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Explain the Math

Mathematics

01. Fundamentals

Quadratic Equation: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ | Completing the Square: $a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

Combinations: $nCk = \binom{n}{k} = \frac{n!}{(n-k)!k!}$ | Permutations: $nPk = \binom{n}{k} = \frac{n!}{(n-k)!}$

Set Notation

$\mathbb{N}$(natural) $\subset \mathbb{Z}$(integers) $\subset \mathbb{Q}$(rational) $\subset \mathbb{R}$(real)

$\mathbb{R}_+$: positive real numbers ; $\mathbb{R}_{++}$: non-negative real numbers

$x \in S$: x is an element of set S | Number of members in set S: $|S|$ or $\sum_{s \in S} 1$

2^S: All subsets of S

$X = \{s \in S \mid s \ldots\}$ [i.e. X is the set of members S which satisfy property…]

Cartesian product of two sets $(A \times B)$ is the set of all **ordered pairs** (a, b)

$\mathbb{R}^2$: set of real numbers (x, y) producing Cartesian graph

$[a, b) = \{x \mid a \leq x < b\}$

Image of a set S under f: $f(S) = \{f(x) \mid x \in S\}$

02. Functions

$\mathbb{R} \to \mathbb{R}$ $[\boldsymbol{f}(\boldsymbol{x})]$: Graph$(\boldsymbol{f}) = \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) \mid \boldsymbol{x} \in \mathbb{R}\}$ which is $\subseteq \mathbb{R}^2$

$\mathbb{R}^n \to \mathbb{R}$ $[\boldsymbol{f}(\boldsymbol{x})]$ *e.g.* Graph$(\boldsymbol{f}) = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^3 \mid \boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})\}$

Homogenous of degree k if $f(\lambda x_1, \lambda x_2 \ldots) = \lambda^k f(x_1, x_2 \ldots)$ for all $\lambda > 0$ and $(x_1, x_2 \ldots) \in \mathbb{R}^n$

Intermediate Value T.: if f is continious on $[(a, b])$it'll take all values between $f(a), f(b)$ [NE

03. Sequence

Sequences with values in set A can be expressed as $f: \mathbb{N} \to A$

Convergence (to L): "there exists K such that $n > K \Rightarrow |a_n - L| < \epsilon$"

(i.e. eventually all terms of the sequence are within the ϵ-neighbourhood of L)

Hence $\lim_{n \to \infty} a_n = L$ or $a_n \to L$

Divergence (to $+\infty$): "for any bound $M > 0$ there exists $K > 0$, such that $n > K \Rightarrow a_n > M$"

04. Limits

"a is a limit point of set S if for every ϵ>0, the ϵ-neighbourhood of a includes elements of S

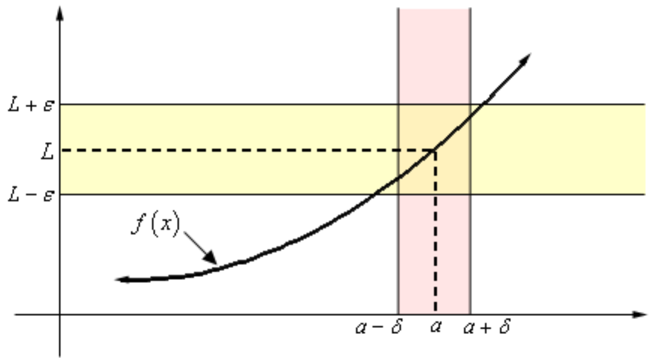
i.e. Numbers that are or are infinitely close to an element in S [NE]

- Bounded: there exists a number N such that $\boldsymbol{x} < |N|$ for all $\boldsymbol{x}$ in S
- Closed: all limit points in S are elements of S [NE]
- Compact: both closed and bounded

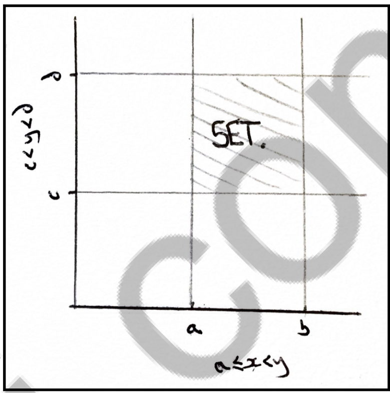
- <u>Open:</u> any point $\boldsymbol{x}$ in S there exist $\epsilon > 0$ such that $(\boldsymbol{x}+\epsilon, \boldsymbol{x}-\epsilon) \subseteq S$ (i.e. all points in S are interior)

Sets have limit points, functions have limits: $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = L$. If $= f(\boldsymbol{a})$ then continuous

L is limit of $f(\boldsymbol{x})$ if, as $\boldsymbol{x}$ approaches to $\boldsymbol{a}$, "given any $\epsilon > 0$ there exists $\delta > 0$ s.t. $\|\boldsymbol{x}-\boldsymbol{a}\| < \delta \Rightarrow |f(\boldsymbol{x}) - L| < \epsilon$" (i.e. points close to $\boldsymbol{a}$ are mapped close to L) [NE]



e.g. $[a,b)\times(c,d)$ has limit points $[a,b]\times[c,d]$



Tip: If $\lim_{x\to a}\frac{f(x)}{g(x)}$ results in $\frac{0}{0}$ use Bernoulli's rule $= \frac{\left(\frac{df(a)}{dx}\right)}{\left(\frac{dg(a)}{dx}\right)}$

05. Differentials

$\frac{\partial}{\partial x} f(x)^n \to n\, f'(x)\, f(x)^{n-1}$ | $f(x) = x^n$ *then* $f^k(x) = \begin{cases} n(n-1)\ldots(n-k+1)x^n - k & \text{if } 0<k\leq n \\ 0 & \text{if } 0<n<k \end{cases}$

Mathematical Intuition

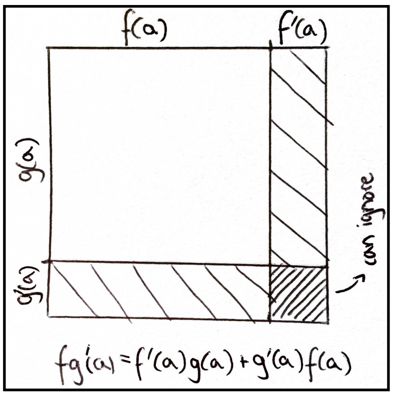
<u>What is differentiation?</u>

- Not "instant rate of change" of $f(x)$ [i.e. $\frac{rise}{run}$ between two close points], that's an oxymoron
- Instead "best constant approx. around a given point" [i.e. slope of line tangent at a single point]
- So dx is not infinitely small or 0 but instead some value h that approaches 0: $\left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Note: Ignore any terms including $dx^{n>1}$ as they are even smaller than dx. Also, must be continuous

<u>Where does the general rule come from?</u>

$\frac{\partial}{\partial x}x^n = \frac{(x+dx)^n - x^n}{dx} = \frac{x^n + nx^{n-1}dx + [\text{terms inc } dx^{n>1}] - x^n}{dx} = \frac{nx^{n-1}dx + [\text{terms inc } dx^{n>1}]}{dx} = \frac{nx^{n-1}dx}{dx} = nx^{n-1}$



Basic Algebra

Sum Rule: $(f+g)'(a) = f'(a) + g'(a)$

Product Rule: $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

Chain Rule: $(g\circ f)(x) = g(f(x))$ and $(g\circ f)'(x) = g'(f(x)) \times f'(x)$

Inverse Product Rule: $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)+f(a)g\prime(a)}{[g(a)]^2}$

Constant Rule: $(cf)'(a) = cf'(a)$

06. First Order Condition

"If $f'(a) = 0$ then a is interior local maximum/minimum" [for single variable, for multi see p7 **Hessian**]

Mathematical Intuition

Extreme Value Theorem: function attains max and min i.e.

