(q_2)$
  - Note only A_1-ICC (need to make them indifferent to pretending) and A_2-PC (need to make them indifferent to participating) are binding. The others hold (A_2 has no incentive to pretend)
    - First let us consider ICC:
      - Rearrange to $w_2 \leq w_1 + c_1(q_2) - c_1(q_1)$ and $w_2 \geq w_1 + c_2(q_2) - c_2(q_1)$
      - If both are satisfied then must have $c_1(q_2) - c_1(q_1) \geq c_2(q_2) - c_2(q_1)$
      - But… recall single crossing $c_2(q_2) - c_2(q_1) \geq c_1(q_2) - c_1(q_1)$ for $q_1 \leq q_2$
    - Now let us consider Agent 1:
      - Rearrange [PC] $w_1 \geq c_1(q_1)$ and [ICC] $w_1 \geq c_1(q_1) + w_2 - c_1(q_2)$
      - Note $w_2 - c_1(q_2) > w_2 - c_2(q_2) \geq 0$ so ICC must be the one that binds
    - Now let us consider Agent 1:
      - First assume ICC binds: $w_2 = w_1 + c_2(q_2) - c_2(q_1) = w_2 + c_1(q_1) - c_1(q_2)$
      - This implies, $c_2(q_2) - c_2(q_1) = c_1(q_2) - c_1(q_1)$ which is false. Thus PC must bind
- Thus rewrite as $P\text{: max}\,\pi_1\big(q_1 - c_1(q_1) - c_2(q_2) + c_1(q_2)\big) + \pi_2\big(q_2 - c_2(q_2)\big)$
- Taking FOC and rearranging: $c'_1(q_1^*) = 1$ and $c'(q_2^*) = 1 + \frac{\pi_1(1 - c'_1(q_2^*))}{\pi_1 + \pi_2}$
  - A_1 produces same q_1^* (MB=MC) but gets more compensation (A+B+D), raising utility
  - A_2 produces lower q_2^* (MB>MC) and gets less compensation (A+D), leaving utility unchanged
  - P smaller utility: A_1 get some surplus (distributional change); output is lower (efficiency change)
- Intuition: We pay A_1 a bit of a surplus and make q_2^* less attractive alternative so they do not lie. q_1^* does not change because A_2 has no reason to lie (no distortion at the top)
- Comparative Statistics:
  - $\uparrow \pi_1, \uparrow$ chance we have an A_1 lying to be A_2, $\uparrow$ need to distort q_2^*, $\uparrow$ distortion
  - $\uparrow \Delta\theta$ (diff in cost-f), $\uparrow A_1$ earning/unit by lying, $\uparrow$ incentive to lie, $\uparrow$ distortion

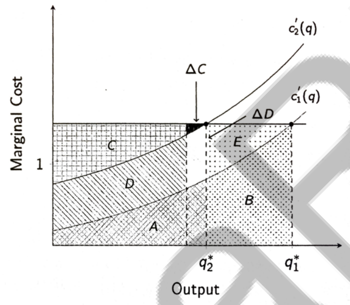


Figure: Lower output target set by the principal

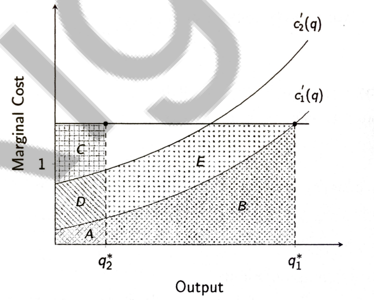


Figure: Optimal production under asymmetric information

Competition Asymmetry

- Zpc means iso-profit line is 45-degree in (w, q) space. Reinterpret reservation utilities:
  - Under monopoly, reservation values determine profit
  - Under competition, zero profit determines reservation value
- Pooling (firms offer both types of workers accept a single contract) is not SPNE, separating (firms offer different types of workers accept different contracts) is…
  - <u>Pooling</u>: Can only ever be optimal for one type (e.g. low-cost). Thus firm can make positive profits by unilaterally deviating and offering a contract preferred by other (e.g. high-cost)
  - <u>Separating</u>: As long as both agents receive at least their reservation level of utility, the only possible equilibrium is separating

