Linear Filters Applied on Images

Computer Vision 2017

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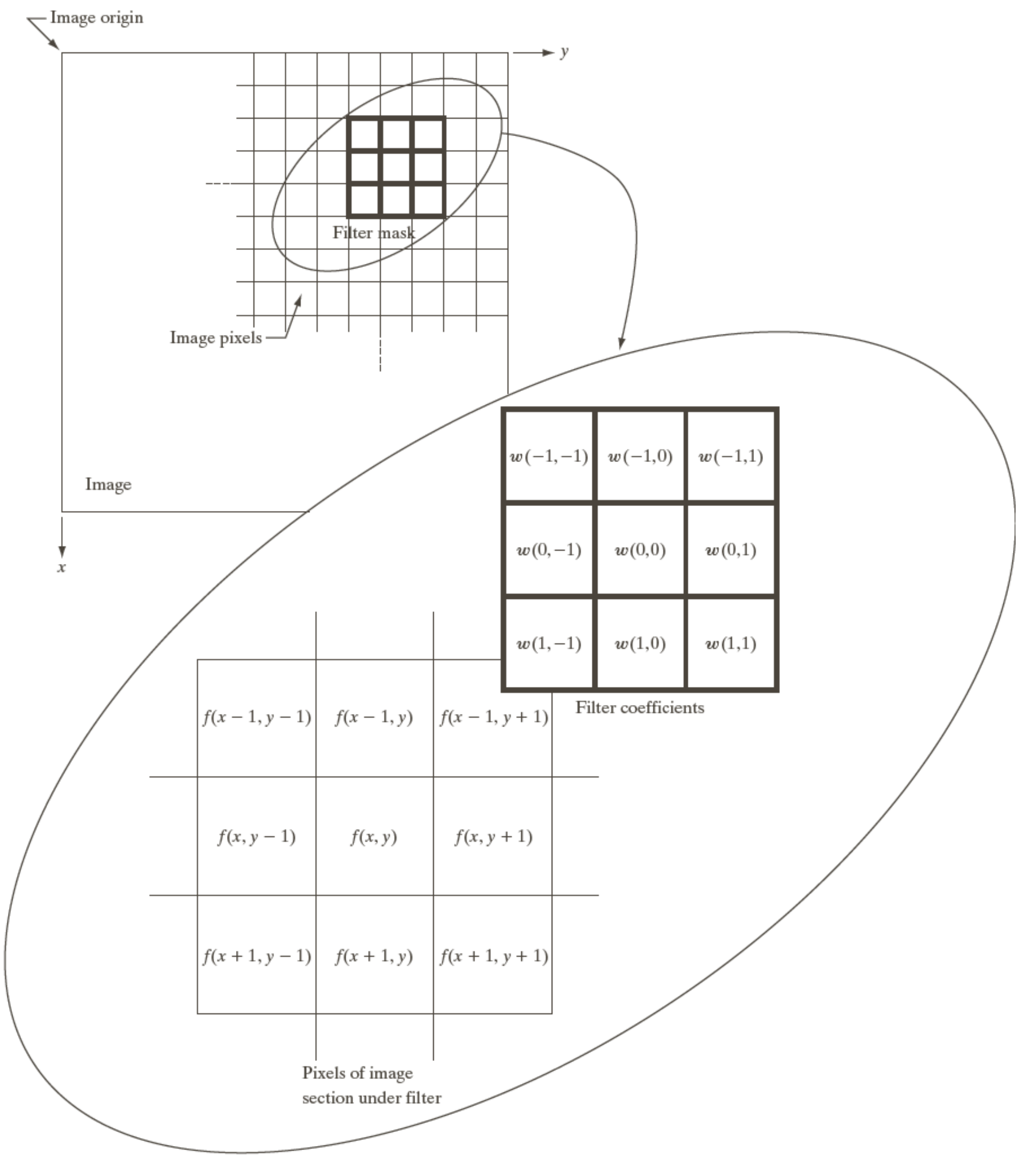
Images from Gonzalez Woods

Spatial Operations

- Geometric transforms
- Single pixel processing
- Processing based on the neighbourhood of a pixel

Spatial Filtering

- The output is typically a function of the samples in a window surrounding the considered pixel
- Filter type:
  1. Linear
  2. Non-linear
- Computation
  1. In the spatial domain
  2. In the frequency domain

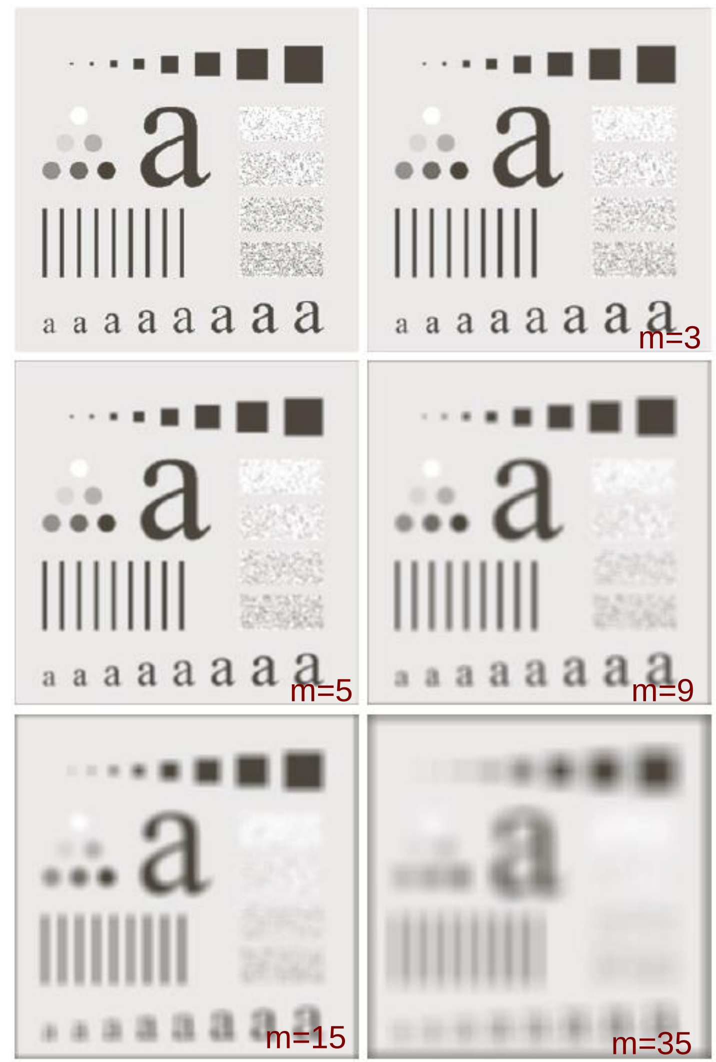


Example: Low-Pass Linear Filter

(simple averaging filter)

FIGURE 3.33 (a) Original image, of size 500 × 500 pixels. (b)–(f) Results of smoothing with square averaging filter masks of sizes $m = 3, 5, 9, 15$, and 35, respectively. The black squares at the top are of sizes 3, 5, 9, 15, 25, 35, 45, and 55 pixels, respectively; their borders are 25 pixels apart. The letters at the bottom range in size from 10 to 24 points, in increments of 2 points; the large letter at the top is 60 points. The vertical bars are 5 pixels wide and 100 pixels high; their separation is 20 pixels. The diameter of the circles is 25 pixels, and their borders are 15 pixels apart; their intensity levels range from 0% to 100% black in increments of 20%. The background of the image is 10% black. The noisy rectangles are of size 50 × 120 pixels.

a b
c d
e f



$$\frac{1}{9} \times \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad \frac{1}{16} \times \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array}$$

a b

FIGURE 3.32 Two 3 × 3 smoothing (averaging) filter masks. The constant multiplier in front of each mask is equal to 1 divided by the sum of the values of its coefficients, as is required to compute an average.

Example: 2D convolution

Original
Cameraman

Cameraman blurred by convolution

Filter impulse response

$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Slides cameraman from EE368 (Bern Girod)

Example: Horizontal Convolution

Original
Cameraman

Cameraman blurred horizontally
Filter impulse response

$$\frac{1}{5}\begin{pmatrix}1 & 1 & [1] & 1 & 1\end{pmatrix}$$

Examples: Vertical Convolution

Original
Cameraman

Cameraman blurred vertically
Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$

Linear Filters

Filters:

- Linear
- Non-linear

Domain of operation:

1. Spatial domain
2. Frequency domain

Filtering in the Spatial Domain

$$\frac{1}{9} \times \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad \frac{1}{16} \times \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array}$$

Linear filter: convolution

$$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

$$g(x,y) = \sum_{s=-a}^{a} w_x(s) \sum_{t=-b}^{b} w_y(t) f(x-s, y-t) \quad \textit{if separable}$$

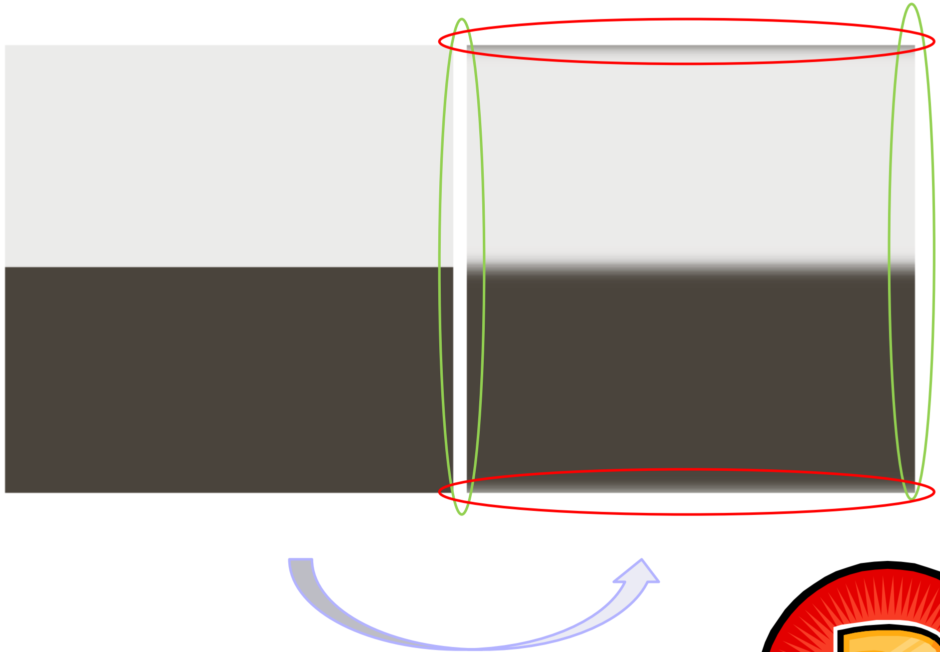
- Linear filter: weighted average of the samples in the window
- Set of weights: *filter mask*
- Separable filters:
  - $w(x,y) = w_x(x) w_y(y)$
  - Apply first on rows then on columns or vice-versa
  - *O(MN(a+b))* instead of *O(MNab)* : *faster specially for large windows*

