Part 1.

What does it mean to be FUZZY?

Fuzzy versus Crisp Concepts

I will call you between 5:20pm and 5:40pm.	I will call you around 5:30pm.
Today between 6am and 6pm the cloud cover averages 10%.	Today is a fairly sunny day.
Beef noodle soup tastes better than 82% of all other food.	Beef noodle soup is very tasty.

Fuzzy versus Crisp Actions

When getting close to a red traffic light, slowly apply the break of your car.

When getting to 20m from a red traffic light, apply the break of your car at 50%.

More (Fuzzy) Examples

spicy food; expensive car; difficult exam; funny movie; big house; [your examples]

drive slowly; study hard; be friendly; drink more water; speak louder; [your examples]

Can you give some "crisp" versions of these examples?

How to "Program" This?

If the subject is difficult, then I will study hard for the exam.

Find a good beef noodle soup restaurant that is close to NCTU and not too expensive.

Modeling of imprecise concepts necessary.

Part 2.

Basics of Fuzzy Sets

First, the Person ...

(Grandpa) **Lotfi A. Zadeh**, who formalized Fuzzy Set Theory in 1965:

Membership

Membership: How much does an "element" belong to a set.

Crisp: $A : X \longrightarrow \{0, 1\}$

Fuzzy: $A : X \longrightarrow [0, 1]$

Crisp sets are special cases of fuzzy sets.

Specifying a Fuzzy Set

By listing the elements:

X = all car brands
A = expensive car brands

A = 1.0/Farrari + 0.9/BMW + 0.3/Honda + ...

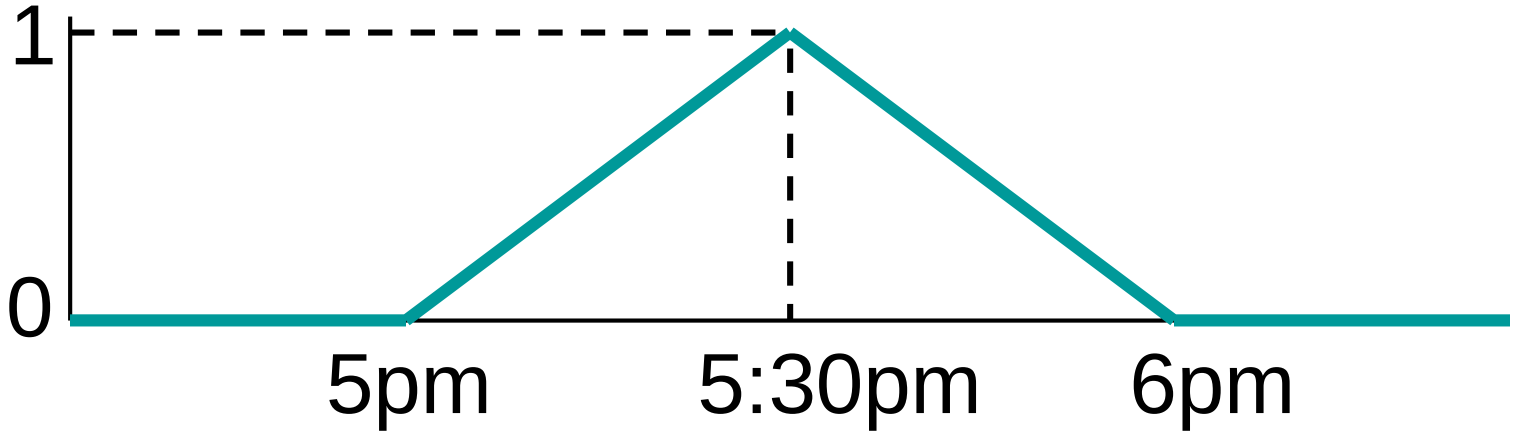
Of course, useful only when there are finite number of elements with non-zero memberships.

Specifying a Fuzzy Set

By a function:

X = all 24-hour time in a day

A = "around 5:30pm"



Examples

Crisp:	X = all people
	A = all males
Fuzzy:	X = all people
	A = young people
Fuzzy:	X = all cars
	A = big cars

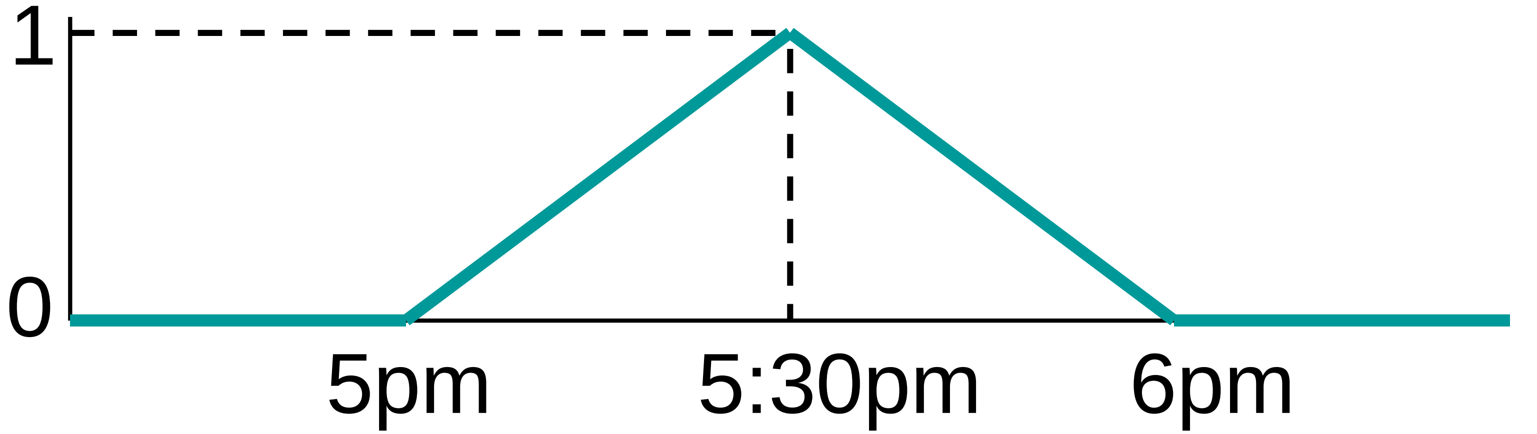
Think of a fuzzy set (e.g., big cars), and assign membership values to 3 elements.

Supports of Fuzzy Sets

Support(A) = {x∈X | A(x)>0}

X = all 24-hour time in a day

A = "around 5:30pm"



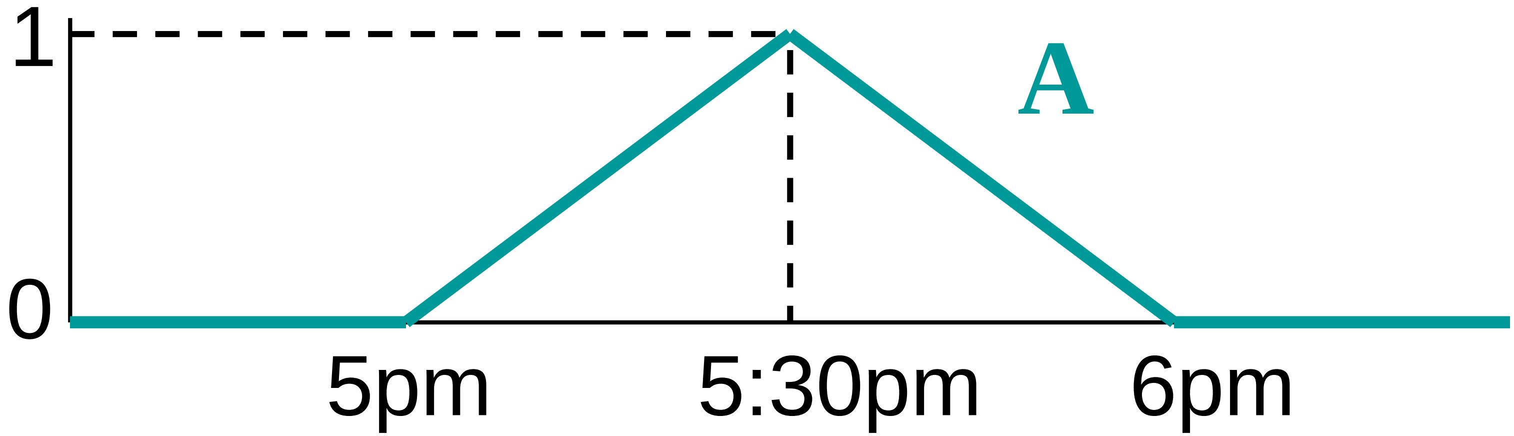
What is Support(A) here?

Supremums and Infimums

Sup(A) = smallest upper bound of A(x)

Inf(A) = largest lower bound of A(x)

(for all $x \in X$)



Sup(A)=1, Inf(A)=0

Supremums and Infimums

$\mathrm{Sup}(A)$ = smallest upper bound of $A(x)$
$\mathrm{Inf}(A)$ = largest lower bound of $A(x)$

$\mathrm{Sup}(A) \overset{?}{=} \max \{ A(x) \}$ → usually true

Asymptotic case:

$A(x) = x / (1+x)$ for all positive $x \in \mathbb{R}$

➔ $\mathrm{Sup}(A) = 1$
$\max(A) = ?$

Scalar Cardinality

Cardinality of a (finite) set A

$|A|$ = the number of elements in A

Fuzzy Set (with finite support):

$$|A| = \sum_{x \in X} A(x)$$

Example

X = the 7 days of last week

A = HOT days

$$\text{The degree of a day being HOT} = \begin{cases} 1 & \text{if } T \geq 30 \\ 0 & \text{if } T \leq 20 \\ (T-20)/10 & \text{otherwise} \end{cases}$$

T: High Temperature (^{o}C)

Assume these temperatures of last week:

Mon:25, Tue:20, Wed:18, Thu:24, Fri:28, Sat:30, Sun:32

A=

Example (continued)

X = the 7 days of last week

A = HOT days

A= 0.5/Mon + 0.4/Thu + 0.8/Fri + 1/Sat + 1/Sun

Support(A) =

Sup(A) =

Inf(A) =

|A| =

The α-cuts

The α-cuts produce <u>crisp</u> sets from fuzzy sets by thresholding the membership values.

$$^{\alpha}A = \{ x \mid A(x) \geq \alpha \}$$

$$^{\alpha+}A = \{ x \mid A(x) > \alpha \} \longleftarrow \text{ strong } \alpha\text{-cuts}$$

For a fuzzy set A, a series of α values such that

$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \ldots$

Give a series of nested crisp sets:

$^{\alpha 1}A \subseteq {}^{\alpha 2}A \subseteq {}^{\alpha 3}A \subseteq {}^{\alpha 4}A \ldots$

The α-cuts – Example

$\mathbf{A}$= 0.5/Mon + 0.4/Thu + 0.8/Fri + 1/Sat + 1/Sun

${}^{0}A$ =

${}^{0+}A$ =

${}^{0.5}A$ =

${}^{0.5+}A$ =

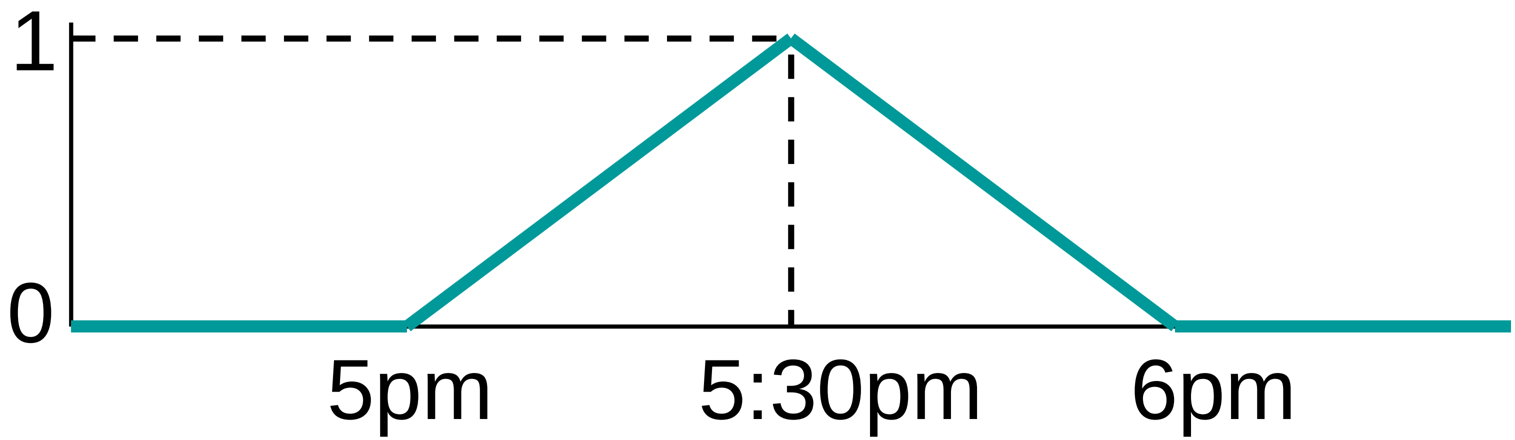
${}^{1}A$ =

${}^{1+}A$ =

More Example of α-cuts

X = all 24-hour time in a day

A = "around 5:30pm"



