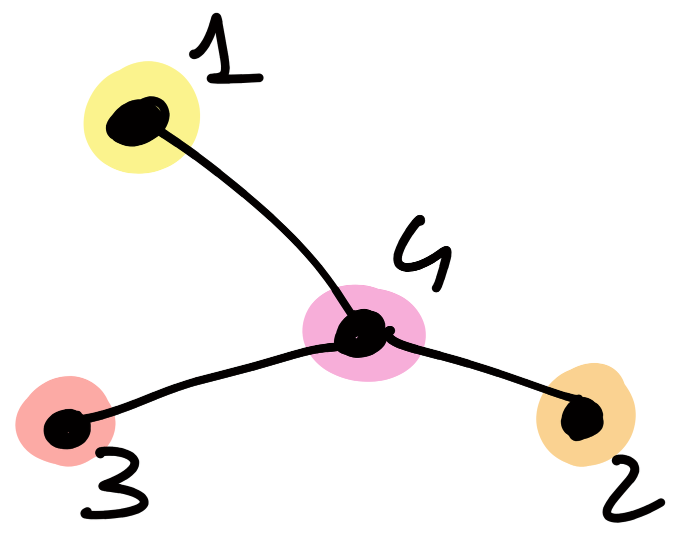
3. Challenge exercise: Counting number of walks using the adjacency matrix (5 pts) (pen and paper)

a) Draw the *induced subgraph* G^* that is induced by vertices $V^* = \{1 \dots 4\}$ of network visualized in Figure 1. Calculate by hand the number of walks of length two between all node pairs (i, j), $i, j \in \{1, \dots, 4\}$ in G^*. The length of a walk is defined as the number of links traveled to get from i to j; a link can be traveled in both directions and the walk can visit a node multiple times. Remember to consider also walks, where $i = j$.

Then, compute the matrix A^2 (you may do this also using a computer), where A is the adjacency matrix of the network G^*. Compare your results; what do you notice?

The induced subgraph G^*, induced by $V^* = \{1, 2, 3, 4\}$ is:



The number of walks of length = 2 between all node pairs (i,j), $i,j \in \{1,2,3,4\}$ in G^* are:

NODE 1	NODE 2	NODE 3	NODE 4
1-4-1	2-4-1	3-4-1	4-1-4
1-4-2	2-4-2	3-4-2	4-2-4
1-4-3	2-4-3	3-4-3	4-3-4

The adjacency matrix for the subgraph G^* is:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \text{ while } A^2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

It is possible to notice that each element a^2_{ij} of the matrix A^2 refers to the number of walks of length 2 from node i to node j

For example $a^2_{44} = 3$ means that there are 3 paths of length 2 that start and finish in node 4. The paths are $\begin{cases} 4-1-4 \\ 4-2-4 \\ 4-3-4 \end{cases}$

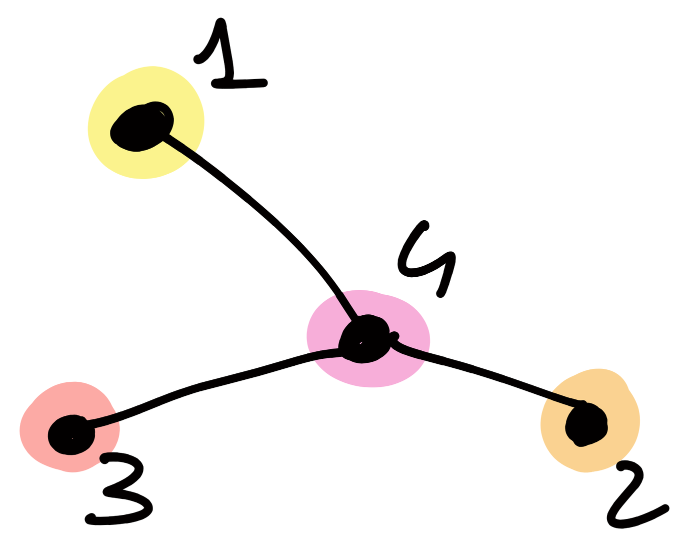
In general it's valid the following theorem

Th: "Let A be the adjacency matrix of a graph G.
The element a^{M}_{ij} of the matrix A^M, where M is a non-negative integer, corresponds to the number of walks of length M between nodes i and j in the graph G"

b) Compute the number of walks of length three from node 3 to node 4 in G^*. Then, starting from matrices A^2 and A, compute by hand the value of $(A^3)_{3,4}$ showing also the intermediate steps for computing the matrix element.

The number of walks of length = 3 from node 3 to node 4 in G^* are:

3-4-1-4
3-4-2-4
3-4-3-4



We can formally say that: $a^{m+1}_{ij} = \sum_{l=1}^{N} a^{m}_{il} \cdot a_{lj}$, where N is the number of nodes

Having

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$a^{3}_{3,4} = \sum_{l=1}^{N} a^{2}_{3l} \cdot a_{l4} = 0\cdot 0 + 0\cdot 0 + 0\cdot 0 + 1\cdot 3 = 3$$

So, $a^{3}_{3,4} = 3$, meaning that there are 3 paths of length = 3 that start in node 3 and end in node 4. The paths, as found out before, are

$$\begin{cases} 3-4-1-4 \\ 3-4-2-4 \\ 3-4-3-4 \end{cases}$$

c) Now, let's consider a general network with adjacency matrix A. Show that the element $(A^m)_{i,j}$, $m \in \mathbb{N}$ corresponds to the number of walks of length m between nodes i and j.

Hint: Make use of mathematical induction: Show first that the statement holds for $m = 1$ by analyzing the elements of the matrix A^1. Next, assume that the statement holds for a general m and prove that it holds also for $m + 1$. To do that, consider the element $a_{i,j}^{(m+1)} (= (A^{m+1})_{i,j})$ assuming that $a_{i,j}^{(m)}$ gives the number of walks of length m.

BASE CASE, $m=1$

We want to show that a^1_{ij} corresponds to the number of walks of length 1 between nodes i and j.

The matrix A^1 is simply the adjeceny matrix, so a^1_{ij} represents the directed connection between nodes i and j.

If there is an edge between i and j, then $a_{ij} = 1$, indicating that there is one walk of length 1 between i and j.

If there is no edge between i and j, then $a_{ij} = 0$, indicating that there are no walks of length 1 between them

INDUCTIVE STEP

Now let's assume that the statement holds for a general m, i.e. a^m_{ij} corresponds to the number of walks of length m between nodes i and j.

In order to prove that it holds for $m+1$, i.e. a^{m+1}_{ij} corresponds to the number of walks of length $m+1$ between nodes i and j

Starting from the definition

$$a^{m+1}_{ij} = \sum_{l=1}^{N} a^m_{il} \cdot a_{lj} = a^m_{i1} \cdot a_{1j} + a^m_{i2} \cdot a_{2j} + \dots + a^m_{iN} \cdot a_{Nj}$$

We can notice that the terms in the sum represents the number of walks of length m from node i to each of its neighbors, and then continuing with an edge from that neighbor to node j.

Therefore, the sum of all thes terms represents the total number of walks of length $m+1$ from node i to node j, because it includes all possible ways to reach node j from node i with an additional step.

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