Economic Growth

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2 Solow-Swan Model

2.1 Assumptions

2.1.1 *Structure*

- Time is continuous: $\dot{X} = \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t}$ thus $\dot{X}(t) = \frac{dX(t)}{dt}$
- If X grows at constant rate k: $\frac{\dot{X}(t)}{X(t)} = k$ thus $X(t) = e^{nt}X(0) = e^{nt}$

2.1.2 *Households*

- Population grows at constant rate n
  - But… "Demographic transition" shows it follows U-shape line
- Save constant and exogenous $[s(.) = s > 0]$ fraction of income $[sY = S]$
  - But… Friedman's PIH savings are function of permanent income not current income

2.1.3 *Economy*

- National Income Accounting and assume $G = NX = 0$ thus $Y = C + I$
  - Since closed economy and perfect capital markets $I(t) = S(t) = sY(t)$
- Capital stock depreciates: $\dot{K}(t) = I(t) - \delta K(t)$
  - K composes of existing stock, depreciation, and investment: $K_{t+1} = (1-\delta)K_t + I_t$
  - Hence $\Delta K = K_{t+1} - K_t = I_t - \delta K$
- Thus equation of motion for capital: $\dot{K}(t) = sY(t) - \delta K(t)$

2.1.4 *Technology*

- Assumes produce and consume single homogenous good (i.e. unit of GDP)
- $Y(t) = F[K(t), L(t), t] = F[K(t), A(t)L(t)] = \left(\alpha K(t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\big(A(t)L(t)\big)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma-1}{\sigma}}$
  - σ is elasticity of subsitution: $\sigma \to 1$ is Cobb-Douglas; $\sigma \to 0$ is Leontief; $\sigma \to \infty$ is perfect substitutes
- Needs to satisfy three conditions of Neoclassical production function. Cobb-Douglas does this: $Y = K^{\alpha}(AL)^{1-\alpha}$
  1. $F(.)$ exhibits positive and diminishing returns wrt each input: $\frac{dF}{dX} > 0; \frac{d^2F}{dX^2} < 0$
  2. $F(.)$ exhibits constant returns to scale, following Robert Lucas' replication argument
  3. Inada conditions: $\lim_{X \to 0}(F_X) = \infty$; $\lim_{X \to \infty}(F_X) = 0$ (guarantee unique steady state)
- Technical progress, $A(t)$, is "manna from heaven", growing at exogenous rate $\frac{\dot{A}}{A} = g$
  - Technology is said to be "labour augmenting" or "Harrod-neutral"
  - Exogeneity justified since many countries just adopt tech that flows free cross-border. And, if not, "income gap" will become infinite
  - But… some countries have R&D that expands technological frontier (see later)

2.1.5 *Firms*

- Maximize profits: $\pi = \max_{K,L} Y - wL - r^K K = \max_{K,L} K^{\alpha}(AL)^{1-\alpha} - wL - r^K K$
- FOC: $w = (1-\alpha)K^{\alpha}A^{1-\alpha}L^{-\alpha} = (1-\alpha)\frac{Y}{L}$ thus total share of labour $\frac{wL}{Y} = (1-\alpha)$
- FOC: $r^K = \alpha K^{-\alpha}A^{1-\alpha}L^{1-\alpha} = \alpha\frac{Y}{K}$ thus total labour of capital $\frac{r^K K}{Y} = \alpha$

- Euler Theorem: Note that capital and labour get paid their marginal product. Hence zero profits, relating to Euler theorem

- Getting from individual representative to aggregate production function:
  - $Y_t = \int_0^1 Y_i \, di = \int_0^1 A_i K_i^\alpha L_i^{1-\alpha} \, di$
  - Since all firms have same tech $A_i = A$: $Y_t = A_t \int_0^1 K_i^\alpha L_i^{1-\alpha}$
  - Since in CD $\frac{K_i}{L_i}$ is constant: $Y_t = A_t \int_0^1 \left(\frac{K}{L}\right)^\alpha L_i = A\left(\frac{K_t}{L_t}\right)^\alpha \int_0^1 L_i = A_t \, K_t^\alpha L_t^{1-\alpha}$

2.2 Balanced Growth Path (BGP)

- Balanced Growth Path: Equilibrium path s.t that all variables (of interest are endog K, Y, C, w, r^K) grow at a constant rate
  - In old models where $g = 0$ could reach a steady state where per capita values are fixed. But with $g \neq 0$ they will always change, thus incorrect to say they are "steady"
  - Instead BGP = "steady state of magnitudes per efficiency unit of labor"
  - The fact that everything grows at a constant rate and all sectors expand equally is a sign that the economy has matured
  - Hirschman: Developing economies may adopt a strategy of unbalanced growth to rectify previous investment decisions

2.2.1 *BGP Equilibrium (i.e. system has **UNIQUE** steady state)*

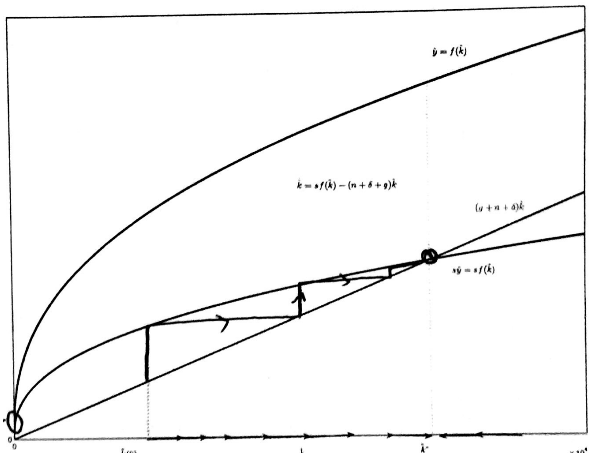
- $\dot{K} = sK^\alpha (AL)^{1-\alpha} - \delta K$ thus $\frac{\dot{K}}{K} = s\left(\frac{AL}{K}\right)^{1-\alpha} - \delta$ thus $g_K + \delta = s\left(\frac{AL}{K}\right)^{1-\alpha}$
- ~~Proof by guess and verification:~~
  - In BGP $g_K + \delta$ is constant so $\frac{AL}{K} = const$
  - $\ln(A) + \ln(L) - \ln(K) = \ln(const.)$ thus $\frac{\dot{A}}{A} + \frac{\dot{L}}{L} + \frac{\dot{K}}{K} = 0$ (differentiate wrt to time)
  - Thus $g + n - g_K = 0$ so $g_K = g + n$
    - Intuitively, if $\frac{AL}{K}$ was not constant, then firms are changing relative use of inputs, changing MR of them (as DMR) so need to re-optimise again?
- Now solving for all other
  - $Y = K^\alpha (AL)^{1-\alpha}$ thus $\frac{Y}{K} = \left(\frac{AL}{K}\right)^{1-\alpha}$ thus, as RHS is constant, $g_Y = g_K = g + n$
  - $C = (1-s)Y$ thus $\frac{C}{Y} = (1-s)$ thus $g_C = g_Y = g_K = g + n$
  - $w = (1-\alpha)A\left(\frac{K}{AL}\right)^\alpha$ thus $g_w = g$ (until 90s these were coupled!)
  - $r^K = \alpha\left(\frac{AL}{K}\right)^{1-\alpha}$ thus $g_{r^K} = 0$