"if f is continious on $[a, b]$ there exist points c and d such that $f(c) \leq x \leq f(d)$"

If $f(c)$ is min and <u>interior</u> then $f(c+h) - f(c) \geq 0$ for $h \neq 0$

$\frac{f(c+h)-f(c)}{h} \geq 0$ for $h > 0$

$\frac{f(c+h)-f(c)}{h} \leq 0$ for $h < 0$

Similar if $f(d)$ is max

Characteristics

Monotonicity is if function's first derivative never changes sign (need not be continuous)

- <u>Increasing:</u> $a > b$ implies $f(a) > f(b)$ hence $f'(x) > 0$
- <u>Decreasing:</u> $a > b$ implies $f(a) < f(b)$ hence $f'(x) < 0$
- <u>Non-Increasing:</u> $a > b$ implies $f(a) \leq f(b)$ hence $f'(x) \leq 0$
- <u>Non-Decreasing:</u> $a > b$ implies $f(a) \geq f(b)$ hence $f'(x) \geq 0$
- <u>Constant:</u> $f(a) = f(b)$ for interval (a, b) hence $f'(x) = 0$

07. Second Order Condition

- If $f'(a) = 0$ and $f''(a) > 0$, a is a local minimum
- If $f'(a) = 0$ and $f''(a) < 0$, a is a local maximum

- <u>Convex function:</u> $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ hence $f''(x) \geq 0$ for all x
- <u>Concave function:</u> $f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b)$ hence $f''(x) \leq 0$ for all x
- <u>Strictly X function:</u> removes the $=$ sign from above. Will mean there is only one max/min
- <u>Quasi X function:</u> $f(\lambda a + (1-\lambda)b)[\leq \max f(x) \ \ or \ \ \geq \min f(x)]$
  [see p6 on **Hessian** for more]

These characteristics are conditional on…

- Continuous and twice differentiable functions
- **Convex domain** (for every $x, y \in S$ and $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y$ is also in S)
- **Note:** Convex preferences are caused by quasi-concave utility functions!

08. Graph Sketching

Sketching $y = f(x)$

[0] figure out the largest possible domain if not give [1] rewrite if top heavy or complete the square if it can't be factorised [2] x, y asymptotes [3] intersection with x, y axis [4] $y, x \to \pm\infty$ [5] x near asymptotes $(+ \text{or} -)$ [6] where increasing and where decreasing [7] max, min if existent [8] where concave and where convex

<u>Algebraic Approach</u>: turn into quadratic with y being coefficient of x^n. Turning point when one solution $(b^2 - 4ac = 0)$, range when two $(b^2 - 4ac > 0)$.

Sketching $y^2 = f(x)$

[1] sketch $y = f(x)$ for reference [2] erase below x axis [3] transfer points on $y = 0,1$ $x = 0$ and value of turning points (sub in) [4] draw $\infty \frac{dy}{dx}$ at x axis intersect unless $f'(x) = 0$ [5] draw above reference for $x < 1$and below for $x > 1$ [6] reflect in x axis. Asymptotes: $y = a$ to $y = \pm\sqrt{a}$; $x = a$ stays $x = a$

09. Multi-variable functions

Typical examples

$\max \pi = pf\big(K^*(p,r,k), L^*(p,r,k)\big) - \big(rK^*(p,r,k) + wL^*(p,r,k)\big)\ s.t. K, L \geq 0$

$B = \{\boldsymbol{x} | \boldsymbol{p} \cdot \boldsymbol{x} \leq m$ and $\boldsymbol{x} \geq 0\}$ Find $(\boldsymbol{a}) \in B$ so that $u(\boldsymbol{a}) \geq u(\boldsymbol{x})$

Vectors

$\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$; Distance $\|\boldsymbol{a} - \boldsymbol{b}\| = \sqrt{(a_1 - b_1)^2 + \cdots (a_n - b_n)^2}$;

r-nneighbourhood $= \{x \in \mathbb{R}^n | \|\boldsymbol{x} - \boldsymbol{a}\| < r\} = (\boldsymbol{a} - r, \boldsymbol{a} + r)$

Mathematical Intuition

$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - f(\boldsymbol{a})\,[\mathbb{R}]}{\boldsymbol{x} - \boldsymbol{a}\,[\mathbb{R}^n]}$ Doesn't make any sense!

Instead take **partial derivative w.r.t variable $\boldsymbol{x_i}$** i.e. $f_i = \frac{\partial f(\boldsymbol{x})}{\partial x_i} = \lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{f(x_1, \dots x_i + h, \dots x_n) - f(\boldsymbol{x})}{h}$ [= 0 for FOC]

Chain Rule: If $t \to f(x(t), y(t))$ [i.e. can be expressed as single variable function], then $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

10. Series

$\sum_{i=0}^{n} a_i = a_0 + a_1 \dots + a_n$; Partial Sum $= S_n$;

Geometric Series: $a_n = a_0 c^n$

$\sum_{i=0}^{n} a_0 c^i = \frac{a(1 - c^{n+1})}{1 - c}$ [if $c \neq 1$]

- $a_n \to 0$ if $|c| < 1$
- $a_n \to a$ if $c = 1$
- a_n diverges [fluctates between a and $-a$] if $c = -1$
- a_n diverges to ∞ if $c > 1$
- a_n diverges with growing magnitude and flipping sign if $c < -1$

11. Integration

$\int f(x)^n \to \frac{1}{n\, f\prime(x)}\, f(x)^{n-1}$

Mathematical Intuition [NE]

Let $f: [a, b] \to \mathbb{R}$ be continuous and $F: [a, b] \to \mathbb{R}$ be defined as $F(x) = \int_a^x f(t)dt$

Since continuous, $\int_a^b f(x)dx$ has max and min: $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$; any $m \leq x \leq M$ attainable i.e. $f(c) = \frac{1}{(b-a)}\int_a^b f(x)dx$ [Mean Value T. [NE]]

$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h} = \lim_{h \to 0} f(c)$[for some $c \in [x, x+h]$] $= f(x)$

