1.4 The Doubling Map: coding and conjugacy

Consider the following map on $[0,1)$, known as *doubling map*:

$$f(x) = 2x \mod 1 = \begin{cases} 2x & \text{if } 0 \le x < 1/2, \\ 2x-1 & \text{if } 1/2 \le x < 1 \end{cases} \tag{1.1}$$

The graph is shown in Figure 1.1. The map is well defined also on $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$. To check

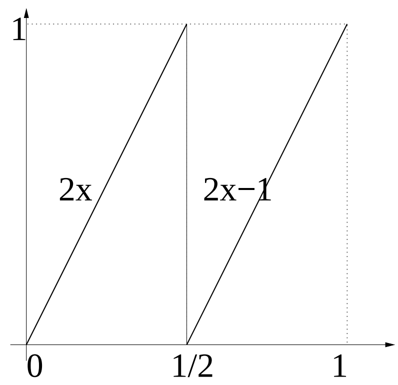

Figure 1.1: The graph of the doubling map.

that, we have to check that the points 0 and 1 which are identified have equivalent images. But this is true since $f(1) = f(0) = 0$. So we can think of f as a map on $\mathbb{R}/\mathbb{Z} = I/\sim$. Since we saw that $\mathbb{R}/\mathbb{Z}$ is identified with S^1 (via the correspondence given by $x \to e^{2\pi i x}$, see § 1.2), we can see f in multiplicative coordinates as a map from $S^1 \to S^1$ given by

$$f(e^{2\pi i\theta}) = e^{2\pi i 2\theta} = (e^{2\pi i\theta})^2. \tag{1.2}$$

Thus the angles are *doubled* and this explains the name *doubling map*. Moreover, one can see that the map f on S^1 is *continuous*.

Remark that f is *not* invertible: each point has two preimages:

$$f^{-1}(y) = \left\{ \frac{y}{2}, \frac{y}{2} + \frac{1}{2} \right\}.$$

Remark also that f *expands* distances. If $d(x,y) < 1/4$, then

$$d(f(x), f(y)) = 2d(x,y).$$

Let us try to answer the following questions for f:

Q. 1 Are there periodic points?

Q. 2 Are there points with a dense orbit?

In order to answer these questions, we will show two powerful techniques in dynamical systems, *conjugacy* and *coding*.

1.4.1 Conjugacy and semi-conjugacy

Let X, Y be two spaces and $f : X \to X$ and $g : Y \to Y$ be two maps.

Definition 1.4.1. *A* conjugacy *between f and g is an* invertible *map $\psi : Y \to X$ such that $\psi g = f\psi$, i.e. for all $y \in Y$*

$$\psi(g(y)) = f(\psi(y)).$$

Since ψ is invertible, we can also write $g = \psi^{-1} \circ f \circ \psi$.

The relation $\psi g = f\psi$ is often expressed by saying that the diagram here below *commutes*, i.e. one can start from a point in $y \in Y$ in the left top corner and indifferently apply first the arrow g and then the arrow ψ on the right side of the diagram or first the arrow ψ on the left side and then the arrow f and the result is the same point in the bottom right corner.

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi} \\ X & \xrightarrow{f} & X \end{array}$$

Lemma 1.4.1. *If f and g are conjugated by ψ, then y is a periodic point of period n for g if and only if $\psi(x)$ is a periodic point of period n for f.*

Exercise 1.4.1. *Check by induction that if $\psi g = f\psi$, then $\psi g^n = f^n\psi$. If ψ is invertible, we also have $g^n = \psi^{-1}g^n\psi$.*

Proof. Assume that $g^n(y) = y$. Then, by the exercise above

$$f^n(\psi(y)) = \psi(g^n(y)) = \psi(y),$$

so $\psi(y)$ is a periodic point of period n for f.

Conversely, assume that $f^n(\psi(y)) = \psi(y)$. Then since ψ is invertible, by the previous exercise we also have $g^n = \psi^{-1}g^n\psi$. Thus

$$g^n(y) = \psi^{-1}(f^n(\psi(y))) = \psi^{-1}(\psi(y)) = y,$$

so y is periodic of period n for g. □

Thus, if the periodic points of the map g are easier to understand than the periodic points of the map f, through the conjugacy one can gain information about periodic points for f. We will see that this is exactly the case for the doubling map.

