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A New Discrete-Time Model for a van del Pol Oscillator

Triet Nguyen Van

Dept. of Intelligent Interaction Technologies
University of Tsukuba
Tsukuba, Japan
nvtriet@digicon-lab.esys.tsukuba.ac.jp

Noriyuki Hori

Dept. of Intelligent Interaction Technologies
University of Tsukuba
Tsukuba, Japan
hori@iit.esys.tsukuba.ac.jp

Abstract— This paper proposes a new discretization model for a second-order nonlinear system whose dynamics are governed by an ordinary differential equation of a van der Pol type, for which an analytical solution is not known. The method is based on a linear-like expression of the nonlinear system and its discrete-time system expression in delta form, where continuous-time system parameters appear directly and discrete-time parameters are contained in the integrator gains. These gains are, in general, functions of continuous-time parameters and the sampling interval for linear systems, but also of system states and inputs for nonlinear systems. The developed model becomes an exact discrete-time model as the van der Pol equation approaches a linear equation. Simulations show that the proposed model gives limit cycles that are more accurate than those of the Mickens and forward-difference models, which can lose limit cycles and numerical stability.

Keywords - van der Pol Oscillator, discretization, limit cycle, discrete-time integrator gains.

I. INTRODUCTION

A van der Pol oscillator has been a subject of extensive research and its discrete-time expressions play an important role in its numerical investigations. The simplest discretization method is the forward-difference model, whose form and parameter values are chosen to be the same as those of the continuous-time equation and only the differentiation is replaced with its discrete-time equivalent. Although very simple and convenient for purposes of designing and analyzing digital control systems, the accuracy of this model is usually poor even for linear cases unless a high sampling frequency is used [1]. This is true also for the design of nonlinear digital control systems that is based on the forward-difference model [2]. A better model called the non-standard discretization method has been proposed by Mickens [3], which uses non-local discretization grids and is applicable to a wide range of nonlinear systems. This method uses constant gains based on the linear portion of the nonlinear equation and has been shown through simulations to have better performances than the forward-difference model. However, a care has to be exercised in determining the order in which state equations are updated. This formulation also makes the

relationship between discrete-time and continuous-time systems less clear.

An approach that is based on the exact discretization of a linear system has been presented for the first-order system governed by a differential Riccati equation [4]. This is based on the use of the discrete-time integrator gain expressed in delta-operator form, where continuous-time system-parameters appear directly and discrete-time parameters are included in the integrator gains. This formulation is independent of the order in which equations are updated and is easier to relate the discrete-time model to the continuous-time original. The integrator gain is a function of continuous-time parameters and a sampling interval for linear systems, but also of system states and an input for nonlinear systems. The choice of the gain is important in creating accurate discrete-time models.

Extensions of the gain in exact discretization methods [4] to non-exact cases have appeared [5],[6],[7], but need further study, especially second and higher-order systems. Based on the idea of linear-like formulation and discrete-time integrator gains, a new discretization method is proposed in the present study. By allowing the gains to be updated at every sampling instant, the accuracy of the discrete-time model is expected to be better than the Mickens and forward-difference models and to increase as the degree of nonlinearity increases.

Section 2 briefly reviews the discrete-time expression in delta form and Section 3 the exact discrete-time model for linear systems. Section 4 explains the proposed discretization method with particular reference to a van der Pol system and Section 5 evaluates the proposed model through simulations to show its numerical performances to be superior to the Mickens and the forward-difference methods in terms of limit cycle fidelity and numerical stability.