Basic Properties

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

$\int kf(x)dx = k\int f(x)dx$

$\int udv = uv - \int vdu$ [always use 'simpler' function for u, unless $\ln x$, then use it as u regardless]

12. (Constrained) Optimisation

$\max f(\boldsymbol{x})\ s.t. \text{"}constraint\text{"}$. By def. $f_{x_i}(\boldsymbol{x}^*) = 0$
Necessary but <u>not sufficient</u> condition for <u>interior</u> solutions [see p6 for more conditions i.e. **Hessian**]

Lagrange: Ensures a constraint binds
$\mathcal{L}(\boldsymbol{x}, \lambda, m) = f(\boldsymbol{x}) - \lambda(\text{"}constraint\text{"})$ [λ-term is punishment for exceeding or reward for 'under'-ceding]
How do we determine λ? By def. $\mathcal{L}_i(\boldsymbol{x}^*, \lambda^*) = 0$ and $\mathcal{L}_\lambda(\boldsymbol{x}^*, \lambda^*) = 0$. Solve equations, compare solutions
λ^* can be thought of as 'willingness to pay to violate constraint by one unit' (e.g. MU of income)
For multiple constraints [i.e. $g^i(\boldsymbol{x})$]: $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{i=0}^m \lambda_i g^i(\boldsymbol{x})$

Envelope Theorem
Let x^* depend on a, which is not a choice variable [i.e. exogenous]
$V(a) = f(x^*(a), a) = \max_x f(x, a)$ s.t. $g(x, a) = 0$
i.e. $V(a) = \mathcal{L}(x^*(a), \lambda^*(a), a) = f(x^*(a), a) - \lambda^* g(x^*(a), a)$
$\frac{\partial V(a)}{\partial a} = \frac{\partial \mathcal{L}^*}{\partial a} = \frac{\partial \mathcal{L}}{\partial x}\Big|_{\substack{x=x^*\\ \lambda=\lambda^*}} \times \frac{\partial x^*}{\partial a} + \frac{\partial \mathcal{L}}{\partial \lambda}\Big|_{\substack{x=x^*\\ \lambda=\lambda^*}} \times \frac{\partial \lambda^*}{\partial a} + \frac{\partial \mathcal{L}}{\partial a}\Big|_{\substack{x=x^*\\ \lambda=\lambda^*}} \times \frac{\partial a}{\partial a}$
Note: By FOC def. $\frac{\partial \mathcal{L}}{\partial x}\Big|_{\substack{x=x^*\\ \lambda=\lambda^*}} = 0$; $\frac{\partial \mathcal{L}}{\partial \lambda}\Big|_{\substack{x=x^*\\ \lambda=\lambda^*}} = 0$; also $\frac{\partial a}{\partial a}=1$
Hence simplifies to $\frac{\partial V(a)}{\partial a} = \frac{\partial \mathcal{L}}{\partial a}\Big|_{\substack{x=x^*\\ \lambda=\lambda^*}}$ where $\mathcal{L}(x, \lambda, a) = f(x, a) - \lambda\, g(x, a)$
"Rate of change of the optimal value (V) with respect to the parameter (a) = rate of change of $\mathcal{L}$ with respect to parameter evaluated at the optimal solution $\left(\frac{\partial \mathcal{L}}{\partial a}\Big|_*\right)$. No indirect effect!

13. Vectors/Matrices

$\boldsymbol{x} = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ | $\boldsymbol{e^1} = (1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | $\boldsymbol{e^2} = (0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
Addition: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ | Scalar: $c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$ | Product: $x \cdot y = x_1 y_1 + x_2 y_2$

Linear Transformations
$T(0) = 0$
$T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
$T(c\boldsymbol{x}) = cT(\boldsymbol{x})$
$(T \circ U)(\boldsymbol{x}) = T(U(\boldsymbol{x}))$
Note: $\boldsymbol{x} = x_1 \boldsymbol{e^1} + x_2 \boldsymbol{e^2}$ so $T(\boldsymbol{x}) = x_1 T(\boldsymbol{e^1}) + x_2 T(\boldsymbol{e^2})$
i.e to describe T it is sufficient to describe what T maps each e^i to
e.g. Rotation 90° clockwise: $x_1 T(\boldsymbol{e^1}) + x_2 T(\boldsymbol{e^2}) = x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Matrix Rules
$A(BC) = (AB)C$ | $AB \neq BA$ | $(AB)^{-1} = B^{-1}A^{-1}$ | $(AB)_{ij} = A_{i1}B_{1j} + \cdots + A_{in}B_{nj}$
Identity Matrix: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ [where $AI = A$] | Transpose Matrix: $\begin{bmatrix} x \\ y \end{bmatrix}^T = \begin{bmatrix} x & y \end{bmatrix}$

If $ax_1 + bx_2 = y_1$ and $cx_1 + dx_2 = y_2$ then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
Hence, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $A^{-1} = \frac{1}{ad-bc} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ [if det $(ad - bc = 0)$ then A is singular]

3x3 Matrix

- Minor: 2x2 determinant when eliminating an element's row & column e.g. $\begin{pmatrix} a & b & c \\ d & \boldsymbol{e} & f \\ g & h & i \end{pmatrix}$
- Cofactor: Minor multiplied by the element's $(-1)^{i+j}$: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$
- Determinant: Sum of a row/column when Cofactor is multiplied by its element
- Inverse: Find Cofactors of elements (do not $\times$ elements); Transpose $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$, divide by det

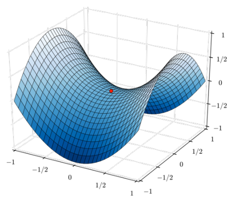
Gaussian Elimination: Apply row operations to $\left[\begin{matrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{matrix} \middle| \begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{matrix}\right]$ until $LS = I$ then $RS = A^{-1}$

14. Hessian

Mathematical Intuition

Multi-variable functions $f'(x^*, y^*) = 0$ insufficient condition for local min/max due to **saddle points** [See 16. Taylor Series]

1. $F(x_1 + h_1, x_2 + h_2) \approx F(x_1, x_2) + F_1(x_1, x_2)h_1 + F_2(x_1, x_2)h_2 + \frac{1}{2}[F_{11}(x_1, x_2){h_1}^2 + 2F_{12}(x_1, x_2)h_1h_2 + F_{22}(x_1, x_2){h_2}^2]$
2. $F(x_1 + h_1, x_2 + h_2) \approx F + \frac{1}{2}[F_{11}{h_1}^2 + 2F_{12}h_1h_2 + F_{22}{h_2}^2]$
3. $F(x_1 + h_1, x_2 + h_2) - F \approx \frac{1}{2}[F_{11}{h_1}^2 + 2F_{12}h_1h_2 + F_{22}{h_2}^2]$
4. $F(x_1 + h_1, x_2 + h_2) - F \approx H_{11}{h_1}^2 + 2H_{12}h_1h_2 + H_{22}{h_2}^2$
5. $F(x_1 + h_1, x_2 + h_2) - F \approx [h_1 \quad h_2]\begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$



