Darcy flow:
Pollution of a drinking water reservoir

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Abstract—**This paper reports the solutions of a problem of drinking water pollution using Finite Elements methods. In particular, since it involves the Darcy Equations, we will explore the use of the Raviart Thomas elements. The corresponding code implemented in FreeFem++ as well as the source of this report can be found on github: https://github.com/LucaZampieri/Darcy_Flow. This project is part of the course MA-468: Numerical Methods for Saddle Point Problems given by Prof. Annalisa Buffa at EPFL.**

I. NOTATIONS

We will use standard notation, complemented by the following (even if some are quite informal):

- $\|\cdot\|_0 = \|\cdot\|_{[L^2(\Omega)]^d}$, $\forall d$ number of dimensions
- $\Gamma_{ij} = \Gamma_i \cup \Gamma_j$ with $i, j < 10$
- $H_0^1(\Omega) = \{x \in H^1(\Omega) : x\rceil_{\partial\Omega} = 0\}$
- $H_{0,\Gamma}^1(\Omega) = \{x \in H^1(\Omega) : x\rceil_\Gamma = 0\}$
- $H(div, \Omega) = \{x \in H^1(\Omega) : div(x) = 0\}$
- $H_0(div, \Omega) = \{x \in H(div, \Omega) : x \cdot n\rceil_{\partial\Omega} = 0\}$
- $H_{0,\Gamma}(div, \Omega) = \{x \in H(div, \Omega) : x \cdot n\rceil_\Gamma = 0\}$

if the domain of the space is omitted, it is assumed to be Ω.

II. PROBLEM (QUESTION 1)

For the complete statement of the problem, refer to the sheet given by the assistants. Here we make a quick reminder of the problem.

Consider a drinking water reservoir with fluid flowing left to right of the domain Ω. There is a pollution accident at C_2 and a pump is placed at the center of the domain in C_1. We are interested in knowing how much pollutant has been extracted by the pump.

Considering $\Omega = [0,1]^2$ with border $\Gamma_i, i = 1 : 4$ denoted in anti-clockwise sense, $C_1 = [0.48, 0.52]^2 \in \Omega$ and $C_2 = [0.28, 0.32]^2 \in \Omega$. We then have to solve the equations:

$$
\begin{cases}
-k\nabla p = \mathbf{u}, & \text{in } \Omega \\
\nabla \cdot \mathbf{u}, & \text{in } \Omega \\
p = p_{in}, & \text{on } \Gamma_4 \\
p = p_{out}, & \text{on } \Gamma_2 \\
\mathbf{u} \cdot \mathbf{n} = 0, & \text{on } \Gamma_1 \cup \Gamma_3
\end{cases}
\tag{1}
$$

$$
\begin{cases}
-\nu\Delta c + \mathbf{u} \cdot \nabla c = g, & \text{in } \Omega \\
c = 0, & \text{on } \Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \\
\nu\nabla c \cdot \mathbf{n} = 0, & \text{on } \Gamma_2
\end{cases}
\tag{2}
$$

Where 1 is a darcy equation, 2 is an advection-diffusion equation and f, g are defined as:

$$
f(\mathbf{x}) = \begin{cases}
-1000, & \text{if } \mathbf{x} \in C_1 \\
0, & \text{if } \mathbf{x} \notin C_1
\end{cases}
\tag{3}
$$

and

$$
g(\mathbf{x}) = \begin{cases}
1000, & \text{if } \mathbf{x} \in C_2 \\
0, & \text{if } \mathbf{x} \notin C_2
\end{cases}
\tag{4}
$$

as such $f, g \in L^\infty(\Omega)$.

We fix $k = I$, $p_{in} = 2$ and $p_{out} = 0$. We are then interested in the quantity $\phi(c) = \int_{C_1} c$.

III. WEAK FORMS (QUESTION 1)

We want to prove the well-posedness of the system of equations. For this we rewrite the sytem of equations (1,2) in the following form:

Find $\mathbf{u} \in V$, $q \in Q$, $c \in W$ such that:

$$
\begin{cases}
a(u,v) + b(p,v) = F_a(v), & \forall \mathbf{v} \in V \\
b(u,q) = F_b(q), & \forall q \in Q \\
a_{ad}(c,w) = G(w), & \forall w \in W
\end{cases}
\tag{5}
$$

where a, b, a_{ad} are bilinear forms and F_a, F_b, G are linear operators.