Filtering in the Frequency Domain

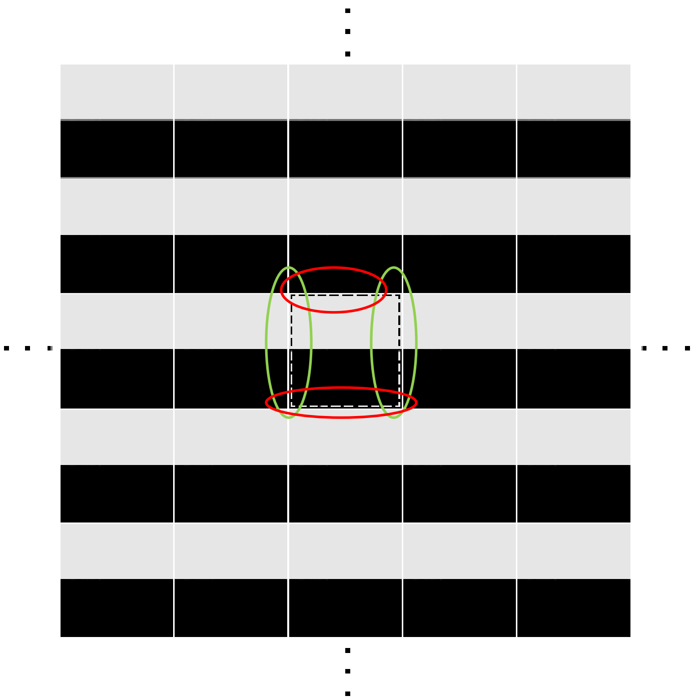
Remember:

- Implicit periodicity: *wraparound error*
- *H(u,v)* typically specified for spectrum centered in *(M/2,N/2)*
- Images are functions with real values

A Critical Issue….



Frequency domain
Gaussian low-pass filter



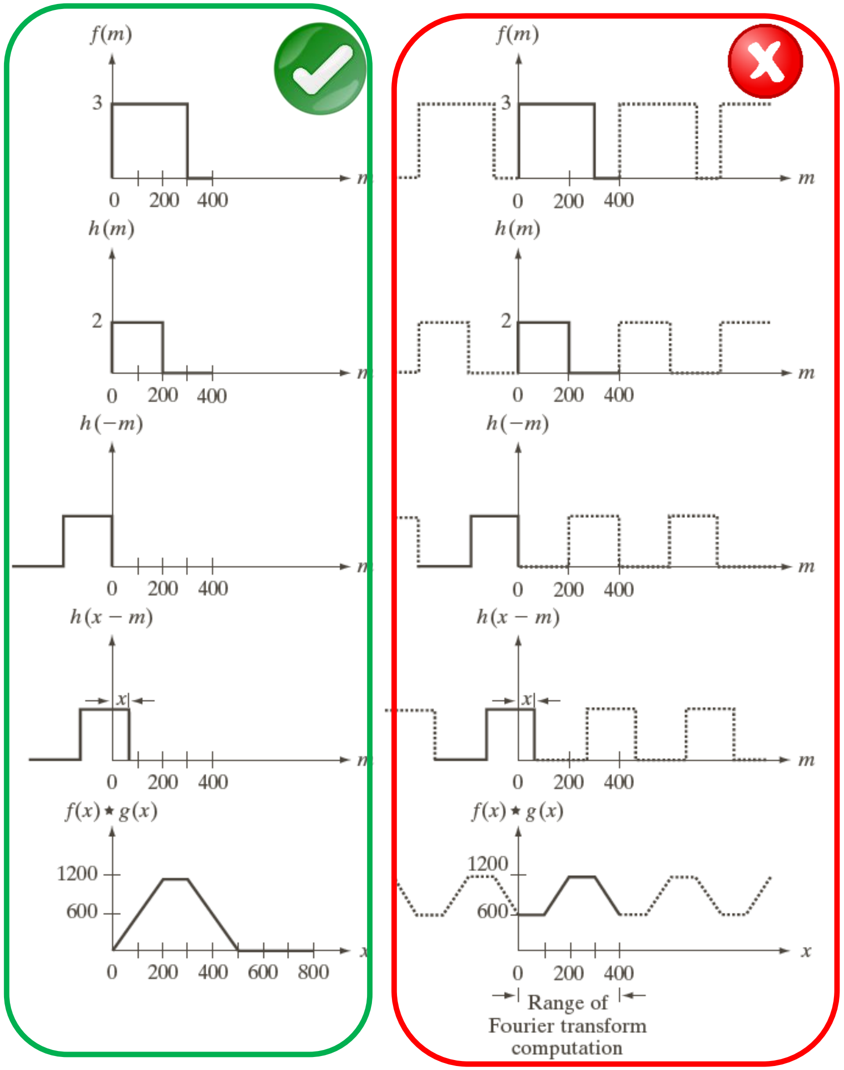
Frequency domain: implict periodicity of the image signal

Convolution in the Spatial and Frequency domains

$$f(x) * h(x) = \sum_{m=0}^{399} f(m)h(x-m)$$

$$f(x) = \begin{cases} 1 & m < 300 \\ 0 & otherwise \end{cases}$$

$$h(x) = \begin{cases} 1 & m < 200 \\ 0 & otherwise \end{cases}$$



a f
b g
c h
d i
e j

FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

Avoid Wraparound Error (1D)

Start from 1D case:

$$f(x)\text{: length } A$$
$$g(x)\text{: length } C$$

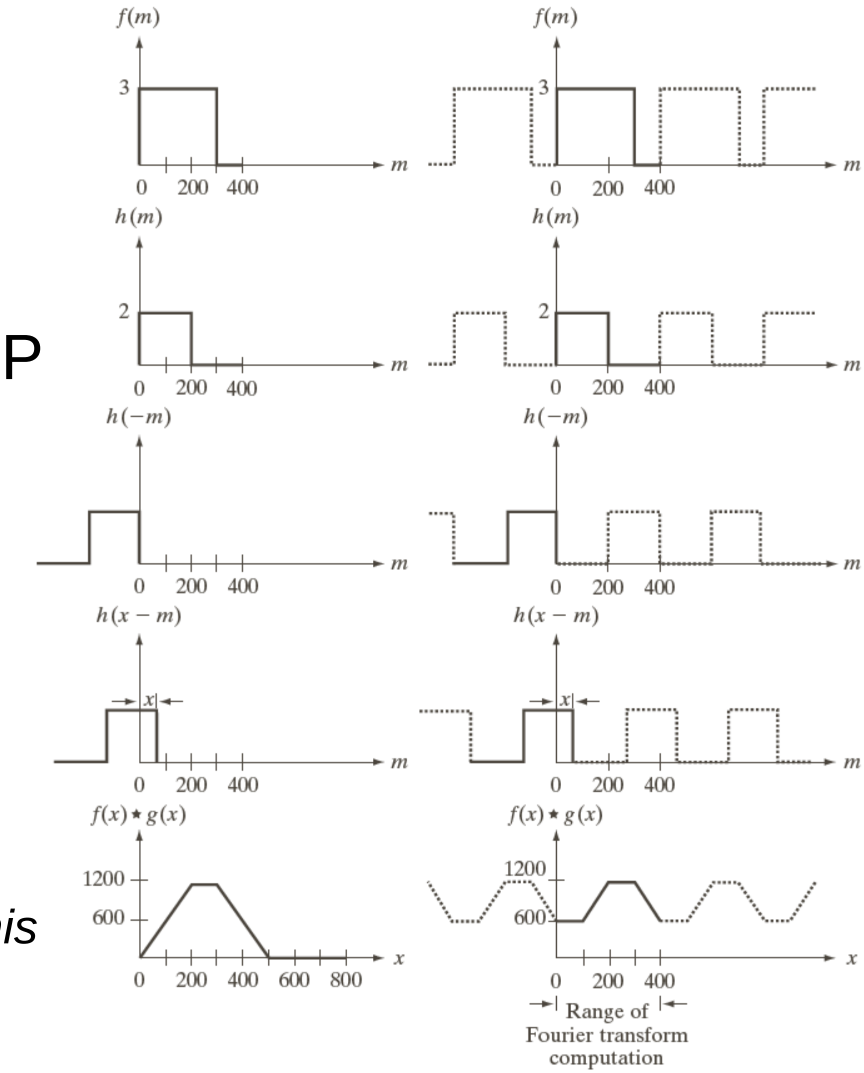
Pad both image and filter to length P

$$P \geq A + C - 1$$

$$f(x) = \begin{cases} f(x) & for\ 0 \leq x \leq A-1 \\ 0 & for\ A \leq x \leq P \end{cases}$$

$$h(x) = \begin{cases} h(x) & for\ 0 \leq x \leq C-1 \\ 0 & for\ C \leq x \leq P \end{cases}$$

(for faster computation pad only image but in this case some wraparound error remains)

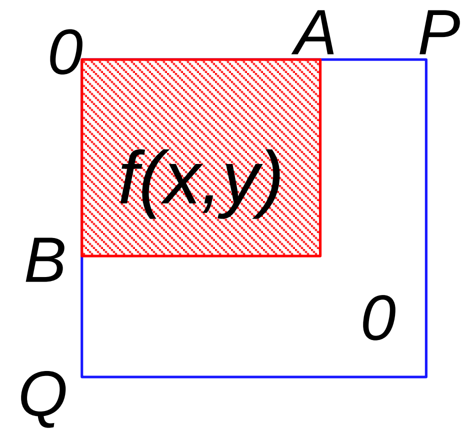


Avoid Wraparound Error (2D)

2D (image) case:

$f(x,y)$: $length\ AxB$

$h(x,y)$: $length\ CxD$



Pad to

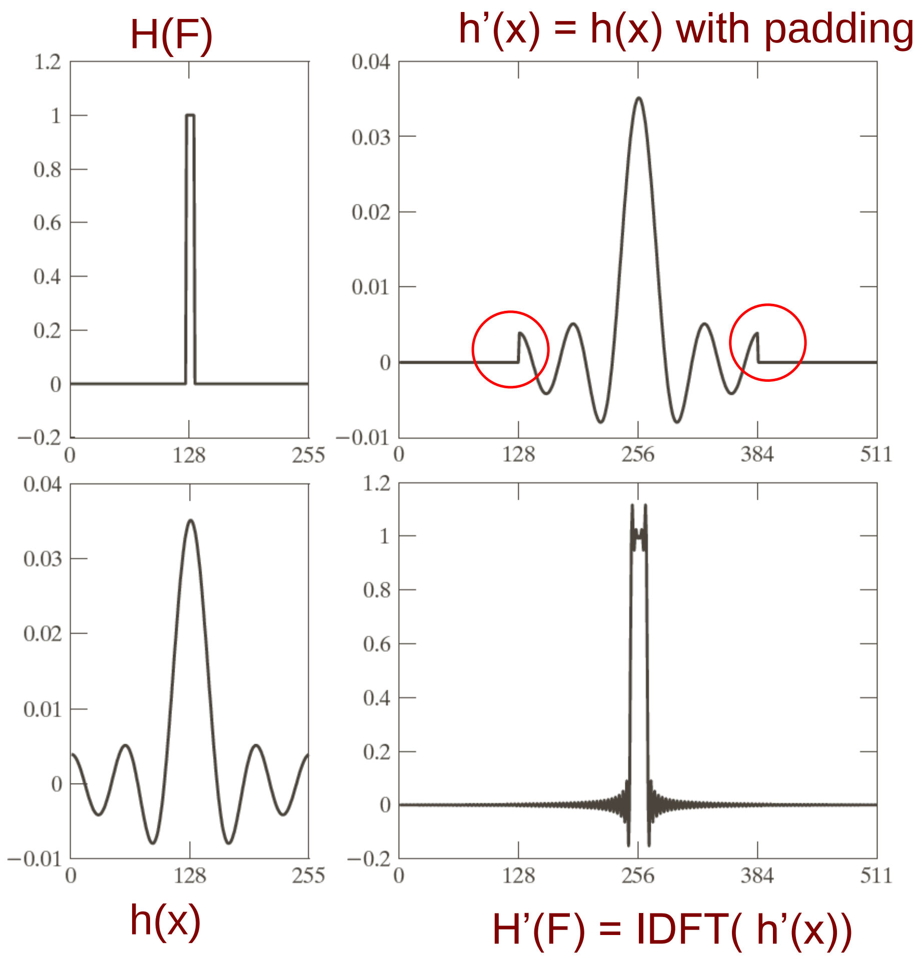
$P \geq A + C - 1$

$Q \geq B + D - 1$

$$f_p(x,y) = \begin{cases} f(x,y) & for\ 0 \leq x \leq A-1\ and\ 0 \leq x \leq B-1 \\ 0 & for\ A \leq x \leq P\ or\ B \leq y \leq Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y) & for\ 0 \leq x \leq C-1\ and\ 0 \leq x \leq D-1 \\ 0 & for\ C \leq x \leq P\ or\ D \leq y \leq Q \end{cases}$$

Padding can Cause Ringing Artifacts



a c
b d

FIGURE 4.34
(a) Original filter specified in the (centered) frequency domain. (b) Spatial representation obtained by computing the IDFT of (a). (c) Result of padding (b) to twice its length (note the discontinuities). (d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). (The curves appear continuous because the points were joined to simplify visual analysis.)

Filtering in the Frequency Domain

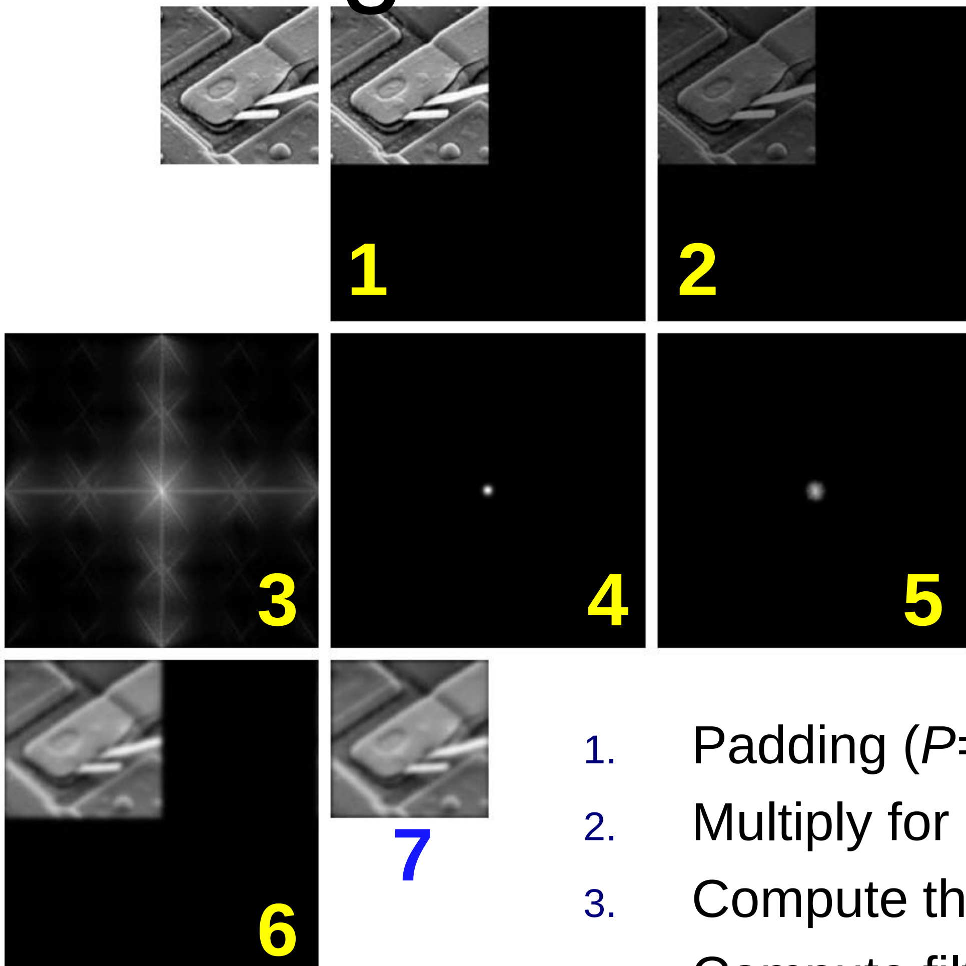
Input:

- Image: $f(x,y)$ size MxN
- Filter $H(u,v)$ size PxQ

Output : $g(x, y) = f(x, y) * h(x, y)$

1. Padding ($P=2M$, $Q=2N$)
2. Multiply for $(-1)^{x+y}$ (center FT)
3. Compute the DFT $F(u,v)$
4. Compute filter response $H(u,v)$ of size PxQ
5. $G = FH$
6. $g_p(x,y) = \{ Re[IDFT(G(u,v))] \} (-1)^{x+y}$
7. $g(x,y)$ (the result) is the upper-left MxN region

Filtering in the Frequency Domain

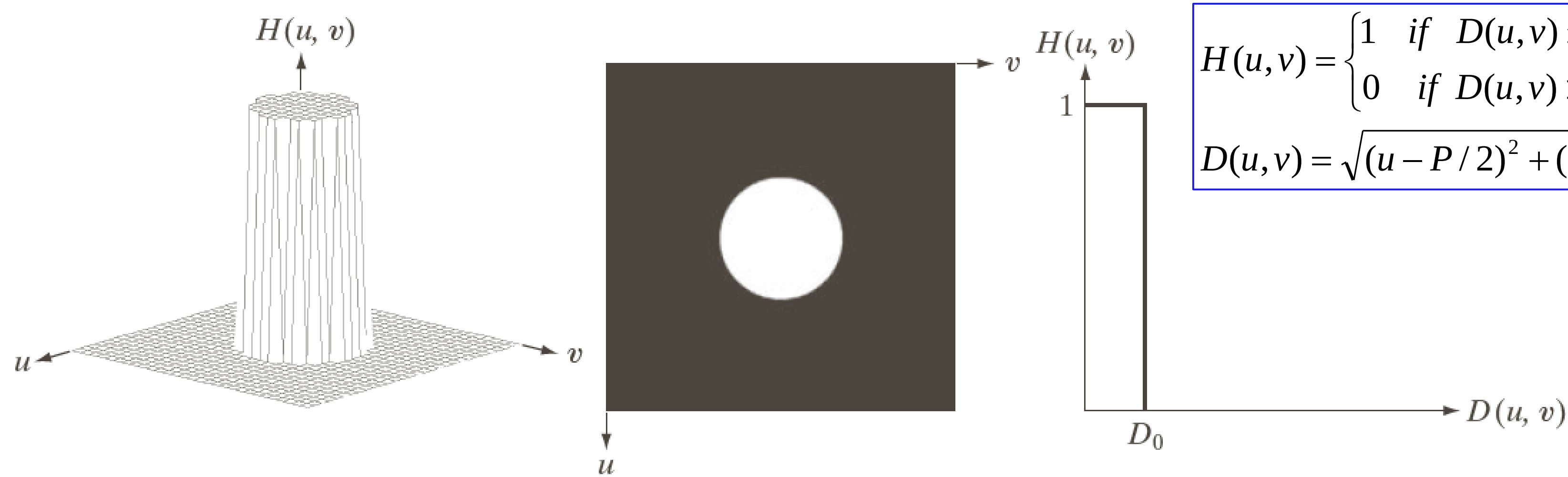


a	b	c
d	e	f
g	h	

FIGURE 4.36 (a) An $M \times N$ image, f. (b) Padded image, f_p of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$. (f) Spectrum of the product HF_p. (g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p. (h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

1. Padding (*P=2M, Q=2N*)
2. Multiply for $(-1)^{x+y}$ (center FT)
3. Compute the DFT $F(u,v)$
4. Compute filter response $H(u,v)$ of size PxQ
5. $G = FH$
6. $g_p(x,y) = \{ Re[IDFT(G(u,v))] \} (-1)^{x+y}$
7. $g(x,y)$ is the upper-left MxN region

Ideal Low-Pass Filter

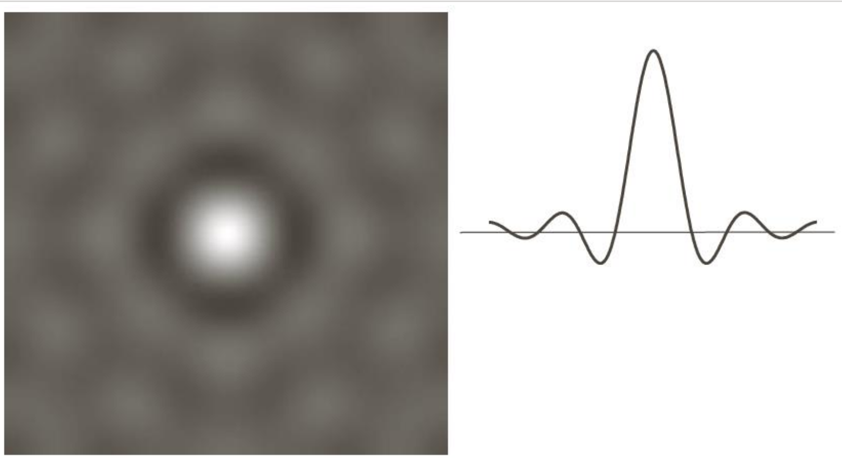


$$H(u,v)=\begin{cases}1 & if \quad D(u,v)\le D_0\\ 0 & if \quad D(u,v)>D_0\end{cases}$$

$$D(u,v)=\sqrt{(u-P/2)^2+(v-Q/2)^2}$$

a b c

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

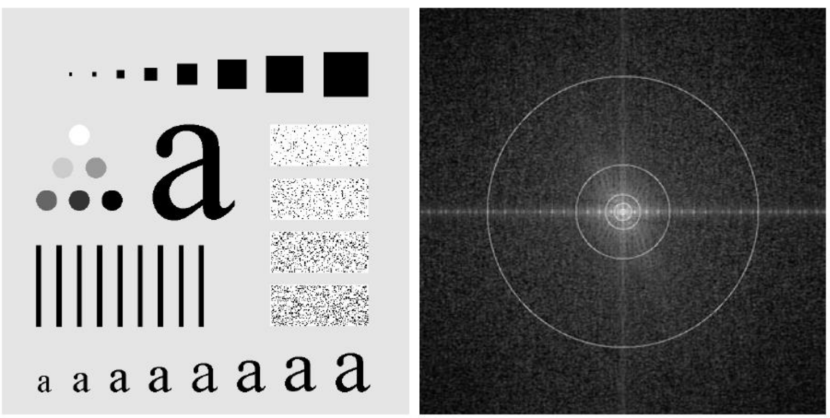


a b

FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000×1000. (b) Intensity profile of a horizontal line passing through the center of the image.