To be definite, graph never crosses axis i.e. quadratic has no solutions

1. $H_{11}{h_1}^2 + 2H_{12}h_1h_2 + H_{22}{h_2}^2$
2. $H_{11}(h_1/h_2)^2 + 2H_{12}(h_1/h_2) + H_{22}$
3. $b^2 - 4ac < 0 \Longrightarrow 4H_{12} - 4H_{11}H_{22} < 0 \Longrightarrow H_{11}H_{22} - (H_{12})^2 > 0 \Longrightarrow \det(H) > 0$

Conditions

Iff **ve+ Definite** i.e. $[h \quad k]H\begin{bmatrix} h \\ k \end{bmatrix} > 0$ for all $(h, k) \neq (0,0)$ (i.e. $det(H) > 0$ and $H_{11}, H_{22} > 0$) then min

Iff **ve- Definite** i.e. $[h \quad k]H\begin{bmatrix} h \\ k \end{bmatrix} < 0$ for all $(h, k) \neq (0,0)$ (i.e. $det(H) > 0$ and $H_{11}, H_{22} < 0$) then max

Iff min then $[h \quad k]H\begin{bmatrix} h \\ k \end{bmatrix} \geq 0$ for all (i.e. $det(H) \geq 0$ and $H_{11}, H_{22} > 0$) then **ve+ Semi-Definite**

Iff min then $[h \quad k]H\begin{bmatrix} h \\ k \end{bmatrix} \geq 0$ for all (i.e. $det(H) \geq 0$ and $H_{11}, H_{22} < 0$) then **ve- Semi-Definite**

If $det(H) = 0$ then indeterminate; If $det(H) < 0$ (i.e. H_{11}, H_{22} opposite signs) then saddle point

f is convex iff H is **ve+ Semi-Definite** for all (x^*, y^*) and strictly convex iff H is **ve+ Semi-Definite**
f is concave iff H is **ve- Semi-Definite** for all $(x^*, y^*)$and strictly concave iff H is **ve- Semi-Definite**

15. Exam Application

Rates of Change

0. $Q(L) = f(a, b, L)$
1. $\frac{\partial Q}{\partial L} = 0$ to get L^*. Don't know if max or min. Check with boundary solutions (or Hessian). Sub for Q^*.
2. $\frac{\partial Q^*}{\partial a}$ [rate of change] $\times$ 0.x[magnitude of change]. Note this is approximate as IRL L^* adjusts.

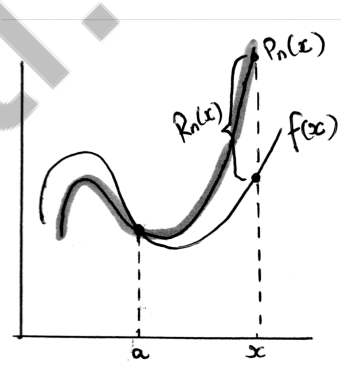
How do inputs (r, w) change factors of production (K^*, L^*)?

0. $\pi = \max pf(K, L) - (wL + rK)$
1. FOC: $\pi_K = pf_K(K^*, L^*) - r = 0 \mid \pi_L = pf_L(K^*, L^*) - w = 0$
2. SOC: $\frac{\partial \pi_K}{\partial w} = pf_{KK}\frac{\partial K^*}{\partial w} + pf_{KL}\frac{\partial L^*}{\partial w} = 0 \mid \frac{\partial \pi_L}{\partial w} = pf_{LL}\frac{\partial L^*}{\partial w} + pf_{KL}\frac{\partial K^*}{\partial w} - 1 = 0$
3. Matrix: $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{KL} & f_{LL} \end{bmatrix}\begin{bmatrix} \frac{\partial K^*}{\partial w} \\ \frac{\partial L^*}{\partial w} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{p} \end{bmatrix} \to \begin{bmatrix} \frac{\partial K^*}{\partial w} \\ \frac{\partial L^*}{\partial w} \end{bmatrix} = \frac{1}{f_{KK}f_{LL} - f^2{}_{KL}}\begin{bmatrix} f_{LL} & -f_{KL} \\ -f_{KL} & f_{KK} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{p} \end{bmatrix}$
4. Evaluate: $-f_{KL}$ shows if substitute or complement (i.e. if income or sub effect is stronger)
   [Note only has a solution for DRTS and this only unique if concave]

From cost function to production function

0. $C(r, w, y) \Longrightarrow \mathcal{L} = Kr + Lw - \lambda[f(K, L) - y]$
1. Via envelope theorem: $\frac{\partial C}{\partial r} = \frac{\partial \mathcal{L}}{\partial r}\Big|_{K^*, L^*, \lambda^*} = K^*$ and $\frac{\partial C}{\partial w} = \frac{\partial \mathcal{L}}{\partial r}\Big|_{K^*, L^*, \lambda^*} = L^*$
2. Combine $C_w = L^*$ and $C_r = K^*$ cancelling r, w terms

16. Taylor Series



$f(x)$ [centred @ $x = a$] $\approx P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$

$R_n(x) = f(x) - P_n(x) \leq \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$ for some c $\in [a, x]$

[Note $f(a) = P_n(a)$; choose c to maximise)

Can check if over or underestimate by looking at sign of next Taylor terms

17. Difference Equations

<u>Linear, First Order, Autonomous</u>: $x_t = bx_{t-1} + a$

$$x_t = b^t x_0 + \frac{1 - b^t}{1 - b}a = b^t\left(x_0 - \frac{a}{1 - b}\right)[\text{deviation}] + \frac{a}{1 - b}[\text{steady state}]$$

Steady state: $x^*{}_t = x_m = x_{m-1}$. Not necessary that a steady state will be reached or converged to

- If $|b| < 1$, converges to $\frac{a}{1-b}$ [Note doesn't depend on x_0!]
- If $b = 1$, constant divergence [Note $x_t = x_0 + ta$]
- If $|b| > 1$ no convergence; If $b = -1$ oscilates.

Growth Case

- <u>Discrete Growth</u>: $A = P(1 + r_d)^t \Longrightarrow \ln(P) + t\ln(1 + r_d) \approx \ln(P) + tr_d$ for small values
- <u>Continuous Growth</u>: $A = Pe^{tr_c} \Longrightarrow \ln(P) + tr_c$ [Discrete Growth approximates Continuous Growth!]