${}^{0}A$ =

${}^{0+}A$ =

${}^{0.5}A$ =

${}^{0.5+}A$ =

${}^{1}A$ =

${}^{1+}A$ =

Level Sets

$$\Lambda(A) = \{\ \alpha \mid \exists x \in X \text{ such that } A(x)=\alpha\ \}$$

Example:

$\mathbf{A}$= 0.5/Mon + 0.4/Thu + 0.8/Fri + 1/Sat + 1/Sun

$\Lambda(A)$ =

Example:

$\mathbf{A}$= "around 5:30pm"

$\Lambda(A)$ =

Convexity

For sets defined on $\mathbb{R}^n$:

Crisp Sets:

$\forall x,y \in A$ and $0 \leq \lambda \leq 1$, $\lambda x+(1-\lambda)y \in A$

Fuzzy Sets:

$\forall x,y \in X$ and $0 \leq \lambda \leq 1$,

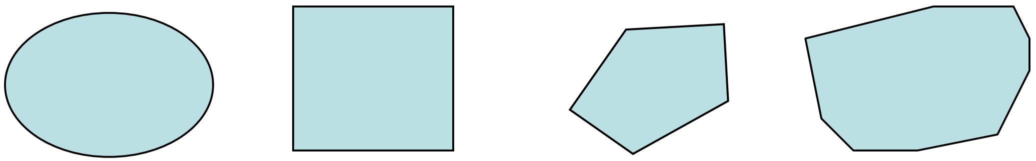
$$A(\lambda x+(1-\lambda)y) \geq \min(A(x),A(y))$$

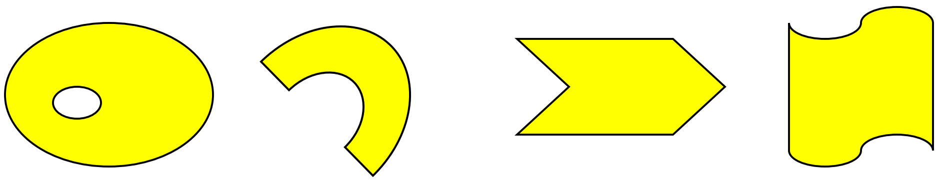
A fuzzy set is convex $\Leftrightarrow$ all its α-cuts are convex

Proof:

Convexity

Examples in $\mathbb{R}^2$ (Crisp Sets)

Convex: 

Non-convex: 

Examples in $\mathbb{R}^1$ (Crisp Sets)

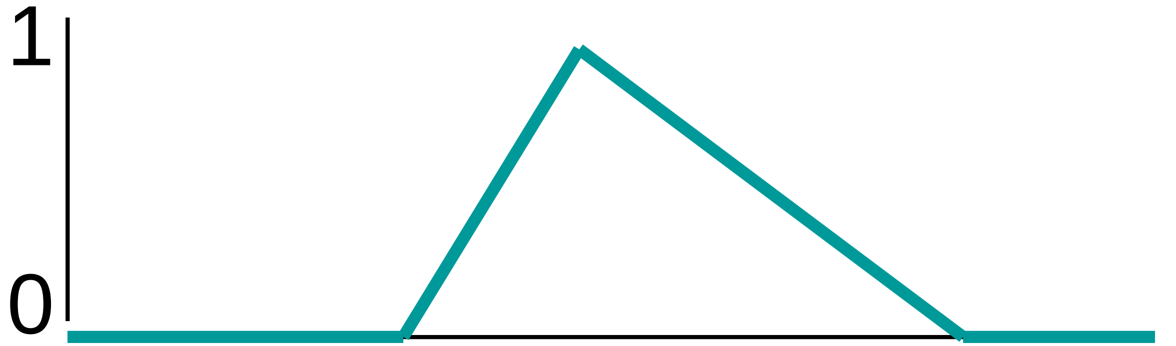
Convex: $[0,1]$, $(0,1)$, $[2,\infty)$

Non-convex: $[0,1] \cup [2,3]$, $\{1,3,8.5\}$

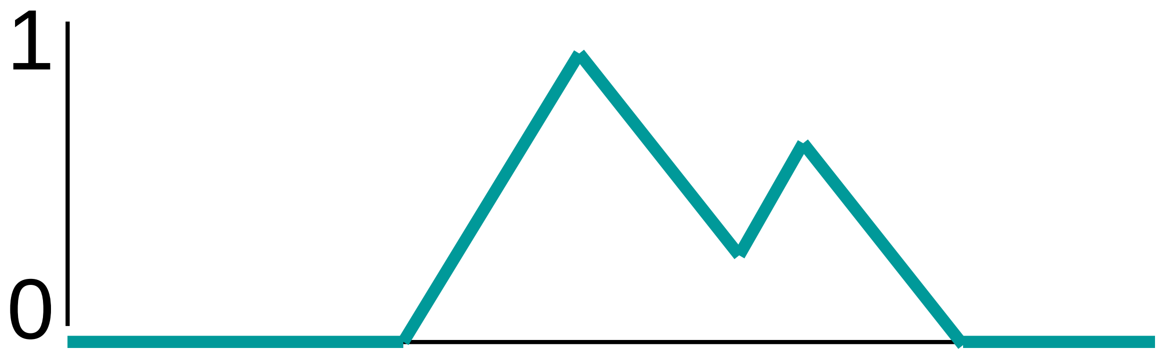
Convexity for Fuzzy Sets

Examples in $\mathbb{R}^1$ (Fuzzy Sets)

Convex



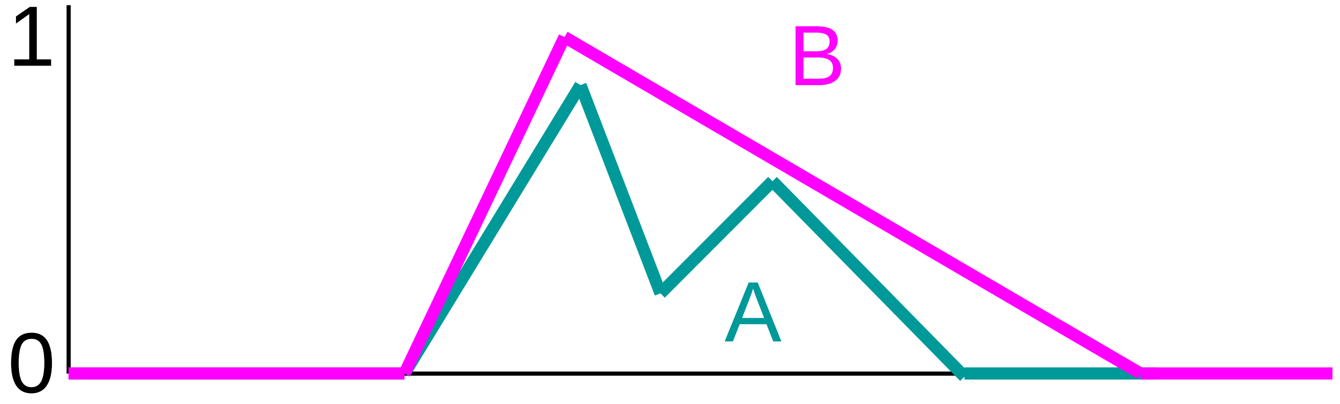
Non-convex



Fuzzy Subset

$$A \subseteq B \iff \forall x \in X, A(x) \le B(x)$$

Example

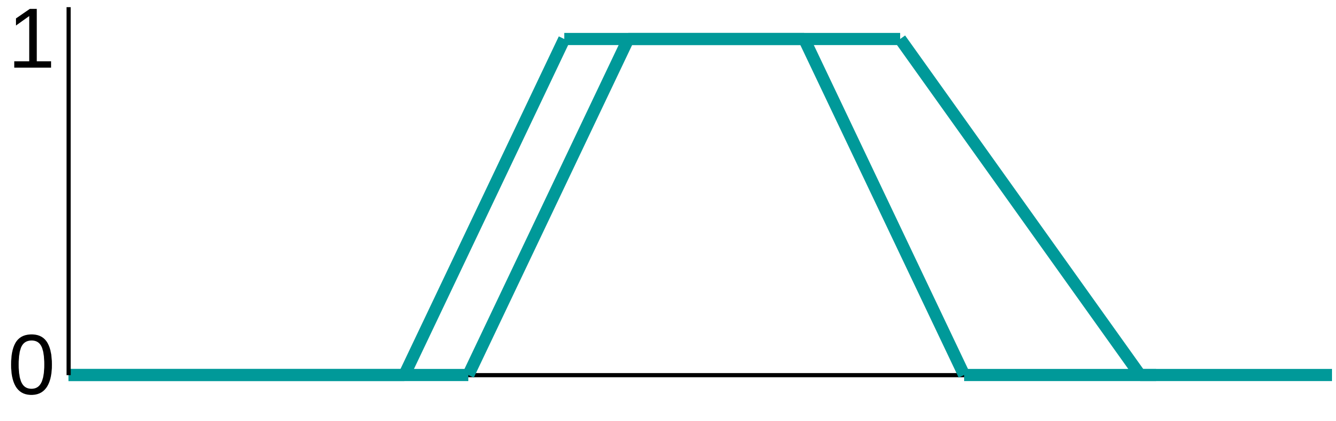


Interval-Valued Fuzzy Sets

$$A : X \longrightarrow \mathcal{E}([0, 1])$$

Intervals in [0,1]

When we are not so sure about memberships



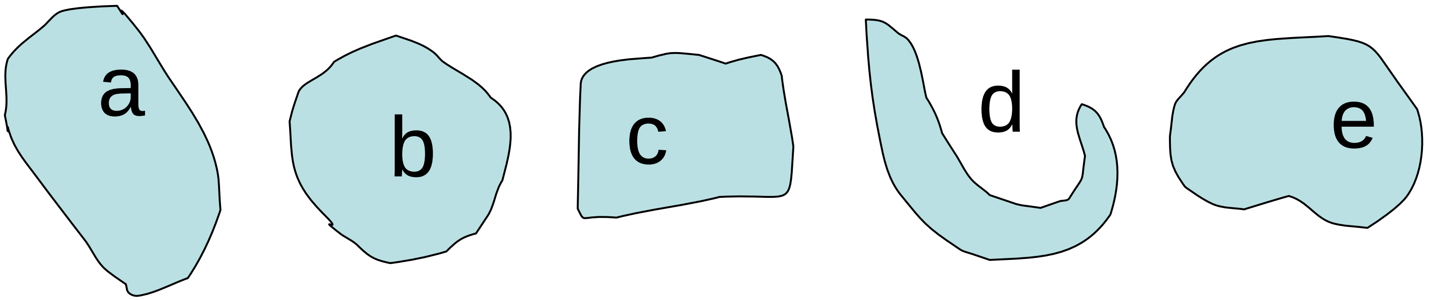
Type 2 Fuzzy Sets

$$A : X \longrightarrow \mathcal{F}([0, 1])$$

All fuzzy sets defined in [0,1]

It is possible to recursively define type 3, type 4, ..., fuzzy sets, but people have not found practical use for those.