Definition 1.4.2. *A* semi-conjugacy *between f and g is a* surjective *map $\psi : Y \to X$ such that $\psi g = f\psi$, i.e. for all $y \in Y$*

$$\psi(g(y)) = f(\psi(y)),$$

or, equivalently, such that the diagram below commutes:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi} \\ X & \xrightarrow{f} & X \end{array}$$

In this case we say that $g : Y \to Y$ is an extension *of $f : X \to X$ and that $f : X \to X$ is a* factor *of $g : Y \to Y$.*

Exercise 1.4.2. *If f and g are semi-conjugated by ψ and y is a periodic point of period n for g, then $\psi(x)$ is a periodic point of period n for f.*

Remark 1.4.1. *There are examples of f and g which are semi-conjugated by ψ and such that $\psi(y)$ is a periodic point of period n for f, but y is not a periodic point for g. We will see such an example using the baker map in a few classes.*

1.4.2 Doubling map: semi-conjugacy and binary expansions

We will define a semi-conjucagy between the doubling map and an abstract space that will help us understand points with periodic and dense orbits.

Given $x \in [0,1]$, we can express x in *binary expansion*, i.e. we can write

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

where x_i are digits which are either 0 or 1. Binary expansions are useful to study the doubling map because if we apply the doubling map:

$$f(x) = 2x \mod 1 = \sum_{i=1}^{\infty} 2\frac{x_i}{2^i} \mod 1 = x_0 + \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}} \mod 1 = \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}}$$

and if we now change the name of the index, setting $j = i - 1$, we proved that

$$\text{if } x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \quad \text{then} \quad f(x) = \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}} = \sum_{j=1}^{\infty} \frac{x_{j+1}}{2^j} \tag{1.3}$$

i.e. the binary expression of $f(x)$ is such that the digits are *shifted by* 1.

Let us construct a map on the space of digits of binary expansion which mimic this behavior.

Let $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ be the set of all sequences of 0 and 1:

$$\Sigma^+ = \{(a_i)_{i=1}^{\infty}, \quad a_i \in \{0,1\}\}.$$

The *points* $(a_i)_{i=1}^{\infty} \in \Sigma^+$ are one-sided sequences of digits 0,1, for example a point is

$$0,0,1,1,0,1,0,0,0,1,1,0,1,1,1,1,0,0,1,0,1,0\dots$$

The *shift* map σ^+ is a map $\sigma^+ : \Sigma^+ \to \Sigma^+$ which maps a sequence to the *shifted* sequence:

$$\sigma((a_i)_{i=1}^{\infty}) = (b_i)_{i=1}^{\infty}, \qquad \text{where} \quad b_i = a_{i+1}.$$

The sequence $(b_i)_{i=1}^{\infty}$ is obtained from the sequence $(a_i)_{i=1}^{\infty}$ by dropping the first digit a_1 and by shifting all the other digits one place to the left. For example, if

$$\begin{aligned}(a_i)_{i=1}^{\infty} &= 0,0,1,1,0,1,0,0,0,1,1,0,1,1,1,1,0,0,1,0,1,0\dots\\ (b_i)_{i=1}^{\infty} &= 0,1,1,0,1,0,0,0,1,1,0,1,1,1,1,0,0,1,0,1,0\dots\end{aligned}$$

Remark that σ^+ is not invertible, because if we know $\sigma((a_i)_{i=1}^{\infty})$ we cannot recover $(a_i)_{i=1}^{\infty}$ since we lost the information about the first digit.

Define the following map $\psi : \Sigma^+ \to [0,1]$. For each $(a_i)_{i=1}^{\infty} \in \Sigma^+$ set

$$\psi((a_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \in [0,1].$$

The map is well defined since the series $\sum_{i=1}^{\infty} \frac{a_i}{2^i} \leq \sum_{i=1}^{\infty} \frac{1}{2^i}$ which is convergent. The map ψ associates to a sequence of 0 and 1 a number in $[0,1]$ which has a_i as digits of the binary expansion.