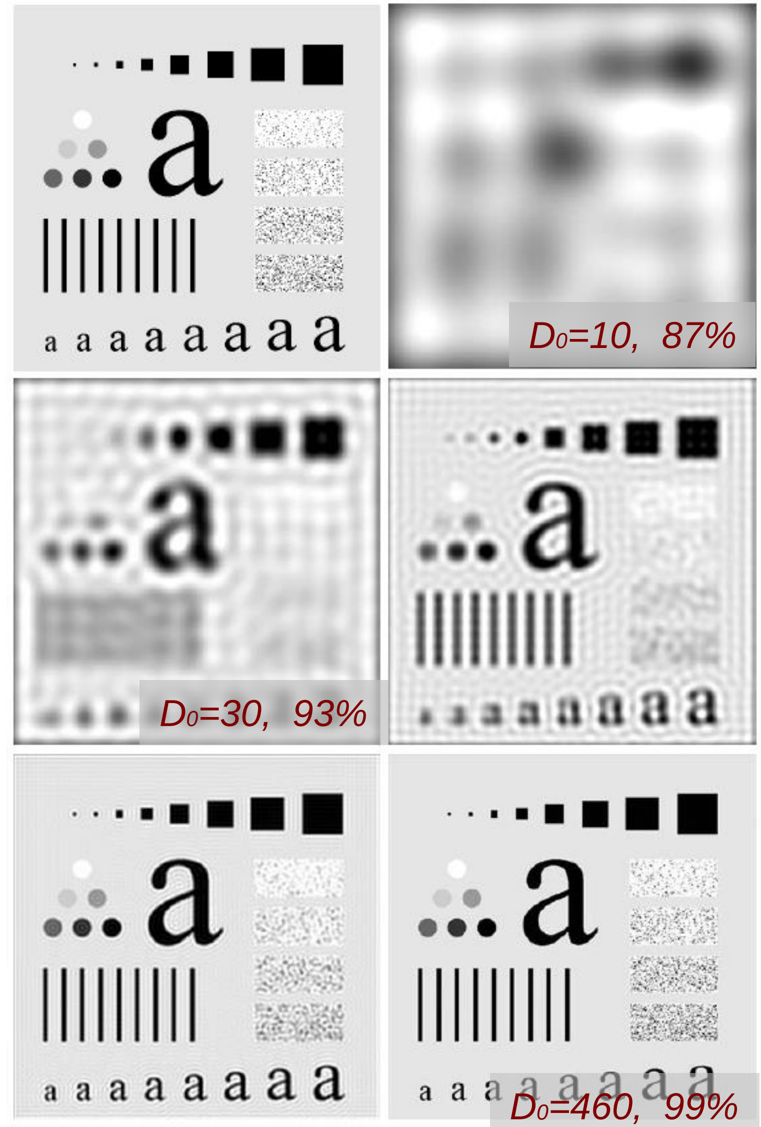
Ideal Low-Pass Filter

(examples of application)



a b

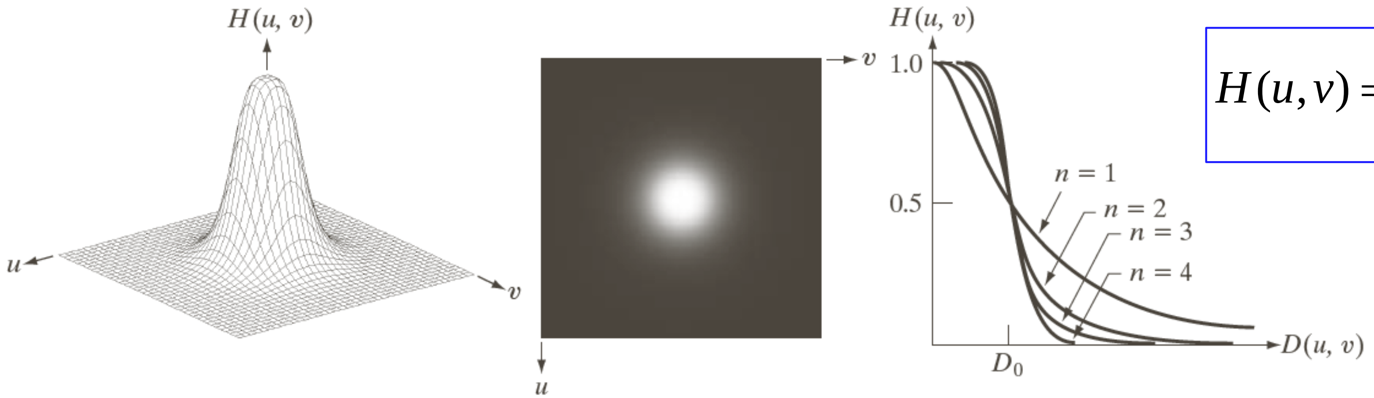
FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

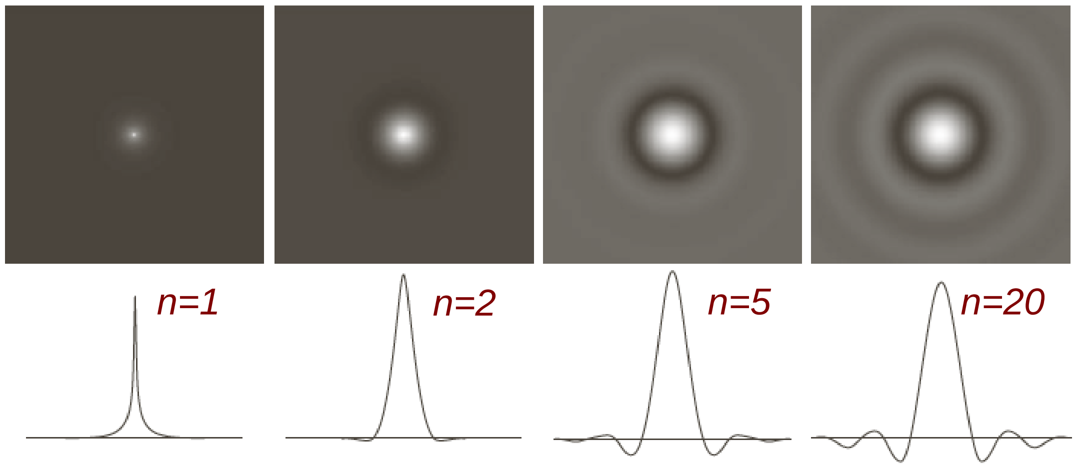
Butterworth Low Pass Filter



$$H(u,v) = \frac{1}{1+(D(u,v)/D_0)^{2n}}$$

a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.



The ringing effect grows with the order of the filter

a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

Butterworth
(examples of application)