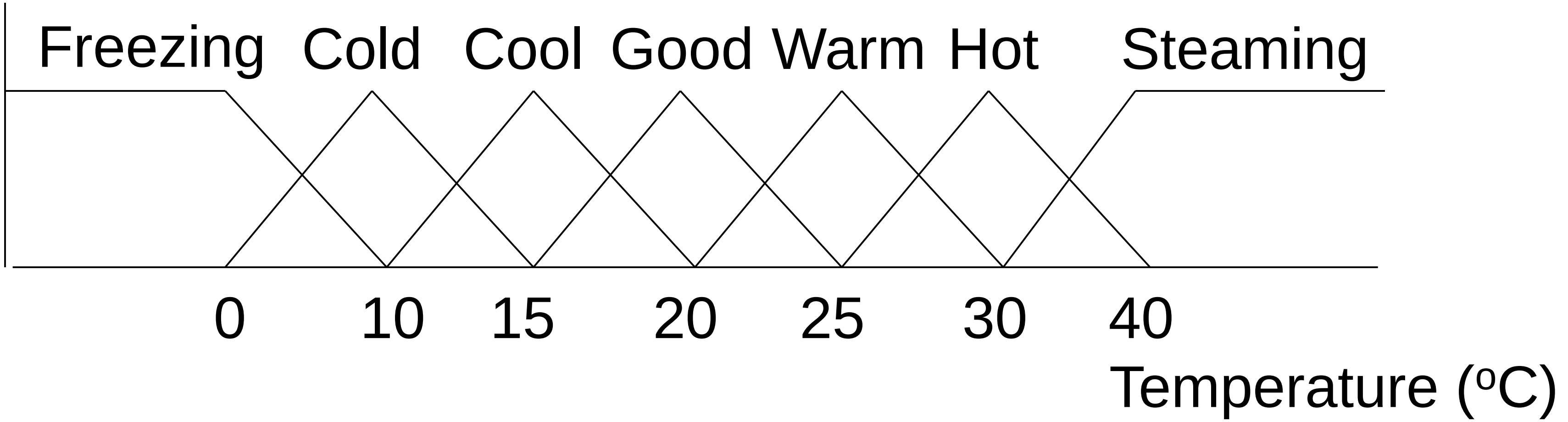
Type 2 Fuzzy Sets – Example



Constructing Fuzzy Sets

- From (single or multiple) expert opinions
- From expert opinions – indirect method (pairwise comparisons)
- Learning

Linguistic Variables



- Usually in continuous X
- Context dependent
- In most applications, results are not very sensitive to actual shapes

Part 3.

Fuzzy Set Operations

Standard Crisp Set Operations

- Complement: A^C
- Intersection: $A \cap B$
- Union: $A \cup B$

Fuzzy Set Operations

These operations are by Zadeh:

- Complement: $(A^C)(x) = 1 - A(x)$
- Intersection: $(A \cap B)(x) = A(x) \wedge B(x)$
- Union: $(A \cup B)(x) = A(x) \vee B(x)$

There are infinite ways to define them (we'll see).

Preserved Properties

Involution: $(A^C)^C = A$

Commutativity: $A \cap B = B \cap A$, etc.

Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$, etc.

Distributivity: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$, etc.

Idempotence: $A \cap A = A,\ A \cup A = A$

Absorption: $A \cap (A \cup B) = A,\ A \cup (A \cap B) = A,$

$$A \cup X = X,\ A \cap \phi = \phi$$

Identity: $A \cap X = A,\ A \cup \phi = A$

De Morgan's Laws:

$$(A \cup B)^C = (A^C \cap B^C)$$

$$(A \cap B)^C = (A^C \cup B^C)$$

Unpreserved Properties

Law of excluded middle: $A \cup A^C = X$

Law of contradiction: $A \cap A^C = \phi$

Example:

Fuzzy Set Operators

Fuzzy set operators are actually functions of membership functions, so they are often stated without the notations of set operators:

- Complement: $c: [0,1] \rightarrow [0,1]$
- Intersection (t-norm):

  $i: [0,1] \times [0,1] \rightarrow [0,1]$
- Union (t-conorm):

  $u: [0,1] \times [0,1] \rightarrow [0,1]$

Axioms: Fuzzy Complement

Required (skeleton) axioms:

c1: $c(1)=0,\ c(0)=1$ (boundary conditions)

c2: $\forall a,b \in [0,1],\quad a \le b \Rightarrow c(a) \ge c(b)$ (monotonicity)

"Nice-to-have" axioms:

c3: c is a continuous function

c4: $\forall a \in [0,1],\quad c(c(a)) = a$ (involution)

Examples

Other than Zadeh's fuzzy complements, there are many other possible definitions:

$$c(a) = 1 - a^2$$

$$c(a) = [1 + \cos(\pi a)] / 2$$

$$c(a) = (1 - a) / (1 + a)$$

$$c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{if } a > t \end{cases} \qquad 0 \le t < 1$$

Do they satisfy the four axioms?

(c2 and c4) → (c1 and c3)

The four axioms are not all independent.

Theorem:

If $c(x)$ satisfies axioms ($c2$) and ($c4$), then $c(x)$ also satisfies axioms ($c1$) and ($c3$).

Proof:

Sugeno Class of Fuzzy Complements

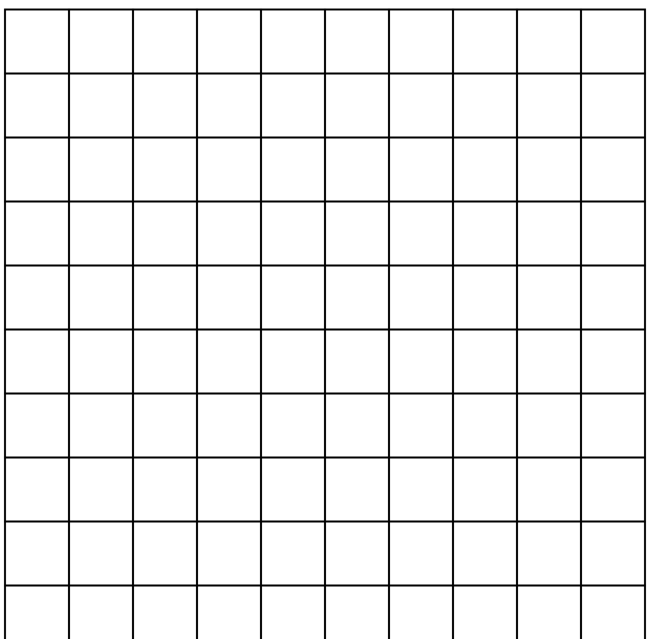
$$c_\lambda(a) = \frac{1 - a}{1 + \lambda a} \qquad \lambda > -1$$

It satisfies all four axioms.

Sugeno Class of Fuzzy Complements

$$c_\lambda(a) = \frac{1 - a}{1 + \lambda a} \qquad \lambda > -1$$

When is it identical to Zadeh's complement?



"Optimistic" complement:

"Pessimistic" complement:

Yager Class of Fuzzy Complements

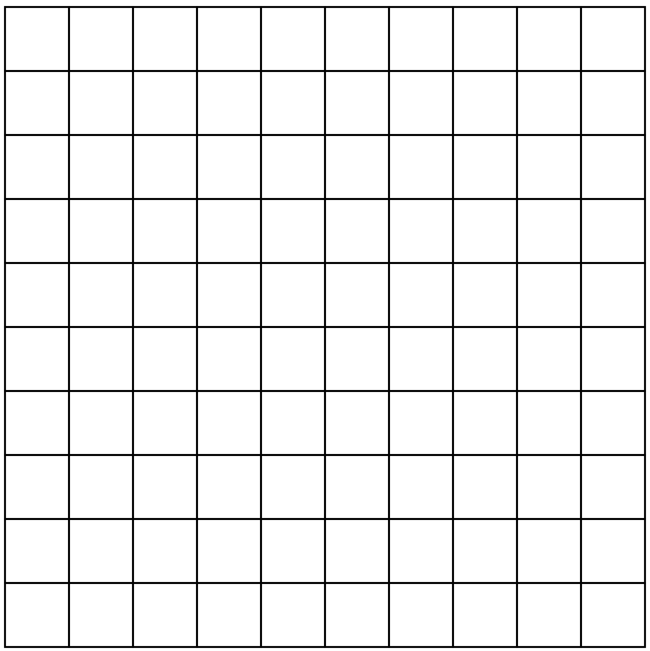
$$c_w(a) = (1 - a^w)^{1/w} \qquad w > 0$$

It satisfies all four axioms.

Yager Class of Fuzzy Complements

$$c_w(a) = (1 - a^w)^{1/w} \qquad w > 0$$

When is it identical to Zadeh's complement?



"Optimistic" complement:

"Pessimistic" complement:

Equilibrium

a is an equilibrium of a fuzzy complement c

$$\Leftrightarrow c(a) = a$$

Examples:

$$c(a) = 1 - a$$

$$c_\lambda(a) = \frac{1 - a}{1 + \lambda a}$$

$$c(a) = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{if } a > t \end{cases} \qquad 0 \leq t < 1$$

Equilibrium

Theorem:

A fuzzy complement has at most one equilibrium.

Proof (by contradiction):

Equilibrium

Theorem:

A continuous fuzzy complement has one equilibrium.

Proof:

Axioms: Fuzzy Intersection

Required (skeleton) axioms:

i1: $i(a,1) = a$ (boundary conditions)

i2: $b \leq d \Rightarrow i(a,b) \leq i(a,d)$ (monotonicity)

i3: $i(a,b) = i(b,a)$ (commutativity)

i4: $i(a,i(b,d)) = i(i(a,b),d)$ (associativity)

"Nice-to-have" axioms:

i5: i is a continuous function

i6: $i(a,a) < a$ (subidempotency)

i7: $a_1 < a_2$ and $b_1 < b_2 \Rightarrow i(a_1,b_1) < i(a_2,b_2)$

(strict monotonicity)

Common Fuzzy Intersections

Zadeh's: $i(a,b) = \min(a,b) = a \wedge b$

Algebraic product: $i(a,b) = ab$

Bounded difference: $i(a,b) = 0 \vee (a+b-1)$

Drastic:
$$i_{min}(a,b) = \begin{cases} a & \text{when } b=1 \\ b & \text{when } a=1 \\ 0 & \text{otherwise} \end{cases}$$

Idempotent Fuzzy Intersections

Theorem:

$\min(a,b)$ is the only idempotent fuzzy intersection.

Proof:

Bounds of Fuzzy Intersections

Theorem:

$$\forall a,b \in [0,1], \quad i_{min}(a,b) \leq i(a,b) \leq \min(a,b)$$

Proof:

Yager's Fuzzy Intersections

$$i_w(a,b) = 1 - 1 \wedge [(1-a)^w + (1-b)^w]^{1/w} \qquad w > 0$$

$$w \to \infty : \quad i_w(a,b) \to i_{min}(a,b)$$

$$w = 1 : \quad i_w(a,b) \to 0 \vee (a+b-1)$$

$$w \to 0 : \quad i_w(a,b) \to a \wedge b$$

Yager's fuzzy intersections cover the whole range between $i_{min}(a,b)$ and $\min(a,b)$.

Yager's Fuzzy Intersections

w=1

a \ b	0	.25	.5	.75	1
1	0	.25	.5	.75	1
.75	0	0	.25	.5	.75
.5	0	0	0	.25	.5
.25	0	0	0	0	.25
0	0	0	0	0	0

w=2

a \ b	0	.25	.5	.75	1
1	0	.25	.5	.75	1
.75	0	.21	.44	.65	.75
.5	0	.1	.29	.44	.5
.25	0	0	.1	.21	.25
0	0	0	0	0	0

w=10

a \ b	0	.25	.5	.75	1
1	0	.25	.5	.75	1
.75	0	.25	.5	.73	.75
.5	0	.25	.46	.5	.5
.25	0	.2	.25	.25	.25
0	0	0	0	0	0

$w \to \infty$

a \ b	0	.25	.5	.75	1
1	0	.25	.5	.75	1
.75	0	.25	.5	.75	.75
.5	0	.25	.5	.5	.5
.25	0	.25	.25	.25	.25
0	0	0	0	0	0

Axioms: Fuzzy Union

Required (skeleton) axioms:

u1: $u(a,0) = a$ (boundary conditions)

u2: $b \leq d \Rightarrow u(a,b) \leq u(a,d)$ (monotonicity)

u3: $u(a,b) = u(b,a)$ (commutativity)

u4: $u(a,u(b,d)) = u(u(a,b),d)$ (associativity)

"Nice-to-have" axioms:

u5: u is a continuous function

u6: $u(a,a) > a \quad (a<1)$ (superidempotency)

u7: $a_1<a_2$ and $b_1<b_2 \Rightarrow u(a_1,b_1)<u(a_2,b_2)$

(strict monotonicity)

Common Fuzzy Unions

Zadeh's: $u(a,b) = \max(a,b) = a \vee b$

Algebraic sum: $u(a,b) = a + b - ab$

Bounded sum: $u(a,b) = 1 \wedge (a+b)$

Drastic: $$u_{max}(a,b) = \begin{cases} a & \text{when } b=0 \\ b & \text{when } a=0 \\ 1 & \text{otherwise} \end{cases}$$

Idempotent Fuzzy Unions

Theorem:

$\max(a,b)$ is the only idempotent fuzzy union.