One can see that ψ is surjective since each real $x \in [0,1]$ has a binary expansion (in the next section, §1.4.3, we will show how to produce a binary expansion). On the other hand, it is not injective, since there are numbers which have two binary expansions. In the same way that in decimal expansion we can write $1.00000\cdots = 0.999999\ldots$, binary expansions which have an infinite tails of 1 yield the same number that an expansion with a tail of 0, for example

$$\frac{1}{2} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{0}{2^i} \quad \text{but also} \quad \frac{1}{2} = \sum_{i=2}^{\infty} \frac{1}{2^i}.$$

One can check that these ambiguity happens only for rational numbers of the form $k/2^n$, whose denominator is a power of 2. For all other numbers ψ is a bijection.

Exercise 1.4.3. *Check that each number of the form $k/2^n \in [0,1]$, where $k \in \mathbb{N}$ has two binary expansions.*

Proposition 1. *The map $\psi : \Sigma^+ \to [0,1]$ is a semi-conjugacy between the shift map $\sigma^+ : \Sigma^+ \to \Sigma^+$ and the doubling map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.*

Proof. Let us check that ψ is surjective. Since every point $x \in [0,1]$ has at least one binary expansion, we can write $x = \sum_{i=1}^{\infty} a_i/2^i$ where $a_i \in \{0,1\}$. Thus, the sequence $\underline{a} = (a_i)_{i=1}^{\infty} \in \Sigma^+$ of its binary digits is such that $\psi(\underline{a}) = x$ by definition of the ψ.

Thus, we are left to prove that the following diagram commutes:

$$\begin{array}{ccc} \Sigma^+ & \xrightarrow{\sigma^+} & \Sigma^+ \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ [0,1] & \xrightarrow{f} & [0,1] \end{array}$$

Take any $(a_i)_{i=1}^{\infty} \in \Sigma^+$. Let us first compute $\psi(\sigma^+((a_i)_{i=1}^{\infty}))$:

$$\psi(\sigma^+((a_i)_{i=1}^{\infty})) = \psi((b_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{b_i}{2^i} = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i},$$

since $b_i = a_{i+1}$. Let us now compare with $f(\psi((a_i)_{i=1}^{\infty}))$:

$$f(\psi((a_i)_{i=1}^{\infty})) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{2^i}\right) = \sum_{i=1}^{\infty} \frac{2a_i}{2^i} \mod 1 = \sum_{j=1}^{\infty} \frac{a_{j+1}}{2^j}$$

in virtue of the computation done in (1.3). Thus the results are the same. This concludes the proof. □

Let us use the correspondence given by ψ to study periodic points of the doubling map.

It is very easy to construct periodic points for σ^+: they are points $(a_i)_{i=1}^{\infty}$ whose digits are repeated periodically. For example, if we repeat the digits $0,1,1$ periodically, we get a periodic point of period 3:

$$\begin{aligned} (a_i)_{i=1}^{\infty} &= 0,1,1,0,1,1,0,1,1,0,1,1,\ldots \\ \sigma^+\left((a_i)_{i=1}^{\infty}\right) &= 1,1,0,1,1,0,1,1,0,1,1,\ldots \\ (\sigma^+)^2\left((a_i)_{i=1}^{\infty}\right) &= 1,0,1,1,0,1,1,0,1,1,\ldots \\ (\sigma^+)^3\left((a_i)_{i=1}^{\infty}\right) &= 0,1,1,0,1,1,0,1,1,\ldots \quad = (a_i)_{i=1}^{\infty} \end{aligned}$$

More in general, if $a_{n+i} = a_i$ for all $i \in \mathbb{N}$, then

$$(\sigma^+)^n\left((a_i)_{i=1}^{\infty}\right) = (a_{n+i})_{i=1}^{\infty} = (a_i)_{i=1}^{\infty},$$

so $(a_i)_{i=1}^{\infty}$ is periodic of period n.

