**Introduction**

The Cox-Ross-Rubinstein (CRR) model is a popular option pricing model used in the field of finance. It provides a discrete-time framework for valuing options and is an extension of the original Black-Scholes-Merton model. The CRR model is particularly well-suited for pricing options on stocks that pay dividends and can handle situations where the underlying asset price fluctuates over time in a binomial manner.

**The Cox-Ross-Rubinstein model for option pricing;**

The Cox-Ross-Rubinstein (CRR) model is a widely used method for pricing options.

The model is a binomial tree model that provides a discrete-time framework for option pricing.

The CRR model assumes that the underlying asset can either go up or down by a certain amount at each time step. This is represented by a binomial tree, where each node represents a possible price of the underlying asset at a particular time. The model then works backwards from the final time period, where the option expires, to calculate the option price at each node.

To calculate the price of the option at a particular node, the model takes into account the probabilities of the underlying asset going up or down, as well as the expected value of the option at the next time step. The expected value is calculated as a weighted average of the option prices at the two possible nodes in the next time step, with the weights being the probabilities of the underlying asset going up or down.

The stock price process in the CRR model is defined via an initial value S0 > 0 and, for 1 ≤ t ≤ T and all ω ∈ Ω,

This formula is commonly used in option pricing models, including the Cox-Ross-Rubinstein (CRR) model, to determine the prices of options and understand the movement of the underlying asset throughout the binomial tree.

**Assumptions of the CRR Model:**

* The underlying asset follows a discrete-time process.
* The price of the underlying asset can only move up or down by specific factors.
* The risk-neutral probability is constant over time and is equal to the probability of an up move.
* There are no transaction costs or taxes.
* The risk-free interest rate is constant over time.

**Risk-Neutral Probability Measure**

The risk-neutral probability measure is a key concept in option pricing theory. It assumes that market participants are risk-neutral, meaning they do not require compensation for bearing risk. Under this assumption, the expected return on any investment should equal the risk-free rate.

The CRR model assumes that the risk-neutral probability of an up move in the underlying asset price is constant over time and is equal to the probability of a down move.

If d< 1+r < u then the CRR market model M = (B, S) is arbitrage-free and complete.

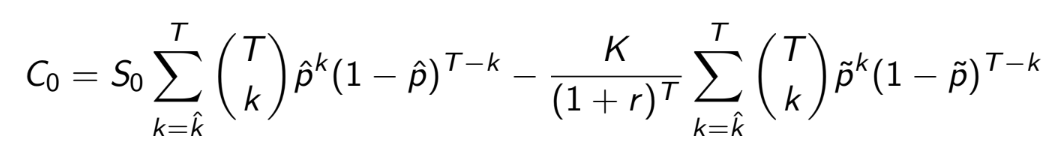
To calculate the expected value of the option payoffs in the model. It provides a probability measure that, when used to discount future cash flows, removes the effects of risk preferences and results in a risk-neutral valuation framework.

The formula calculates the risk-neutral probability πt(X) by dividing the discounted expected value of X at time t by the price of the zero-coupon bond at its maturity.

Since the CRR model is complete, the unique arbitrage price of any European contingent claim can be computed using the risk-neutral valuation formula.

**CRR Call Option Pricing Formula**

The arbitrage price at time t = 0 of the European call option CT = (ST − K) + in the binomial market model M = (B, S) is given by the CRR call pricing formula



**CRR Parametrization**

* [, where σ is the standard deviation of the underlying asset's returns and Δt is the length of each time period.](https://chat.openai.com/c/c1ac744d-996c-4f82-b021-74bbc4dcc7bd)
* [, representing the downward price movement.](https://chat.openai.com/c/c1ac744d-996c-4f82-b021-74bbc4dcc7bd)

**Put-Call Parity**

The Put-Call Parity relationship states that the difference between the price of a call option and the price of a put option is equal to the difference between the current price of the underlying asset and the present value of the strike price.

Put-Call Parity is derived from the principle of arbitrage. If a discrepancy arises between the prices of call and put options that violates the parity relationship, an arbitrage opportunity can be exploited.

It is possible to derive explicit pricing formula for the call option at any date t = 0, 1, . . .,T. Since *CT* − *PT* = *ST* − *K,* we see that the following put-call parity holds at any date *t* = 0, 1, . . . , *T*

*Ct* − *Pt* = *St* − *K* (1 + *r* )−(*T* −*t*) = *St* − *KB* (*t*, *T* )

*Where B* (*t*, *T)* = (1 + *r)*−(*T* −*t*) is the price at time *t* of zero-coupon bond maturing at *T* .