II. A DISCRETE-TIME SYSTEM EXPRESSED IN DELTA FORM

Let a continuous-time system be given by the following state space equation:

$$\dot{x} = f(x, u), \quad (1)$$

where f is, in general, a nonlinear function of state x and input u . It is assumed that this system is to be under digital control and thus the input u is piecewise-constant through a zero-order hold with the time period of T . To discuss discrete-time models of (1), let a discrete-time system be expressed [4] as

$$\delta x_k = \Gamma f(x_k, u_k), \quad (2)$$

where x_k and u_k are discrete-time versions of the state and the input, the discrete-time operator is defined as

$$\delta = \frac{q-1}{T}, \quad (3)$$

with q being the usual shift-left operator such that $qx_k = x_{k+1}$. Γ is called the discrete-time integrator gain and is a function of system parameters, time period T , as well as the state x_k and the input u_k , in general. When the discrete-time integration gain satisfies the condition given by

$$\lim_{T \rightarrow 0} \Gamma = I, \quad (4)$$

system (2) is said to be a discrete-time model of the continuous-time system (1).

For a linear system given by

$$\dot{x} = Ax + Bu, \quad (5)$$

a large number of discrete-time models are known [1]. Of those, the exact discrete-time model, whose state matches that of the continuous-time original exactly at discrete-time instants for any sampling interval, is well known. Such a model is given by [5]

$$\delta x_k = \Gamma_A (Ax_k + Bu_k), \quad (6)$$

where the discrete-time integrator gain Γ_A is given by

$$\Gamma_A = \frac{1}{T} \int_0^T e^{A\tau} d\tau, \quad (7)$$

which approaches unity as T approaches zero. In this case, the gain is an average of state transition matrix between successive sampling instants and is constant for given system parameters and sampling interval T . When A is non-singular, the gain can be calculated as

$$\Gamma_A = \frac{e^{AT} - I}{T} A^{-1}, \quad (8)$$

where I is the identity matrix. For nonlinear systems, however, the integrator gain Γ is a function of x_k and u_k , in addition to T and system parameters contained in f .

III. AN EXACT DISCRETE-TIME MODEL FOR LINEAR SYSTEMS

The method of discretization to be introduced for a van der Pol oscillator is based on the following second-order linear system. In particular, consider the linear system given by (5) where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = 0 \quad (9)$$

with a_{ij} being constant. Written in explicit form, it is given by

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y = f(x, y) \\ \dot{y} &= a_{21}x + a_{22}y = g(x, y). \end{aligned} \quad (10)$$

Its exact discrete-time model for the general case is given by (6). When the eigenvalues λ_1 and λ_2 of A are distinct, it can be shown that

$$\begin{aligned} \Gamma_A A &= \frac{1}{T} \int_0^T e^{A\tau} d\tau \\ &= \phi A + \varphi I = A \Gamma_A \end{aligned} \quad (11)$$

where ϕ and φ are scalars defined by

$$\phi = \frac{\lambda_1 \gamma_{\lambda_1} - \lambda_2 \gamma_{\lambda_2}}{\lambda_1 - \lambda_2} \rightarrow 1 \text{ as } T \rightarrow 0 \quad (12)$$

$$\varphi = \frac{\lambda_1 \lambda_2 (\gamma_{\lambda_2} - \gamma_{\lambda_1})}{\lambda_1 - \lambda_2} \rightarrow 0 \text{ as } T \rightarrow 0 \quad (13)$$

and

$$\gamma_{\lambda_i} = \frac{e^{\lambda_i T} - 1}{\lambda_i T}. \quad (14)$$

Using these notations, the exact model given by (6) and (11) can be expressed as

$$\begin{aligned} \delta x_k &= \phi f(x_k, y_k) + \varphi x_k \\ \delta y_k &= \phi g(x_k, y_k) + \varphi y_k, \end{aligned} \quad (15)$$

which approaches the continuous-time equation (10) as $T \rightarrow 0$. In this expression, ϕ and φ are the discrete-time integration gains.

IV. A DISCRETE-TIME MODEL FOR VAN DEL POL OSCILLATOR

The van der Pol equation is given by

$$\ddot{x} = -x + \varepsilon(1 - x^2)\dot{x}, \quad (16)$$

which can be written in a form of

$$\begin{aligned} \dot{x} &= y = f(x, y) \\ \dot{y} &= -x + \varepsilon y - \varepsilon x^2 y = g(x, y) \end{aligned} \quad (17)$$

where ε is a positive parameter that characterizes a degree of nonlinearity.