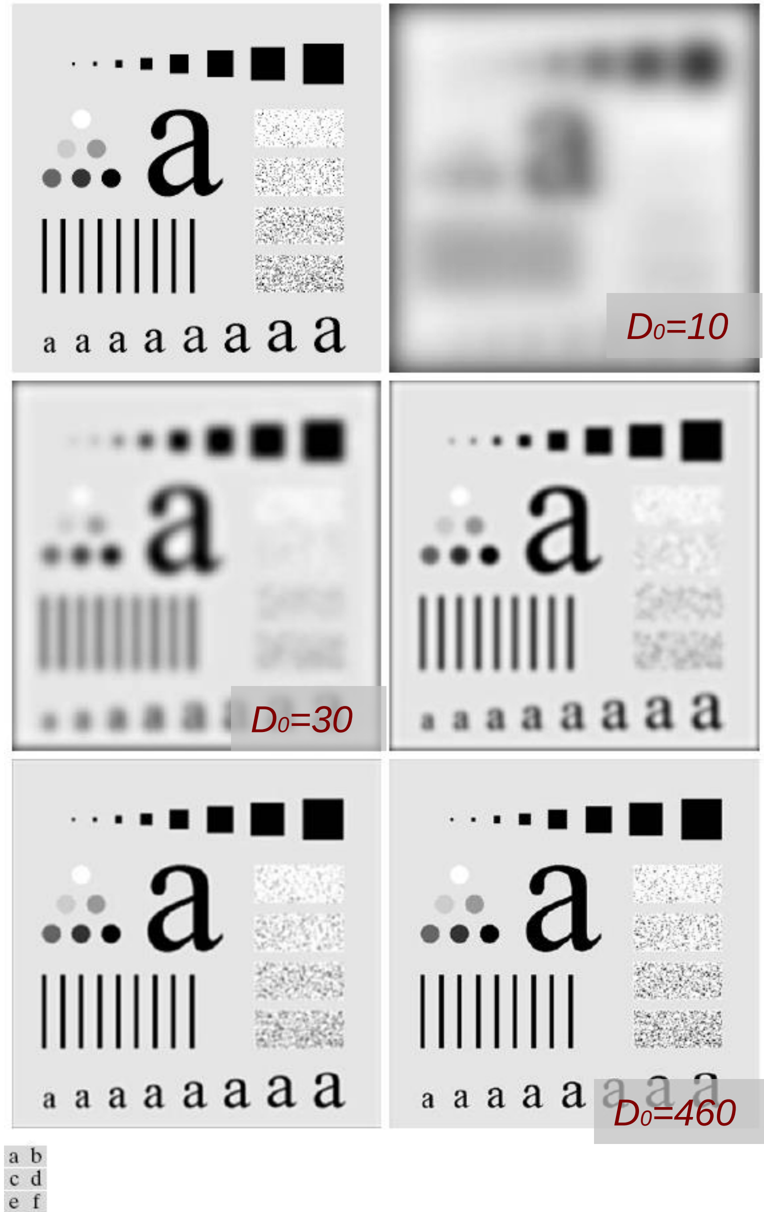


FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Gaussian Low Pass Filter

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

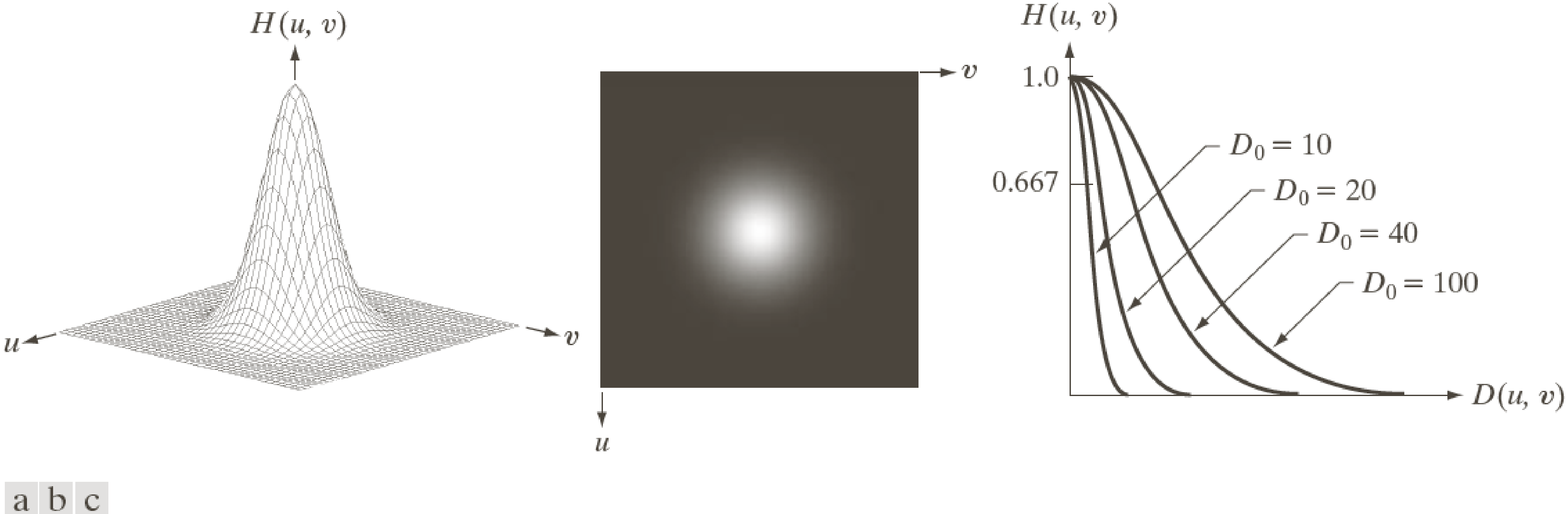
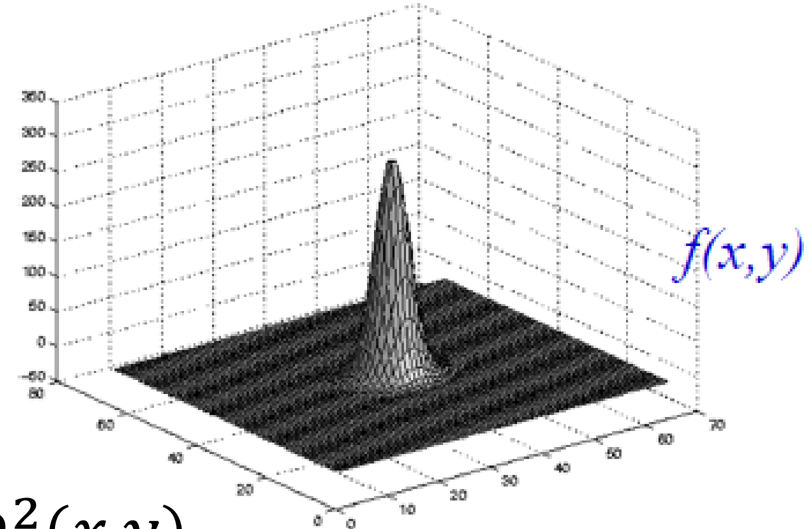


FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

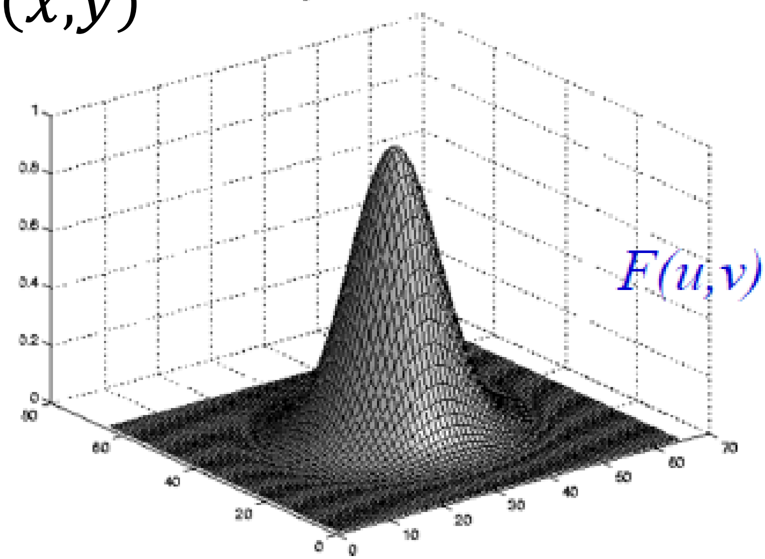
There's no ringing !

Gaussian Filter: Frequency Domain

Gaussian centred on origin

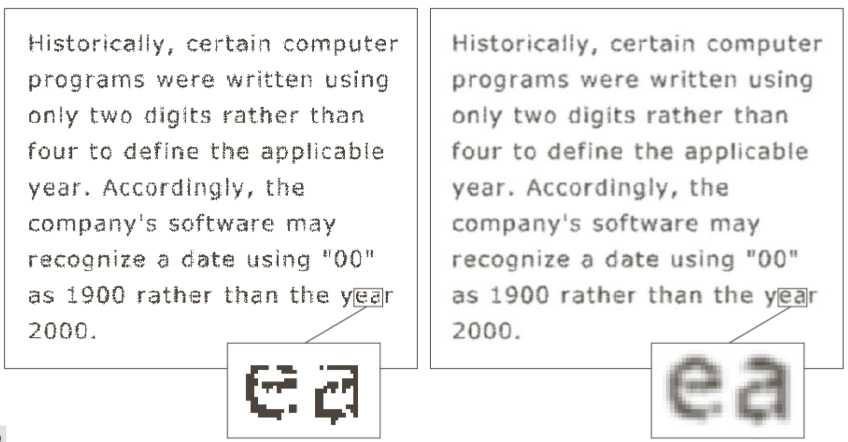


$$e^{-\frac{D^2(u,v)}{2\sigma^2}} \overset{\mathcal{F}}{\Leftrightarrow} 2\pi\sigma^2 e^{-2\pi^2\sigma^2 D^2(x,y)}$$



- FT of a Gaussian is a Gaussian
- Note inverse scale relation

Gaussian Filter:
examples of application (1)



a b

FIGURE 4.49 (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

a b
c d
e f

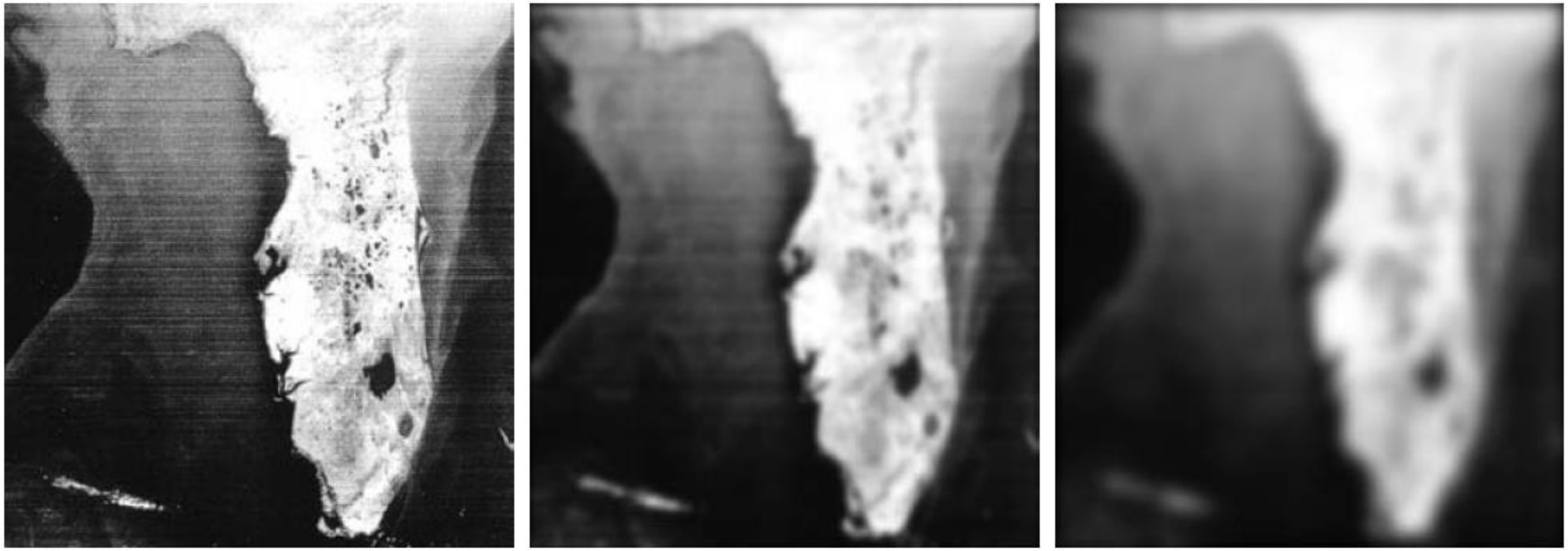
FIGURE 4.48 (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

Gaussian: Examples of Application (2)

a b c

FIGURE 4.50 (a) Original image (784×732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Gaussian: Examples of Application (3)



a b c

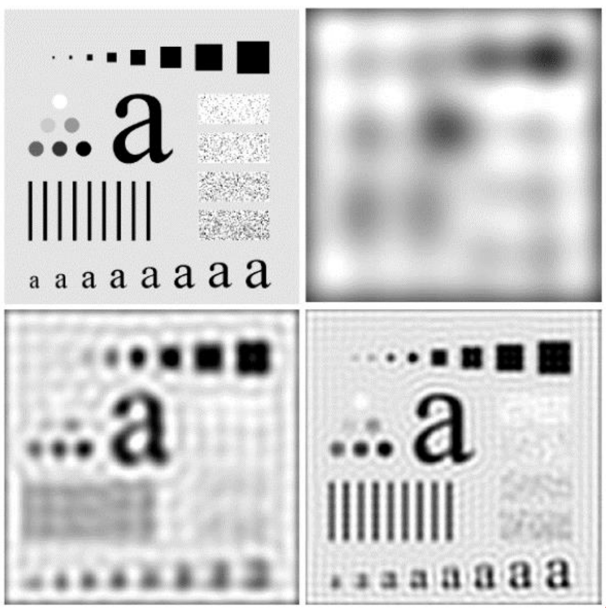
FIGURE 4.51 (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