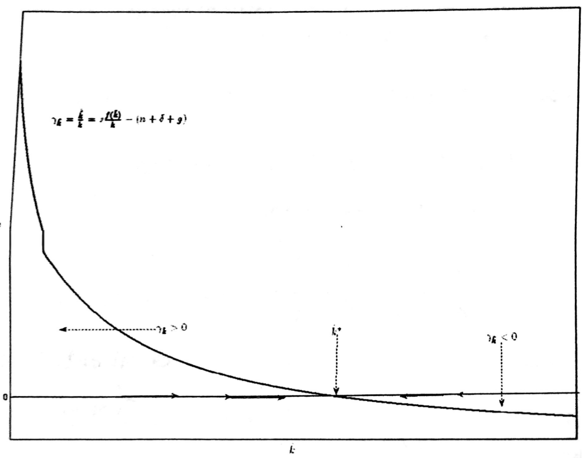
2.2.2 *BGP Convergence (i.e. system is uniquely stationary as $g_{\tilde{k}}$ tends to 0)*

- Let $\tilde{x} = \frac{X}{AL}$, that is in per efficient unit of labour. Now have $\tilde{y} = \tilde{k}^\alpha$
- **Easy mistake is to forget that $\frac{\dot{K}}{AL} \neq \dot{\tilde{k}} = \left(\frac{\dot{K}}{AL}\right)$. Instead…**
- **Note quotient rule $\frac{d}{dt}\tilde{k} = \dot{\tilde{k}} = \frac{\dot{K}AL - (\dot{A}L + A\dot{L})K}{(AL)^2} = \frac{\dot{K}}{AL} - \left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right)\tilde{k}$ thus $\frac{\dot{K}}{AL} = \dot{\tilde{k}} + (g+n)\tilde{k}$**
  - Intuitively, does not just depend on capital but also size relative to A and L
- Hence can show economy has unique BGP stable equilibrium $\tilde{k}^*$…
  - $\dot{K} = sY - \delta K$ thus $\frac{\dot{K}}{AL} = s\frac{Y}{AL} - \frac{\delta K}{AL}$ thus $\dot{\tilde{k}} + (g+n)\tilde{k} = s\tilde{k}^\alpha - \delta\tilde{k}$

- That is $\dot{\tilde{k}} = s\tilde{k}^{\alpha} - (n+g+\delta)\tilde{k}$ i.e. investment (concave) – depreciation (linear)
    - Intuitively, growth of capital per effect labour (K/AL) increases with investment (K) and falls with depreciations (K), tech (A) and population (L)
- $\tilde{k}^*$ such that $\dot{\tilde{k}}^* = 0$ thus $s{\tilde{k}^*}^{\alpha} - (n+g+\delta)\tilde{k}^* = 0$ thus $\tilde{k}^* = \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}$

- … and this will be converged to…
    - $\gamma_{\tilde{k}} = \frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (n+g+\delta)$
        - $\frac{\dot{\tilde{k}}}{\tilde{k}}$ is proportional growth rate
        - Note since $\alpha - 1 < 0$ get $\lim_{\tilde{k}\to 0} \gamma_{\tilde{k}} = \infty$ and $\lim_{\tilde{k}\to\infty} \gamma_{\tilde{k}} = -(n+g+\delta) < 0$
        - Unique $\dot{\tilde{k}}^* = 0$ and thus $\frac{\dot{\tilde{k}}}{\tilde{k}} = 0$. Hence if $\tilde{k} < \tilde{k}^*$; savings > depreciation; $\gamma_{\tilde{k}} > 0$ (and vice versa). Will always be moving in direction of BGP.
    - $\frac{d\gamma_{\tilde{k}}}{d\tilde{k}} = s(1-\alpha)\tilde{k}^{\alpha-2} < 0$. So growth rate is decreasing in $\tilde{k}$





2.2.3 BGP Speed Of Convergence

- Take first order Taylor approximation of $G(\tilde{k})$ around steady state k^*
    - $G(\tilde{k}) \approx G(\tilde{k}^*) + G'(\tilde{k}^*)(\tilde{k} - \tilde{k}^*)$
- In case of $G(\tilde{k}) = \dot{\tilde{k}}|_{\tilde{k}^*}$ we get:
    - $G(\tilde{k}) \approx s(\tilde{k}^*)^{\alpha} - (n+g+\delta)\tilde{k}^* + \left[\alpha s(\tilde{k}^*)^{\alpha-1} - (n+g+\delta)\right](\tilde{k} - \tilde{k}^*)$
        - Note that at $\dot{\tilde{k}}^* = 0$ we have $s(\tilde{k}^*)^{\alpha-1} = (n+g+\delta)$
    - $G(\tilde{k}) \approx -(1-\alpha)(\delta+g+n)(\tilde{k} - \tilde{k}^*) = -\beta(\tilde{k} - \tilde{k}^*)$
- Also see
    - supervision for terms of $g_{\tilde{k}}$ and $\frac{\tilde{k}-\tilde{k}^*}{\tilde{k}}$, useful for $\left(\frac{\tilde{k}-\tilde{k}^*}{\tilde{k}^*}\right) \approx \ln\tilde{k} - \ln\tilde{k}^*$ interpretation
    - Lecture 3 Slide 14 for solutions
- The speed of convergence depends on β and difference between $\tilde{k}$ and $\tilde{k}^*$
    - E.g. Germany and Japan had capital stocks wiped out by WWII and hence grew rapidly until they reached steady state and then slower

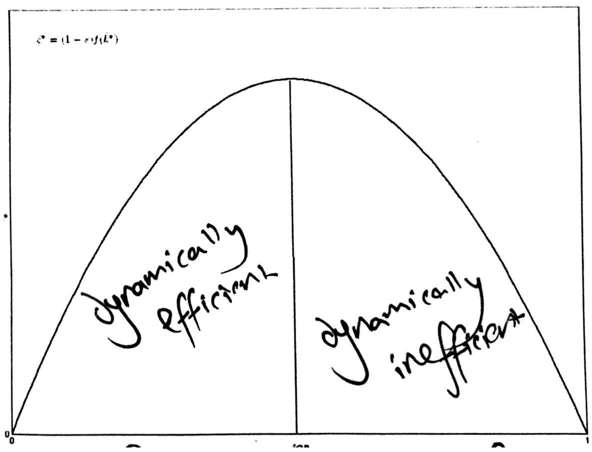
2.2.4 BGP Properties

- Aggregate variables grow at $g+n$ and per capita variables at g
- Changes in s, n, δ affect k^*, y^*, c^* but not their respective growth rates

- Along BGP, y/c will be higher in countries with high investment rate and low population growth – but neither factor has impact on LR growth rate (as per capita variables grow at g)

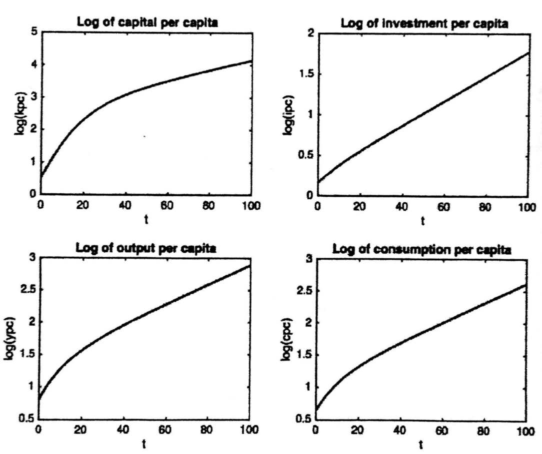
2.2.5 Golden Rule (i.e. the best BGP: Phelps 1966)