A. The Proposed Model

Although (17) is a nonlinear equation, it can be written in a seemingly linear form as

$$\dot{x} = A(x)x \quad (18)$$

where the state dependent system matrix is

$$A(x) = \begin{bmatrix} 0 & 1 \\ -1 - (1 - \mu)\varepsilon xy & \varepsilon(1 - \mu x^2) \end{bmatrix} \quad (19)$$

with μ being a design parameter that determines how the contribution of nonlinear term $-\varepsilon x^2 y$ is spread between

variables x and y in the second equation of (17). In a symbolic form, its characteristic polynomial may be defined as

$$\det(\lambda A - I) = \lambda^2 - \varepsilon(1 - \mu x^2)\lambda + \{1 + (1 - \mu)\varepsilon xy\} \quad (20)$$

and its eigenvalues as

$$\lambda_{1,2} = \alpha \pm j\beta \quad (21)$$

where α and β are given by

$$\alpha = \frac{\varepsilon}{2}(1 - \mu x^2) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (22)$$

$$\beta = \sqrt{1 + (1 - \mu)\varepsilon xy - \frac{\varepsilon^2}{4}(1 - \mu x^2)^2} \rightarrow 1 \text{ as } \varepsilon \rightarrow 0. \quad (23)$$

The role of these eigenvalues when ε is nonzero in determining the behavior of the solution is not clear. However, response behaviors near the limit of ε being zero should be reflected on this set of values and it is worth studying this form of nonlinearity expression to see how the accuracy of discrete-time models obtained from such studies would be.

With this in mind, let α_k and β_k be the discrete-time version of (22) and (23), i.e.,

$$\alpha_k = \frac{\varepsilon}{2}(1 - x_k^2) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (24)$$

$$\beta_k = \sqrt{1 + (1 - \mu)\varepsilon x_k y_k - \frac{\varepsilon^2}{4}(1 - \mu x_k^2)^2} \rightarrow 1 \text{ as } \varepsilon \rightarrow 0. \quad (25)$$

where states x and y are replaced with x_k and y_k . It can be shown then, assuming $\beta \neq 0$, that (12) and (13) can be rewritten as

$$\phi_k = \frac{\sin(\beta_k T)}{\beta_k T} e^{\alpha_k T} \rightarrow 1 \text{ as } T \rightarrow 0 \quad (26)$$

$$\varphi_k = \frac{e^{\alpha_k T} \{\cos(\beta_k T) - \frac{\alpha_k}{\beta_k} \sin(\beta_k T)\} - 1}{T} \rightarrow 0 \text{ as } T \rightarrow 0. \quad (27)$$

The proposed discrete-time model is based on the exact model for the linear system (15) and is given by

$$\begin{aligned} \delta x_k &= \phi_k f(x_k, y_k) + \varphi_k x_k \\ \delta y_k &= \phi_k g(x_k, y_k) + \varphi_k y_k, \end{aligned} \quad (28)$$

which approaches the van der Pol equation (17) as $T \rightarrow 0$. When parameter ε is zero, this model becomes linear and is the exact discrete-time model.

B. Mickens Model

Mickens model proposed in [2] has a form similar to (28), but with constant discretization gain and the state x_{k+1} appearing on the right-hand-side of the second state equation. This model is given by

$$\begin{aligned} \delta x_k &= \phi y_k + \varphi x_k \\ \delta y_k &= \phi \{-x_k + \varepsilon(1 - x_{k+1}^2)y_k\} + \varphi y_k, \end{aligned} \quad (29)$$

where ϕ and φ are constant gains, which are obtained by

setting $\alpha = \frac{\varepsilon}{2}$ and $\beta = \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2}$ in (26) and (27), as

$$\phi = \frac{e^{\frac{\varepsilon T}{2}} \left\{ \cos \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} T - \frac{\varepsilon \sin \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} T}{2 \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2}} \right\} - 1}{T} \quad (30)$$

$$\varphi = e^{\frac{\varepsilon T}{2}} \frac{\sin \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} T}{\sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} T}. \quad (31)$$

C. Forward Difference Model

The forward difference model is obtained from (17) by replacing the differentiator with the delta operator on the left-hand-side, as

$$\begin{aligned} \delta x_k &= f(x_k, y_k) \\ \delta y_k &= g(x_k, y_k), \end{aligned} \quad (32)$$

which is (28) with $\phi_k = 1$ and $\varphi_k = 0$.