A. Darcy equation

For the Darcy equation we seek a mixed-form weak formulation. We set the spaces $V = H(div, \Omega)$ and $Q = L^2(\Omega)$. Multiplying the first equation of 1 by a test function $v \in V$, the second equation by a test function $q \in Q$ and integrating we get:

$$
\begin{cases}
\int_\Omega -k^{-1}\mathbf{u} \cdot \mathbf{v} - \int_\Omega \nabla p \cdot \mathbf{v} = 0, & \forall \mathbf{v} \in V \\
\int_\Omega (\nabla \cdot \mathbf{u})q = \int_\Omega fq, & \forall q \in Q
\end{cases}
\tag{6}
$$

Integrating by parts the second term of the first equation of 6 and applying boundary conditions we get the mixed-form weak formulation:

Find $\mathbf{u} \in V$, q $\in Q$ such that :

$$
\begin{cases}
\int_\Omega -k^{-1}\mathbf{u} \cdot \mathbf{v} + \int_\Omega p(\nabla \cdot \mathbf{v}) = \int_{\partial\Omega} \tilde{p}(\mathbf{n} \cdot \mathbf{v}), & \forall \mathbf{v} \in V \\
\int_\Omega (\nabla \cdot \mathbf{u})q = \int_\Omega fq, & \forall q \in Q
\end{cases}
\tag{7}
$$

with $\tilde{p}$ is such that $\tilde{p}\rceil_{\Gamma_{13}} = 0$, $\tilde{p}\rceil_{\Gamma_2} = p_{out}$ and $\tilde{p}\rceil_{\Gamma_4} = p_{in}$
Which is the form we desired (5) since a, b are clearly bilinear and F_a, F_b are trivialy linear.

From the theory on mixed-forms, to prove well-posedness we additionally need:

- Coercivity on the Kernel i.e. $a(u_0, u_0) \geq \alpha \|u_0\|_{H(div,\Omega)}^2$, $\forall u_0 \in Ker(B)$ where $Ker(B) = \{v \in H_{0,\Gamma_{13}} : div(v) = 0\}$
- Inf-Sup conditions i.e: $inf_{q \in Q} sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta > 0$

We then proceed to prove these two properties:

proof: Coercivity on the kernel. We have:

$$a(u_0, u_0) = \int k^{-1}|u_0|^2 \geq k_{min} \|u_0\|_{L^2(\Omega)}^2 = \|u_0\|_{H(div,\Omega)}^2$$

where k_{min} is a minimal eigenvalue of k and where we used both the facts that $k = I$ and that $u_0 \in Ker(B)$ thus $\|div(v)\|_{L^2(\Omega)} = 0$, which concludes the proof. $\square$

Proof of Inf-Sup condition. Consider the separate problem on the same domain:
Let $f \in L^2(\Omega)$:

$$\begin{cases} -\Delta z = f_L, \text{ in } \Omega \\ \partial_n z = 0, \text{ on } \Gamma_{13} \end{cases} \quad (8)$$

which finds solution $z \in H^1(\Omega)$ with $\|z\|_{H^1(\Omega)} \leq C \|f_L\|_0$. If we now consider $\nabla z \in L^2(\Omega)^2$ of the previous problem, we have $\nabla z \in H_{0,\Gamma_{13}}(div, \Omega)$ and from the solution of the previous problem we get:

$$\|\nabla z\|_{H_{0,\Gamma_{13}}(div)} \leq C \|f_L\|_0 \quad (9)$$

and shows that $B = div : H_{0,\Gamma_{13}}(\Omega) \longrightarrow L^2(\Omega)$ is surjective. However, by VI.6, this implies that the inf-sup condition holds. $\square$

In parallel we could have done as below III-A.

Proof of Inf-Sup condition BIS. We recall that $V = H_{0,\Gamma_{13}}(\Omega)$, $Q = L^2(\Omega)$. By integrating 8 and multiplying by $q \in Q$

$$\int_\Omega q \cdot div(\nabla z) \geq \|f_L\|_0 \|q\|_Q \geq \|\nabla z\|_V \|q\|_Q \quad (10)$$

Since $\nabla z, u \in V$ we can set $u = \nabla z$. Taking the supremum of the LHS we get:

$$sup_{u \in V} \int_\Omega q \cdot div(u) \geq \|f_L\|_0 \|q\|_Q \geq \|u\|_V \|q\|_Q \quad (11)$$

which can be rewritten as:

$$sup_{u \in V} \frac{\int_\Omega q \cdot div(u)}{\|u\|_V} \geq \|q\|_Q \quad (12)$$

which is equivalent to the Inf-Sup condition. $\square$

We can finally conclude on the well-posedness of equation 7 by the main theorem of the course VI.5.