Summary: Low Pass Filters

TABLE 4.4

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$



High Pass Filters (*sharpening*)

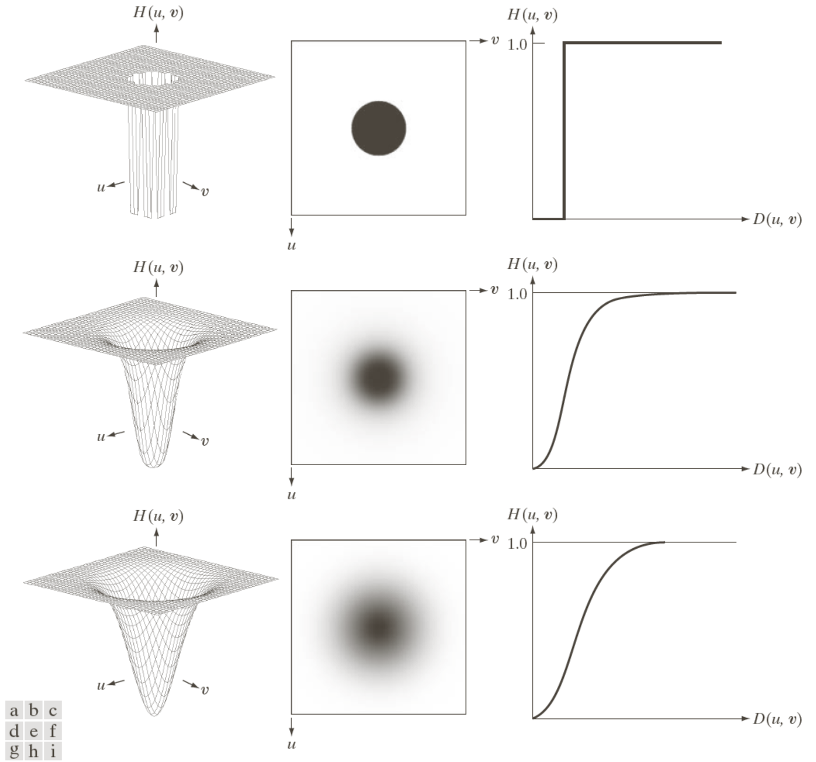


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

TABLE 4.5
Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

High pass filtering in the spatial domain

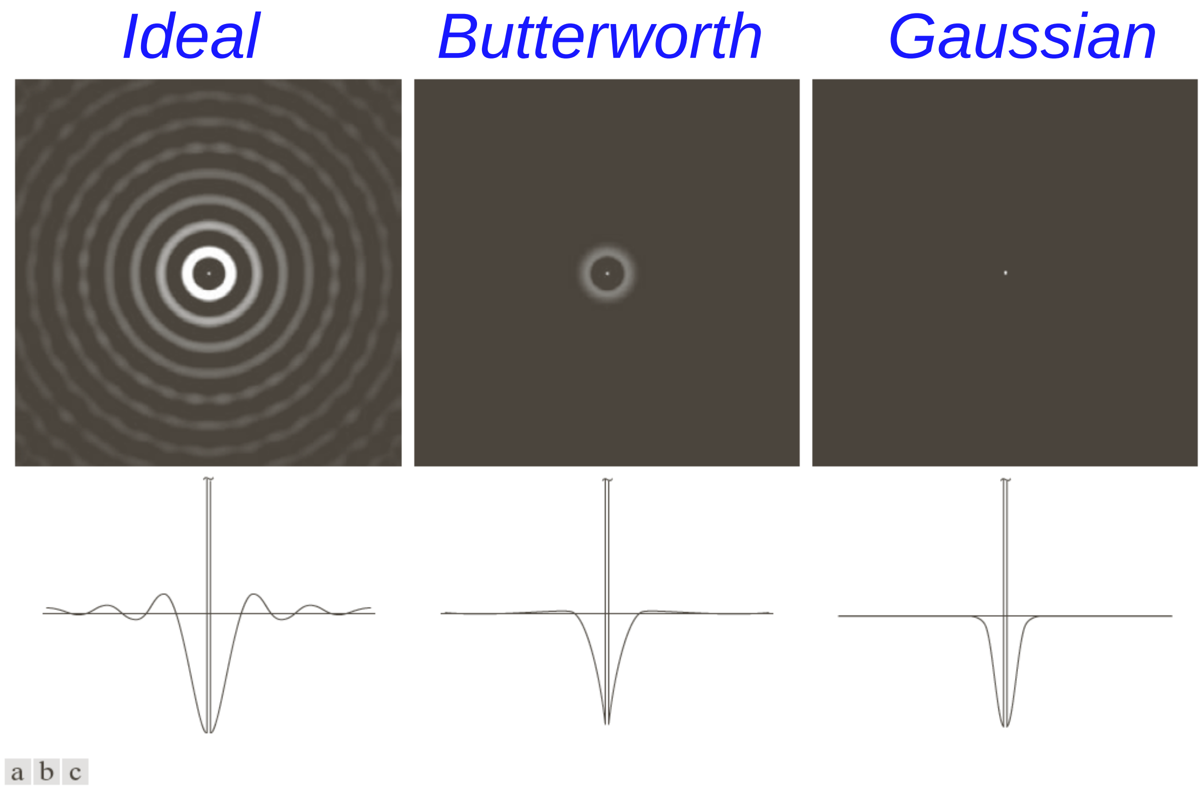


FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Ringing as in the low pass case!

Examples: Ideal High-Pass Filter

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60,$ and 160.

$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \le D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$

Examples: Butterworth High-Pass Filter

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

$$H(u, v) = \frac{1}{1 + [D_0/D(u, v)]^{2n}}$$

Examples: Gaussian High-Pass Filter

a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

$$H(u,v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}}$$

Low-Pass and High-Pass Filters (Summary)

TABLE 4.4
Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

low pass

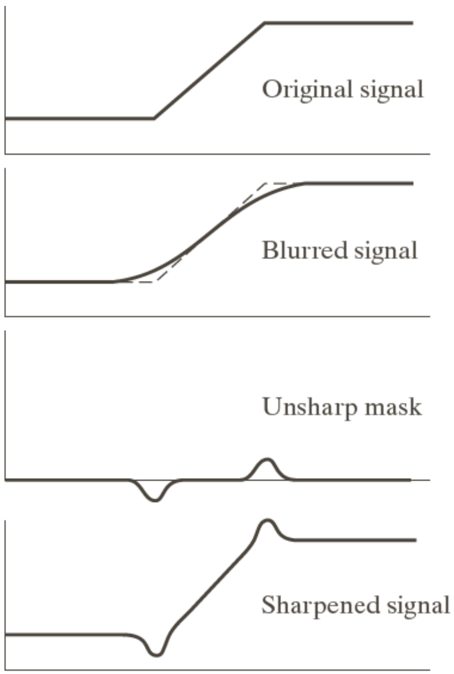
Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

TABLE 4.5
Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

high pass

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

Unsharp Masking



a
b
c
d

FIGURE 3.39 1-D illustration of the mechanics of unsharp masking. (a) Original signal. (b) Blurred signal with original shown dashed for reference. (c) Unsharp mask. (d) Sharpened signal, obtained by adding (c) to (a).

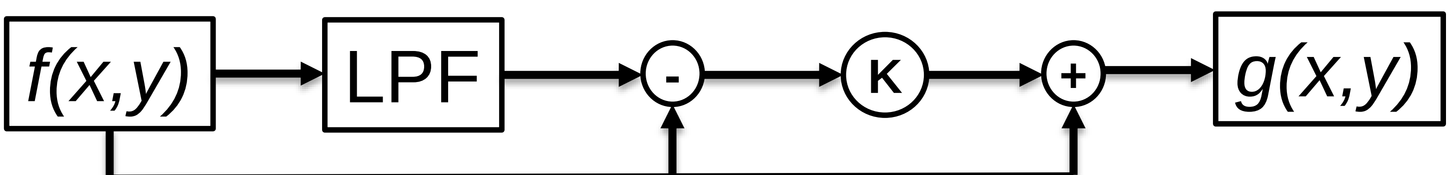
f(x,y) : image to be filtered
h_{lp}(x,y) : low-pass filter
k : constant
- *k = 1 : unsharp mask*
- *k>1 : highboost*

$$1.\ f_{lp}(x,y) = f(x,y) * h_{lp}(x,y)$$

$$2.\ m(x,y) = f(x,y) - f_{lp}(x,y)$$

$$3.\ g(x,y) = f(x,y) + km(x,y)$$

f
f_{lp}
m
g



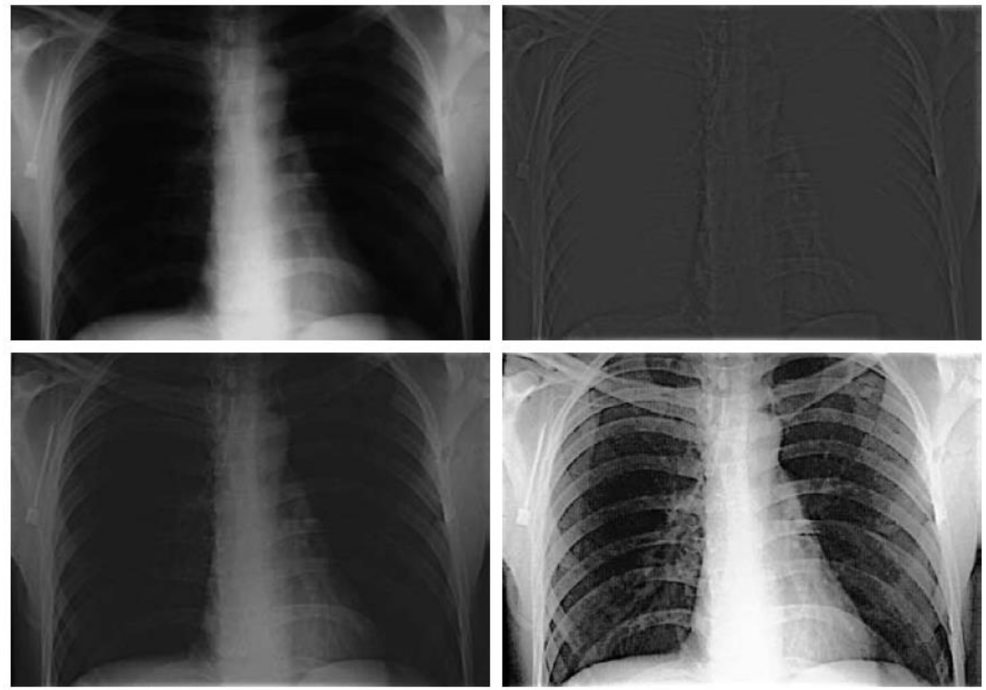
1. Low-pass filtering of the image
2. Subtract filter output from the original image (result = mask)
3. Add mask to the original image

Gonzalez 3.6.3 and 4.9.5

Unsharp Masking: Examples

a
b
c
d
e

FIGURE 3.40 (a) Original image. (b) Result of blurring with a Gaussian filter. (c) Unsharp mask. (d) Result of using unsharp masking. (e) Result of using highboost filtering.