Using the put-call parity, one can derive an explicit pricing formula for the European put option with the payoff *PT =* (*K* − *ST* )+.

Once the price of the call (put) has been determined, relation can be used to calculate the price of the put (call) having the same characteristics.

**Jarrow-Rudd approach**

The Jarrow-Rudd approach is an extension of the original Cox-Ross-Rubinstein (CRR) model for option pricing. The Jarrow-Rudd approach aims to improve the accuracy and convergence of option prices compared to the CRR model, especially when the time intervals between steps are small. By incorporating continuous-time dynamics and a smoothing technique, it provides a more refined and precise estimation of option prices, making it a valuable tool for option pricing and risk management in financial markets.

This formula is derived from the continuous-time geometric Brownian motion equation used to model the underlying asset's price dynamics. It incorporates the risk-free interest rate, volatility, and the length of the time step to calculate the factor u or d, which represents the upward/downward movement in the Jarrow-Rudd model.

**Comparing American and European Call and Put Options**

The main difference between American and European options in the CRR model is the method used to calculate their value. American options require backward induction to determine the optimal exercise strategy, while European options can be valued using a closed-form equation. Additionally, because American options can be exercised at any time before expiration, they tend to be more valuable than European options with the same strike price and expiration date.

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| **American Call Options** | **European Call Options** |
| American Call/Put Option: An American call/put option gives the holder the right, but not the obligation, to buy/sell an underlying asset at a specified price (strike price) at any time before the option's expiration date. In the CRR model, the value of an American call/put option can be calculated using backward induction, which involves working backwards from the expiration date to the present date, and at each step determining whether to exercise the option or hold it.  Pros of American Call Options:   1. Flexibility: American call options offer the flexibility to exercise the option at any time before expiration, which can be advantageous if the underlying asset's price increases significantly before expiration. 2. Hedging: American call options can be used as a hedging tool to protect against price increases in the underlying asset. 3. Higher Premium: American call options tend to have higher premiums than European call options, which can provide the seller with higher potential profits.   Here we considered call options but for put options it will be the same. The advantages of the European call/put option are the disadvantages of the American and vice versa. | European Call/Put Option: A European call/put option also gives the holder the right, but not the obligation, to buy/sell an underlying asset at a specified price (strike price) but can only be exercised on the option's expiration date. In the CRR model, the value of a European call/put option can be calculated using a closed-form equation, which does not involve backward induction.  Pros of European Call Options:   1. Lower Premium: European call options tend to have lower premiums than American call options, which can make them more attractive for buyers. 2. Certainty: The fixed expiration date of European call options provides certainty for the option holder, which can help eliminate confusion and stress. 3. No Market Timing Risk: Since the option can only be exercised on the expiration date, the option holder does not face market timing risk. |

**How its Limiting Case converges to the Black-Scholes model**

One of the major advantages of the Cox, Ross and Rubinstein (CRR) model is its relative mathematical simplicity. This makes it easier to approach than the Black and Scholes (BS) model. However, if, in CRR, we divide time into a number of periods that tend to inﬁnity, then both models converge. Another major advantage that both these models offer is that the common method underlying both can be extended to any asset that results in random ﬁnancial ﬂows. The CRR and BS models were constructed to evaluate European buying options (calls), which can only be exercised at the time of maturity. The underlying support or asset for these options is a share that does not yield any dividends between the time the option is created and the time it matures. We also assume that the interest rates are constant over this period. In the CRR model, the period between the date the option was created, at *t* = 0, and its date of maturity, at *t* = *T*, can be divided into *n* number of periods that are arbitrarily chosen. The CRR model (constructed using the discrete time hypothesis) then becomes equivalent to the BS model. We propose the hypothesis that at each instant *t*∈[0*, T*], the support can change in only two ways. It may increase, being multiplied by a factor u>1, or it may diminish being multiplied by a factor d ∈[0*,*1]. This hypothesis is chosen not only for the CRR model but also, in an analogous form, in the BS model, where it is believed that the continuous process of changes in prices is a Brownian motion. Moreover, CRR and BS are operational models used by all ﬁnancial operators.