V. SIMULATION RESULTS

Simulations have been carried for the van der Pol equation (16) for 10,000 iterations; i.e., from 0 to 1,000 seconds with $T = 0.1$ s. In all these runs, it was found that the proposed discrete-time model consistently gave results closer to the van der Pol model than the forward-difference and Mickens models. Fig. 1 shows the phase plane of the continuous-time model, which is obtained using the Runge-Kutta method, and three discrete-time models, which are the forward-difference, the Mickens, and the proposed models, for $\varepsilon = 0.1$, $\mu = 1$, and $T = 0.1$ s, all starting from the initial condition of $x_0 = 1.5$ and $y_0 = 0.5$. For relatively small values of ε , both the Mickens and proposed discrete-time models yield the phase plane plots that are fairly close to the continuous-time plot. However, the forward-difference model does not produce a limit cycle.

As the value of ε increases, i.e., as a degree of nonlinearity increases, the accuracy of the Mickens model starts to deteriorate. Fig. 2(a) shows the phase planes for $\varepsilon = 1.5$, $T = 0.1$ s, and the initial condition of $x_0 = 1.5$ and $y_0 = 0.5$, which is inside the limit cycle. It can be seen from these plots that the forward-difference model is inaccurate and the proposed model gives a trace that is much closer to the continuous-time model than Mickens model. Fig. 2(b) is for the same system but with a different initial condition of $x_0 = 5.5$; i.e., outside the limit cycle. In both Figs. 2(a) and 2(b), both Mickens and the proposed methods yield limit cycles. However, Mickens model yields larger deviations.

When ε is increased to $\varepsilon=1.55$, as shown in Fig. 3(a) and 3(b), even Mickens model does not preserve a limit cycle, whereas the proposed model still retains the general shape of the limit cycle. Fig. 4 shows the phase-plane plots for $\varepsilon=1.998$, where the proposed model is fairly accurate even when the Mickens model is numerically unstable. It may be observed that the performance of the proposed model is more or less invariant to changes in ε .

So far, design parameter μ , which determines how the nonlinear term $-\varepsilon x^2 y$ is split between x and y , is fixed at $\mu=1$, i.e., this term is included only on y . Fig. 5 shows the state trajectories of the continuous and discrete-time systems for different values of μ . It can be seen that as μ is increased, the limit cycle broadens along the x axis, while changes in y is rather small.

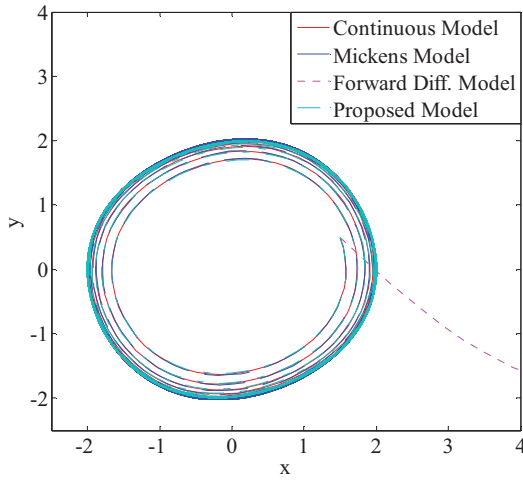


Fig. 1: Phase plane of the continuous-time, forward-difference, Mickens, and the proposed models for $\varepsilon = 0.1$, $\mu = 1$, $T = 0.1$ s, and the initial condition of $x_0 = 1.5$, and $y_0 = 0.5$.