Theorem 1.4.1. *The doubling map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has $2^n - 1$ periodic points of period n. Moreover, periodic points are dense.*

The second part of the theorem states that there are periodic points arbitrarily close to any other point. More precisely, for any $x \in \mathbb{R}/\mathbb{Z}$ and any $\epsilon > 0$ there exists a periodic point y such that $d(x, y) < \epsilon$, where d is the distance on $\mathbb{R}/\mathbb{Z}$.

Proof. Since the shift map σ^+ and the doubling map f are semi-conjugated, periodic points of period n for $\sigma^+ : \Sigma^+ \to \Sigma^+$ are mapped to periodic points of period n for f (not necessarily distint). Periodic points of period n for σ^+ are all sequences whose digits repeat periodically with period n. Thus, there are 2^n such sequences, since we can choose freely the first n digits in $\{0, 1\}$ and then repeat them periodically. Remark that the sequence $0, 0, 0, \dots$ is mapped to 0 and the sequence $1, 1, 1, \dots$ is mapped to 1, which are the same point in $\mathbb{R}/\mathbb{Z} = I/\sim$. Moreover, one can check that these are the only two periodic sequences which, applying Ψ, give identical periodic points[1]. Thus, there are $2^n - 1$ periodic points of period n.

We leave the second part as an exercise (see Exercise 1.4.5 below). □

Example 1.4.1. *The periodic points of period 3 for σ^+ are the periodic sequences obtained repeating the blocks of digits:*

$$000 \quad 001 \quad 010 \quad 011 \quad 100\dots, \quad 101\dots, \quad 110\dots, \quad 111$$

Let us take the corresponding binary expansions. Since $a_{i+3} = a_i$ for all $i \in \mathbb{N}$ we have

$$\sum_{i=1}^{\infty} \frac{a_i}{2^i} = \left(\frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_3}{2^3}\right) + \left(\frac{a_0}{2^4} + \frac{a_1}{2^5} + \frac{a_3}{2^6}\right) + \cdots = \sum_{j=0}^{\infty} \left(\frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_3}{2^3}\right) \frac{1}{(2^3)^j}$$

So, for example, starting from the sequence obtained repeating the block 101 we obtain

$$\sum_{j=0}^{\infty} \left(\frac{1}{2} + \frac{0}{4} + \frac{1}{8}\right) \frac{1}{8^j} = \frac{5}{8} \sum_{j=0}^{\infty} \frac{1}{8^j} = \frac{5}{8} \frac{1}{\left(1 - \frac{1}{8}\right)} = \frac{5}{7}.$$

Thus, we find the $7 = 2^3 - 1$ periodic points of period 3 for the doubling map are

$$0, \ \frac{1}{7}, \ \frac{2}{7}, \ \frac{3}{7}, \ \frac{4}{7}, \ \frac{5}{7}, \ \frac{6}{7}.$$

Exercise 1.4.4. *Periodic points for the doubling map can also be found directly solving the equation $f^n(x) = x$. List the fractions which correspond to periodic points of period n.*

Exercise 1.4.5. *Prove that the periodic points for the doubling map are dense.*
[Hint: use the previous exercise.]

The map $\psi : \Sigma^+ \to [0, 1]$ sends a sequence in 0 and 1 to the corresponding binary expansion. Conversely, given a point $x \in [0, 1]$, we are going to see how one can construct a binary expansion for x using symbolic coding.

[1] To see that, one can use that two binary expansions that produce the same number have one an infinite tail of digits 0 and the other an infinite tail of digits 1 and the only periodic sequences which are eventually equal to 0 or eventually equal to 1 are the sequence where all digits are 0 and the sequence where all digits are 1.