Proof:

Bounds of Fuzzy Unions

Theorem:

$$\forall a,b \in [0,1], \quad \max(a,b) \le u(a,b) \le u_{max}(a,b)$$

Proof:

Yager's Fuzzy Unions

$$u_w(a,b) = 1 \wedge (a^w + b^w)^{1/w} \qquad w > 0$$

$$w \to \infty : \quad u_w(a,b) \to a \vee b$$

$$w = 1 : \quad u_w(a,b) \to 1 \wedge (a+b)$$

$$w \to 0 : \quad u_w(a,b) \to u_{max}(a,b)$$

Yager's fuzzy unions cover the whole range between $\max(a,b)$ and $u_{max}(a,b)$.

Yager's Fuzzy Unions

w=1

a \ b	0	.25	.5	.75	1
1	1	1	1	1	1
.75	.75	1	1	1	1
.5	.5	.75	1	1	1
.25	.25	.5	.75	1	1
0	0	.25	.5	.75	1

w=2

a \ b	0	.25	.5	.75	1
1	1	1	1	1	1
.75	.75	.79	.9	1	1
.5	.5	.56	.71	.9	1
.25	.25	.35	.56	.79	1
0	0	.25	.5	.75	1

w=10

a \ b	0	.25	.5	.75	1
1	1	1	1	1	1
.75	.75	.75	.75	.8	1
.5	.5	.5	.54	.75	1
.25	.25	.27	.5	.75	1
0	0	.25	.5	.75	1

$w \to \infty$

a \ b	0	.25	.5	.75	1
1	1	1	1	1	1
.75	.75	.75	.75	.75	1
.5	.5	.5	.5	.75	1
.25	.25	.25	.5	.75	1
0	0	.25	.5	.75	1

De Morgan's Laws

Crisp Set Operations:

$$(A \cup B)^C = (A^C \cap B^C)$$

$$(A \cap B)^C = (A^C \cup B^C)$$

Fuzzy Set Operations:

$$c(\ u(a,b)\ ) = i(c(a),\ c(b))$$

$$c(\ i(a,b)\ ) = u(c(a),\ c(b))$$

They are NOT satisfied by all combinations of fuzzy complement, intersection, and union operators.

Operator Triples

Operators u and i are "dual" with respect to c if they satisfy the De Morgan's laws. They form a "dual triple" $\langle \mathrm{i},\mathrm{u},\mathrm{c}\rangle$.

Example:

$$\mathrm{i}(a,b) = a \wedge b.$$
$$\mathrm{u}(a,b) = a \vee b.$$
$$\mathrm{c}(a) = 1 - a.$$

Operator Triples

The De Morgan's Laws allow us to derive a union operator given an intersection and an involutive complement operator, or an intersection operator given a union and an involutive complement operator.

$$u(a,b) = c(c(u(a,b))) = c(\ i(c(a), c(b))\ )$$

$$i(a,b) = c(c(i(a,b))) = c(\ u(c(a), c(b))\ )$$

Examples:

$u(a,b) = 1 \wedge (a+b)$

$c(a) = 1 - a$

$i(a,b) = ?$

$i(a,b) = ab$

$c(a) = (1-a)/(1+a)$

$u(a,b) = ?$

Operator Triples

- Usually we can not obtain a unique complement operator given a union and an intersection operators.

- For any fuzzy complement c, $\langle \min, \max, c \rangle$ satisfies the De Morgan's laws.

- For any fuzzy complement c, $\langle i_{min}, u_{max}, c \rangle$ satisfies the De Morgan's laws.

Aggregation Operators

$$h: [0,1]^n \rightarrow [0,1]$$

Required axioms:

h1: $h(0,0,\ldots,0) = 0$, $h(1,1,\ldots,1)=1$ (boundary conditions)

h2: $(\forall i,\ a_i \leq b_i) \Rightarrow h(a_1,a_2,\ldots) \leq h(b_1,b_2,\ldots)$ (monotonicity)

h3: h is continuous

Other common properties:

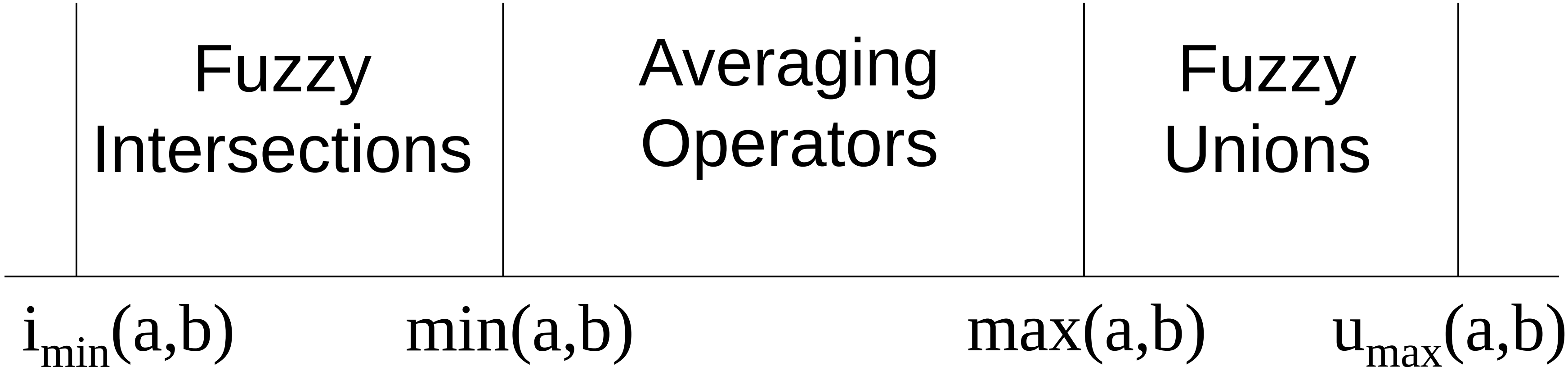
h4: $h(a_1,a_2,\ldots) = h(a_{p(1)},a_{p(2)},\ldots)$ (symmetry)

h5: $h(a,a,\ldots,a) = a$ (idempotency)

Mean/Average Operators

(h2 and h5) $\Rightarrow$

$$\min(a_1,a_2,\ldots) \le h(a_1,a_2,\ldots) \le \max(a_1,a_2,\ldots)$$



Generalized Mean

$$h_\alpha(a_1,a_2,\ldots,a_n) = \left[\frac{a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha}{n}\right]^{1/\alpha}$$

h_1: arithmetic mean

h_0: geometic mean

h_{-1}: harmonic mean

h_∞: max

$h_{-\infty}$: min

Generalized Weighted Mean

When the inputs are not equally important ...

(h4) not satisfied.

$$h_\alpha(a_1,a_2,\ldots,a_n) = (\ w_1 a_1^{\alpha} + w_2 a_2^{\alpha} + \ldots + w_n a_n^{\alpha}\ )^{1/\alpha}$$

$$\Sigma\, w_i = 1$$

Ordered Weighted Mean (OWA)

When the relative importance of the inputs depend on their orders. (h4) is not satisfied.

$$h(a_1,a_2,\ldots,a_n) = w_1 b_1 + w_2 b_2 + \ldots + w_n b_n \qquad \Sigma\, w_i = 1$$

$(b_1,b_2,\ldots,b_n)$ are the sorted values of $(a_1,a_2,\ldots,a_n)$

Examples of OWA

Everyday examples (not necessarily fuzzy sets):

Final grade = 0.8 x (better of two exams)
+ 0.2 x (worse of two exams)

Average of the 6 "middle" scores from 8 judges

Majority voting

OWA can be used to model linguistic concepts such as "MOST", "SOME", "AT LEAST ...", etc.

Hybrid Operators

Combination of a union and an intersection operators

Additive type: $r\, u(a,b) + (1-r)\, i(a,b)$

Multiplicative type: $[u(a,b)]^{r} \bullet [i(a,b)]^{1-r}$

$$0 \leq r \leq 1$$

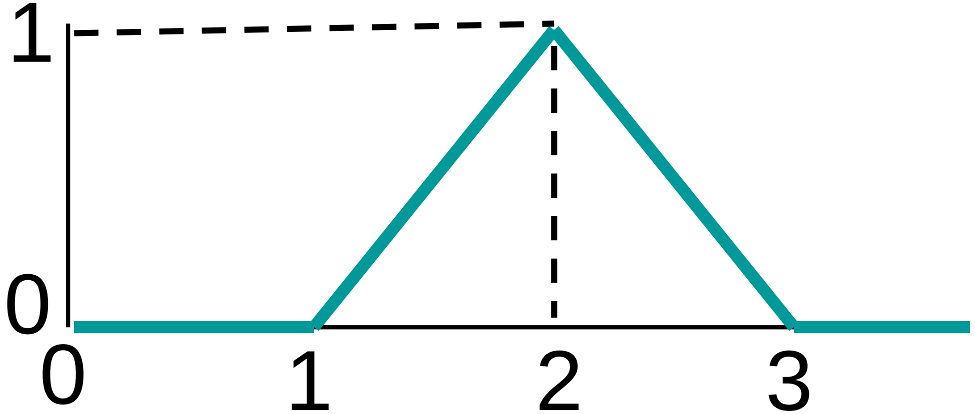
Part 4.

Fuzzy Numbers
Fuzzy Arithmetic
Extension Principle

Fuzzy Numbers

Idea: A fuzzy number is a fuzzy set defined on $\mathbb{R}$ that captures the idea of "being close to r", where r is an ordinary real number.

For example, the concept of "a number close to 2" may be represented by this fuzzy set:



Properties of Fuzzy Numbers

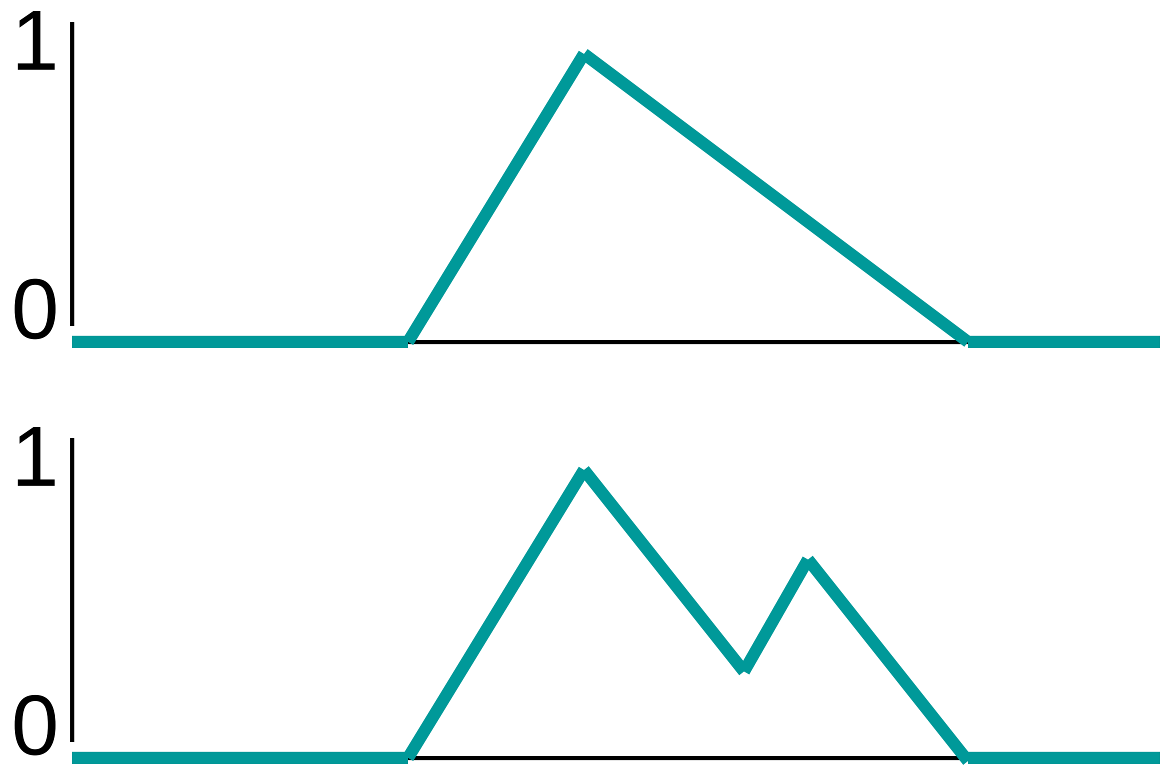
- It is a normal fuzzy set (height = 1).
- It is a convex fuzzy set (each α-cut is an interval).