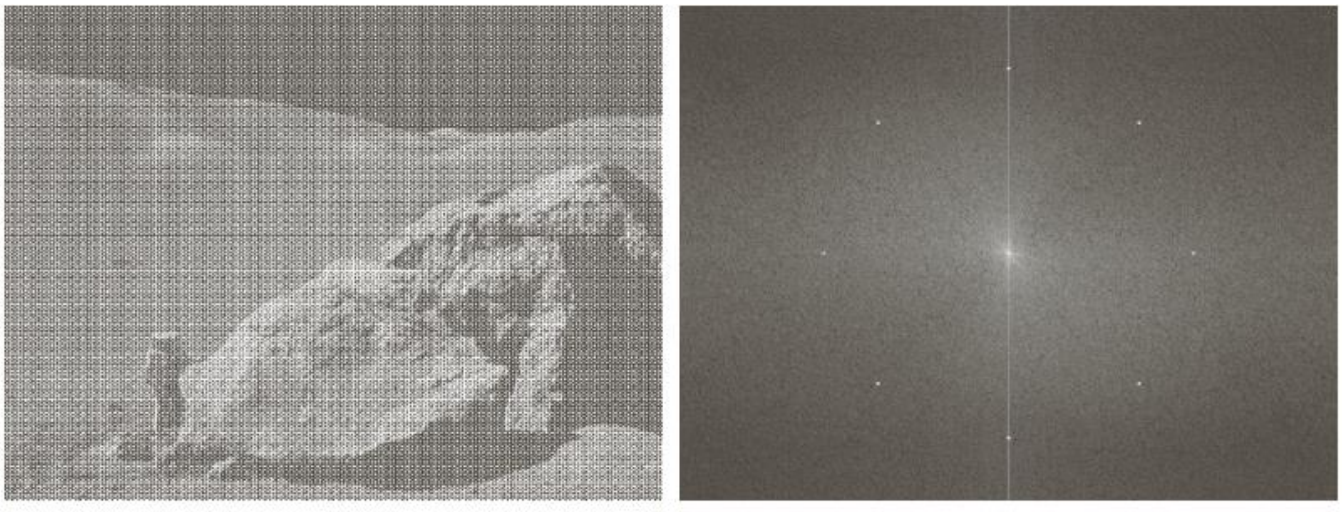


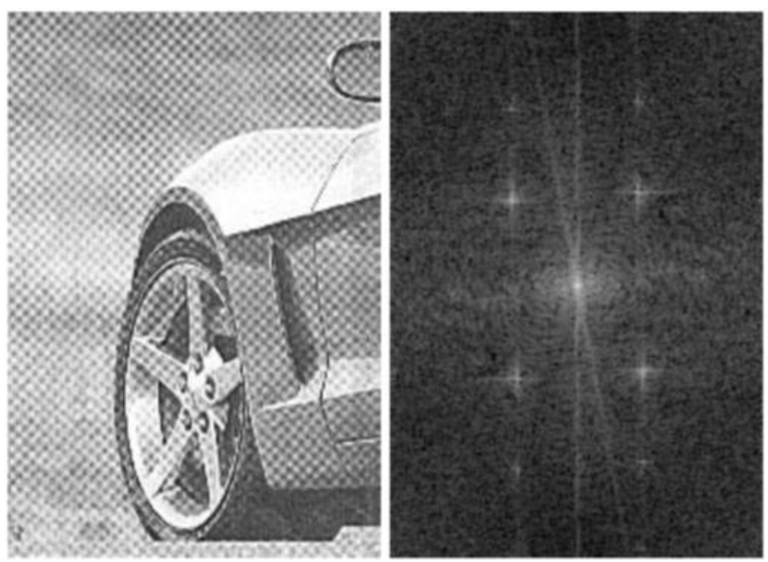
a b
c d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Emphasis HF: $G(u,v) = F(u,v)* [1+k[1-H_{lp}(u,v)]] = F(u,v)*[1+kH_{hp}(u,v)]$

What Should We Do… ?





Band-Pass and Band-Reject Filters

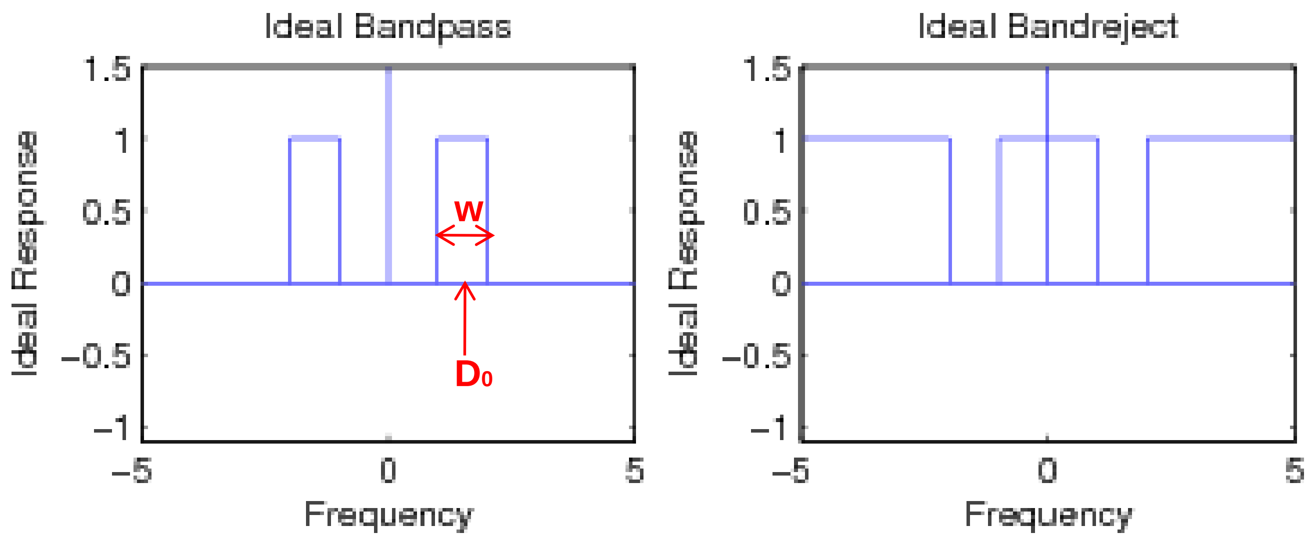


TABLE 4.6
Bandreject filters. W is the width of the band, D is the distance $D(u, v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u, v)$ to simplify the notation in the table.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

Gonzalez 4.10

Band-Pass and Band-Reject Filters

a b c Ideal Butterworth (n=1) Gaussian

FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.

TABLE 4.6
Bandreject filters. W is the width of the band, D is the distance $D(u, v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u, v)$ to simplify the notation in the table.

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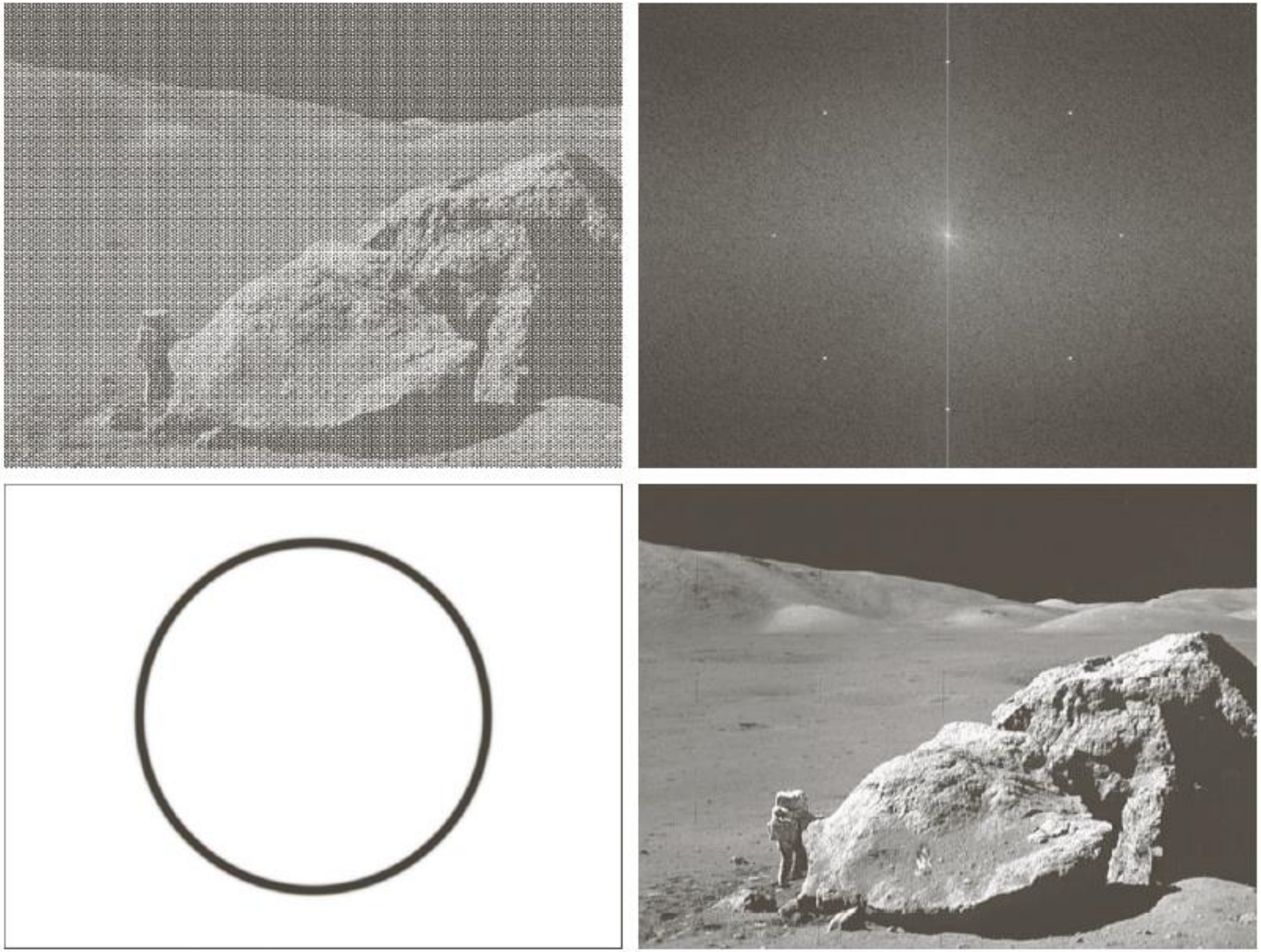
$$H_{bp}(u,v) = 1 - H_{br}(u,v)$$