18. Differential Equations

Differential Equation: Relates function to its derivatives

<u>Separable</u>: $\frac{\partial y}{\partial t} = y'(t) = F'(y)G'(t) \Longrightarrow \int F'(y)\partial y = \int G'(t)\partial t \Longrightarrow F(y) = G(t) + C$

<u>Linear, First Oder, Autonomous.</u>: $\frac{\partial y}{\partial t} = y'(t) = by(t) + a \Longrightarrow y(t) = Ae^{bt} - \frac{a}{b}$ [solve for $t = 0$ to find A]

- General Solution: $y(t) = Ae^{bt} - \frac{a}{b}$
- Let $z(t) = y_1(t) - y_2(t)$. Note Complementary solution $z'(t) = bz(t)$
- Stationary Solution: y^P s.t. $y^{P\prime}(t) = 0$ for all t. Hence in form $by^P + a = 0 \Rightarrow y^P = -\frac{a}{b}$
- Stable Solution: $y^P = -\frac{a}{b}$ if unique and always converged to as $t \to \infty$. Requires $b < 0$

Worked Example

0. $q^D = a_0 - a_1 p(t)$; $q^S = b_0 - b_1 p(t)$; $\frac{\partial p}{\partial t} = \lambda(q^D - q^S)$
1. Equilibrium when $q^D = q^S \Rightarrow a_0 - a_1 p^* = b_0 - b_1 p^* \Rightarrow p^* = \frac{a_0 - b_0}{a_1 + b_1}$ [Particular solution]
2. $\frac{\partial p}{\partial t} = \lambda\{[a_0 - b_0] - [a_1 + b_1]p(t)\} = \lambda[a_1 + b_1][p^* - p(t)] = c[p^* - p(t)] \Rightarrow \frac{\partial p}{\partial t} + cp(t) = cp^*$
3. $p = p^* + Ae^{-ct}$ [General solution]

Statistics

01. Fundamentals

Sample statistics are not population statistics | Correlation does not imply causation

- **Frequentists**: observed f represent p of future outcomes: $\hat{p} = \frac{\#correct}{\#total}$
- **Bayes-ists**: $prior + new\ data \Longrightarrow improved\ belief$: $\hat{p} = P(A|prior) = \frac{P(prior|A)P(A)}{P(prior)}$

$P(X$[outcome of random variable] $= x$[a number in sample space]$)$

Measures of Central Tendency: Arithmetic mean $\left(\frac{\sum x_i}{n}\right)$; Geometric mean $\left(\sqrt[n]{x_1 \dots x_n}\right)$; median; mode

Measures of Dispersion: range $(L - S)$; IQR $(Q_3 - Q_1)$; var (see 04. Moments); skewness= $\frac{\frac{1}{n}\sum(x_i - \bar{x})^3}{\sigma^3}$ [how grouped]; kurtosis= $\frac{\frac{1}{n}\sum(x_i - \bar{x})^4}{\sigma^4}$ [how peaked]

Measures of Relationships: co-var $Cov = E([X - E(X)][Y - E(Y)]) = E(XY) - E(X)E(Y) = \sigma_{XY}$, for sample $= \frac{1}{n[-1]}\sum([x_i - \mu_x][y_i - \mu_y])$or $\iint (x_i - \mu_x)(y_i - \mu_y)\, f(x,y)dxdy$ [Note: = 0 if independent]; correlation $r = \frac{Cov(X,Y)}{\sqrt{Var(x)Var(y)}}$ for sample $= \frac{S_{XY}}{S_X S_Y}$ where $S_{XY} = \frac{1}{n-1}\sum([X - \bar{X}][Y - \bar{Y}])$

02. Single Probability (and Distribution)

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [if mutually exclusive $\cap = 0$]
- $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$ [if independent also $= P(A)P(B)$]
- Hence $P(B|A)$[i. e. $P(B)$ given A] $= \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$

  Use last term for P(T1-error) questions e.g. $P(ve^+\ test\ |\ disease) = 0.99$, $P(disease) = 0.01$

  $P(not\ disease|ve^+\ test) = \frac{P(ve^+\ test\ |\ disease)P(disease)}{P(ve^+ test)} = \frac{0.99 \times 0.11}{0.99 \times 0.11 + 0.99 \times 0.11} = 0.5$

- **Permutations** (i.e. order relevant) $= nPr\ \frac{n!}{(n-r)!} = \frac{\#objects!}{\#repeats!}$
- **Combinations** (i.e. order irrelevant) $= nCr\ \frac{n!}{(n-r)!r!}$

- **Probability Density Function**: gives p for a <u>continuous</u> random variable: $P(a \le X \le b) = \int_a^b f(u)du$
- **Probability Mass Function**: gives p for a <u>discrete</u> random variable: $P(X = x) = f(x)$
- **Cumulative Distribution Function:** $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du$ [$p \to 1$ as $x \to \infty$]

Note: $F(x) = \int_{-\infty}^{x} f(x)dx$ is incoherent! Must be $F(x) = \int_{-\infty}^{x} f(u)du$

Types of PMFs

- **Uniform:** $P(X = x|N) = \frac{1}{N}$
- **Bernoulli:** $P(X = x|p) = p^x(1-p)^{1-x}$ $[x = 0,1]$
- **Poisson:** $P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$ $[x = 0, \dots, n]$
- **Geometric:** $P(X = x|n) = p^x(1-p)^{n-x}$ [" "]
- **Bino:** $P(X = x|n,p) = \binom{n}{x} p^x(1-p)^{1-n}$ [" "]

Types of PDFs

- **Uniform:** $f(x|a,b) = \frac{1}{b-a}$ $[a \le x \le b]$
- **Exponential:** $f(x|\beta) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$ $[x \ge 0, \beta > 0]$
- **Normal:** $f(x|\mu,\sigma^2) =$ see tables

03. Joint Probability

$P(x,y) = P(X=x, Y=y)$ [if independent $= P(X=x)P(Y=y)$]
Can be PMassF (discrete) or PDensityF (continuous)

		Y_k			$\sum_{j=1}^{J} p_{jk}$
		1	2	3	
X_j	1	0.2	0.1	0.05	0.35
	2	0.05	0.2	0.05	0.3
	3	0.05	0.1	0.2	0.35
$\sum_{k=1}^{K} p_{jk}$		0.3	0.4	0.3	1.0

- Joint: $P\left(X=X_j, Y=Y_k\right) = p_{jk}$
- Marginal: $P\left(X=X_j\right) = \sum_{k=1}^{K} p_{jk}$ or $\int_{\text{lower } y}^{\text{upper } y} f_{XY}(x,y)dy$
- Conditional: $P\left(Y|X=X_j\right) = \frac{p_{jk}}{\sum_{k=1}^{K} p_{jk}}$ or $\frac{f_{XY}(x,y)}{f_Y(y)}$

Note: Converting back depends. If IID observations, then j-PDF= $f(x,y) = f(x)f(y)$ and j-PMF= $P(x,y) = P(x)P(y)$. If not, much more complicated

04. Moments

	population	sample	discrete	continuous
$E(x)$	$\mu = \frac{1}{N}\sum(X)$	$\bar{x} = \frac{1}{n}\sum X$	$\frac{1}{N}\sum xP(X=x)$	$\int xf(x)\,dx$
$Var(x)$	$\sigma^2 = \frac{1}{N}\sum(X-\mu)^2$ $= \frac{1}{N}\sum(X) - \mu^2$	$\bar{\sigma}^2 = \frac{1}{n-1}\sum(X-\bar{x})^2$ $= \frac{1}{n-1}\sum(X) - \bar{x}^2$	$\frac{1}{N}\sum x^2 P(X=x)$	$\int x^2 f(x) - \mu^2\,dx$