Review of Convex Fuzzy Sets

Examples in $\mathbb{R}^1$ (Fuzzy Sets)

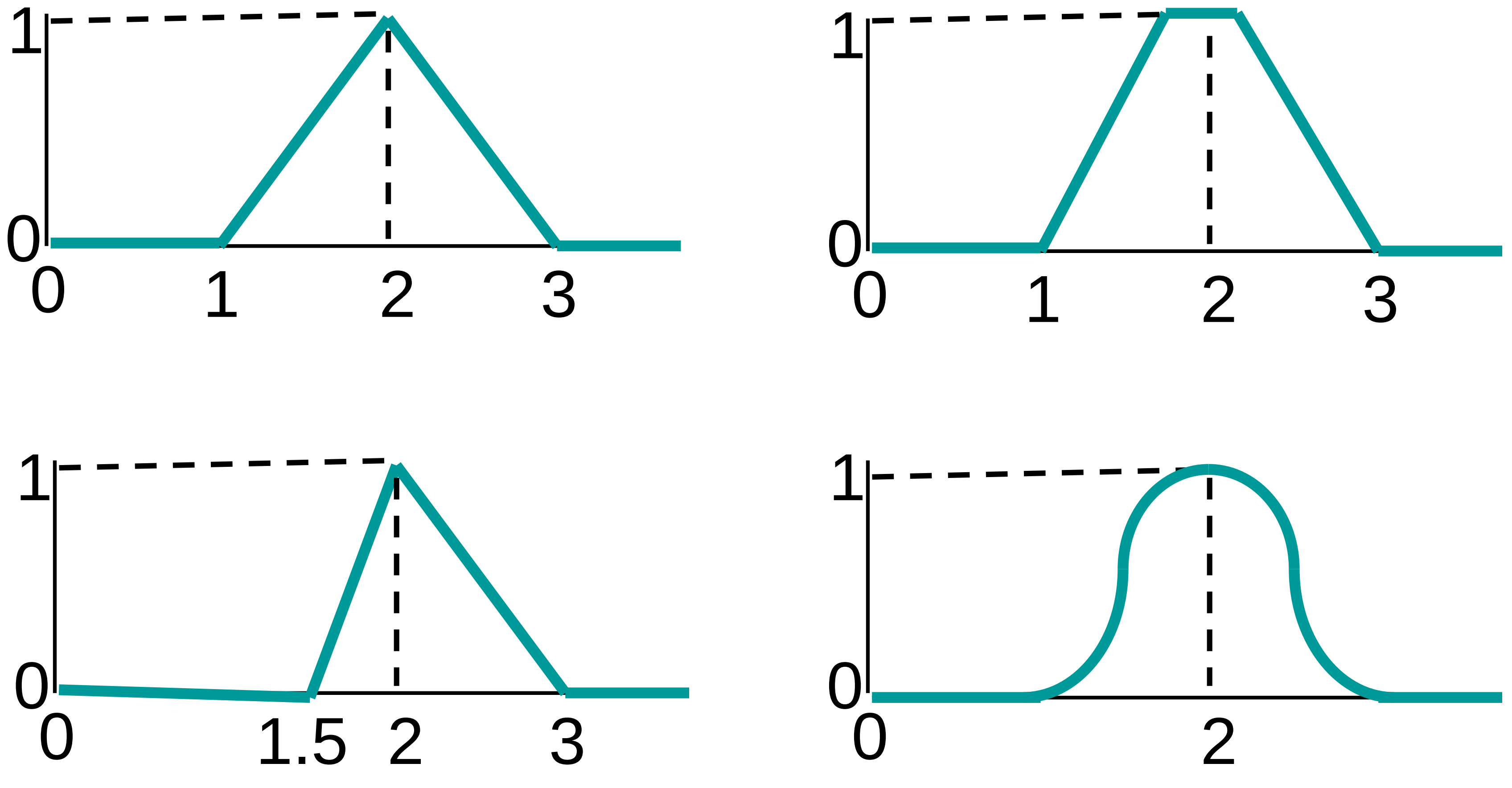
Convex

Non-convex



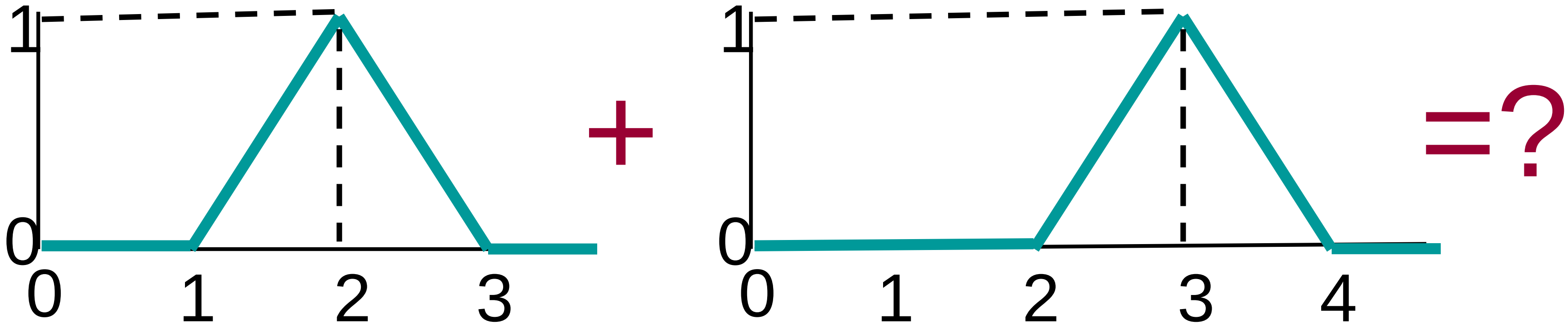
Shapes of Fuzzy Numbers

For "Close to 2"



Arithmetic of Fuzzy Numbers

Q: What is "a number close to 2" plus "a number close to 3"?



We will expect it to be like "a number close to 5". The problem is in getting the actual membership function.

Arithmetic of Fuzzy Numbers

We can do "interval arithmetic" on all α-cuts of the two fuzzy numbers and then combine their results to form a final output fuzzy set.

Let A and B be two fuzzy numbers:

$^{\alpha}(A \otimes B) = (^{\alpha}A) \otimes (^{\alpha}B)$ where $\otimes$ is +, –, x, ÷

$(A \otimes B) = \bigcup_{\alpha} {}_{\alpha}(A \otimes B)$

where $_{\alpha}(A \otimes B)(x) = \begin{cases} \alpha & \text{if } x \in {}^{\alpha}(A \otimes B) \\ 0 & \text{otherwise} \end{cases}$

The resulting $(A \otimes B)$ is also a fuzzy number.