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Define the nth CRR model as a viable discrete-time market model with the following parameters, S0, n, un, dn, rn where n is the number of time steps to maturity that divides the time [0,T] into n equidistant parts and the change in time is given by Δn =.

On the other hand, the Black-Scholes model is a continuous-time market model in which the price of a stock St at any time t∈[0, T] is given by

, (1)

where r is the continuous time riskless rate, σ > 0 is the volatility of the stock and (is a Brownian motion under probability measure Q.

Time in the nth CRR model time is given by the sequence 0, Δn, 2Δn, …, nΔn (where nΔn = T), and the price of a bond at any time t = kΔn where k∈{0,1,2....,n} is given by

(2)

where rn > -1 is a fixed rate of return for period [0, kΔn].

The price of the same bond in the BS model at time t∈[0, T] and again, t= kΔn is given by

Bt = exp(rkΔn) (3)

Therefore rn+1 = exp(rΔn). To investigate the convergence of the CRR model to the Black-Scholes model we will approximate underlying stock prices St (where t∈[0, T]) in the Black-Scholes model with underlying stock prices (where t=kΔn and k=0,1,2,...,n) in the nth CRR model.

According to equation (1), the price of a stock St in the BS model given St-1 is given by

St = St-1exp(. (4)

To approximate the above stock price, Cutland and Roux (2013, p. 197) proposed approximating the Brownian increment in the above equation by a symmetric random walk that takes values or - at each time step. Therefore, the approximate stock price in the Black-Scholes model given is

= exp(. (5)

On the other hand, the price of a stock in the n\_{th} CRR model given is

= (6)

The “up” and “down” parameters of the nth CRR model are then given by

=exp( =exp( (7)

We can now use the above results to write the price of a stock in the nth CRR model given the initial price >0. This is given by

= exp(, (8)

where is a symmetric random walk that approximates the Brownian motion in equation (1) and

= (9)

The following theorem from Cutland and Roux (2013, p. 197) answers questions about the viability and the existence of one step risk-neutral probabilities in the nth CRR model.

Theorem 6.2.

1. The nth Cox-Ross-Rubinstein model is viable if and only if n N, where N =.

2. For n > N the unique one-step conditional risk-neutral probabilities for the nth Cox-Ross-Rubinstein model are (qn,1-qn), where

== (10)

3. For n N the unique equivalent martingale measure Qn in the nth Cox-Ross-Rubinstein model satisfies

Qn ({: ( (11)

for s n.

Proof.

1. According to Theorem 2.1, the nth CRR model is viable if and only if

1+dn < 1+rn < 1+un, (12)

and substituting equations in (7) and (1+rn) = exp( we get the following

exp( exp( exp( (13)

>0, therefore, n < and with a bit of arithmetic we can prove that n

2. A viable nth CRR model admits a unique equivalent martingale measure Qn with one-step conditional risk-neutral probabilities at every non-terminal node given by (qn,1-qn) where qn is given by equation (10).

Now we have to find the prices of plain puts and calls in the nth CRR model that approximates prices in the Black-Scholes model. Secondly, we have to investigate the convergence of plain options prices in the nth CRR model to prices in the Black-Scholes model.

To address the first issue, we can modify the pricing formula in Cox-Ross-Rubinstein Formula such that the initial price of a plain call option in the nth CRR model is given by

([ (14)

where and are a complementary binomial distribution functions of obtaining at least An "upward moves" in the underlying asset in n trials.

To find the initial price of a put option in the nth CRR model we modify Cox-Ross-Rubinstein Formula such

([ (15)

where and are binomial distribution functions of obtaining at least An "upward moves" in n trials.

To implement the above pricing formulas we adjust the parameters in Cox-Ross-Rubinstein Formula accordingly.

The final step in this article is to investigate the convergence of the nth CRR model to the Black-Scholes model. Let

= ), (16)

where (z) = S0exp), for z , = is given in equation (9).

Theorem 6.3

Let be a bounded continuous function. For each n, define the path-independent derivative Dn in the nth Cox-Ross-Rubinstein model by Dn = (17)

and let be its unique fair price at time 0. Then =exp(-rT)( (18)

Proof

Define a bounded and continuous function g such that g(z) := ( for z (19)

We apply the weak convergence of ( to for n and achieve following results =()=( (20)

To complete the proof, we perform the following

==exp(-rT)( (21)

In conclusion, the approach analysed in this article boils down to approximating the Brownian motion term in the Black-Scholes model with a symmetric random walk.

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