B. Advection Diffusion

Due to the previous subsection, we have $\mathbf{u} \in V$. Set $W = H_{0,\Gamma_{134}}^1$. Multiplying the equation in 2 by a test function $w \in W$, integrating and using integration by part we can reach the weak formulation:
Find $c \in W$ such that:

$$a_{ad}(c, w) = \int_\Omega \nu \nabla c \nabla w + \int_\Omega (\mathbf{u} \cdot \nabla c) w = \int_\omega gw = G(w), \forall w \in W \quad (13)$$

where we used the fact that $\int_{\partial\Omega} \nu(\nabla c \cdot \mathbf{n})w = 0$ since on Γ_2 we have homogeneous Neumann conditions and on the rest of $\partial\Omega$ we have $w = 0$ since it belongs to W. We note that $a_{ad}(c, w)$ is a bilinear form and $G(w)$ a linear operator.
We want to conclude using the Lax-Milgram Lemma thus we desire:

- Continuity of G(w) which is implicit since $g \in L^\infty(\Omega)$
- Continuity of the bilinear form $a_{ad}(c, w)$
- Coercivity of the bilinear form $a_{ad}(c, w)$

Note that thanks to the Pointcare inequality VI.4, the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H_0^1} = |\cdot|_{H^1} = \|\nabla\cdot\|_0$ are equivalent meaning there exist $m, M > 0$ s.t. $m \|\cdot\|_{H^1} \leq \|\cdot\|_{H_0^1} \leq M \|\cdot\|_{H^1}$.
We can proceed:

1) Continuity of the bilinear form: Rewriting

$$|a_{ad}(c, w)| \leq \nu \|\nabla c\|_0 \|\nabla w\|_0 + |\mathbf{u} \cdot \nabla cw| \quad (14)$$

$$\leq \nu \|\nabla c\|_0 \|\nabla w\|_0 + \|\mathbf{u}\|_{L^4(\Omega)} \|\nabla c\|_0 \|w\|_{L^4(\Omega)} \quad (15)$$

$$\leq (\nu + \|\mathbf{u}\|_{L^4(\Omega)}) \|c\|_{H^1} \|w\|_{H^1} \quad (16)$$

and since $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H_0^1}$ are equivalent norms, we have continuity.

2) Coercivity of the bilinear form:

$$a_{ad}(c, c) = \nu \|\nabla c\|_0^2 + \int_\Omega (\mathbf{u} \cdot \nabla c)c \quad (17)$$

To handle the last term of the equation, we need an assumption. The weakest we found is $\mathbf{u} \cdot \mathbf{n} \geq 0$. We can rewrite the last term as:

$$\int_\Omega (\mathbf{u} \cdot \nabla c)c = \frac{1}{2} \int_\Omega \mathbf{u} \cdot \nabla(c^2) \quad (18)$$

$$= -\int_\Omega (\nabla \cdot \mathbf{u})c^2 + \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})c^2 \quad (19)$$

$$\quad (20)$$

The first term vanishes due to $u \in V$ which is divergence free, and the second term vanishes on Γ_{134} since it has diriclet boundary conditions. Finally, using $\mathbf{u} \cdot \mathbf{n} \geq 0$:

$$a_{ad}(c, c) \geq \nu \|\nabla c\|_0^2 + \int_{\Gamma_2} (\mathbf{u} \cdot \mathbf{n})c^2 \geq \nu \|\nabla c\|_0^2 \quad (21)$$

Which proves coercivity of $a_{ad}(\cdot, \cdot)$.

Using Lax-Milgram Lemma VI.2 we conclude on the well-posedness of the advection-diffusion equation 13 thus concluding on the well-posedness of the system of equations.

IV. Numerical Discretization (Question 2)

We should now find a suitable numerical discretization for the weak formulation of both the mixed-form of Darcy problem and the diffusion-advection equation for the pollutant agent, i.e. we want to find the spaces $V_h \subseteq V$, $Q_h \subseteq Q$ and $W_h \subseteq W$ such tha tthe discrete problem is well-posed. Finally, we will state the expected rate of convergence with these elements.

A. Discrete Darcy

Lets begin with the Darcy equation. Setting $V_h =$ and $Q_h =$ the discrete problem writes
Find $u_h \in V_h$, $p_h \in Q_h$ s.t.:

$$\begin{cases} \int_{\tau_h} k^{-1} u_h \cdot v_h + \int_{\tau_h} div(v_h) p_h = ..., & \forall v_h \in V_h \\ \int_{\tau_h} div(u_h) q_h = \int_{\tau_h} f \ q_h, & \forall q_h \in Q_h \end{cases} \quad (22)$$

and again, to prove well-posedness we need the two properties:

- $a(u_h, u_h) = \int_{\tau_h} k^{-1} |u_h|^2 \geq \alpha \|u_h\|_{V_h}^2, \forall u_h \in Ker(B_h)$
- $sup_{u_h \in V_h} \frac{\int_{\tau_h} div(u_h) p_h}{\|u_h\|_{V_h}} \geq \beta \|p_h\|_{Q_h}$

where $\|\cdot\|_{V_h} = \|\cdot\|_V$, $\|\cdot\|_{Q_h} = \|\cdot\|_Q$ and $Ker(B_h) = \{u_h \in V_h : \int_{\tau_h} div(u_h) q_h = 0 \forall q_h \in Q_h\}$.