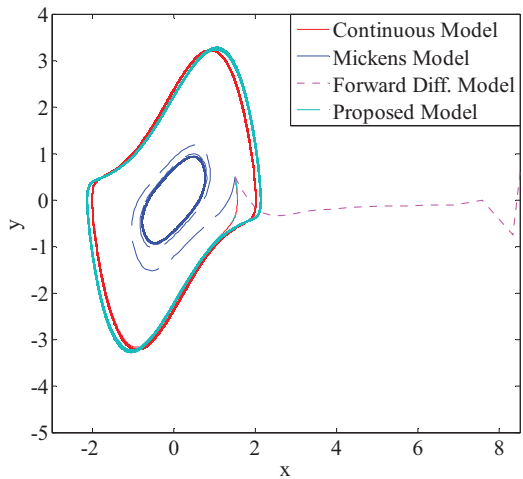


Fig. 2(a): Phase plane of the four models for $\varepsilon = 1.5$, $\mu = 1$, $T = 0.1$ s, $x_0 = 1.5$, and $y_0 = 0.5$.

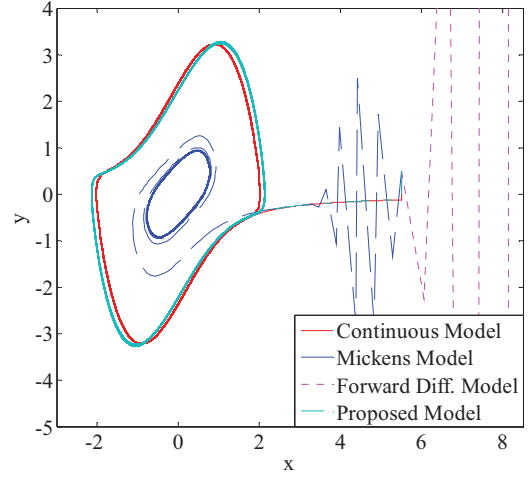


Fig. 2(b): Phase plane of the four models for $\varepsilon = 1.5$, $\mu = 1$, $T = 0.1$ s, $x_0 = 5.5$, and $y_0 = 0.5$.

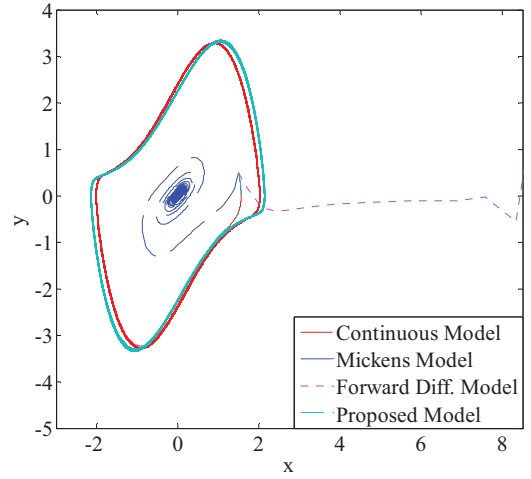


Fig. 3(a): Phase plane of the four models for $\varepsilon = 1.55$, $\mu = 1$, $T = 0.1$ s, $x_0 = 1.5$, and $y_0 = 0.5$

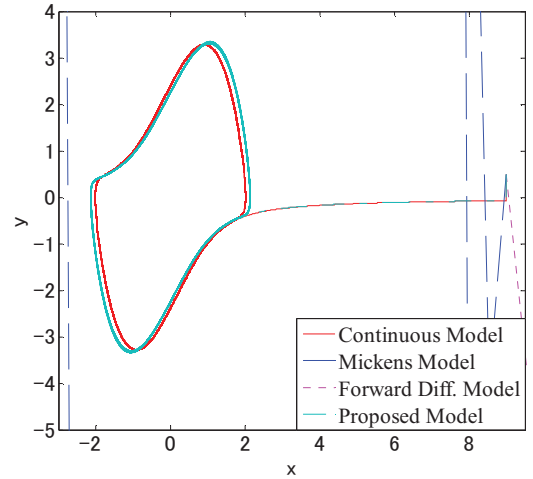


Fig. 3(b): Phase plane of the four models for $\varepsilon = 1.55$, $\mu = 1$, $T = 0.1$ s, $x_0 = 9.0$, and $y_0 = 0.5$.