Notes: $E(x)$ always a number; $\not\equiv \bar{x}$ [$E(x)$ value constant, sample mean changes if repeated]

Basic Rules

$E(a) = a$	$Var(a) = 0$
$E(bX) = bE(X)$	$Var(bX) = b^2 Var(X)$
$E(X+Y) = E(X) + E(Y)$	$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X,Y)$

[Note: $Cov = 0$ if independent

05. Sampling

Observation: known value [i.e. real number] of variable $(X_1 = x_1)$
Sample: collection of variables from pop. $(X_1, \ldots, X_n)$, if IID then random. Hence, it has a distribution…
$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ [latter term requires full IID to be gained from def. of $Var(\bar{X})$]

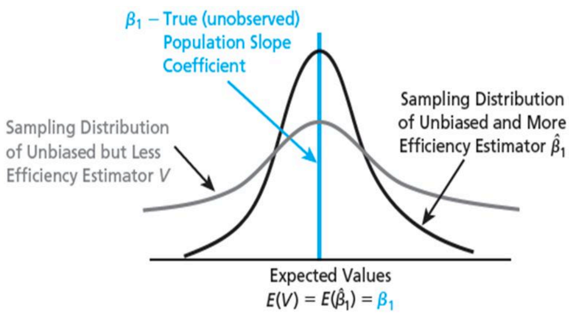
Note: μ is the population mean $\left(= \frac{1}{N}\sum X_N\right)$; $\bar{X}$ is the sample mean $\left(= \frac{1}{n}\sum X_n\right)$
$\bar{X}$[variable] $\neq \mu$[number] but $E(\bar{X})$[number] $= \mu$[number] if IID;

IRL issues: confidentiality, lying, non-response, attrition, survivorship bias (i.e. sampling on outcome)

06. Estimators

$\hat{\theta}$[estimator] is good if <u>unbiased</u>, <u>efficient</u>, and <u>random</u>

- $bias(\hat{\theta}) = E(\hat{\theta}) - \theta$. The lower, the less biased.
  If bias is consistent term, you can multiply it away
  Exam Q: Does $E(\hat{\theta}) = \int_{-\infty}^{\infty} xf(x)\,dx$
  Asymptotic unbias: as $n \to \infty, E(\hat{\theta}) \to \theta$
- Mean Squared Error= $E\left[\left(\hat{\theta} - \theta\right)^2\right] = Var(\hat{\theta}) + bias\left(\hat{\theta}\right)^2$. The lower, the more efficient.



Common Estimators

$\hat{\mu} = \bar{X} = \frac{1}{n}\sum X_i$ [sample mean; unbiased]

$\hat{\sigma}^2 = s^2$ [sample variance; biased] so adjust to…

$\hat{\sigma}^2 = S_{n-1}^2 = \frac{n}{n-1}s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$ [unbiased]

A note on notation

σ^2 = pop Var	μ = pop E
$\hat{\sigma}^2$ = pop Var est.	$\hat{\mu}$ = pop E est.
$\bar{\sigma}^2$ = sample dist. Var	$\bar{\mu}$ = sample dist. E
S^2 = sample Var [bias]	$\bar{X}$ = sample E
$\bar{\sigma}^2 = \frac{\sigma^2}{\sqrt{n}} \approx \frac{S_{n-1}^2}{\sqrt{n}}$	

07. Standard Error & CI

Confidence Intervals: range of values so $P(LC < X < UC) = p$. Use standardisation to easier calculate. Two tailed: need to divide significance level accordingly [use common sense]

Mean

$P(\bar{x} - SE < \mu < \bar{x} + SE) = p$ [where SE is Standard Error at some Confidence Level]

Note that $P(X < x) = P(Z < z) = \phi(z)$ where $X \sim N(\mu, \sigma^2)$; $Z \sim N(0,1)$; $z = \frac{x-\mu}{\sigma/\sqrt{n}}$ [$n = 1$ for single obser!]

- If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\bar{\mu} = \mu, \bar{\sigma}^2 = \left(\frac{\sigma}{n}\right)^2\right)$. Use $z = \frac{x-\mu}{\sigma/\sqrt{n}}$ with relevant substitutions to get $\bar{x} \pm SE$
  - can sub in $\hat{\mu}$ (i.e. $\bar{x}$) as it just 'shifts' everything
  - <u>can't</u> sub in $\hat{\sigma}$ (i.e. S_{n-1}^2) as it affects interval width (unless n is sufficiently large)
    - Use t-distribution instead of Normal to account for this
    - df = #"IID observations" − #"est. parameters for $\hat{\sigma}$" $= n - 1$ [usually have to est. $\hat{\mu}$]
- If $X \sim ?()$ or testing sample dist. apply CLT if n is sufficiently large so that $\bar{X} \sim N\left(\bar{\mu} = \mu, \bar{\sigma}^2 = \left(\frac{\sigma}{n}\right)^2\right)$
  - can sub in $\hat{\mu}$ as it just 'shifts' everything
  - <u>can</u> sub in $\hat{\sigma}$ (i.e. S_{n-1}^2) as n has to be sufficiently large to have applied CLT!
- If comparing two samples use $z = \frac{\bar{x}_a - \bar{x}_b}{\sqrt{\frac{s_a{}^2}{n_a} - \frac{s_b{}^2}{n_b}}} \sim (0,1)$

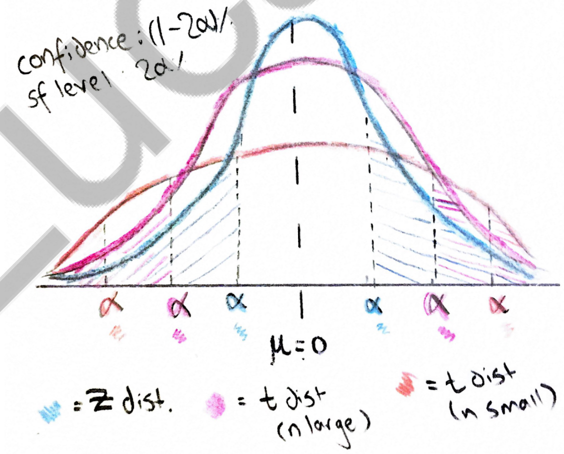
Variance

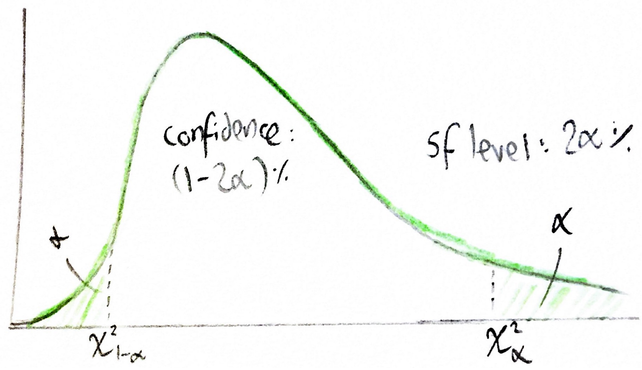
If $X \sim N(\mu, \sigma^2)$then $\frac{(n-1)S_{n-1}^2}{\sigma^2} \sim \chi_{n-1}^2$ [i.e. chi-squared (asymmetric!) with $n-1$ d.o.f.]

