Darcy flow: Pollution of a drinking water reservoir

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***Abstract*—This paper reports the solutions of a problem of drinking water pollution using Finite Elements methods. In particular, since it involves the Darcy Equations, we will explore the use of the Raviart Thomas elements. The corresponding code implemented in FreeFem++ can be found on github: https://github.com/LucaZampieri/Darcy_Flow. This project is part of the course MA-468: Numerical Methods for Saddle Point Problems given by Prof. Annalisa Buffa at EPFL.**

I. Notations

We will use standard notation, complemented by the following:

- $\|\cdot\|_0 = \|\cdot\|_{L^2(\Omega)}$
- $H_0^1(\Omega) = \{x \in H^1(\Omega) : x\restriction_{\partial\Omega} = 0\}$
- $\Gamma_{ij} = \Gamma_i \cup \Gamma_j$ with $i, j < 10$
- $H^1_{0,\Gamma_{ij}}(\Omega) = \{x \in H^1(\Omega) : x\restriction_{\Gamma_{ij}} = 0\}$

II. Problem (Question 1)

For the complete statement of the problem, refer to the sheet given by the assistants. Here we make a quick reminder of the problem.
Consider a drinking water reservoir with fluid flowing left to right of the domain Ω. There is a pollution accident at C_2 and a pump is placed at the center of the domain in C_1. We are interested in knowing how much pollutant has been extracted by the pump.
Considering $\Omega = [0,1]^2$ with border $\Gamma_i, i = 1 : 4$ denoted in anti-clockwise sense, $C_1 = [0.48, 0.52]^2 \in \Omega$ and $C_2 = [0.28, 0.32]^2 \in \Omega$. We then have to solve the equations:

$$\begin{cases} -k\nabla p = \mathbf{u}, & \text{in } \Omega \\ \nabla \cdot \mathbf{u}, & \text{in } \Omega \\ p = p_{in}, & \text{on } \Gamma_4 \\ p = p_{out}, & \text{on } \Gamma_2 \\ \mathbf{u}\cdot\mathbf{n} = 0, & \text{on } \Gamma_1 \cup \Gamma_3 \end{cases} \tag{1}$$

$$\begin{cases} -\nu\Delta c + \mathbf{u}\cdot\nabla c = g, & \text{in } \Omega \\ c = 0, & \text{on } \Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \\ \nu\nabla c\cdot\mathbf{n} = 0, & \text{on } \Gamma_2 \end{cases} \tag{2}$$

Where 1 is a darcy equation and 2 is an advection-diffusion equation.We are then interested in the quantity $\phi(c) = \int_{C_1} c$.

III. Weak forms (Question 1)

We want to prove the well-posedness of the system of equations. For this we rewrite the sytem of equations (1,2) in the following form:

$$\begin{cases} a(u,v) + b(p,v) = F_a(v), \forall \mathbf{v} \in V \\ b(u,q) = F_b(q), \forall q \in Q \end{cases} \tag{3}$$

$$a_{ad}(c,w) = G(w)\forall w in W \tag{4}$$

where a, b, a_{ad} are bilinear forms and F_a, F_b, G are linear operators.

A. Darcy equation

For the Darcy equation we seek a mixed-form weak formulation. We set the spaces $V = H(div, \Omega)$ and $Q = L^2(\Omega)$. Multiplying the first equation of 1 by a test function $v \in V$, the second equation by a test function $q \in Q$ and integrating we get:

$$\begin{cases} \int_\Omega -k^{-1}\mathbf{u}\cdot\mathbf{v} - \int_\Omega \nabla p\cdot\mathbf{v} = 0, & \forall \mathbf{v} \in V \\ \int_\Omega (\nabla\cdot\mathbf{u})q = \int_\Omega fq, & \forall q \in Q \end{cases} \tag{5}$$

Integrating by parts the second term of the first equation of 5 and applying boundary conditions we get the mixed-form weak formulation:
Find $\mathbf{u} \in V$, q $\in Q$ such that :

$$\begin{cases} \int_\Omega -k^{-1}\mathbf{u}\cdot\mathbf{v} + \int_\Omega p(\nabla\cdot\mathbf{v}) - \int_{\partial\Omega} p(\mathbf{n}\cdot\mathbf{v}) = 0, & \forall \mathbf{v} \in V \\ \int_\Omega (\nabla\cdot\mathbf{u})q = \int_\Omega fq, & \forall q \in Q \end{cases} \tag{6}$$

From the theory on mixed-forms, to prove well-posedness we need:

- Coercivity on the Kernel
- Inf-Sup conditions

1) Coercivity on the kernel:

2) Inf-Sup Condition:

B. Advection Diffusion

Set $W = H^1_{0,\Gamma_1\cup\Gamma_3\cup\Gamma_4}$. Multiplying the equation in 2 by a test function $w \in W$, integrating and using integration by part we can reach the weak formulation: Find $c \in W$ such that:

$$a_{ad}(c,w) = \int_\Omega \nu\nabla c\nabla w + \int_\Omega (\mathbf{u}\cdot\nabla c)w = \int_\omega gw = G(w), \forall w \in W \tag{7}$$

where we used the fact that $\int_{\partial\Omega} \nu(\nabla c\cdot\mathbf{n})w = 0$ since on Γ_2 we have homogeneous Neumann conditions and on the rest of $\partial\Omega$ we have $w = 0$ since it belongs to W. We note that

$a_{ad}(c,w)$ is a bilinear form and $G(w)$ a linear operator.
We want to conclude using the Lax-Milgram Lemma thus we search for:

- Continuity of G(w) which is implicit if $g \in XXXXXXX$
- Continuity of the bilinear form $a_{ad}(c,w)$
- Coercivity of the bilinear form $a_{ad}(c,w)$

we then proceed:

1) Continuity of the bilinear form: Rewriting

$$|a_{ad}(c,w)| \leq \nu \left\|\nabla c\right\|_0 \left\|\nabla w\right\|_0 + \tag{8}$$

2) Coercivity of the bilinear form: For the coercivity we can make two parallel assumptions:

$$a_{ad}(c,c)| = \nu \left\|\nabla c\right\|_0^2 + + \tag{9}$$

IV. Numerical Discretization (Question 2)

V. figures

Fig. 1: lala

Fig. 2: im2

Fig. 3: im1

VI. Appendix

code:

References

[1] lalala