# Patterns in the sine map bifurcation diagram

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# Patterns in the sine map bifurcation diagram

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## INTRODUCTION

A onedimensional map $f$ is defined here as any $f : [a, b] \to [a, b]$. The logistic map $\phi_r$

$$\phi_r : [0, 1] \to [0, 1], \phi_r(x) = rx(1 - x) \qquad (1)$$

is well known for it's chaotic properties. As $r$ is varied through different intervals, the logistic map goes from having a fixed point to having stable cycles of different orders $n$ (there is at least an $x$ so that $\phi_r^n(x) = x$, where the exponent refers to composition).

This type of behaviour is generic for a large number of chaotic systems, and it is best viewed in a bifurcation diagram, see figure 1. The initial fixed point degenerates into a cycle with period 2, and then gradually the period of the stable cycle keeps doubling, up to a point where the map becomes fully chaotic. A reference paper discussing the universality of the period doubling cascade is [1]. One obvious feature of this bifurcation diagram is that patterns can be observed in it — discontinuities in the density of points.
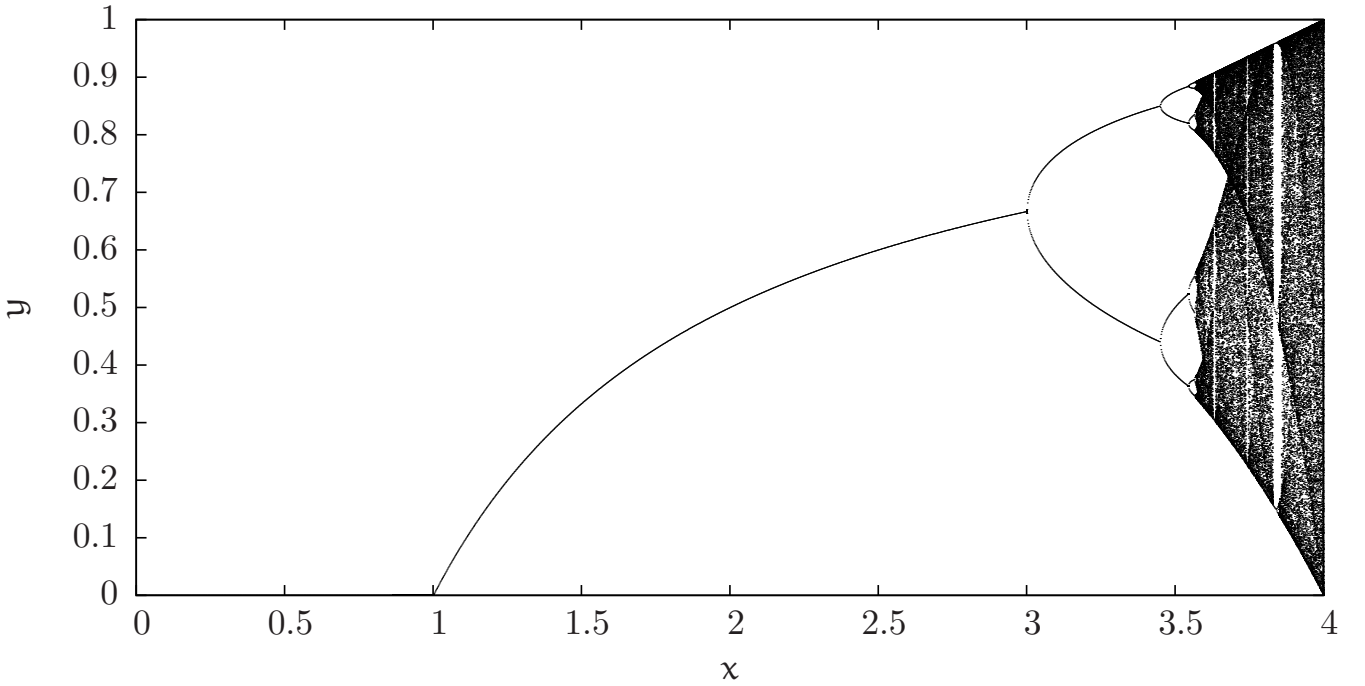


FIG. 1. Bifurcation diagram for the logistic map

## SINE MAP

Consider the map

$$\xi_r : \mathbb{R} \to \mathbb{R}, \xi_r(x) = r \sin(x) \qquad (2)$$

This map generates a bifurcation diagram that is symmetric with respect to both axis of the $(r, x)$ plane. A twodimensional map can be defined:

$$\psi : \mathbb{R}^2 \to \mathbb{R}^2, \psi(x, y) = (x, x \sin(y)) \qquad (3)$$

The bifurcation diagram (viewed as a subset of $\mathbb{R}^2$) is simply the largest subset of $\mathbb{R}^2$ that is invariant to $\psi$:

$$\mathcal{B} = \lim_{n \to \infty} \psi^n(\mathbb{R}^2) \qquad (4)$$
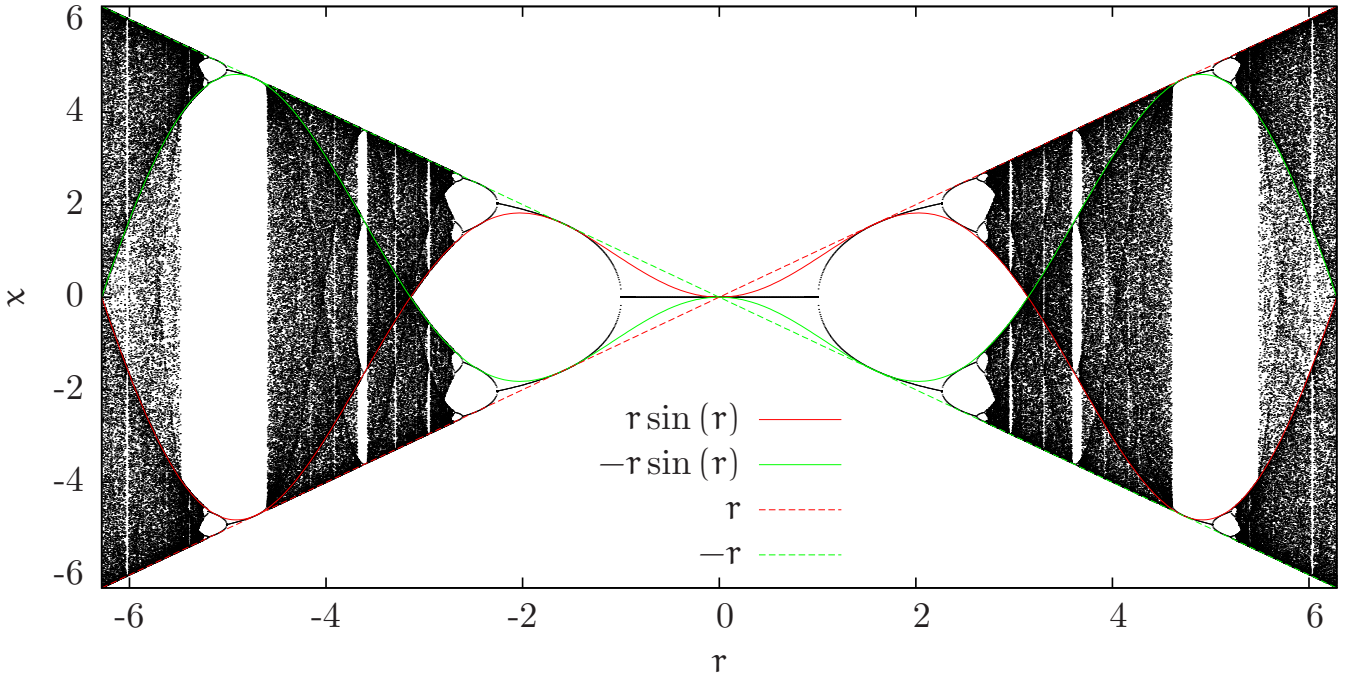
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FIG. 2. Bifurcation diagram for the sine map, for $r \in [-2\pi, 2\pi]$.

## PATTERNS

It can be seen directly in figure 2 that the following two functions appear as patterns:

$$\pm r \sin(r), \text{ with the graphs given by } \psi(\pm r, r) \quad (5)$$

More generally, the bifurcation diagrams suggest that the array of curve pairs

$$C^{(n)} = \{\psi^n(\pm x, x) | x \in \mathbb{R}\} \quad (6)$$

is in fact the succession of all patterns. The convention $\psi^0(x, y) \equiv (x, y)$ is adopted here, so $C^{(0)} = \{(x, x), (-x, x) | x \in \mathbb{R}\}$. Figure 3 further suggests that $C^{(n)} \cap C^{(m)} \subset \mathcal{B}$, for all $n \neq m \in \mathbb{N}$.

## EXACT RESULTS

Let $(a, b) \in C^{(n)} \cap C^{(m)}$. It is convenient to introduce the notation $\psi_{\pm}^n(x, y) \equiv \psi^n(\pm x, y)$. This means that

$$(a, b) \in \{\psi_{\pm}^n(a, a), \psi_{\pm}^m(a, a)\} \quad (7)$$

Consider the case $(a, b) = \psi_+^n(a, a) = \psi_+^m(a, a)$. Assume $m > n$, and $m - n = p$. This means that

$$\psi_+^n(a, a) = \psi_+^p(\psi_+^n(a, a)) \quad (8)$$

so, by definition, $\psi_+^n(a, a) \equiv (a, b)$ is part of a cycle with period $p$, which means it is in $\mathcal{B}$. The alternative is that

$\psi_+^n(a, a) = \psi_-^m(a, a)$. This means that

$$\psi_+^n(a, a) = \psi_+^p(\psi_-^n(a, a)) \quad (9)$$
$$\psi_-^p(\psi_+^n(a, a)) = \psi_-^p(\psi_+^p(\psi_-^n(a, a))) \quad (10)$$
$$\psi_+^p(\psi_-^n(a, a)) = \psi_+^p(\psi_+^p(\psi_+^n(a, a))) \quad (11)$$
$$\psi_+^n(a, a) = \psi_+^{2p}(\psi_+^n(a, a)) \quad (12)$$

so, again by definition, $(a, b)$ is part of a cycle with period $2p$, so $(a, b) \in \mathcal{B}$.