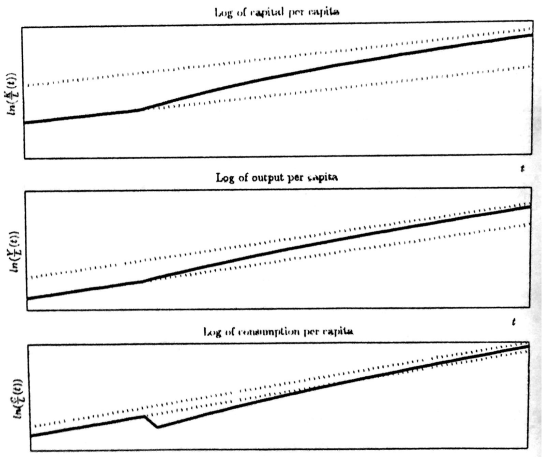
- See that s affects level of c* in two ways: increases y* and thus c*, decreases share of c* and thus c*. How to balance these out?
- Savings rate maximizing long-run consumption: $\max_s \tilde{c}^* = \max_s \tilde{c}^* \, (1-s)f(\tilde{k}^*)$
- Let us solve assuming standard $\tilde{k}^* = \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}$ and $f(\tilde{k}^*) = (\tilde{k}^*)^\alpha$
  - FOC: $\frac{d\tilde{c}^*}{ds} = -\left[\frac{s}{n+g+\delta}\right]^{\frac{\alpha}{1-\alpha}-1}\left[\frac{s-\alpha}{n+g+\delta}\right] = 0$
  - Thus $s_{GR} = \alpha$ and $\tilde{k}^*_{GR} = \left[\frac{s_{GR}}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}$
- Note rise in s has two counter effects: lowers fraction of income consumed, but raises capital accumulation rate and hence total income
  - If $s < s_{GR}$ then increase in s would increase $\tilde{c}^*$ in long run
    - Trade-off between current and future welfare. In this sense Pareto efficient
  - If $s > s_{GR}$ then decrease in s would increase $\tilde{c}^*$ in long run
    - Economy is dynamically inefficient because *all* generations can be made better off



2.2.6 Transition Dynamics (i.e. changing BGP)

- Suppose initially economy is in BGP equilibrium $\tilde{k}_1^*$, so that $\dot{\tilde{k}} = sf(\tilde{k}_1^*) - (n+g+\delta)\tilde{k}_1^* = 0$
- If s increases $s_{new}f(\tilde{k}_1^*) > (n+g+\delta)\tilde{k}_1^*$ thus $\dot{\tilde{k}} > 0$
- Capital stock $\tilde{k}$ grows until reaches new higher BGP
- Along transition, $\tilde{k}$ and $\tilde{y}$ rises but growth rate slows down
- In new BGP, per capita variables grow again at rate g





			Level Effects	Growth Rate
Capital per effective worker:	$\hat{k}^* = \frac{K}{AL}$	$\hat{k}^* = \left(\frac{s}{\delta+g+n}\right)^{\frac{1}{1-\alpha}}$	s δ g n + − − −	$g_{\hat{k}} = 0$
Output per effective worker:	$\hat{y}^* = \frac{Y}{AL}$	$\hat{y}_t^* = \left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}}$	s δ g n + − − −	$g_{\hat{y}} = 0$
Capital per worker:	$k^* = \frac{K}{L} = \hat{k} * A$	$k^*(t) = A(t)\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$	s δ n + − −	$g_k = g$
Output per worker:	$y^* = \frac{Y}{L} = \hat{y} * A$	$y^*(t) = A(t)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$	s δ n + − −	$g_y = g$
Investment per worker	$i^* = s * \hat{y} * A$		s δ n + − −	$g_k = g$
Consumption per worker	$c^* = (1-s)\hat{y} * A$		s δ n + − −	$g_y = g$
Labor, L_t				$g_L = n$
Capital, K_t^*	$K = \hat{k} * A * L$	$K(t)^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}} A(t)L(t)$	s δ + −	$g_K = g + n$
Output, Y_t^*	$Y = K^\alpha (AL)^{1-\alpha}$ $= \hat{y} * A * L$	$Y(t)^* = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t)$	s δ + −	$g_Y = g + n$
Investment	$I = s * \hat{y} * A * L$		s δ + −	$g_Y = g + n$
Consumption	$C = (1-s) * \hat{y} * A * L$		s δ + −	$g_Y = g + n$

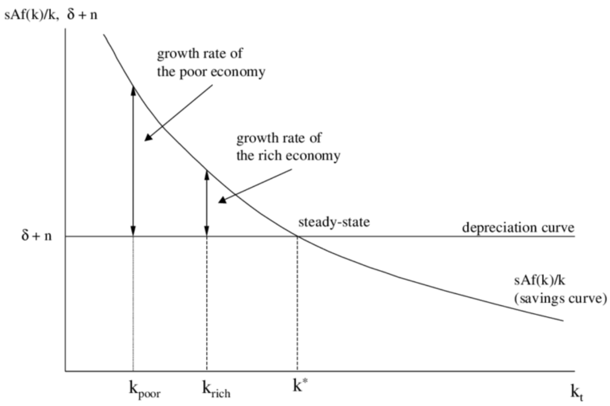
Dan S9 immigration

3 Solow-Swan Evidence

3.1 Conditional Convergence

3.1.1 Prediction

- Distinguish between absolute and conditional convergence
- Recall equation of motion for capital: $\dot{\tilde{k}} = s\tilde{k}^\alpha - (n+g+\delta)\tilde{k}$
- First term (investment) is concave; second (depreciation) is linear. Hence function decreases in $\tilde{k}$
- Ceteris paribus, lower $\tilde{k}$ country has higher $\dot{\tilde{k}}$ and thus $\dot{\tilde{y}}$



3.1.2 EVIDENCE

For

- Mankiw, Romer & Weil (1992): Estimate $\ln y = gt + \ln A(0) + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(\delta + n + g)$
  - Note, assumes g and δ are common to all countries (since tech flows across borders)
  - Explains a 60% of cross-country income variation (80% when include HK)
- Barro & Sala-i-Martin (1992): Find similar results when controlling for national characteristics (a proxy for steady-states) by examining US states, regions of France, and prefectures in Japan

Against

- Rate of convergence is too slow in real life than what the Solow model predicts
  - MRW: $t_{half,US} = \frac{\ln 2}{\beta_{US}} = 13y$, which is too fast for many countries (but… not Singapore)
  - Barro & SiM (1992): For 'correct' convergence rate 2%p.a. need $\alpha = 0.75$ irl $\alpha = 0.3$
- Half of US growth 1948-2010 (i.e. 1.4%pa) due TFP and is thus not explained by Solow ("residual" or "measure of our ignorance")
  - But… Young (1995): Solow can explain a lot of East Asian Tiger growth where there wasn't much TFP. In Singapore its was even slightly negative!
- Inequality does not seem to be decreasing in the long run
  - Pritchett (1997): "Divergence, Big Time" as between 1870-1990 ratio of per capita incomes between the richest and the poorest countries increased ~5x
  - Quah (1996): Twin peaks hypothesis as middle income countries become relatively richer but poorest relatively (though not absolutely) poorer

3.2 CAPITAL FLOWS

3.2.1 PREDICTION

- Note lower $\tilde{k}$ has relatively higher rate of return (i.e. long-run interest rate) in K due to diminishing marginal returns
  - $\frac{r_{poor}}{r_{rich}} = \frac{\alpha\left(\frac{y_{poor}}{A_{poor}}\right)^{\frac{\alpha-1}{\alpha}}}{\alpha\left(\frac{y_{rich}}{A_{rich}}\right)^{\frac{\alpha-1}{\alpha}}}$
- Thus expect flow of capital investment from rich to poor countries

3.2.2 *Evidence*

- Lucas (1990) tests specification by comparing the US and India, assuming $\alpha = \frac{1}{3}$ for both (as per the literature); $A_{India} = \mathrm{A}_{US}$ (due to diffusion of knowledge); and $10y_{India} = \mathrm{y}_{US}$ (observation)
  - $\frac{r_{poor}}{r_{rich}} = 10^2 \left(\frac{A_{India}}{A_{US}}\right) = 100$ Implies very large capital flows, not observe in real life
  - $\frac{r_{poor}}{r_{rich}} = 10^2 \left(\frac{A_{India}}{A_{US}}\right)\left(\frac{h_{India}}{h_{US}}\right)$ Cannot be fixed by incorporating human capital
  - Unlikely further adjustments (e.g. risk, institutions, tech differences) will correct this
- But… does not disprove Solow per se, since it assumes closed economy!!