1.4.3 Symbolic coding for the doubling map

Consider the two intervals

$$P_0 = \left[0, \frac{1}{2}\right), \qquad P_1 = \left[\frac{1}{2}, 1\right).$$

They give a *partition*[2] $\{P_0, P_1\}$ of $[0,1]/\sim$, since $P_0 \cap P_1 = \emptyset$ and $P_0 \cup P_1 = [0,1]/\sim$. Let $\phi : I/\sim \to \Sigma^+$ be the map

$$x \to \phi(x) = (a_k)_{k=0}^{\infty}, \qquad \text{where} \quad \begin{cases} a_k = 0 & \text{if } f^k(x) \in P_0, \\ a_k = 1 & \text{if} f^k(x) \in P_1. \end{cases}$$

The sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ is called the *itinerary* of $\mathcal{O}^+{}_f(x)$ with respect to the partition $\{P_0, P_1\}$: it is obtained by iterating $f^k(x)$ and recording which interval, whether P_0 or P_1, is visited at each k. In particular, if $a_0, a_1, a_2, \ldots, a_k, \ldots$ is called the *itineary* of $\mathcal{O}_f^+(x)$ we have

$$x \in P_{a_0}, \ f(x) \in P_{a_1}, \ f^2(x) \in P_{a_2}, \ldots \ f^k(x) \in P_{a_k}, \ldots$$

Remark 1.4.2. *The idea of* coding *an orbit by recording its* itineary *with respect to a partition is a very powerful technique in dynamical systems. It often allow to conjugate a dynamical system to a shift map on a space of symbols. These symbolic spaces will be studied in Chapter 2 and, even if at first they may seem more abstract, they are well studied and often easier to understand then the original system.*

Itineraries of the doubling map produce the digits binary expansions, in the following sense:

Proposition 2. *If $a_0, a_1, \ldots, a_n, \ldots$ is the itinerary of the point $x \in [0,1]$, one has*

$$x = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots = \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i}, \tag{1.4}$$

that is $a_0, a_1, \ldots, a_n, \ldots$ are the digits of a[3] *binary expansion of x.*

Before proving the proposition let us remark that another equivalent way to express that the doubling map gives the digits of a binary expansion of a point is the following:

Corollary 1.4.1. *The mao $\phi : \mathbb{R}/\mathbb{Z} \to \Sigma^+$ is a right inverse for the map $\psi : \Sigma^+ \to I/\sim$ constructed before, i.e. $\psi \circ \phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the identity map.*

Indeed, given a point $x \in [0,1]$, first $\phi(x) = (a_i)_{i=1}^{\infty}$ gives its itinerary $a_0, a_1, a_2, \ldots$ and then $\psi((a_i)_{i=0}^{\infty})$ produces a point whose binary expansion has digits a_i, which, by (1.4) yield back the same point x.

[The map ϕ is not a left inverse, i.e. the map $\phi \circ \psi : \Sigma^+ \to \Sigma^+$ is not necessarily the identity map. For most points, it is indeed identity, but the problem is related to points in X which have two binary expansions: the coding map ϕ yields only one of the two expansions.]

Exercise 1.4.6. *Give an example a point $(a_i)_{i=1}^{\infty}$ such that $\phi(\psi((a_i)_{i=1}^{\infty}) \neq (a_i)_{i=1}^{\infty}$.*

[2] A finite (countable) *partition* of a set X is a subdivision of X into finitely many (respectively countably many) subsets $X_i \subset X$ that are pairwise *disjoint*, that is $X_i \cap X_j = \emptyset$ for all $i \neq j$, and *cover* X, that is such that $\cup_i X_i = X$.

[3] Remark that if x has two binary expansions, the itinerary will produce only one of them.

Proof of Proposition 2. Let $a_0, a_1, \ldots, a_n, \ldots$ be the itinerary of $\mathcal{O}_f^+(x)$ with respect to x. We have to check that it gives a binary expansion for x, that is that we can write

$$x = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots = \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i}.$$

If the first digit of the itinerary is $a_0 = 0$, $x \in P_0$, that is $0 \le x < 1/2$. Thus the first digit of the binary expansion of x is 0 (since, if it were 1, we would have $x = \frac{1}{2} + \ldots$ which is larger than 1/2). Similarly, if $a_0 = 1$, $x \in P_1$, that is $1/2 \le x < 1$ and we can take 1 as first digit of the binary expansion of x.