Gonzalez 4.10

Band-Reject Filters



a b
c d

FIGURE 5.16 (a) Image corrupted by sinusoidal noise. (b) Spectrum of (a). (c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

Band-Reject and Notch Filters

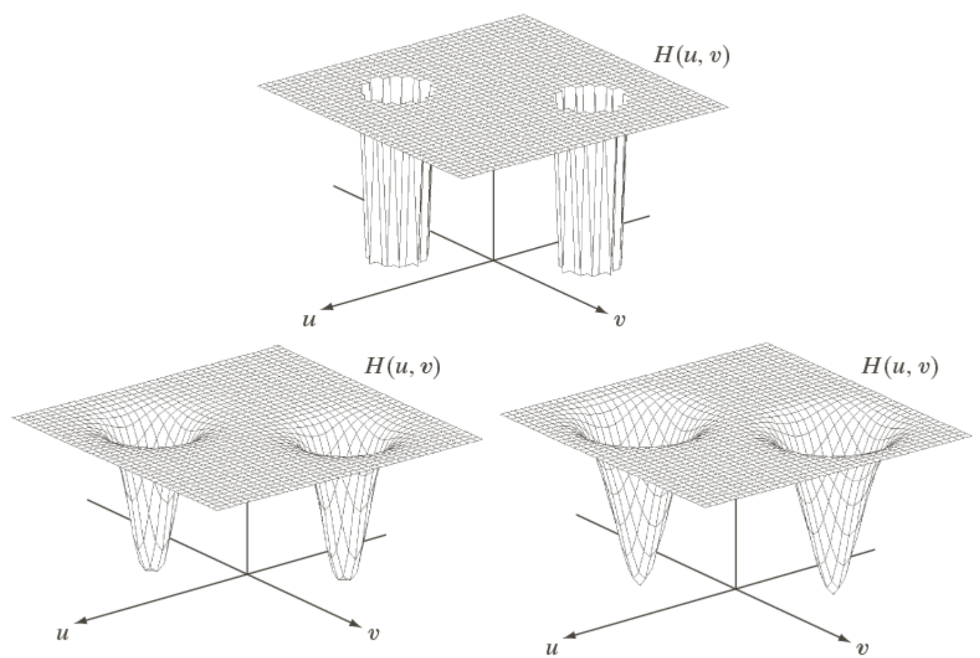
Ideal Butterworth (n=1) Gaussian

a b c

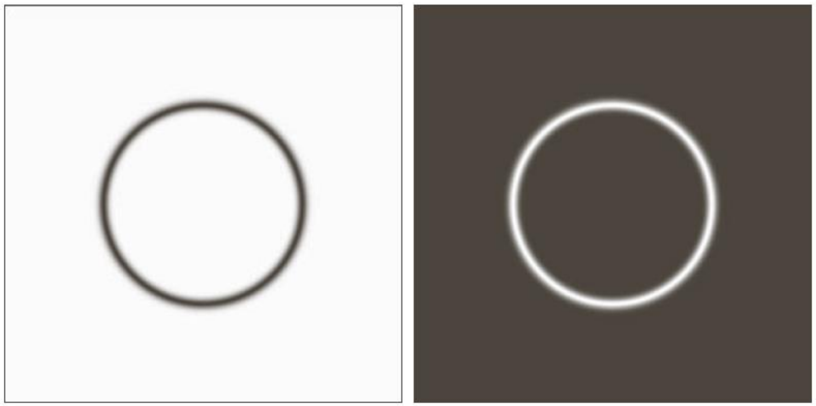
FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.

a
b c

FIGURE 5.18
Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.



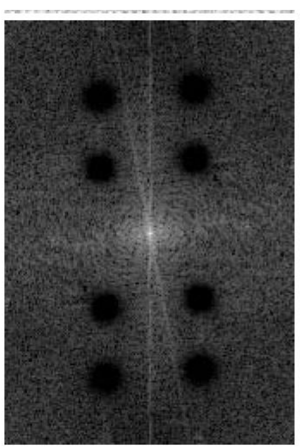
Band-Pass and Notch Filters: Examples



Band reject *Band pass*

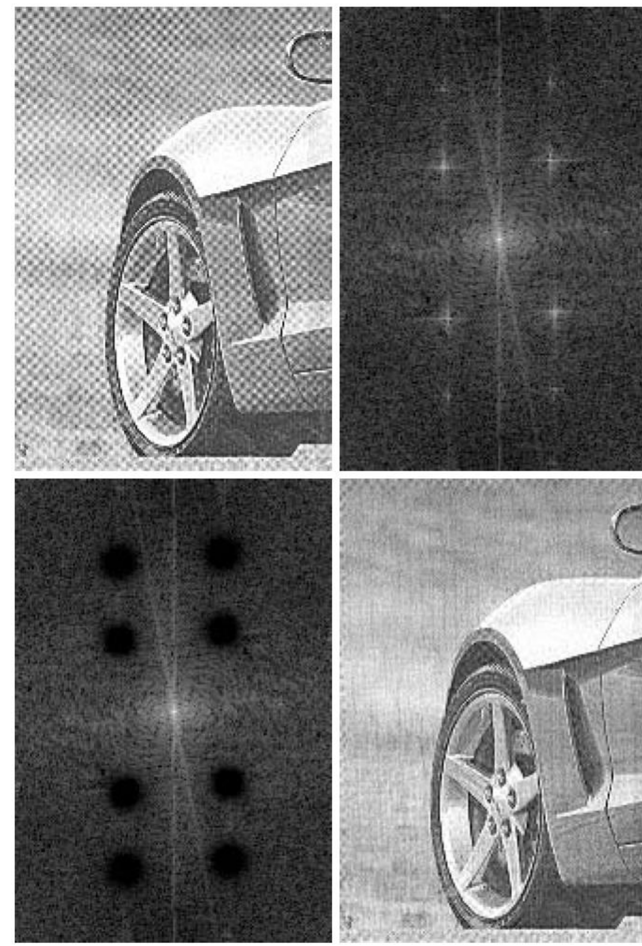
a b

FIGURE 4.63
(a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.



Notch

$$H_{NR}(u,v) = \prod_{k=1}^{4} \left[\frac{1}{1+\left[\frac{D_{0k}}{D_k(u,v)}\right]^{2n}}\right]\left[\frac{1}{1+\left[\frac{D_{0k}}{D_{-k}(u,v)}\right]^{2n}}\right]$$

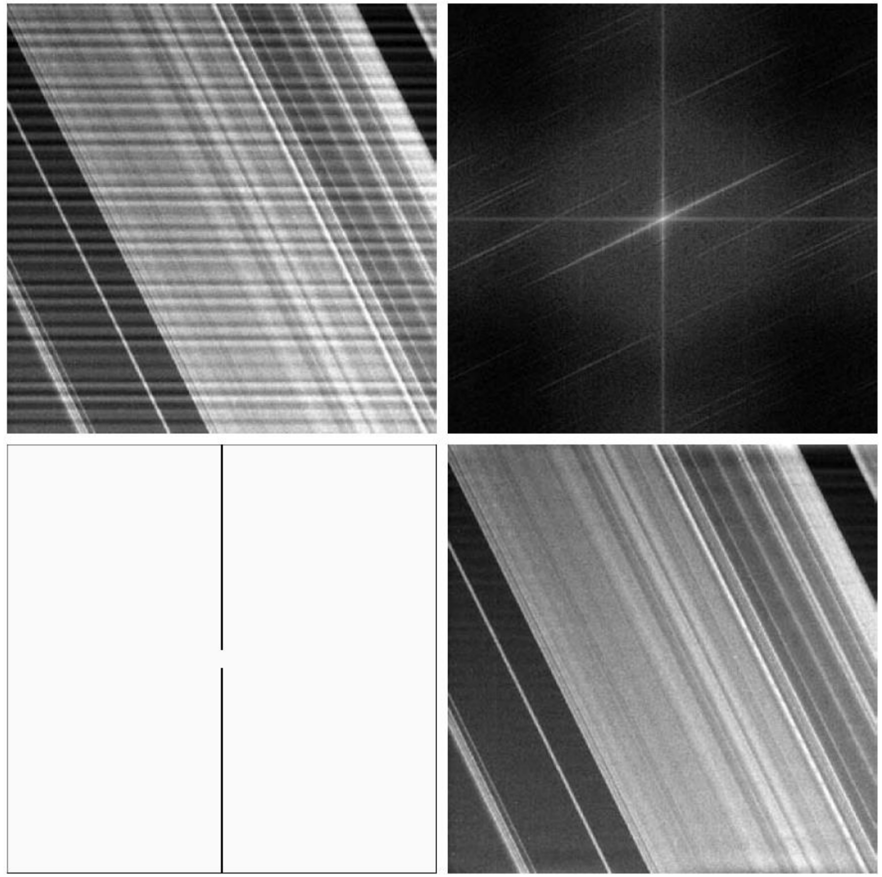


a b
c d

FIGURE 4.64
(a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

Notch Filter

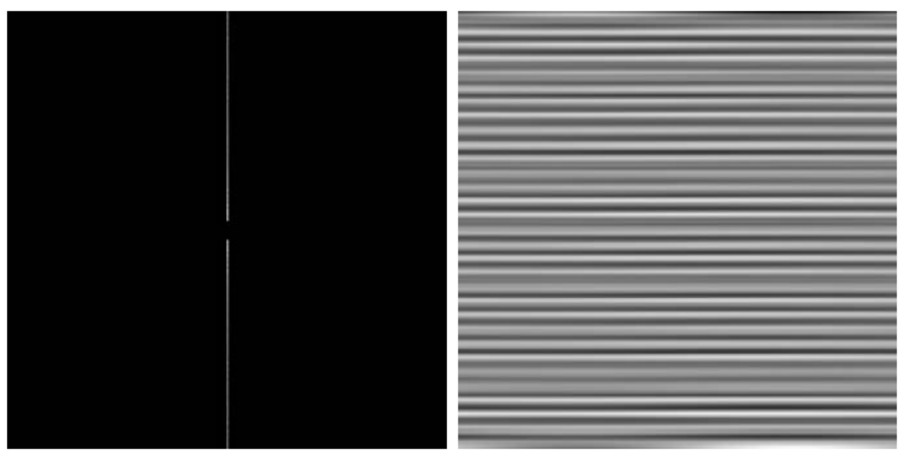
notch reject



a b
c d

FIGURE 4.65
(a) 674 × 674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

notch pass



a b

FIGURE 4.66
(a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).