Concerning $\lim_{n \to \infty} C^{(n)}$, it is sufficient to remark that $C^{(n)} = \psi^n(C^{(0)}$, so the limit $C^{(\infty)}$ is a subset of $\mathcal{B}$.

## GENERALIZATION

Consider a general mapping

$$\theta : \mathbb{R}^2 \to \mathbb{R}^2, \theta(x, y) = (x, x f(y)) \quad (13)$$

For $f : \mathbb{R} \to [-1, 1]$ (with $f(\mathbb{R}) = [-1, 1]$), the invariant set of $\theta$ is well defined as

$$\mathcal{D} \equiv \lim_{n \to \infty} \theta^n (\mathbb{R}^2) \quad (14)$$

and the reasoning from the previous section can be applied. Call $\theta_\pm(x, y) \equiv \theta(\pm x, y)$ and

$$D^{(n)} \equiv \{\theta_\pm^n(x, x) | x \in \mathbb{R}\} \quad (15)$$

Then for any $n \neq m \in \mathbb{N}$, $D^{(n)} \cap D^{(m)} \subset \mathcal{D}$ and the limit $D^{(\infty)}$ is a subset of $\mathcal{D}$.
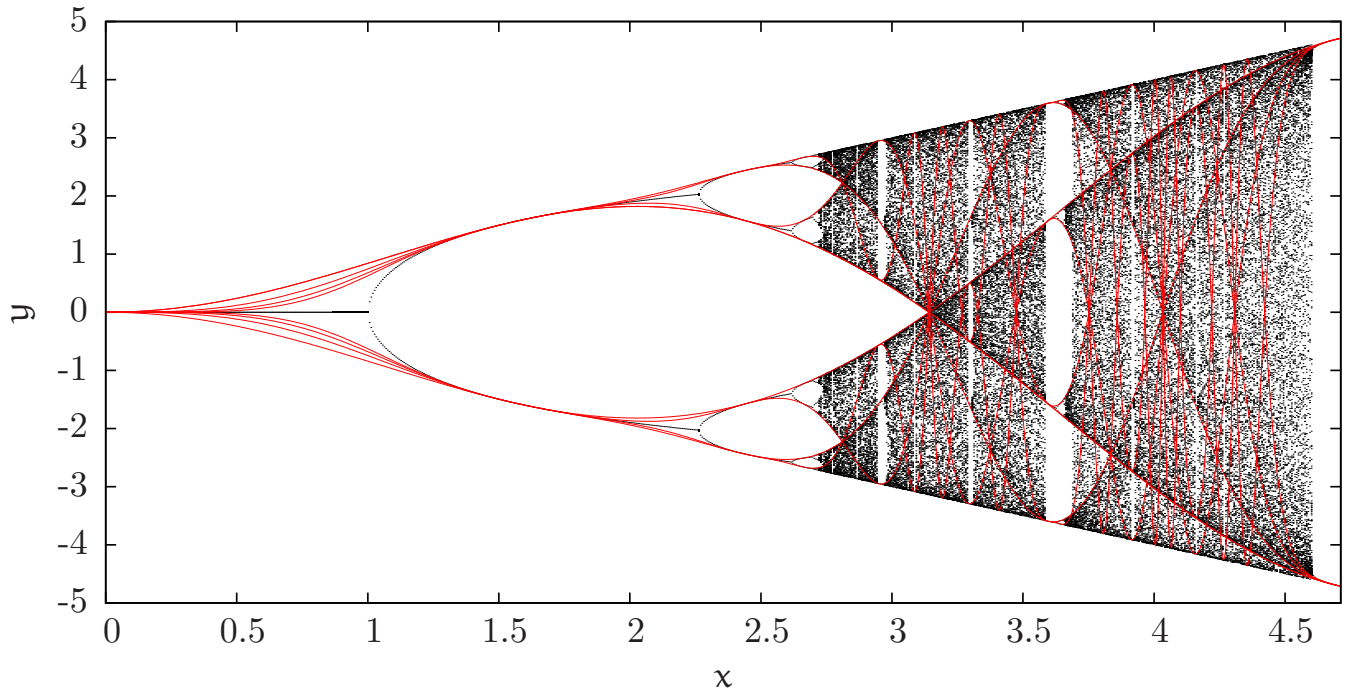
FIG. 3. The curve pairs $C^{(n)}$ ($n = 1, 2, 3, 4$) superposed over the bifurcation diagram.

## NOTES

The author would like to thank Gy. Steinbrecher and D. Constantinescu from the University of Craiova for their valuable comments.

The author is unaware if the ideas presented are new; the webpage [2] does discuss a family of curves that appears to have the properties mentioned in this work. The figures in this article were generated with a simple Python script[1].

[1] Mitchell J. Feigenbaum. Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics*, 19:25–52, 1978. 10.1007/BF01020332.
[2] Evgeny Demidov. http://www.ibiblio.org/e-notes/Chaos/bifurcat.htm

[1] http://chichi.lalescu.ro/files/bc.py