To show that the k^{th} entry a_k gives the k^{th} digit of the binary expansion of x, we can apply the doubling map k times and recall that if $x_1, x_2, \ldots, x_k, \ldots$ are digits of a binary expansion of x, since the doubling map acts as a shift on binary expansions, the digits of $f^k(x)$ are $x_{k+1}, x_{k+1}, \ldots$. Moreover, by definition of itinerary, the itinerary of $f^k(x)$ is a_k, $a_{k+1}, a_{k+1}, \ldots$. Now, we can reason as before: if $a_k = 0$ (respectively $a_k = 1$), then $f^k(x) \in P_0$ (respectively P_1) and $0 \le f^k(x) < 1/2$ (respectively $1/2 \le f^k(x) < 1$). Thus, we see that the first digit of the binary expansion of $f^k(x)$, that is x_{k+1}, is 0 (respectively, can be taken to be 1). $\square$

Let $a_0, a_2, \ldots, a_n$ be an $(n+1)-$tuple of digit 0 or 1. Let

$$\begin{aligned} I(a_0, a_1, \ldots, a_n) &= \{x \in [0,1] \text{ such that } \phi(x) = (a_0, a_1, \ldots, a_n, \ldots)\} \\ &= \{x \in [0,1] \text{ such that } f^k(x) \in P_{a_k} \text{ for all } 0 \le k \le n\} \end{aligned}$$

These are all points whose itineary (and also whose binary expansion) starts with $a_1, a_2, \ldots, a_n$. In order to construct them, one can use that

$$I(a_0, a_1, \ldots, a_n) = P_{a_0} \cap f^{-1}(P_{a_1}) \cap \cdots \cap f^{-n}(P_{a_n}).$$

If x belongs to the intersection in the righ hand side, clearly $x \in P_{a_0}$, $f(x) \in P_{a_1}, \ldots, f^n(x) \in P_{a_n}$, so by definition the itinerary starts with the block $a_0, a_1, \ldots a_n$.

For example we have $I(0) = P_0$, $I(1) = P_1$ and (see Figure 1.2)

$$f^{-1}(P_1) = \left[\frac{1}{4}, \frac{1}{2}\right) \cup \left[\frac{3}{4}, 1\right), \quad \text{so that} \quad I(0,1) = P_0 \cap f^{-1}(P_1) = \left[\frac{1}{4}, \frac{1}{2}\right).$$

Repeating for the other paris of digits we find:

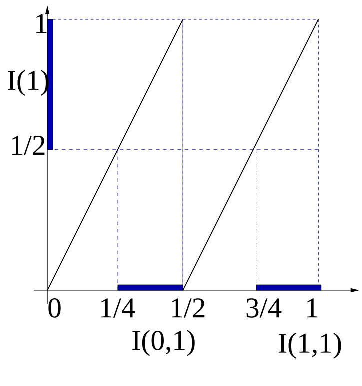

Figure 1.2: The intervals $I(0,1)$ and $I(1,1)$ for the doubling map.

$$I(0,0) = \left[0, \frac{1}{4}\right), \quad I(0,1) = \left[\frac{1}{4}, \frac{1}{2}\right), \quad I(1,0) = \left[\frac{1}{2}, \frac{3}{4}\right), \quad I(1,1) = \left[\frac{3}{4}, 1\right).$$

More in general, one can prove by induction that

(P1) each $I(a_0, a_1, \dots, a_n)$ is an interval of length $1/2^{n+1}$;

(P2) as $a_0, a_1, \dots, a_n$ ranges through all possible $(n+1)-$tuples of digit 0 or 1 (which are 2^{n+1}, as many as choices of $n+1$ digits in $\{0,1\}$), one obtains a partition of $[0,1)$ into 2^n intervals of length $1/2^{n+1}$ (as in the example for $n = 2$ above):

$$\bigcup_{(a_0,\dots,a_n)\in\{0,1\}^{n+1}} I(a_0, a_1, \dots, a_n) = [0,1).$$

Each interval is a dyadic interval of the form

$$\left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right), \qquad \text{where} \quad 0 \le k < 2^{n+1}.$$

Let us use conjugacy and coding to construct a dense orbit for the doubling map.

Theorem 1.4.2. *Let f be the doubling map. There exists a point $\overline{x}$ whose forward orbit $\mathcal{O}^+_f(\overline{x})$ under the doubling map is* dense.