Interval Arithmetic

$[a,b] + [c,d] = [a+c,\ b+d]$

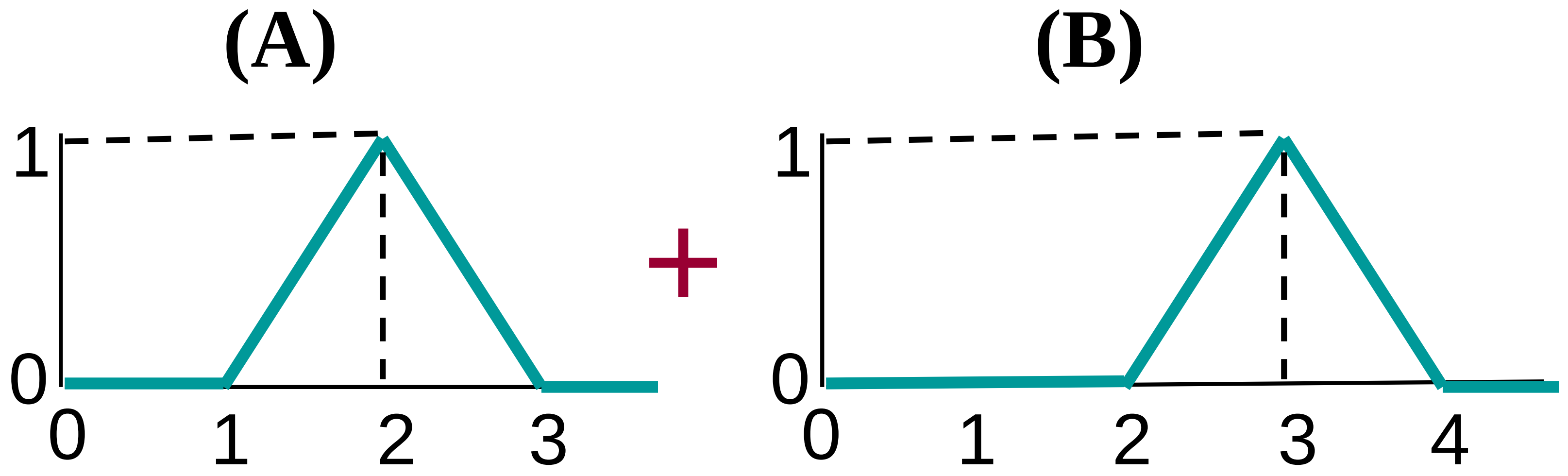
$[a,b] – [c,d] = [a-d,\ b–c]$

$[a,b] \times [c,d] = [\ \min(ac,ad,bc,bd),\ \max(ac,ad,bc,bd)\ ]$

$1\ /\ [a,b] = [1/b,\ 1/a]$ if $0 \notin [a,b]$

$[a,b]\ /\ [c,d] = [a,b] \times (1/[c,d])$ if $0 \notin [c,d]$

Example



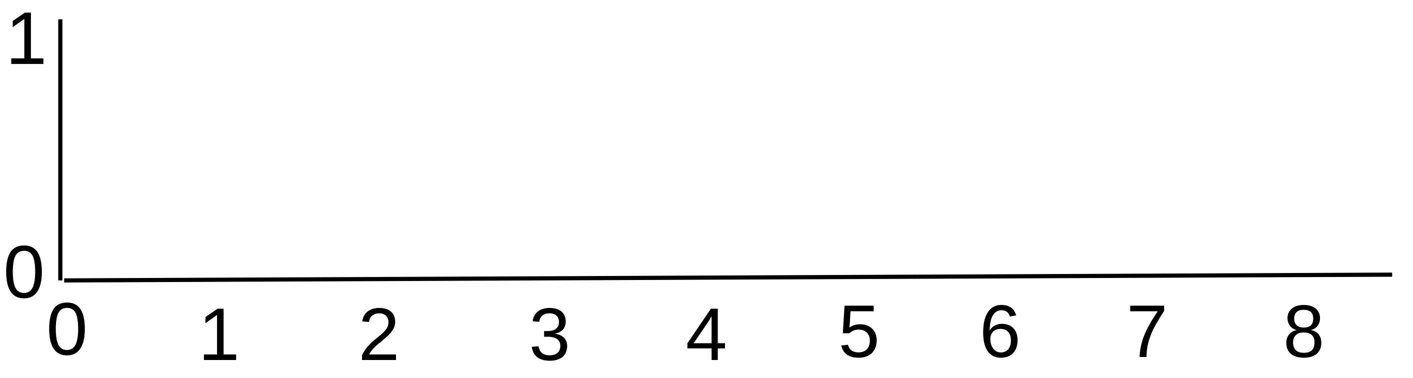
$^{\alpha}A =$

$^{\alpha}B =$

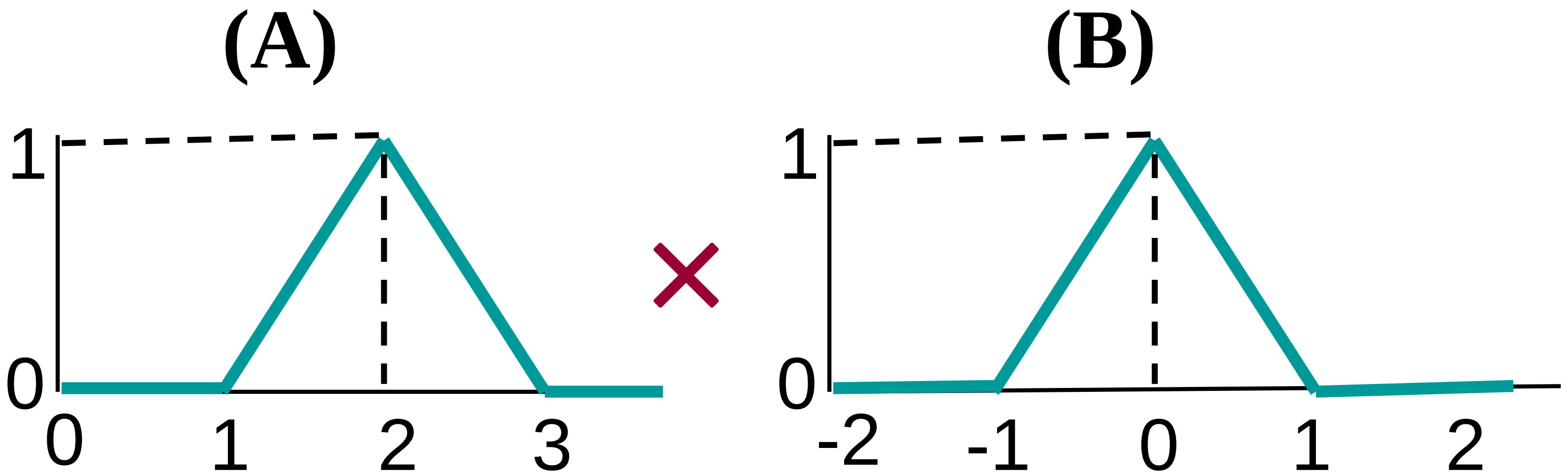
$^{\alpha}(A+B) =$

Example

$(A+B)(x) =$



Example



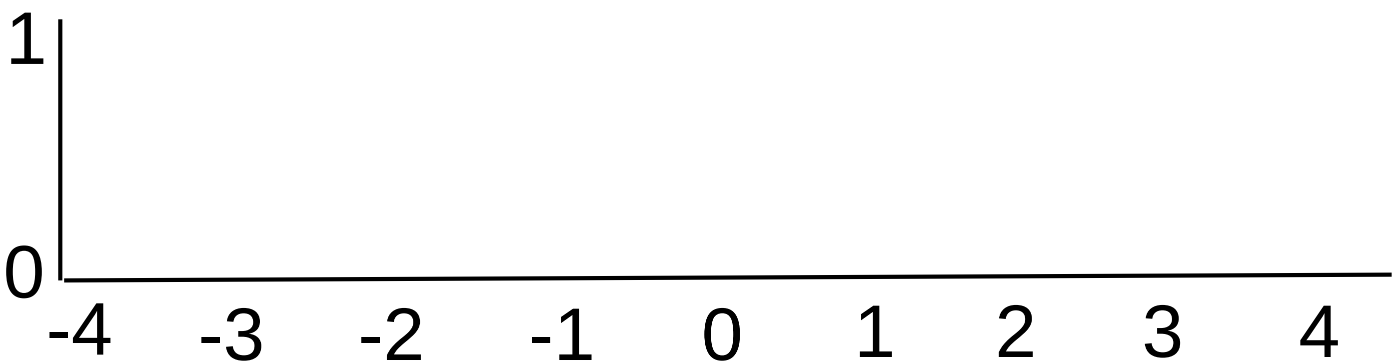
$^{\alpha}A =$

$^{\alpha}B =$

$^{\alpha}(A \times B) =$

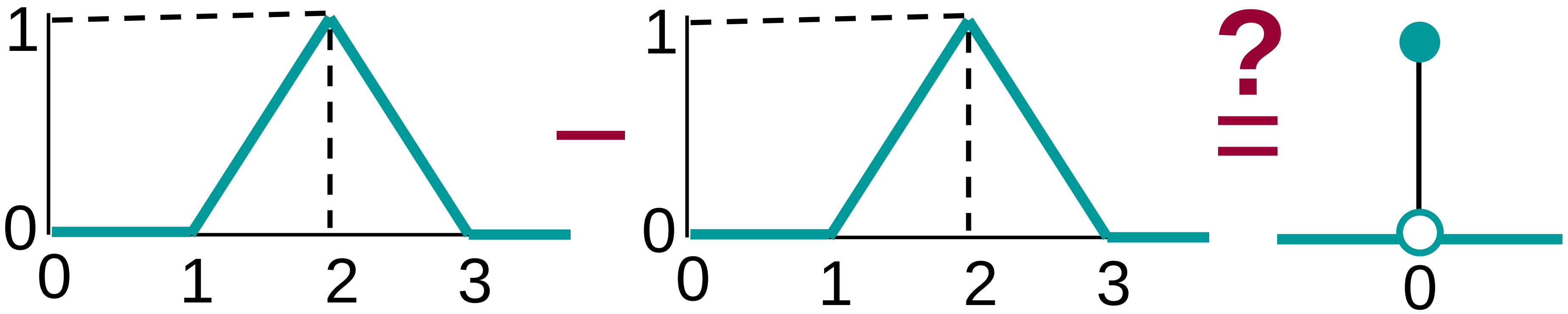
Example

$(A \times B)(x) =$



Duplicate Variables

What should $(A - A)$ be?
Should we expect it to be zero?



$^{\alpha}A =$

$^{\alpha}(A–A) =$

$(A–A)(x) =$

Duplicate Variables

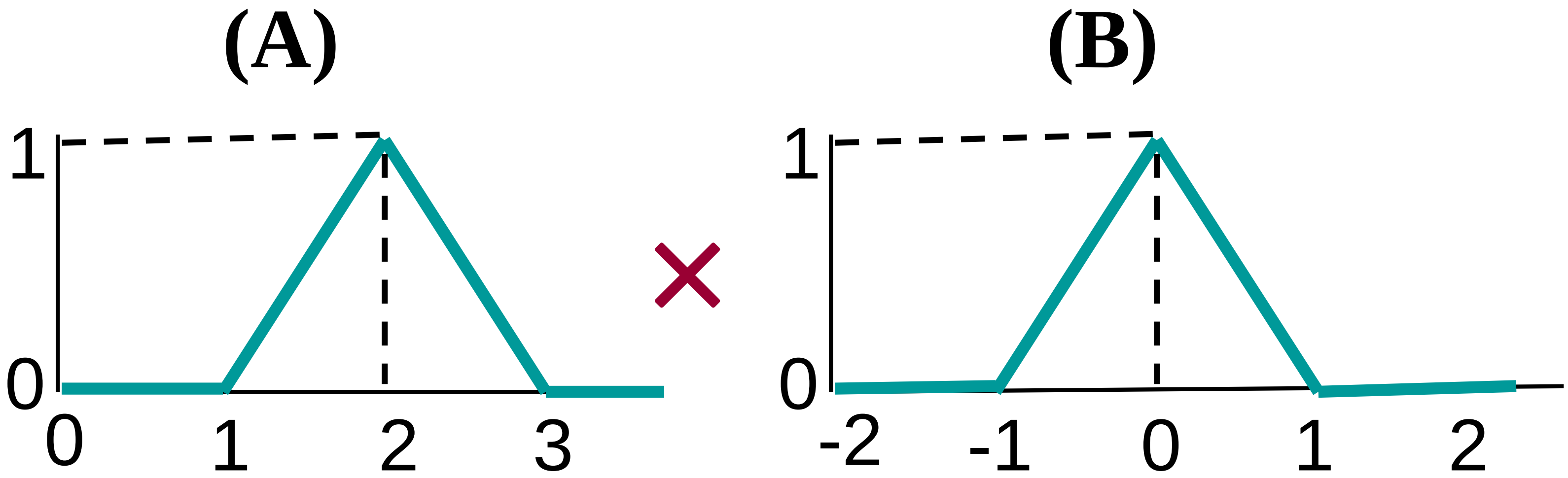
When computing $^{\alpha}(A–A)$, instead of using interval arithmetic and treating the two A's independently, just calculate all permutations of A's end points, and use the largest range as the answer.

$$^{\alpha}(A–A) =$$

$$(A–A)(x) =$$

Duplicate Variables

Consider the example of $(A \times B)/A$:



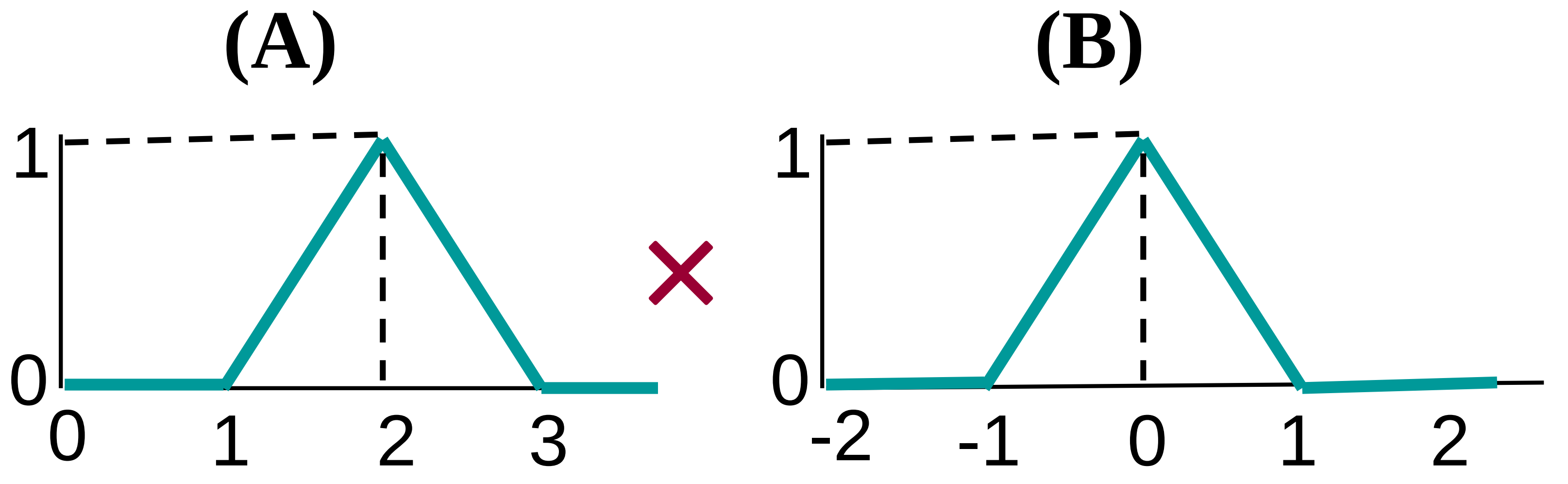
Interval arithmetic:

$^{\alpha}(A \times B) =$

$^{\alpha}((A \times B)/A) =$

$((A \times B)/A)(x) =$

Duplicate Variables



Permutation method:

$^{\alpha}A =$

$^{\alpha}B =$

$^{\alpha}(A \times B) =$

$^{\alpha}((A \times B)/A) =$

$((A \times B)/A)(x) =$

Duplicate Variables

Permutation method (continued):

$^{\alpha}A =$ $^{\alpha}B =$

Permutations

$^{\alpha}((A \times B)/A) =$

$((A \times B)/A)(x) =$

Duplicate Variables

Permutation method for more complicated operations, such as

$$\frac{(w_1 A + w_2 B)}{(w_1 + w_2)}$$

There are 4 variables $\Rightarrow$ 2^4=16 permutations.