4 Piketty

4.1 Empirical Observation

- Collects wealth data by looking at inheritance tax paid and deducing generational wealth
- Capital-output ratios $\frac{K}{Y}$ across all rich economies follow u-shaped trend
  - 1871-1913:~650-750% 1913-1945: ~200-300% 1970-now:~400-600%
    - Why temporary decline? War's physical destruction; political/budgetary shocks; inflation eroding bonds; decolonisation collapsing foreign portfolios; Great Depression forced to sell-off capital to maintain living standards
  - Some variation due to national characteristics e.g. US was less exposed to the World Wars and never had Empire to lose, thus smaller downturn
  - Implies inequality follows same pattern, as capital owned by small elite
- [Weak version] SS Model predicts long-run economies along BGP have at most 300-400% $\frac{K}{Y}$
  - SS states $\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta$ and BGP $\frac{\dot{K}}{K} = g_K = g_A + n = g$ [note different g notation]
  - Thus $\frac{K}{Y} = \frac{s}{g+\delta}$ (rearranging above law of motion)
  - Sub in observed parameters and limit $g = 0$ we get $\left(\frac{K}{Y}\right)_{MAX} = 3(4)$ as upper bound
  - Thus SS model cannot explain inequality that we observe
- Post-tax rate of return to capital (r_{post}) relative to the growth rate (g) also follows u-shape
  - r_{post} historically higher than g; fell sharply 1913-1950; gradually catching up
  - Soon will reach $r_{post} > g$, as shown by 'secular stagnation'
- Some criticisms of empirical findings (Magness & Murphy, 2014) but generally accepted

4.2 [Strong Version] Theoretical Model See After Piketty Chapter 4

- Can explain dramatically higher ceiling of capital-output ratio (i.e. capital's share of income) and thus also "endless inegalitarian spiral"
  - Let $I - \delta K = \tilde{s}[F(K, AL)] - \delta K$ so net investment = fixed fraction of net output (instead of constant fraction of income)
  - Thus $I = \tilde{s}F(K, AL) + (1 - \tilde{s})\delta K$ where $\tilde{s}$ is now net rather than gross savings rate
  - Thus $\dot{K} = \tilde{s}F(K, AL) + (1 - \tilde{s})\delta K - \delta K = \tilde{s}(F(K, AL) - \delta K)$
  - Thus along BGP equilibrium $\frac{\dot{K}}{K} = g_K = g_A + n = g$ get $\frac{K}{Y} = \frac{\tilde{s}}{g+\tilde{s}\delta} = \frac{\tilde{s}}{\tilde{g}}$

        - This gives us other fundamental law $\alpha_K = r^K \frac{\tilde{s}}{\tilde{g}}$ [recall $\alpha_K = r^K \frac{K}{Y}$ from firm maximization problem. Accounting identity: $\alpha = r^K \beta$]
        - Generally, if $\uparrow r^K$ outweighs $\downarrow g$ then capital's share of income α_K is set to rise
    - Sub in observed parameters and limit $g = 0$ we get $\left(\frac{K}{Y}\right)_{MAX} = 12.5(16.7)$
        - If g falls to half, as it is forecast to do in some economies, capital's share doubles and we return to Belle Epoque levels of inequality
        - "The reason why wealth today is not as unequally distributed as in the past is simply that not enough time has passed since 1945"
    - If we assume $\delta = 0$ then as $g \to 0, \frac{K}{Y} = \alpha_K \to \infty$

- This is linked to a complete decline in consumption
    - Note consumption rate in terms of gross savings rate
        - Let $C = Y - I = F - \tilde{s}F - (1-\tilde{s})\delta K = (1-\tilde{s})(Y - \delta K)$
        - Thus $\frac{C}{Y} = (1-\tilde{s})\left(Y - \delta\frac{K}{Y}\right)$ and along BGP equ. $\frac{C}{Y} = (1-\tilde{s})\frac{\tilde{s}}{g+\tilde{s}\delta}$
    - Note net savings rate: $s = \frac{Y-C}{Y} = 1 - \frac{C}{Y}$
    - Combined get $s(g) = \frac{\tilde{s}(g+\delta)}{g+\tilde{s}\delta}$ and thus $s'(g) = \frac{\tilde{s}\delta(1-\delta)}{(g+\tilde{s}\delta)^2} < 0$
    - As $g \to 0$ then that $s(g) \to 1$ and $\frac{C}{Y} \to 0$. Dubious this is true (see K&S)

4.3 CRITICISMS

4.3.1 KRUSELL & SMITH (2015)

- Critical difference is that Piketty assumes different savings behaviour:
    - Piketty: constant <u>net</u> saving rate $\tilde{s}$
        - i.e. economy increases its capital stock from year to year by constant fraction of (net) national income
    - SS: constant <u>gross</u> saving rate s
        - i.e. gross investment (including depreciation) as a fraction of (gross) national income, is constant.
- Very dubious that this is justified, as shown by $g \to 0$
    - SS: net savings also fall to zero, thus $g_k \to 0$
        - $s = constant$ thus $s'(g) = 0$
    - Piketty: K continues to grow at cost of evermore consumption until only savings
        - $s(g) = \frac{\tilde{s}(g+\delta)}{g+\tilde{s}\delta}$ thus $s'(g) = -\frac{\tilde{s}\delta(1-\delta)}{(g+\tilde{s}\delta)^2} < 0$ where $s(g \equiv 0) = 1$
- Against micro-founded Friedman (1957) who state at zero growth net savings rate is zero
- Not consistent with post-war US data where low growth decades had low/negative s

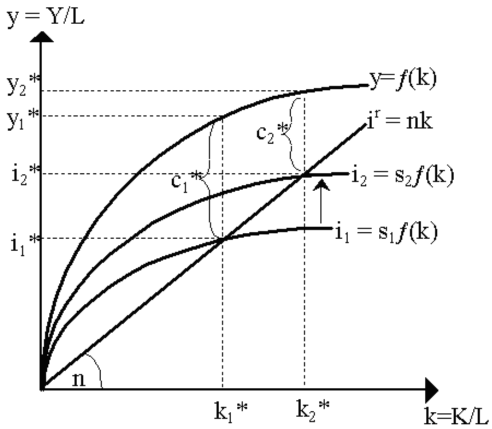
4.3.2 MANKIW (2015)

- Interpretation of r is misguided so $r > g$ is not applicable
    - Mankiw accepts premise that in the US $r = 5\%$ and $g = 3\%$ so $r > g$ by 2%-points
    - To have effect on cross-generational effects must consider further conditions
        - Marginal propensity to consume out of wealth (~3%)
        - Dynastic wealth is spread out across multiple inheritors (~2%)
        - Estate and capital income taxes (~2%)
        - Using very conservative estimates rooted in the literature, Mankiw thus notes that to have an effect on cross-generational inequality, we must have

- To have $r - 7 > g$ need secular decline, not secular stagnation, which is not true. Even secular stagnation is not clear if it will last!