Hence $P\left(\frac{(n-1)S_{n-1}^2}{\sigma^2} < \chi_{\alpha,n-1}^2\right) = a$ which rearranges to $P\left(\frac{(n-1)S_{n-1}^2}{\chi_{\alpha,n-1}^2} < \sigma^2\right) = a$

Proportion

- If $X \sim B(n, p)$ and $np, n(1-p)$ are large (i.e. > 10) can approx. $X \sim N(np, npq)$; divide by n [so we get a proportion not number]
- Hence, $z = \frac{x - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ [sub in $\hat{p}$ i.e. $\bar{p}$ for x]





If H_0 is in Critical region (i.e. $|H_0| > t_{crit}$), as shown by α it's rejected

08. Hypothesis Tests

(1) Define H_0 and H_1; (2) Assume H_0 is true; (3) Define rejection rule; (4) Test and draw conclusion

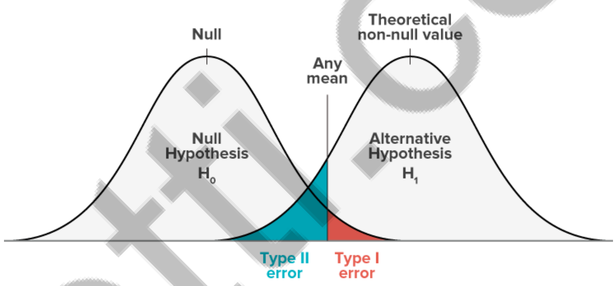
Need to know distribution | Acceptance region + Critical region $= 1$ | two-tailed tests has sig level $\div 2$ |
$\Rightarrow$ Use standardization, with $\hat{?}$ replacing x [i.e. sample data] and $?_0$ replacing $?$; see diagram above

<u>About the mean</u>: Use $\hat{\mu}$ (and $\hat{\sigma}^2$) to see if $\mu_0 = \mu$: Reject if $\left|\frac{\hat{\mu}-\mu_0}{\sqrt{\frac{\hat{\sigma}^2}{n}}}\right| > t_{crit}$ [normal or t tables]

"At the $x\%$ sig. level, the X-tailed hypothesis level is [$<$ *or* $> t_{crit}$], hence [do *or* don't]reject H_0"
<u>About the variance:</u> Use $\hat{\sigma}^2$ to see if $\sigma_0 = \sigma$: Reject if $\left|\frac{(n-1)S_{n-1}^2}{\sigma^2}\right| > t_{crit}$ [chi-squared tables]
<u>About proportions:</u> Use $\hat{p}$ to see if $p_0 = p$. If assume normal [continuity correction].
If not... Reject if $\left|\frac{\hat{p}-p_0}{\left(\frac{p_0(1-p_0)}{N}\right)}\right| > t_{crit}$ [normal tables]

Errors

	H_0 true	H_0 false
Don't reject H_0	No error	Type II error
Reject H_0	Type I error	No error



H_0: specific claim about population | H_1: alternative
$P(TI) = P(\text{reject } H_0 | H_0 \text{ false}) = \alpha$ [i.e. sig. level]
power$= 1 - P(TII) = 1 -$ ['Acceptance Range' of H_0 in H_1 distribution] $= P(\text{reject } H_0 | H_1 \text{ true})$

09. OLS Regression: Basics

Correlation: X causes Y; units matter | Regression: X related to Y; units free
$Y_i = \alpha + \beta X_i + \mu_i$ can be estimated with $\widehat{Y_i} = \hat{\alpha} + \hat{\beta} X_i$
Residual: $e_i = Y_i - \widehat{Y_i}$ with standard deviation $\sqrt{\frac{\sum(e_i)^2}{n-1}}$
Standardize: $X_i \rightarrow \frac{X_i - \bar{X}}{S_X} = \tilde{X}$ [Note: now $\tilde{\beta} = r$]

10. OLS Regression: Deriving $\hat{\alpha}, \hat{\beta}$

Want to minimise residuals i.e. (i) $\sum(Y_i - \hat{\alpha} - \hat{\beta}X_i)$; (ii) $\sum|Y_i - \hat{\alpha} - \hat{\beta}X_i|$; (iii) $\sum(Y_i - \hat{\alpha} - \hat{\beta}X_i)^2$
(i) Offsetting effect, infinite number of solutions
(ii) Not differentiable, harder to observe stat properties
(iii) Weight extremes more but... unique solution, will also equilibrate others. Use this

$$\min_{\hat{\alpha},\hat{\beta}} SSR = \sum(e_i)^2 = \sum\left(Y_i - \widehat{Y_i}\right)^2 = \sum\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right)^2$$

- $\frac{\partial SSR}{\partial \alpha} = \frac{\partial SSR}{\partial e_i} \times \frac{\partial e_i}{\partial \alpha} = 2\sum(e_i) \times -1 = 0$
- $\sum(e_i) = 0$
- $\sum(Y_i - \hat{\alpha} - \hat{\beta}X_i) = 0$
- $n\bar{Y} - n\hat{\alpha} - \hat{\beta}n\bar{X} = 0$
- $\bar{Y} - \hat{\alpha} - \hat{\beta}\bar{X} = 0$
- $\bar{Y} = \hat{\alpha} + \hat{\beta}\bar{X}$

- $\frac{\partial SSR}{\partial \beta} = \frac{\partial SSR}{\partial e_i} \times \frac{\partial e_i}{\partial \beta} = 2\sum(e_i) \times -\sum(X_i) = 0$
- $\sum(e_i X_i) = 0$
- $\sum(X_i Y_i - \hat{\alpha}X_i - \hat{\beta}X_i^2) = 0$
- $\sum(X_i Y_i) = \hat{\alpha}\sum(X_i) + \hat{\beta}\sum(X_i^2)$
- $\sum(X_i Y_i) = (\bar{Y} - \hat{\beta}\bar{X})\sum(X_i) + \hat{\beta}\sum(X_i^2)$
- $\sum(X_i Y_i) = \bar{Y}\sum(X_i) - \hat{\beta}\bar{X}\sum(X_i) + \hat{\beta}\sum(X_i^2)$
- $\sum(X_i Y_i) - \bar{Y}\sum(X_i) = \hat{\beta}\left[\sum(X_i^2) - \sum(X_i)\right]$
- $\sum(X_i Y_i) - n\bar{X}\bar{Y} = \hat{\beta}\left[\sum(X_i^2) - n\bar{X}^2\right]$
- $\hat{\beta} = \frac{S_{XY}}{S_X{}^2}$ [Intuition: slope $= r\frac{S_y}{S_x} = \frac{S_{xy}}{S_x S_y} \times \frac{S_y}{S_x} = \frac{S_{xy}}{S_x^2}$