1) Discrete coercivity on the kernel:

Proof. Take $q_h = div(u_h)$ (possible since $div(V_h) = Q_h$):

$$\int_{\tau_h} |div(u_h)|^2 = 0 \implies div(u_h) = 0$$

and $Ker(B_h) \subset Ker(B)$. Which implies , $\forall u_h \in Ker(B_h)$:

$$a(u_h, u_h) = \int_{\tau_h} k^{-1} |u_h|^2 \geq \alpha(\|u_h\|_0^2 + \|div(u_h)\|_0^2) \quad (23)$$

since here $\|u_h\|_0^2 = \|u_h\|_0^2 + \|div(u_h)\|_0^2 = \|u_h\|_{V_h}^2$ as $\|div(u_h)\|_0^2 = 0$ $\qquad\square$

2) Discrete Inf-Sup conditions:

Proof. From the continuous Inf-Sup condition, with $p_h \in L^2(\Omega)$:

$$\exists u \in H(div, \Omega) : \frac{\int_\Omega div(u) \cdot p_h}{\|u\|_V} \geq \|p_h\|_0$$

by the definition of the L^2 projection Π^s we have

$$\int_{\tau_h} div(u) p_h = \int_{\tau_h} \Pi^s(div(u)) p_h$$
$$= \int_{\tau_h} div(I_{RT} u) p_h$$
$$= \int_{\tau_h} div(u_h) p_h$$

and by the continuity of I_{RT}

$$\|I_{RT} u\|_{H(div, \Omega)} \leq C \cdot \|u\|_{H(div, \Omega)}$$

we can then write:

$$\frac{\int_{\tau_h} div(u_h) \cdot p_h}{\|u_h\|_{H(div,\Omega)}} = \frac{\int_{\tau_h} div(u) \cdot p_h}{\|u_h\|_{H(div,\Omega)}} \geq \frac{1}{C} \frac{\int_{\tau_h} div(u) \cdot p_h}{\|u\|_{H(div,\Omega)}} \geq \frac{\beta}{C} \|p_h\|_0$$

proving that the Inf-Sup condition holds $\qquad\square$

Thus the Discrete Darcy problem 22 is well-posed and verifies optimal error estimates in the energy norms:

$$\|u - u_h\|_V + \|p - p_h\|_Q \leq$$
$$C(inf_{u_h \in V_H} \|u - vh\| + inf_{q_h \in Q_h} \|p - p_h\|_Q)$$

B. Discrete Advection-Diffusion

Setting $W_h =$, the discrete problem writes
Find $c_h \in W_h$ s.t.:

$$a_{ad}(c_h, w_h) = \int_{\tau_h} \nu \nabla c_h \nabla w_h + \int_{\tau_h} (u_h \cdot \nabla c_h) w_h = \int_{\tau_h} g_h w_h = G(w_h) \quad (24)$$

$\forall w_h \in W_h$.
Repeating the steps done for the continuous case we use the Lax-Milgram Lemma to prove existence and uniqueness of the solution. Moreover, Cea's lemma gives us the error estimate:

$$\|c - c_h\|_W \leq \frac{M}{\alpha} inf_{w_h \in W_h} \|c - w_h\|_W$$

V. Figures

Fig. 1: lala

Fig. 2: im2   Fig. 3: im1

VI. Appendix

A. Code

code:

B. Properties, Lemmas & Theorems

Some of these theorems are in the course, the others can be found in [1] and [2].

Theorem VI.1 (Sobolev Embeddings [1]). *Let $u \in H^1(\mathbb{R}^2)$ Then $u \in L^p(\mathbb{R}^2)$ for $2 \leq p < \infty$, and*

$$\|u\|_{L^p(\mathbb{R}^2)} \leq c(p) \|u\|_{H^1(\mathbb{R}^2)}$$

Lemma VI.2 (Lax-Milgram)**.**

Lemma VI.3 (Cea's Lemma)**.**

Lemma VI.4 (Pointcare)**.**

Theorem VI.5 (Existence and uniqueness for mixed form)**.**

Theorem VI.6 (Equivalence for inf-sup Condition)**.**

REFERENCES

[1] Partial Differential Equations in Action - From Modelling to Theory — Sandro Salsa — Springer." Accessed June 7, 2018. https://www.springer.com/gp/book/9783319150932.

[2] "Numerical Models for Differential Problems — Alfio Quarteroni — Springer." Accessed June 7, 2018. https://www.springer.com/gp/book/9788847058835.