Proof. To prove that an orbit $\mathcal{O}^+_f(x)$ is dense, it is enough to show that for each $n \ge 1$ it visits all intervals of the form $I(a_0, a_1, \dots, a_n)$. Indeed, if this is the case, given $y \in I$ and $\epsilon > 0$, take N large enough so that $1/2^{N+1} \le \epsilon$ and take the interval $I(a_0, a_1, \dots, a_N)$ which contains y (one of them does since they partition $[0,1)$ by (P2) above). If we showed that there is a point $f^k(x)$ in the orbit $\mathcal{O}^+_f(x)$, which visits $I(a_0, a_1, \dots, a_N)$, since both y and $f^k(x)$ belong to $I(a_1, \dots, a_N)$ (which has size $1/2^{N+1}$ by (P1) above), we have $d(f^k(x), y) \le 1/2^{N+1} < \epsilon$. This shows that $\mathcal{O}^+_f(x)$ is dense.

To construct an orbit which visits all dyadic intervals, let us list for each n all the possible sequences $a_0, a_1, \dots, a_n$ of length n (there are 2^{n+1} of them) and create a sequence $(\overline{a}_i)_{i=0}^{\infty}$ by just apposing all such sequences for $n = 0$, then $n = 1$, then $n = 2$ and so on:

$$0,1, \quad 0,0,\ 0,1,\ 1,0,\ 1,0,\ 1,1, \quad 0,0,0,\ 0,0,1,\ 0,1,0,\ 0,1,1,\ 1,0,0,\dots$$

Then, let us see that the orbit of the point $\overline{x} := \psi((\overline{a}_i)_{i=0}^{\infty})$ visits all intervals of the form $I(a_0, a_1, \dots, a_n)$. To see that, it is enough to find where the block $a_0, a_1, \dots, a_n$ appears inside $(\overline{a}_i)_{i=0}^{\infty}$, for example at $\overline{a}_k = a_0, \overline{a}_{k+1} = a_1, \dots, \overline{a}_{k+n} = a_n$. Then, since the itinerary of $f^k(\overline{x})$ by definition of itinerary is $\overline{a}_k, \overline{a}_{k+1}, \dots, \overline{a}_{k+n}, \dots$, this shows that

$$f^k(\overline{x}) \in I(\overline{a}_k, \overline{a}_{k+1}, \dots, \overline{a}_{k+n}) = I(a_0, a_1, \dots, a_n),$$

so $f^k(\overline{x})$ is the point in $\mathcal{O}^+_f(\overline{x})$ which visits $I(a_0, a_1, \dots, a_n)$. This concludes the proof that $\mathcal{O}^+_f(\overline{x})$ visits all dyadic intervals and hence that it is dense. □

Exercise 1.4.7. *Draw all intervals of the form $I(a_1, a_2, a_3)$ where $a_1, a_2, a_3 \in \{0,1\}$.*

Linear expanding maps We remarked earlier that the doubling map doubles distances: if $x, y \in \mathbb{R}/\mathbb{Z}$ are any two points such that $d(x,y) < 1/4$, then $d(f(x), f(y)) = 2d(x,y)$, that is, the distance of their images is doubled. The doubling map is an example of an *expanding map*[4]

[4] A one-dimensional map $g : I \to I$ of an interval $I \subset \mathbb{R}$ is called an *expanding* if it is piecewise differentiable, that is we can decompose I into a finite union of intervals on each of which g is differentiable, and the derivative g' satisfies $|g'(x)| > 1$ for all $x \in I$.

More precisely, the doubling belongs to the family of *linear expanding maps of the circle*: for each $m \in \mathbb{Z}$ with $|m| > 1$ the map $E_m : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is given by

$$E_m(x) = mx \mod 1 \qquad (\text{or equivalently} \quad E_m(z) = z^k \quad \text{on } S^1).$$

The doubling map is the same than E_2. These maps can be studied analogously, by considering expansion in base m instead than binary expansions. One can prove that they are semi-conjugated with the shift σ^+ on the space

$$\Sigma_m^+ = \{0, 1, \ldots, m-1\}^{\mathbb{N}}$$

of one-sided sequences in the digits $0, \ldots, m-1$.