4.4 Evaluation

- See also here (2;4;14) and here. Work needs to be done to reconcile criticisms.
- Critical if policymakers base decisions on Piketty's "second fundamental law" or a neoclassical growth model.
    - Drastically different methods to tackle inequality (the former favours a wealth tax, the latter is associated with a progressive tax on consumption)
    - Wealth tax attempts to break fact that K/Y means all goes to capital owners
- Vastly contrasting interpretations of what $r > g$ means.
    - Piketty: Represents rise of rentier class and inequality
    - SS: under steady-state we are not dynamically inefficient wrt to golden rule



5 Human Capital

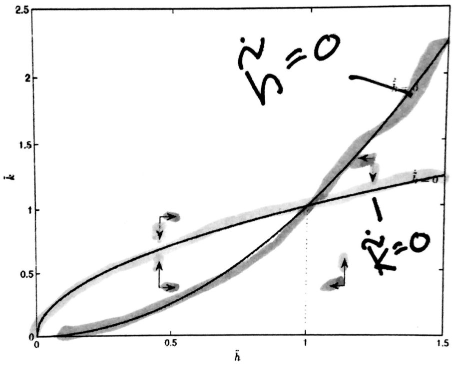
5.1 Theory

- Two ways to consider richer definition of capital. We will assume MRW but interchangeable
    - MRW (1992): $Y_t = K_t^{\alpha} H_t^{\gamma} (A_t L_t)^{1-\alpha-\gamma}$
    - Lucas (1988): $Y_t = K_t^{\alpha} (A_t H_t)^{1-\alpha}$ where $H_t = e^{\phi u_t} L_t$; u_t share of labour in training

- Assume that these depreciate at the same rate δ (makes maths easier, treat HK as another form of K, – relate this to education for example)
- Now have three unknowns and two accumulation equations for capital
    - $\dot{K} = s_K Y - \delta K \qquad \dot{H} = s_H Y - \delta H$
- Can find existence of BGP as follows:
    - $g_K = \frac{\dot{K}}{K} = \frac{s_K Y}{K} - \delta$ thus $g_K + \delta = s_K K_t^{\alpha-1} H_t^{\gamma} (A_t L_t)^{1-\alpha-\gamma}$ thus $(\alpha - 1) g_K + \gamma g_H + (1 - \alpha - \gamma)(n + g) = 0$
    - $g_H = \frac{\dot{H}}{H} = \frac{s_H Y}{H} - \delta$ thus $g_H + \delta = s_K K_t^{\alpha} H_t^{\gamma-1} (A_t L_t)^{1-\alpha-\gamma}$ thus $\alpha g_K + (\gamma - 1) g_H + (1 - \alpha - \gamma)(n + g) = 0$
    - Subtracting $(\alpha - 1) g_K + \gamma g_H + (1 - \alpha - \gamma)(n + g) - [\alpha g_K + (\gamma - 1) g_H + (1 - \alpha - \gamma)(n + g)] = g_H - g_K = 0$. Thus along BGP $g_H = g_K$
    - Sub back in $\alpha g_K + (\gamma - 1) g_K + (1 - \alpha - \gamma)(n + g) = 0$ so $g_K = g_H = n + g$

    - Technically need go on to show C,Y,r^K,r^H, w all growing at constant rate too

- Can find convergence by showing stationary system:
    - (i) $\dot{\tilde{k}} = s_K \tilde{k}^\alpha \tilde{h}^\gamma - (\delta + n + g)\tilde{k}$ $\qquad \tilde{k}^* = \left[\frac{s_K}{(\delta+n+g)}\right]^{\frac{1}{1-\alpha}} \tilde{h}^{\frac{\gamma}{1-\alpha}}$
    - (ii) $\dot{\tilde{h}} = s_h \tilde{k}^\alpha \tilde{h}^\gamma - (\delta + n + g)\tilde{h}$ $\qquad \tilde{k}^* = \left[\frac{(\delta+n+g)}{s_H}\right]^{\frac{1}{\alpha}} \tilde{h}^{\frac{1-\gamma}{\alpha}}$
- Note transitional dynamics:
    - (i) concave locus ($\gamma < 1 - \alpha$ due to CRTS); if $\tilde{k}$ above (below) line $\dot{\tilde{k}} = 0$, $g_{\tilde{k}} < (>)0$
    - (ii) convex locus ($1 - \gamma > \alpha$ due to CRTS); if $\tilde{h}$ left (right) line $\dot{\tilde{h}} = 0$, $g_{\tilde{h}} > (<)0$



Thus naturally inclined to move along 'saddle path'

- Hence have steady state at $\dot{\tilde{k}} = \dot{\tilde{h}} = 0$
    - $\tilde{k}^* = \left[\frac{s_K^{1-\gamma} s_H^\gamma}{(\delta+n+g)}\right]^{\frac{1}{1-\alpha-\gamma}}$ $\qquad \tilde{h}^* = \left[\frac{s_K^\alpha s_H^{1-\alpha}}{(\delta+n+g)}\right]^{\frac{1}{1-\alpha-\gamma}}$ $\qquad \tilde{y}^* = \left[\frac{s_K^\alpha s_H^\gamma}{(\delta+n+g)^{\alpha+\gamma}}\right]^{\frac{1}{1-\alpha-\gamma}}$
    - Note investment in physical/human capital increases the marginal productive of the other as they are complementary
    - Thus higher saving rate in physical K increases k and also h
- Get new rate of convergence
    - If firm borrows to increase K[H] increases by 1, then Y increases by MPK[H]; but will also depreciate by δ; so net MB is MPK[H] – δ; marginal cost of borrowing I is r
        - Stems from assumption that have same depreciation, which doesn't seem warranted
    - $R_K = r + \delta = \alpha \frac{\tilde{y}}{\tilde{k}}$ and $R_H = r + \delta = \gamma \frac{\tilde{y}}{\tilde{h}}$
    - Thus $\tilde{h} = \frac{\alpha}{\gamma}\tilde{k}$ and $\dot{\tilde{k}} = s_K \left(\frac{\alpha}{\gamma}\right)^\gamma \tilde{k}^{\alpha+\gamma} - (\delta + n + g)\tilde{k}$
    - Thus $\beta = (1 - \alpha - \gamma)(n + g + \delta)$

5.2 Evidence

- Mankiw, Romer, & Weil (1992) compare
    - Rise in R^2: from 0.6 to 0.8 and more realistic implied α: from 0.75 to 0.31
    - Hence also more realistic rate of convergence: from 13y to 23y
- But… there are many possible issues in their methodology
    - Endogeneity: s_K, s_H, n could depend on Y/L
    - OVB: Caselli et al (1996) note if $A(0)$ systematically larger in rich countries and correlated to y_0
    - See also Lucas (1990) again. Helps but doesn't resolve this

6 TECHNOLOGY AND ENDOGENOUS GROWTH

In first generation tech is accidental by-product; In second generation it is consciously developed
What is technology A?: Ideas are non-rival and (partially) excludable

6.1 GROWTH ACCOUNTING

- Solow (1957) allows us to decompose output changes into input and "technological" change
  - Decomposition: $Y = BK^{\alpha}L^{1-\alpha}$ thus $g_Y = g_B + \alpha g_K + (1-\alpha)g_L$
    - Hence TFP (i.e. g_B) is known as the Solow residual
  - Development accounting: y-ratio = factor-ratio × productivity-ratio
    - $\frac{y_{it}}{y_{jt}} = \left(\frac{k_{it}}{k_{jt}}\right)^{\alpha}\left(\frac{h_{it}}{h_{jt}}\right)^{1-\alpha}\left(\frac{A_{it}}{A_{jt}}\right)$
- Technology appears critical in many cases (but not all!)
  - Penn World Table: 50-66% of in cross-country y cannot be explained by K or H variation
  - Barro & Sala-I-Martin: In Germany 42% of 1960-95 growth; in Singapore 2% of 1966-90
  - Barrell et al (2010): In terms of TFP Japan is fine; ageing population is the real problem
- Primary determinant of long run growth cannot itself be explained by Solow!