11. OLS Regression: Proof that $\hat{\alpha}, \hat{\beta}$ are unbiased

- Let deviation of mean be $X_i^* = X_i - \bar{X}$ where $\sum(X_i^*) = 0$
- Note that $\sum(X_i^* X) = \sum(X_i^*[X_i^* + \bar{X}]) = \sum(X_i^*)^2 + \bar{X}\sum(X_i^*) = \sum(X_i^*)^2$
- Hence $\hat{\beta} = \frac{S_{xy}}{{S_x}^2} = \frac{\sum([X_i-\bar{X}][Y_i-\bar{Y}])}{\sum(X_i-\bar{X})^2} \times \frac{n-1}{n-1} = \frac{\sum(X_i^* Y_i) - \bar{Y}\sum(X_i^*)}{\sum(X_i^*)^2} = \frac{\sum(X_i^* Y_i)}{\sum(X_i^*)^2} = \frac{\sum(X_i^*[\alpha+\beta X_i+\mu_i])}{\sum(X_i^*)^2} = \beta + \frac{\sum(X_i^*\mu_i)}{\sum(X_i^*)^2}$
- $E(\hat{\beta}) = E(\beta) + E\left(\frac{\sum(X_i^*\mu_i)}{\sum(X_i^*)^2}\right) = E(\beta) + \frac{E(\mu_i)\sum(X_i^*)}{\sum(X_i^*)^2} = E(\beta)$ [Unbiased!]
- $E(\hat{\alpha}) = E(\bar{Y} - \hat{\beta}\bar{X}) = \bar{Y} - \bar{X}E(\hat{\beta}) = \alpha + \beta\bar{X} - \beta\bar{X} = \alpha$ [Unbiased!]

Distribution of $\hat{\beta} \sim N[E(\beta), V(\beta)]$ can be used to set up Hypothesis test: $zort = \frac{(b-\beta_{H_0})}{V(\hat{\beta})}$

$$V(\hat{\beta}) = V\left(\beta + \frac{\sum(X_i^*\mu_i)}{\sum(X_i^*)^2}\right) = V\left(\frac{\sum(X_i^*\mu_i)}{\sum(X_i^*)^2}\right) = \frac{1}{\left[\sum(x_i^*)^2\right]^2} V\left(\sum(X_i^*\mu_i)\right) = \frac{1}{\sum(x_i^*)^2} V(\mu_i) = \frac{\sigma^2}{\sum(X_i-\bar{X})^2} = \frac{\left[\frac{\sum(e_i^2)}{n-2}\right]}{\sum(X_i-\bar{X})^2}$$

12. OLS Regression: Extras

Transformation/Function-Forms

Non-Linear:

- Linear: $E(Y|X) = \alpha + \beta X$ ($\beta = \frac{\partial y}{\partial x}$ Elastic: constant unit between Y and X)
- Non-1 $E(\ln Y\,|X) = \alpha + \beta X$ ($\beta = \frac{\partial y}{\partial x}\frac{1}{y}$ is Semi-elastic: constant unit between $\ln Y$ and X)
- Non-2 $E(\ln Y\,|\ln X) = \alpha + \beta \ln X$ ($\beta = \frac{\partial y}{\partial x}\frac{x}{y}$ is Semi-elastic: constant unit between $\ln Y$ and $\ln X$)

Conditional Expectation Function: $E(Y|X_1, X_2)$

- Normally $E(\hat{Y}_i) = a = \bar{Y}$. Now $E(\hat{Y}_i) = a_K = \bar{Y}|(X_1, X_2)$ where K is distinct combination of X
- $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2$
- Using $\frac{\partial SSR}{\partial \hat{\beta}_i} = 0$ get $\hat{\beta}_0 = \hat{Y}_i - \hat{\beta}_1\bar{X}_1 + \hat{\beta}_2\bar{X}_2$

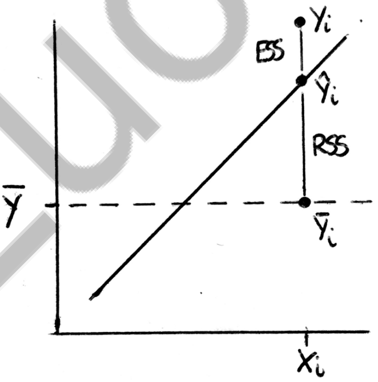
Dummy Variable: $X_i = 0,1$

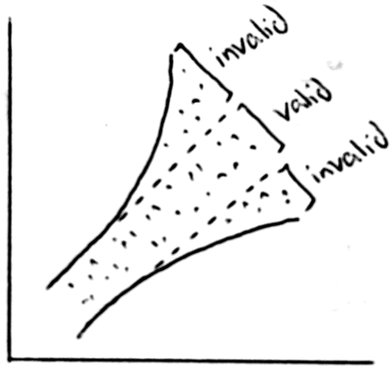
- E.g. $E(wage|[edu, sex]) = \hat{\beta}_0 + \hat{\beta}_1 edu + \hat{\beta}_2 sex$
- Where $\hat{\beta}_2 = E(wage|[edu, fem]) - E(wage|[edu, mal])$ [Note: same slope, different intercept]

R-Squared

$R^2 = \frac{RSS}{TSS}$ = % of variation explained by regression model

- *TSS*: Total Sum of Squares: $\sum(Y_i - \bar{Y})^2 = \sum(Y_i^*)^2$. Can be broken down to…
- *RSS*: Regression Sum of Squares: $\sum(\hat{Y}_i - \bar{Y})^2$ | *ESS*: Error Sum of Squares: $\sum(Y_i - \hat{Y}_i)^2 = \sum {e_i}^2$





13. OLS Regression: Assumptions

- If upheld least square estimate is BLUE (Best Linear Unbiased Estimator i.e. least sample variance)

A1. Linear Functional Form: $Y_i = \alpha + \beta X_i + \mu_i$ NOT $Y_i = \alpha + \beta X_i^2 + \mu_i$ or $Y_i = \alpha + \alpha\beta X_i + \mu_i$

A2. X_is are Non-stochastic with variation in X: [i] Sample variation in X_i (i.e. not same value) and [ii] $X_i s$ are fixed in repeated sampling and Y_is randomly drawn (i.e. treat X_is as constant)

A3. Zero conditional mean of error: $E(\mu_i|X_i) = E(\mu_i)X_i = 0$

A4. Homoskedastic errors: constant conditional variance: Errors (μ_i) don't increase as X_i increases: $Var(\mu_i|X_i) = \sigma^2 = E(\mu^2|X) - [E(\mu|X)]^2 = E(\mu^2|X)$

A5. Independent Ys: No correlation in errors (e.g. square foot and square meters)