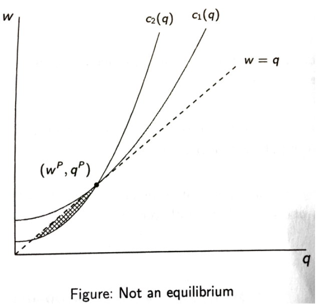


Figure: Not an equilibrium

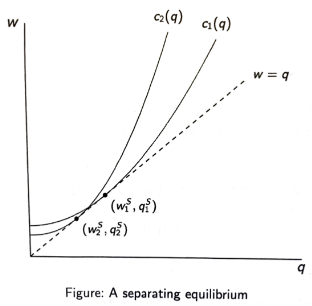
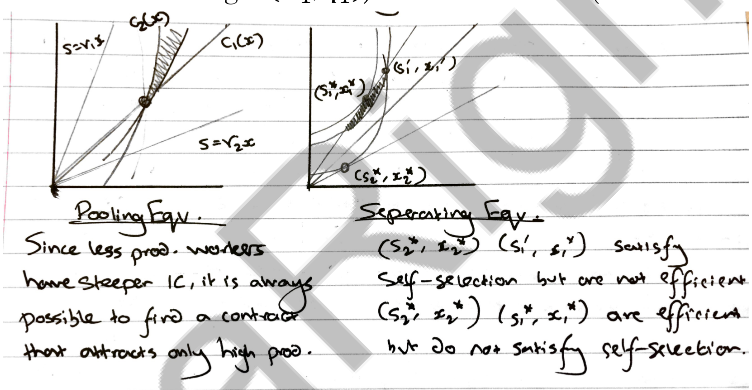


Figure: A separating equilibrium

Extensions

Type/Productivity Differences

- Now consider types also differ in productivity (output of an agent: $v_t q_t$ where $v_1 > v_2$). Hence have three possible iso-profit lines: $v_1 q_1$, $v_2 q_2$, and $[\pi_1 v_1 + \pi_2 v_2]q$
  - Any contract that attracts both types must lie above $[\pi_1 v_1 + \pi_2 v_2]q$
  - Any contract that attracts only the high/low-type must lie above $v_1 q / v_2 q$
- Under asymmetry, pooling is not SPNE and separating can be depending on conditions:
  - Pooling: All workers earn same w^P so by zpc $(p_1 v_1 + p_2 v_2)x = w$ must make profit/loss on high/low type workers. At any point on that line, IC of high type is flatter thus firm could profitably deviate by constructing contract to attract only high-type.
  - Separating: In SPNE any contract chosen by worker must provide firm with zero profit, with zero profit line of A_1being steeper. Let low- (w_2^*, q_2^*) and high-type inefficient (w_1', q_1'), constructed so that low-type is just indifferent to pretending.
    - Either unique SPNE (if $[\pi_1 v_1 + \pi_2 v_2]q$ is everywhere below IC of high type that passes through (w_1', q_1')) or no SPNE at all (if intersects twice)



Graphic Intuition

- Profit from agent A_t: $\Re_t = q_t - w_t$ (so isoprofit line: $w_t = q_t - \Re_t$) and total profit $\pi_1 \Re_1 + \pi_2 \Re_2$
- We know isoprofit line is tangent to A_1's IC and cuts through A_2's IC
  - If A_1 was not there, both P and A_2 would be better off my A_2 working more. But in presence of A_1 P would also have to pay A_1 more so they keep producing q_1^*
- Negative externality between A_1 and A_2 and inability to offer distinct wage leads to inefficient equ.

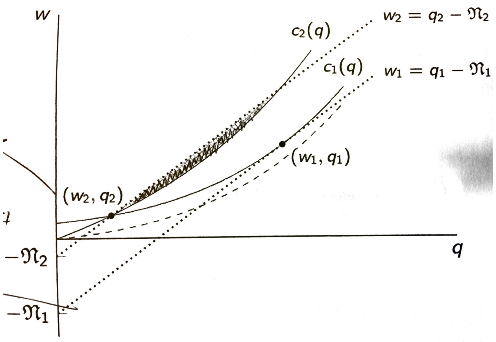


Figure: Asymmetric information and optimal contracts

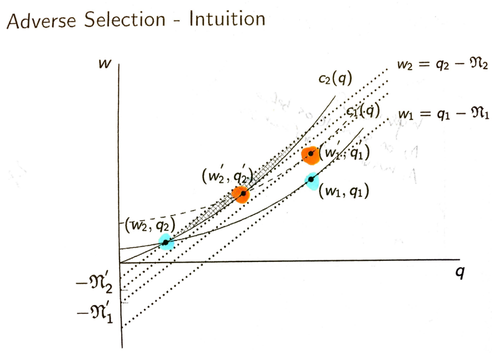


Figure: ... thus, requiring that the principal increases w_1.