6.2 LEARNING BY DOING MODEL

- Informed by Arrow (1962) and Romer (1986), who see knowledge spillovers as accidental by-products of economic activity
- Hence from aggregate to allow for positive externalities

6.2.1 SEPARATE FIRM PRODUCTION FUNCTION

- Economy consist of J identical firms — Thus $Y_j = \bar{A}K_j^{\alpha}L_j^{1-\alpha}$
- All firms are price takers (i.e. perfect competition) — Thus $w = (1-\alpha)Y_j$ and $r^K = \alpha Y_j$
- In equilibrium the economy is described by… — $Y = JY_j = \bar{A}K^{\alpha}L^{1-\alpha}$
- Don't have to assume identical firms but makes it easier. So far just like Solow

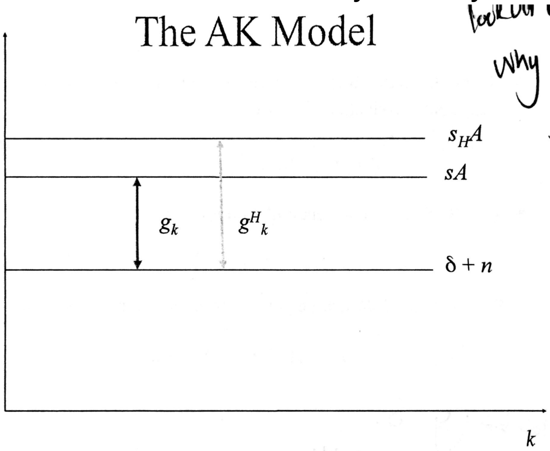
6.2.2 ALL FOR POSITIVE EXTERNALITIES

- Assume $\bar{A} = A\left(\frac{K}{L}\right)^{\mu}$ where A is Solow component, $\frac{K}{L}$ is externality, and μ strength of this
- Interpret as learning by doing or agglomeration effect. Note these are not micro-founded!
- Allen: capital intensive technology is linked to TFP; no micro-foundation for what this mechanism is
- As firms are small they do not consider positive externality they create

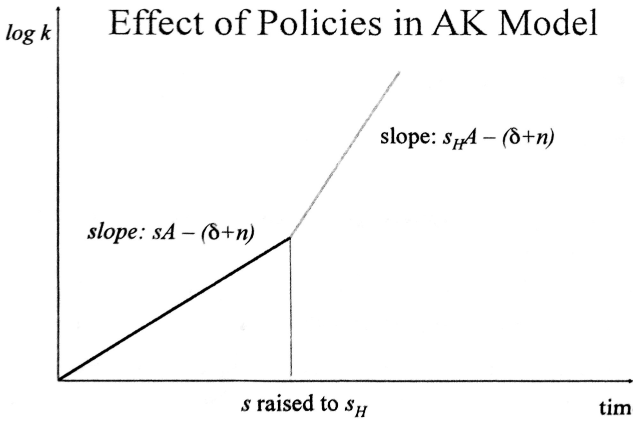
6.2.3 SOLVE FOR EQUILIBRIUM

- Solving like Solow:
  - Individual firms face CRTS but at aggregate: $Y = A\left(\frac{K}{L}\right)^{\mu}K^{\alpha}L^{1-\alpha}$ and $y = Ak^{\alpha+\mu}$
  - Thus, $\dot{K} = sY - \delta K$ and $\dot{k} = sAk^{\alpha+\mu} - (\delta+n)k$
  - Thus $g_y = (\alpha+\mu)g_k = (\alpha+\mu)[sAk^{\alpha+\mu-1} - (\delta+n)]$
  - $\frac{dg_k(t)}{dt} = (\alpha+\mu-1)g_k(t)$
- Hence have three possible results:
  - If $\alpha + \mu > 1$: No BGP as explosive growth. This is rejected by data

- If $\alpha + \mu < 1$: BGP with $g_y = 0$. Like Solow but externality slows down convergence
- If $\alpha + \mu = 1$: BGP with equilibrium $g_y = (\alpha + \mu)[sA - (\delta + n)]$.
  - This is the 'AK Model'
  - But very unlikely coincidence to be exactly one so "on a knife edge"



Why is S. Grow



6.3 Romer (1990) Model (i.e. R&D Model of Economic Growth)

- Assume CD production function $Y = K^\alpha (AL_Y)^{1-\alpha}$. Note two distinctions to Solow:
  - Incorporating ideas A we have IRTS, not CRTS
  - Labour split between output Y and research input R so $L_Y + L_A = L$ and $s_R = \frac{L_A}{L}$
- Need to deviate from perfect competition so have profits to pay R&D

6.3.1 *Structure of Economy (i.e. micro-foundations)*

Final Goods Sector

- Buys available intermediate goods x_j at p_j and labour L_Y at w as input. Then produce homogenous output Y. Characterized by perfectly competitive markets.
- Firms produce according to CRTS production function: $Y = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^\alpha$
  - $\frac{dY}{dx_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha - 1}$ with $\lim_{x_j \to 0} \frac{dY}{dx_j} = \infty$. [That is x_j is input with diminishing returns]
  - A is number of capital inputs used in production (instead of single K, have variety A)
  - Note x_j are not perfect substitutes (unless $\alpha = 1$) but complements
    - $x_i = L_Y \left[\frac{\alpha}{p_i}\right]^{\frac{1}{1-\alpha}}$ thus $\frac{x_i}{x_j} = \left[\frac{p_j}{p_i}\right]^{\frac{1}{1-\alpha}}$ thus $\ln \frac{x_i}{x_j} = \frac{1}{1-\alpha} \ln \frac{p_j}{p_i} = -\frac{1}{1-\alpha} \ln \frac{p_i}{p_j}$
    - Hence elasticity of substitution $\sigma_{i,j} = -\frac{d \ln \frac{x_i}{x_j}}{d \ln \frac{p_i}{p_j}} = \frac{1}{1-\alpha}$
- Maximize profits, hence $\max\{Y_i - wL_Y - \sum_{j=1}^{A} p_j x_j\} = \{L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^\alpha - wL_Y - \sum_{j=1}^{A} p_j x_j\}$
  - FOC: $\frac{d\pi}{dL_Y} = (1-\alpha) L_Y^{-\alpha} \sum_{j=1}^{A} x_j^\alpha - w = 0$ thus $w = (1-\alpha) L_Y^{-\alpha} \sum_{j=1}^{A} x_j^\alpha = MPL$
  - FOC: $\frac{d\pi}{dx_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} - p_j = 0$ thus $p_j = \alpha L_Y^{1-\alpha} x_j^{\alpha - 1} = MPK_j \ \forall j$

Intermediate Capital Goods Sector

- Firms purchase patents from R&D sector and rent capital at rate r. Act as monopolists in production of each intermediate good x_j at p_j to sell to FGS (as imperfect substitutes!)
- Maximize profits, hence $\max_{x_j}\{p_j(x_j)x_j - rx_j\}$

- FOC: $p_j'(x_j)x_j + p_j(x_j) - r = 0$
- Thus $p_j = \frac{1}{1+\frac{p'(x)x}{p_j}} r = \frac{1}{\alpha} r$ [sub in p_j from above]
  - Note mark up $\frac{1}{\alpha}$ over MC as not perfect competition
  - Higher α, capital more important, more market power, more profit
- Since $p_j = p$ for all j then $x_j = x$ for all j
  - Same profits: $\pi = px - rx = px - \alpha px = (1-\alpha)px = \alpha(1-\alpha)L_Y^{1-\alpha}x^\alpha = \alpha(1-\alpha)\frac{Y}{A}$
    - As $\alpha \to 1, \pi \to 0$ and we return to perfectly competitive world
- In equ. in the capital market: $\sum x_j\Big|_{j=0}^{A} = K$ thus $x = \frac{K}{A}$
  - $Y = AL_Y^{1-\alpha}x^\alpha = AL_Y^{1-\alpha}\left(\frac{K}{A}\right)^\alpha = K^\alpha(AL_Y)^{1-\alpha}$
  - This is akin to Solow Model!