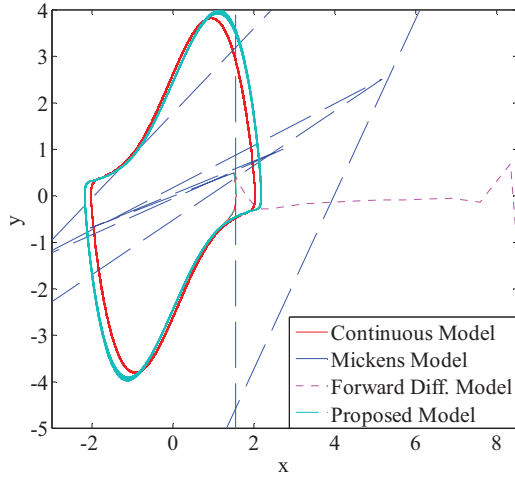


Fig. 4: Phase plane of the four models for $\varepsilon = 1.998$, $\mu = 1$, $T = 0.1$ s, $x_0 = 1.5$, and $y_0 = 0.5$.

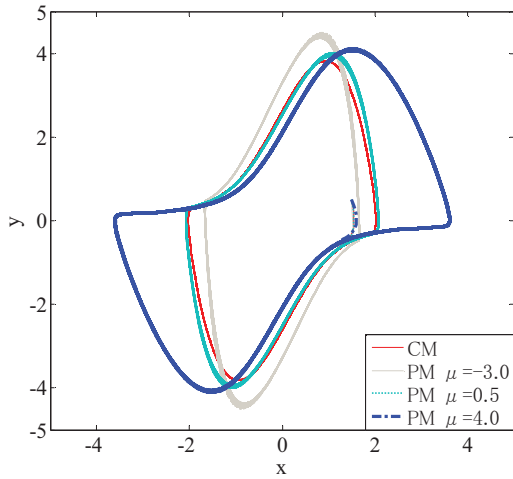


Fig. 5: Proposed model with different values of μ and continuous model when $\varepsilon = 1.5$, $T = 0.1$ s, and $x_0 = 1.5$, $y_0 = 0.5$.

To assess simulation results more quantitatively, a measure is introduced such that the difference between the continuous-time and the discrete-time responses at some representative points are taken into account. For example, the following performance index may be used based on the four particular points in the phase-plane:

$$\begin{aligned}
 PI = & \sqrt{(x_k - x)^2 + (\delta x_k - \dot{x})^2} \Big|_{\max\{x_k\}, \max\{x\}} \\
 & + \sqrt{(x_k - x)^2 + (\delta x_k - \dot{x})^2} \Big|_{\min\{x_k\}, \min\{x\}} \\
 & + \sqrt{(x_k - x)^2 + (\delta x_k - \dot{x})^2} \Big|_{\max\{\delta x_k\}, \max\{\dot{x}\}} \\
 & + \sqrt{(x_k - x)^2 + (\delta x_k - \dot{x})^2} \Big|_{\min\{\delta x_k\}, \min\{\dot{x}\}}
 \end{aligned} \quad (33)$$

Fig. 6(a) shows the performance index for the proposed models for $T=0.1$ s and a range of μ . It can be seen that this index is smaller with the proposed method than Mickens model and is minimal at around $\mu = 0.2$. As expected from Fig. 5, parameter μ does have an effect on the accuracy of the discrete-time model. Fig. 6(b) summarizes results of the same system for a range of discrete-time period T . It was found that the value of μ with which the performance index was minimal increased from 0.2 for $T=0.1$ s to about 1.0 for $T=0.4$, 0.7, and 1.0s. In general, the accuracy of the proposed model improves as the discrete-time period decreases. The response of the proposed model can diverge as the magnitude of μ increases and the range of μ for which stable responses would result shrinks as T increases; for instance, the response diverges for $\mu > 2.0$ when $T=1.0$ s.