Akerlof's Lemons

- There are sellers S and buyers B, peaches P and lemons L. Let $p_P^S = 5; p_L^S = 1; p_P^B = 6; p_L^B = 2$
  - Full information: $p_P \in [5,6]$ and $p_L \in [1,2]$. Market functions
  - No-information: Using 50:50 expectations, $p_{P,L} \in [3,4]$. Market functions
  - Asymmetric information: B will pay up to expected valuation 4. S only accept prices higher than their valuations. Only L are traded. Once buyers recognize this the market unravels!
- Note the degree of unravelling is highly sensitive based on gap in valuation between buyers and lemons, differences in quality of lemons and peaches, number of possible quality types in the market

Exam-Example: Insurance

Basic Contract

- DM is strictly risk-averse, has initial wealth W of which they could lose M with probability p. Able to purchase insurance with premium α, paying q if loss occurs.
- Assume perfect competition thus zpc thus $(1-p)\alpha q - p(1-\alpha)q = 0$ thus $\alpha = p$
- Solve as follow:
  - $U = pu(W_{Bad}) + (1-p)u(W_{Good}) = pu(W - M - \alpha q + q) + (1-p)u(W - \alpha q)$
  - FOC $pu'[W - M + q^*(1-\pi)](1-\alpha) - (1-p)u'(W - \alpha q^*)\alpha = 0$
  - Rearrange and substituting in $\alpha = p$ $\frac{u'(W-M+(1-\alpha)q^*)}{u'(W-\alpha q^*)} = \frac{1-p}{p}\frac{\alpha}{1-\alpha} = 1$
  - $u'(W - M + (1-\alpha)q^*) = u'(W - \alpha q^*) \rightarrow W - M + (1-\alpha)q^* = W - \alpha q^* \rightarrow q^* = M$

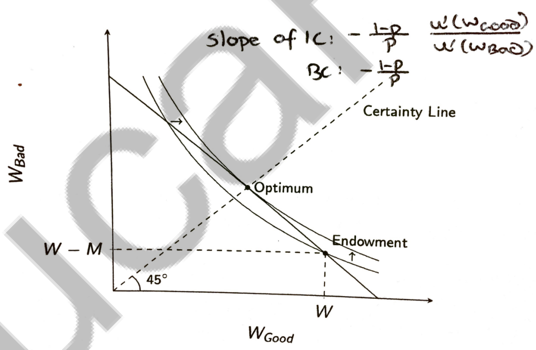
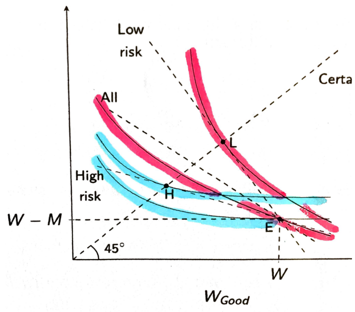


Figure: Purchase of insurance

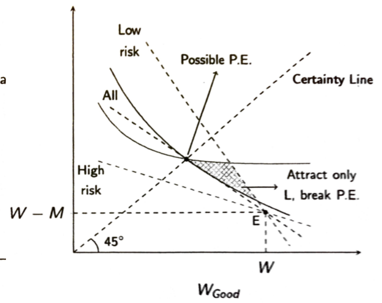
Adverse Selection

- Now let there be high- and low-risk types ($p_H > p_L$) with population fraction of high-risk being π_H
  - Insurer's probability of payout M is $\bar{p} = \pi_H p_H + (1-\pi_H)p_L$
  - Insuree's IC $W_{Bad} = \frac{W - p_i M}{p_i} - \frac{1-p_i}{p_i} W_{Good}$, being steeper for low-risk
- H has an incentive to pretend to be L since doing so gets them on a higher [blue] IC

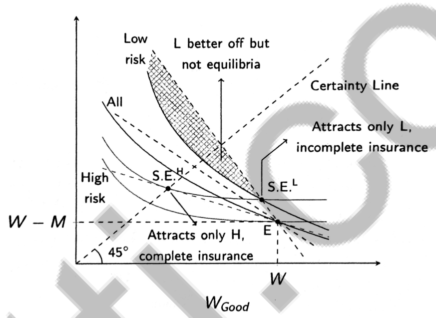
- Pooling is not SPNE but separating is:
  - <u>Pooling</u>: Always possible to find contract preferred only by L and firms (i.e. limit coverage at lower cost). L migrate leaving other firms with negative profit.
    - Intuition: L will drop out of plans that fairly insure average risk. H will self-select into plans that fairly insure average risk
  - <u>Separating</u>: Let H be fully insured (at efficient bundle) and L get best possible coverage such that H have no incentive to pretend (i.e. underinsured).
    - Note inefficiency: L prefers to buy more at profitable price but doing so means attracting H



ıre: Purchase of insurance - High- and low-risk



ure: Purchase of insurance - No pooling equilibrium



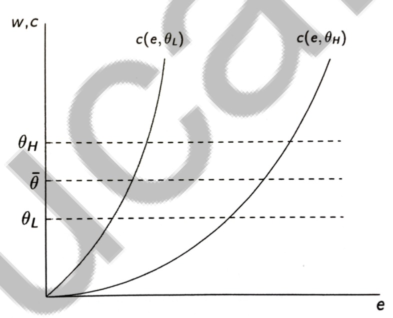
gure: Purchase of insurance - Separating equilibrium