Research Sector

- Generates new ideas A at price P_A, using existing knowledge and scientists L_A
- "Knowledge stock" given $\dot{A} = \bar{\delta}L_A$ where $\bar{\delta} = BL_A^{\lambda-1}A^\phi$. Scientist take $\bar{\delta}$ as fixed so do not consider externality
    - How many scientists we have * how productive they are
  - $0 \leq \lambda \leq 1$: rate of discoveries decreases with # of researchers due to duplication
  - ϕ: relationship between research productivity and ideas stock i.e. rate of discovery $\dot{A}$
    - If $\phi > 0$ then "standing on the shoulders of the giants" and $\frac{d\dot{A}}{dA} > 0$
    - If $\phi < 0$ there is a "fishing out" as the easy ideas are used up and $\frac{d\dot{A}}{dA} < 0$
    - If $\phi = 0$ then independent and $\frac{d\dot{A}}{dA} = 0$
- Like model of creative destruction but instead of destroying existing, expand variety of capital goods with some complementarity
- Can show that $w_A = \bar{\delta}P_A$ (stock of new ideas / number of scientists) * price of new ideas

Interpretations

- Price of a patent is subject to arbitrage as investors can choose between capital and patents
  - If purchase unit of K earn r; if purchase patent earn π and sell it
  - Thus $rP_A = \pi + \dot{P}_A$. Rearrange to $r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}$ and $P_A = \frac{\pi + \dot{P}_A}{r}$
  - In BGP, r is constant
    - Recall $r = \alpha p$ and $p = \alpha L_Y^{1-\alpha}x^{\alpha-1}$ thus $r = \alpha^2 L_Y^{1-\alpha}x^{\alpha-1}$
    - Recall $Y = L_Y^{1-\alpha}\frac{K}{x}x^\alpha = L_Y^{1-\alpha}Kx^{\alpha-1}$ thus $\frac{Y}{K} = L_Y^{1-\alpha}x^{\alpha-1}$
    - Put together and $r = \alpha^2\frac{Y}{K}$ in equilibrium. Know $g_Y = g_K$
  - Thus $\frac{\pi}{P_A}$ has to be constant thus π and P_A grow at same rate
  - As $\pi = \alpha(1-\alpha)\frac{Y}{A} = (1-\alpha)\frac{y}{A}L$ we know $g_\pi = \frac{\dot{P}_A}{P_A} = g_y - g_A + g_L = g_L = n$
  - Subbing back in we get $r = \frac{\pi}{P_A} + n$ and $P_A = \frac{\pi}{r-n}$
    - Like Gordon Growth model where P_A is discounted value of dividends

- Note $r < MPK$ as $\alpha^2\frac{Y}{K} < \alpha\frac{Y}{K}$ and $\alpha < 1$ (again, if $\alpha = 1$ return to perfect comp.)
  - In Solow, under CRTS and perfect competition, factors paid marginal products

- In Romer, this is what necessitates imperfect competition (patent protection): Capital is paid less so as to compensate researches for new ideas

6.3.2 *EQUILIBRIUM*

General

- Define $a_L = \frac{L_A}{L}$. In aggregate $\frac{\dot{A}}{A} = BL_A^\lambda A^{\phi-1} = Ba_L^\lambda L(t)^\lambda A^{\phi-1}$
- Also note goods market equilibrium similar to Solow
  - $\dot{K} = sY - \delta K = sK^\alpha (AL_y)^{1-\alpha} - \delta K = sK^\alpha (A(1-a_L)L)^{1-\alpha} - \delta K$
  - $\dot{\tilde{k}} = s\tilde{k}^\alpha (1-a_L)^{1-\alpha} - (\delta + n + g_A)K$
  - $\tilde{k}^* = \left(\frac{s}{\delta+n+g_A}\right)^{\frac{1}{1-\alpha}} (1-a_L)$

$\phi = 1$: BGP only if $n = 0$

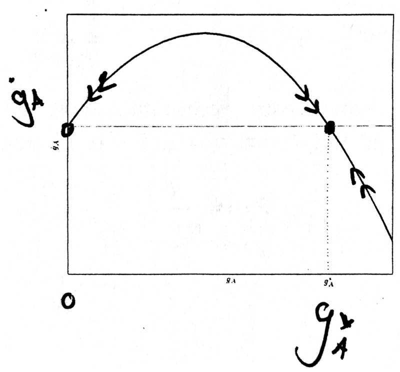
- Thus $g_A = \frac{\dot{A}}{A} = BL_A^\lambda = Ba_L^\lambda L(t)^\lambda$
- Two cases:
  - If $n > 0$, g_A grows over time (i.e. no BGP)
  - If $n = 0$, $g_y = g_A = Ba_L^\lambda L^\lambda$
- Similar to AK model and scale effect (↑pop, ↑ R&D investment, grow faster)
  - Some link to Einstein effect in Economic History

$\phi > 1$: No BGP

- $g_A = \frac{\dot{A}}{A} = BL_A^\lambda A^{\phi-1}$ then $\frac{\dot{g_A}}{g_A} = \lambda n + (\phi - 1)g_A$ so $\dot{g_A} = g_A[\lambda n + (\phi-1)g_A]$
  - Explosive growth as $\dot{g}_A$ increasing in g_A; increase in a_L leads to divergent path
  - But… not seem consistent with data!

$\phi < 1$: BGP

- $g_A = \frac{\dot{A}}{A} = BL_A^\lambda A^{\phi-1}$ then $\frac{\dot{g_A}}{g_A} = \lambda n + (\phi - 1)g_A$ so $\dot{g_A} = g_A[\lambda n + (\phi-1)g_A]$
- Along BGP $\dot{g}_A = 0$ so either converge to $g_A = 0$ or $g_A^* = \frac{\lambda n}{1-\phi}$
  - Long-run growth rate increasing in population growth rate (↑pop, ↑ R&D investment, permanently grow faster)
    - Intuitively, need positive population growth to sustain growth of Y/L because diminishing returns to knowledge production
    - Also consider demand side: larger market, more incentives for ideas
  - Policies only have level effects, not growth effects



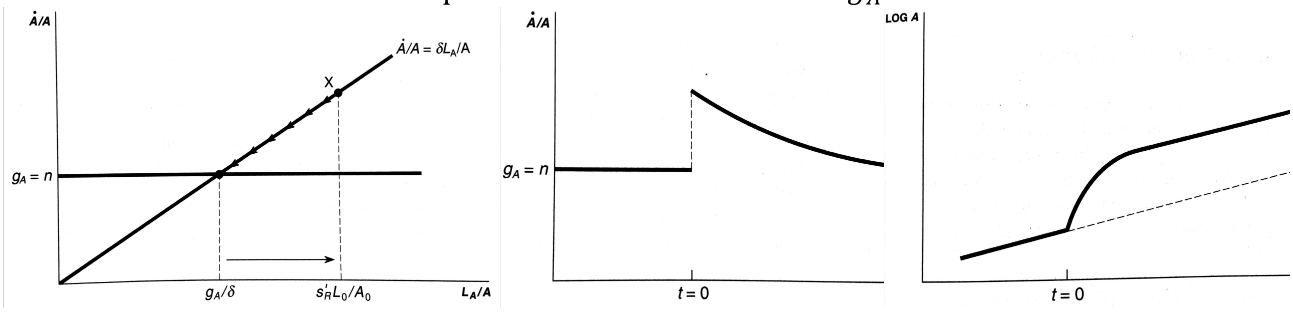
6.3.3 *DYNAMICS*

- Model permanent increase in the share of inputs devoted to domestic R&D as $s_R' > s_R$