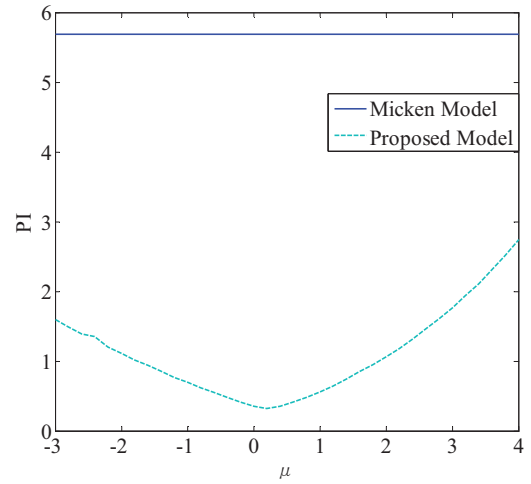


Fig. 6(a): Performance index for the proposed and Mickens models for different values of μ with $\varepsilon = 1.5$, $T = 0.1$ s, $x_0 = 1.5$, $y_0 = 0.5$.

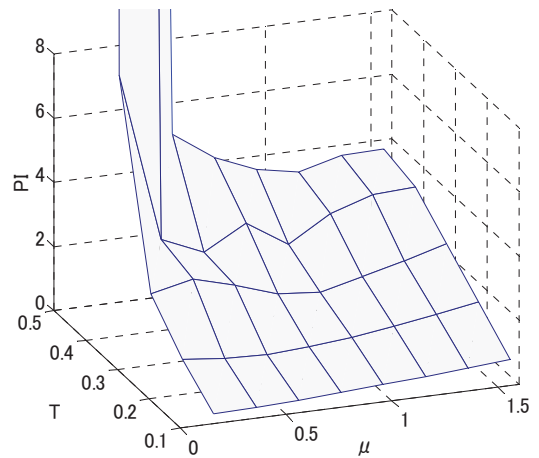


Fig. 6(b): Effect of T on the performance index.

Performances of discrete-time models depend on T and deteriorate as T becomes larger. Fig. 7 shows a phase-plane plot for $T = 0.5$ s. Although the performances of all discrete-time models are poor, the proposed model still shows a limit-cycle-like trajectory.

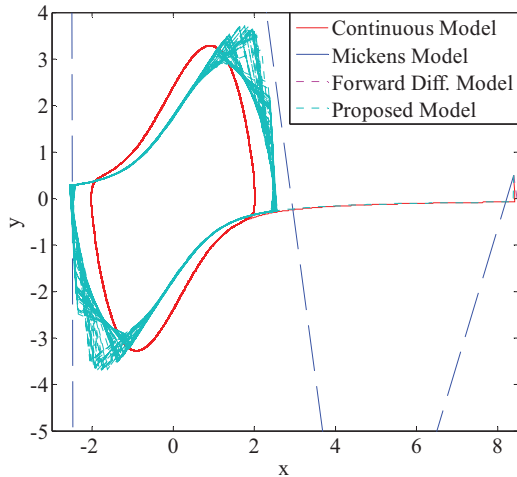


Fig. 7: Continuous, Mickens, forward diff. and proposed model when $\varepsilon = 1.55$, $\mu = 1$, $T = 0.5$ s, $x_0 = 8.4$, and $y_0 = 0.5$.

VI. CONCLUSIONS

The proposed discrete-time model, which used the linear-like formulation with state-dependent system-matrix, was found to retain a limit-cycle even for large values of parameter

ε , initial states, and discrete-time period, for which the simple forward-difference model and the accurate Mickens model could not. It is important to investigate further why and how this is so.

A preliminary simulation study indicates that a similar good performance can be obtained when the proposed method is applied to other nonlinear systems, such as Duffing and Lewis oscillators. It is expected, therefore, that the proposed method has good potential to be a useful tool for creating accurate discrete-time models for other classes of nonlinear systems.

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