Signaling

- <u>Screening</u>: Uniformed party P designed different contracts in order to incentive agents to self-select according to their productivity
- <u>Signaling</u>: Informed party H cannot simply reveal itself, since this is not credible, but can costly reveal itself if doing so is even more costly for L

Set Up

- Two types of prospective employees with different productivities ($0 < \theta_L < \theta_H$) and different costs to acquire education [i.e. $c(e, \theta_L)$ is steeper than $c(e, \theta_H)$]. Have $u_i(w, e, \theta_i) = w - c(e, \theta_i)$
- Firms believe education e indicates ability θ_i with probability $\mu(e)$. Assume zpc [$\pi = \theta - w = 0$ so $\theta = w$] and they know fraction of pop that is H is λ
- Nature chooses θ, employee chooses e, firms observes e and offers w, employee decides if to accept
  - Key that informed party moves first!



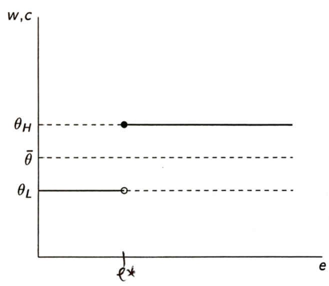


Figure: Beliefs and associated wage-schedules

Solving

- Look for Perfect Bayesian Equilibrium: All strategies $[e_L, e_H, w(e)]$ are optimal; beliefs are consistent i.e. no systematic errors [state this in exams!]; wage-offers (w, e) constitute NE.
- Use backwards induction: zpc thus firms will offer $w(e) = \mu(e)\theta_H + [1 - \mu(e)]\theta_L$. Thus prospective employee look to $\max w(e) - c_i(e) = \mu(e)\theta_H + [1 - \mu(e)]\theta_L - c_i(e)$

- There are multiple pooling and separating equilibria. Whether best pooling or best separating is better depends on how low average productivity is (and thus how strong the incentive to signal)
  - Pooling: prospective employees choose same levels of education
    - All choose e_P^* thus $\mu(e_P^*) = \lambda$ and $w(e_P^*) = \bar{\theta} = \lambda\theta_H + (1-\lambda)\theta_L$
    - All $e_P^* > 0$ are Pareto dominated by $e_P^* = 0$. No one invests in education: $(w(0), 0)$
    - Happens when H has to undertake so much edu to separate from L they end up not much better off. More likely under high p_H (few L, so little need to separate from)
    - Generally seen to be unstable; if H increased its education slightly, then the market can identify agents and collapse back into the dominant separating equilibrium.
  - Separating: prospective employees choose different levels of education
    - Each employee is paid marginal product: $w^*\big(e^*(\theta_i)\big) = \theta_i$.
    - L does not deploy e as a signal at all $(w(0), 0)$
    - H over-invests: receive (w, e) s.t. lies on H-zpc, preferred to $(w(0), 0)$ by H not L
    - Intuitively, we must have that:
      - L acts as if they are L (i.e. H has done enough to separate themselves so L gains more utility from undertaking their optimal education than with w_H)
      - H must not want to give up signaling H (i.e. utility they get from being H is more than any utility they could gain from receiving w_L)
    - We note there is a single e_L and a range of e_H. The e_H which gives H the highest utility is known as the dominant separating equilibrium.
  - Check/show that there is no incentive to unilaterally deviate using off-equilibrium beliefs. This is what gives us $e_L = 0$ in pooling equilibrium (unless peahens believe $f_L = f_H = f^P$!!)

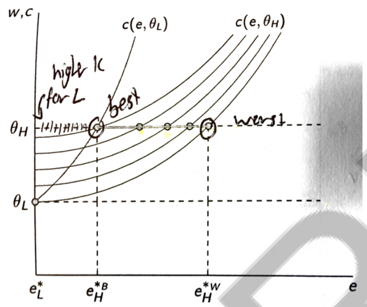


Figure: Multiple separating equilibria

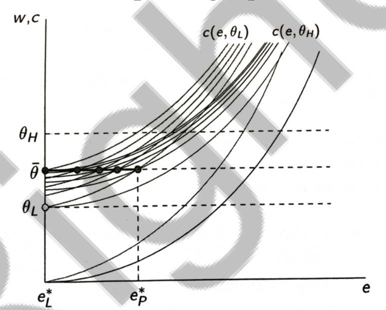


Figure: Set of pooling equilibria and associated beliefs

TIP: Always think timeline!