- For simplicity, but without losing generality, assume $\phi = 0, \lambda = 1$
  - $\dot{A} = \delta s_R L$; $g_A = \frac{\dot{A}}{A} = \delta s_R \frac{L}{A}$; along BGP $g_y = g_k = g_A = \frac{\lambda n}{1-\phi} = n$

Comparative Statistics

- Short Run: ↑ s_R immediately ↑ g_A
  - With a population of L_0 number of researchers increases so ratio $\frac{L_A}{A}$ jumps up
  - Additional researchers produce more new ideas so ↑ g_A



- Long Run: Even permanent increase in s_R allocated to tech progress cannot raise growth
  - g_A exceeds n (or in general case $\frac{\lambda n}{1-\phi}$). Thus ratio $\frac{L_A}{A}$ declines over time and with it g_A
  - Continues until return to BGP. But.. permanent level effect…
- Empirically, despite huge changes in R&D (Gordon, 2016) growth stable at 1.8%. Suggests governments should not be active agent

Evaluation Points:

- Because A is non-rivalrous and hard to regulate we expect "free flow" and hence treat as global. This has two key implications:
  - Scale effect, whereby a larger economy provides a larger market for non-rivalrous ideas, raising return to research (a demand effect). Hence should eliminate barriers.
  - Developing country not on "technology frontier" it should not engage in R&D and instead rely on 'transfers' from advanced economies (can afford "breakthroughs")

7 LIMITS TO GROWTH

7.1 BACKGROUND

- Long-existing notion in literature
  - Malthusian Hypothesis (1798) and Ehrlich (1968) identified fertility/"The Population Bomb"
  - Club of Rome (1968) identified non-renewable resources (esp. rising oil prices)
- Nordhaus (1992): "Because boys have mistakenly cried "wolf" in the past does not mean that the woods are safe"

- Climate change poses particular challenges:
  - Global externality: Independent of where it is emitted
  - Temporal externality: Unborn generations are also affected
  - Uncertainty about its impact
- Gordon (2016): Notion that growth is persistent has only been true of last 250y. Appears to be slowing down.

- Innovation is discrete process where have General Purpose Technology followed by incremental improvements. Many of these can only happen once
    - 1750-1830: steam engine; cotton gin; railroads
    - 1870-1900: electric light; internal combustion engine; telephone
    - 1960-today: electronic computers; web

7.2 MODEL

7.2.1 GENERAL

- Production: $Y(t) = K(t)^{\alpha} E(t)^{\beta} \big(A(t)L(t)\big)^{1-\alpha-\beta}$
- Capital: $\dot{K}(t) = sY(t) - \delta K(t)$
- Technical change and population growth: $\frac{\dot{A}(t)}{A(t)} = g; \frac{\dot{L}(t)}{L(t)} = n$
- Rearranging terms:
    - $\frac{\dot{K}}{K} = s\frac{K^{\alpha}E^{\beta}(AL)^{1-\alpha-\beta}}{K} - \delta$
    - $g_K + \delta = sK^{\alpha-1}E^{\beta}(AL)^{1-\alpha-\beta}$
    - Can show $g_Y = g_K; g_y = g_k; g_x = g_X - n$ via standard BGP

7.2.2 LAND (I.E. FIXED RESOURCE)

- Let $E(t) = \bar{T}$, that is fixed
- Taking logs, differentiating wrt time and considering BCG:
    - $g_K + \delta = sK^{\alpha-1}E^{\beta}(AL)^{1-\alpha-\beta} \quad 0 = (\alpha - 1)g_K + (1 - \alpha - \beta)(n + g)$
    - $g_K = \frac{1-\alpha-\beta}{1-\alpha}(n + g)$
        - If $\beta = 0$ then $g_K = n + g$; If $\beta > 0$ then $g_K < n + g$
- Growth in capital per capita $g_k = g_K - n$
    - $g_k = \frac{1-\alpha-\beta}{1-\alpha}(n + g) - n = \frac{\beta}{1-\alpha}n + \frac{1-\alpha-\beta}{1-\alpha}g$
    - If $g \geq \frac{\beta}{1-\alpha-\beta}n$ then $g_k \geq 0$ If $g \leq \frac{\beta}{1-\alpha-\beta}n$ then $g_k \leq 0$
        - Note also for income as $g_k = g_y$
- Interpret:
    - Population pressure on the fixed resource leads the marginal product of labour to fall and even accumulation of capital cannot fully offset this effect
    - Technological progress has potential to offset these effects and lead to sustained growth in per capita income
    - Hence 'growth' depends on size of tech progress relative to population growth and importance of land (↑ importance, sharper diminishing returns, ↓ growth)
    - Expect land price to rise as it becomes more valuable (i.e. overtime)

7.2.3 NON-RENEWABLES (I.E. DEPLETED RESOURCE)

- Let $E(t)$ be characterized as follows:
    - Initial stock R_0 so $\dot{R}(t) = -E(t)$
    - Extraction rate constant s_E (Dasgupta and Heal, 1974)
    - Thus $E = s_E R$ (use constant fraction of stock left) so $s_E = \frac{E}{R}$ and $\dot{R} = -s_E R$ (stock depleted by that amount used)
    - Resources will eventually be depleted $E = s_E R_0 e^{-s_E t}$ so $\lim\limits_{t\to\infty} E = 0$
- Taking logs both sides and differentiate with respect to time. Considering BGP:

- $g_K + \delta = sK^{\alpha-1}E^{\beta}(AL)^{1-\alpha-\beta} \quad 0 = (\alpha-1)g_K + \beta(-s_E) + (1-\alpha-\beta)(n+g)$
- $g_K = \frac{1-\alpha-\beta}{1-\alpha}(n+g) - \frac{\beta}{1-\alpha}s_E$
  - If $\beta = 0$ then $g_K = n + g$

- Growth in capital per capita $g_k = g_K - n$
  - $g_k = \frac{1-\alpha-\beta}{1-\alpha}g - \frac{\beta}{1-\alpha}(n + s_E)$
  - If $g \geq \frac{\beta}{1-\alpha-\beta}(n+s_E)$ then $g_y = g_k \geq 0$ If $g \leq \frac{\beta}{1-\alpha-\beta}(n+s_E)$ then $g_y = g_k \leq 0$
- Interpret:
  - 'Growth' depends on size of technical progress relative to population growth rate and use of non-renewable resources
  - Similar to land, faster population growth leads to increased pressure on finite resource stick, reducing per capita growth
  - Increase in the depletion rate s_E reduces long-run growth rate of the economy. Fundamental trade-off between using energy today or in the future
    - One can raise the economy's LR growth by reducing depletion rate permanently and accepting a lower level of income in SR
  - Prices adjust to reflect scarcity of resources. Dasgupta & Heal (1974): Thus constant fraction of remaining stock of the energy is used in production each period $s_E = E/R$

7.2.4 NORDHAUS: EMPIRICAL EVIDENCE

- Instead of CRTS to K and L, production function now exhibits diminishing returns to K, L (i.e. excluding land and energy)
- Solving get $g_y = g - (\bar{\beta} + \bar{\gamma})n - \bar{\gamma}s_E$
  - Terms other than g is "growth drag" resulting from (1) population pressure on finite stock implying diminishing returns and (2) depletion of non-renewables
- US economy growth is about 0.3%-points lower than this (i.e. 15% reduction than possible)
- Note, assumes constant shares of factor of production. Appears true of labour but land and renewables is falling over time