Extension Principle

Extension Principle is how operations and functions of ordinary numbers are extended to inputs that are fuzzy sets defined on $\mathbb{R}$, such as fuzzy numbers.

$$y = f(x)$$

$$B = f(A)$$

Extension Principle

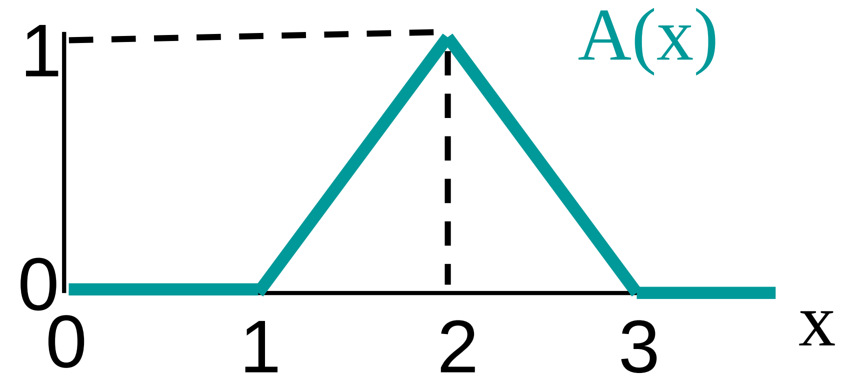
$$y = f(x)$$

$$B = f(A)$$

$$B(y) = \bigvee_{f(x)=y} A(x)$$

Example

$y = f(x) = x^2$

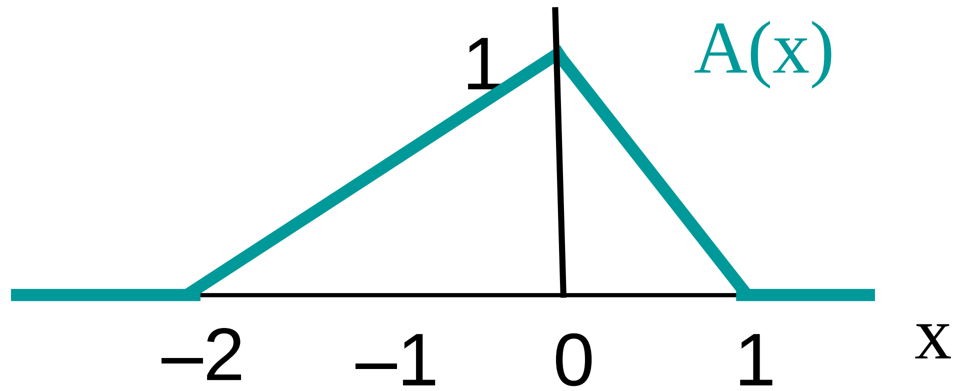


$B(y) = ?$

B(0)=
B(1)=
B(2)=
B(4)=
B(6)=
B(8)=
B(9)=

Example

$y = f(x) = x^2$



y:	0	1	2	3	4	.

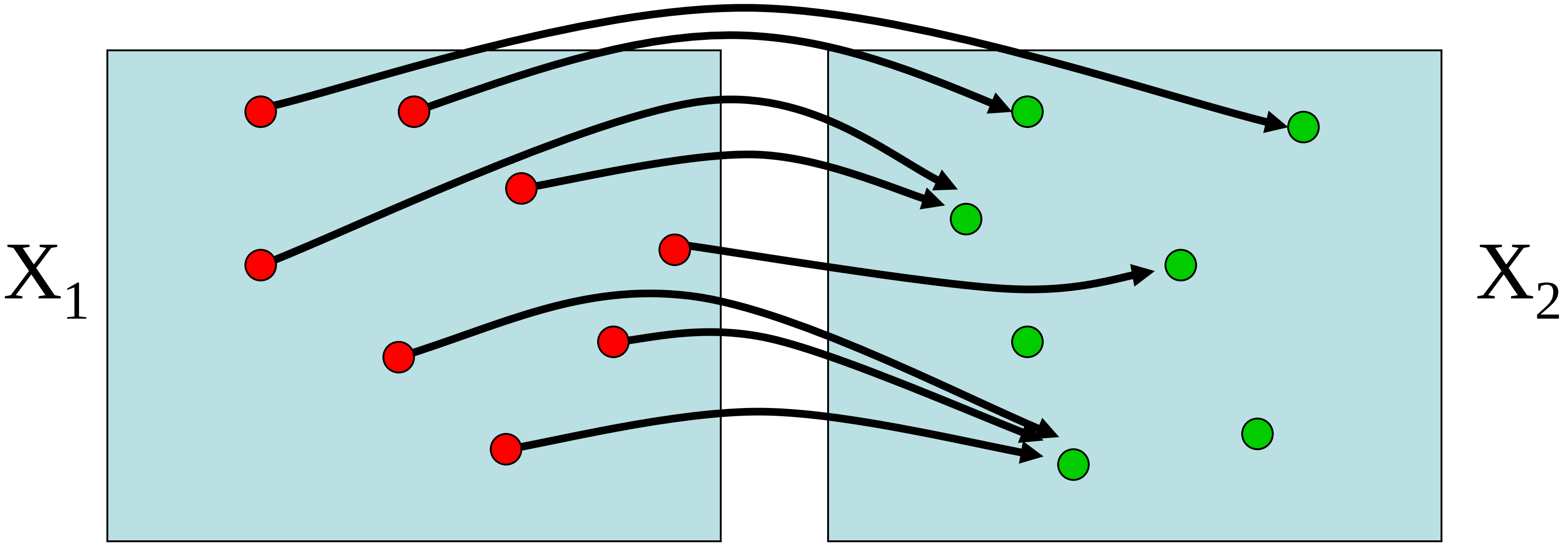
$\{x|f(x)=y\}$:

B(y):

General One-Variable Case

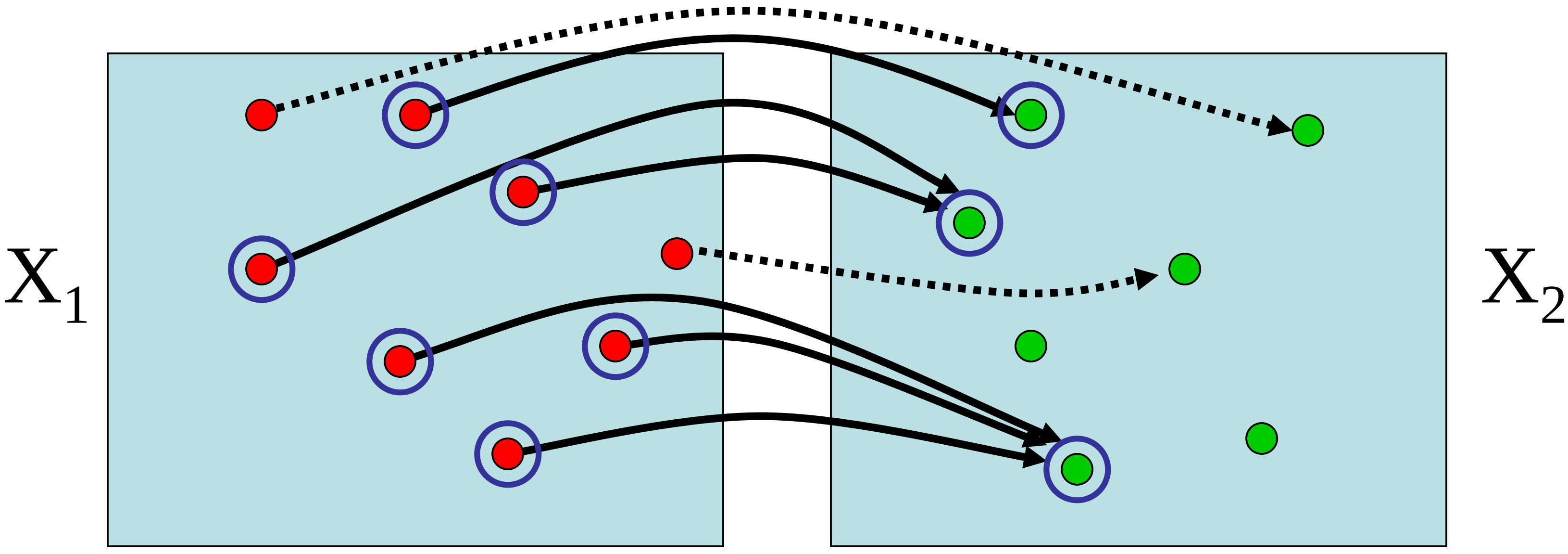
Extension Principle is not limited to only sets defined on real numbers, although these are the most common ones.

Crisp Case: $f: X_1 \rightarrow X_2$



General One-Variable Case

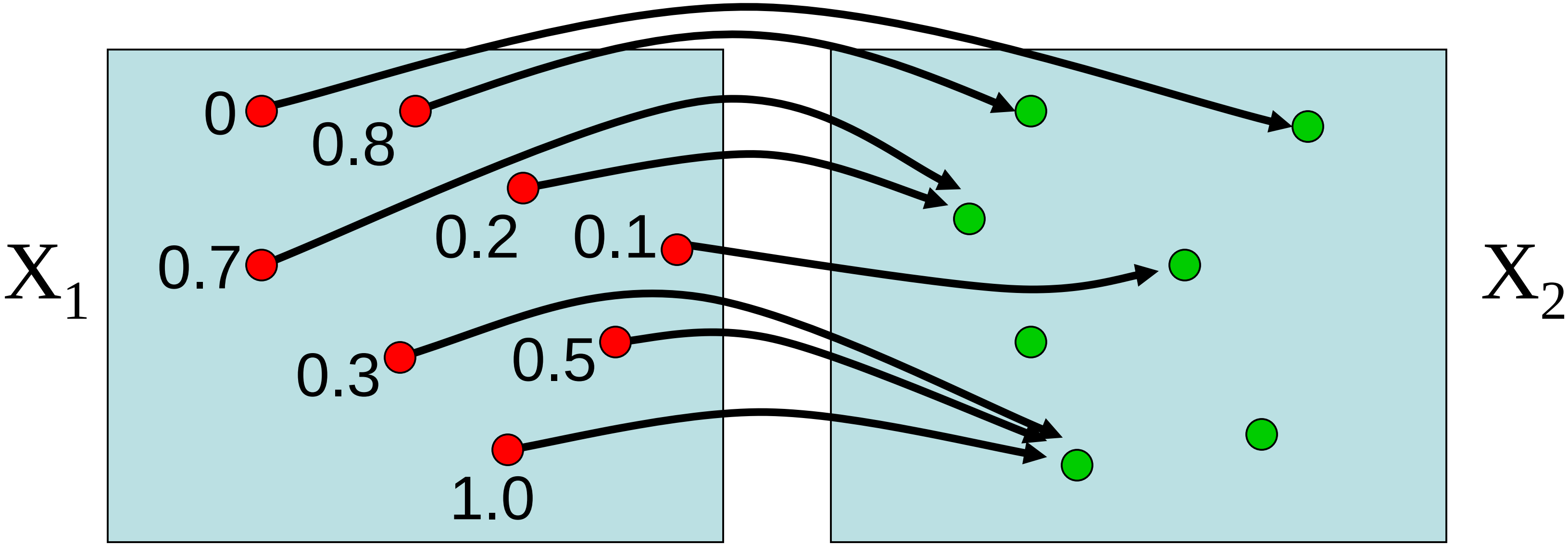
Crisp Case:



A crisp subset in X_1 A crisp subset in X_2

General One-Variable Case

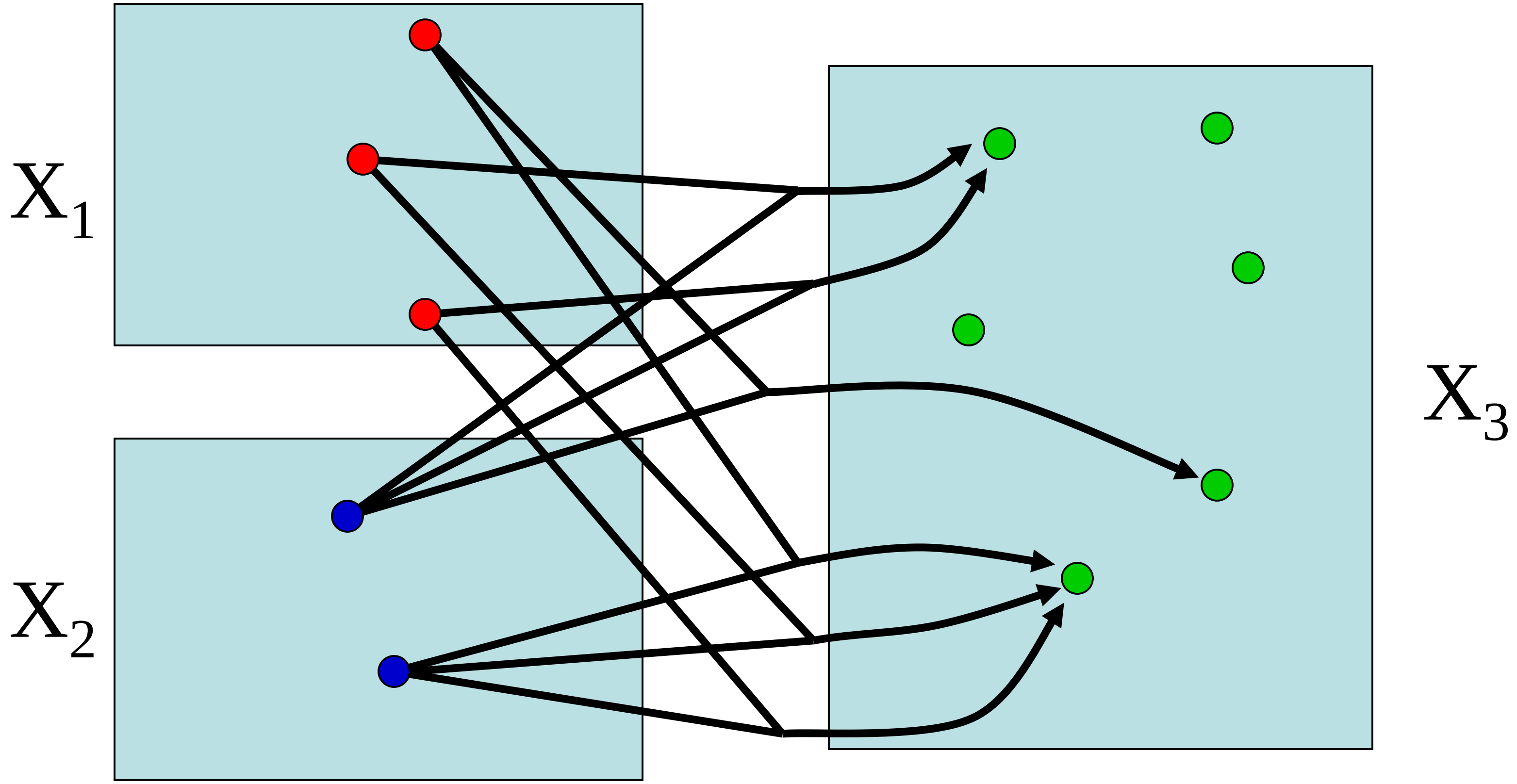
Fuzzy Case: Best Agreement



A fuzzy set in X_1 A fuzzy set in X_2

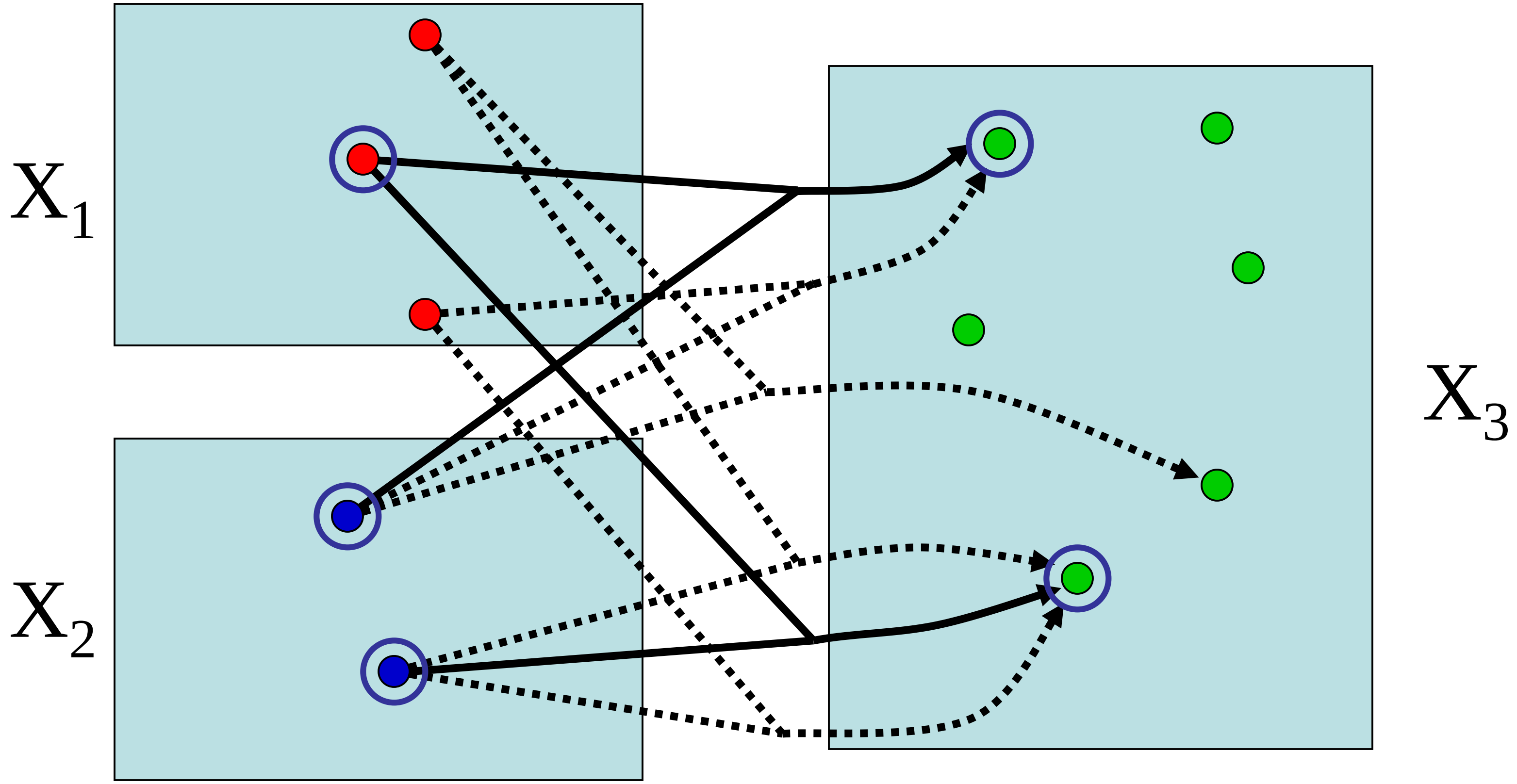
General Multi-Variable Case

Crisp Case: $f: X_1 \times X_2 \rightarrow X_3$



General Multi-Variable Case

Crisp Case:

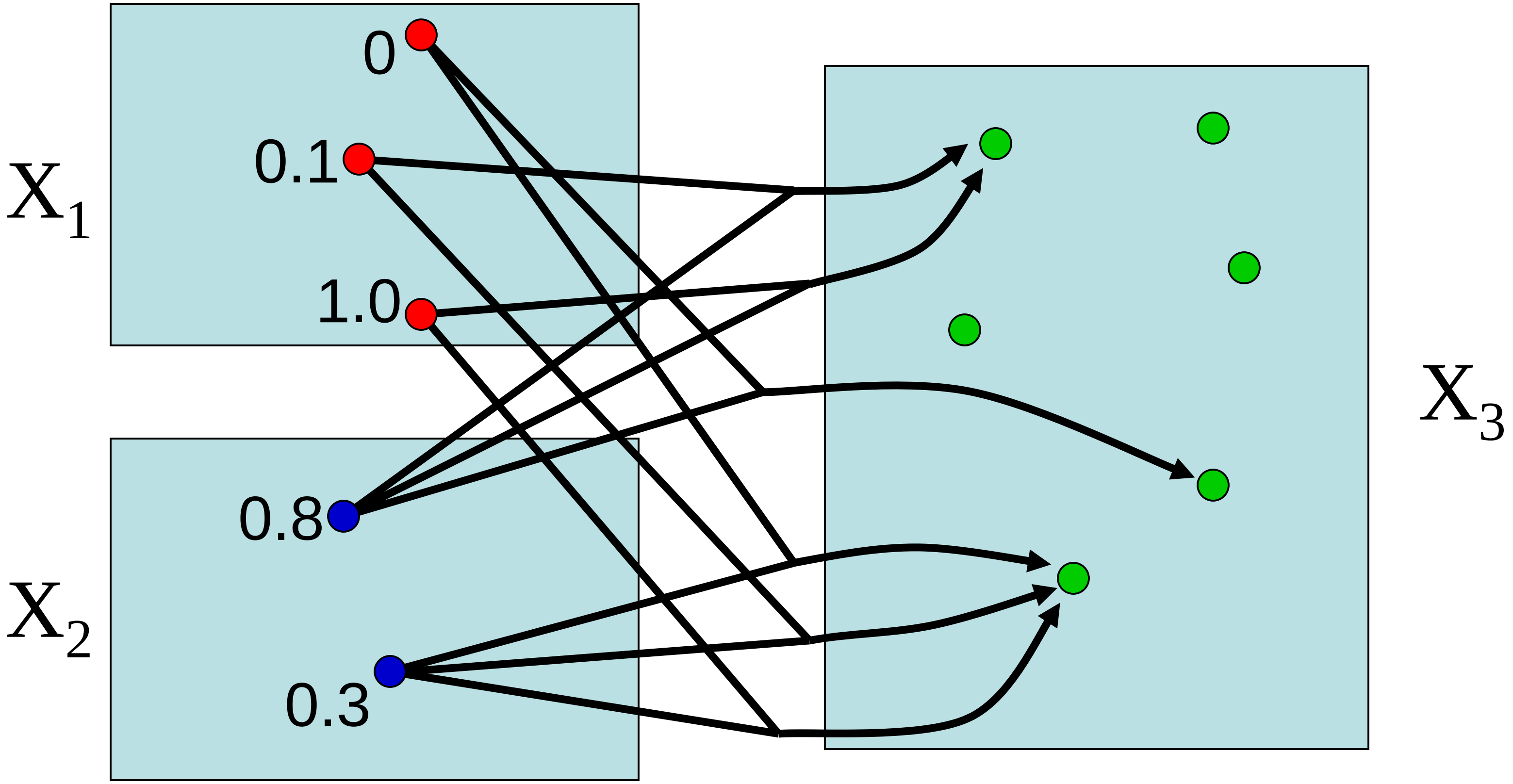


Subsets in X_1 and X2

A subset in X_3

General Multi-Variable Case

Fuzzy Case: Best Pessimistic Agreement



Fuzzy sets in X_1 and $X2$

A fuzzy set in X_3

Arithmetic As Functions

Standard arithmetic operations (+, –, $*$, ÷) can be treated as 2-variable functions.

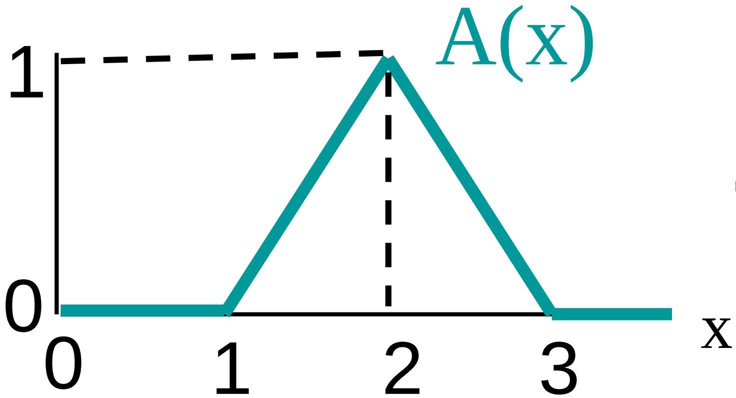
$$y = f(x,y) = x + y$$

$$C = f(A,B)$$

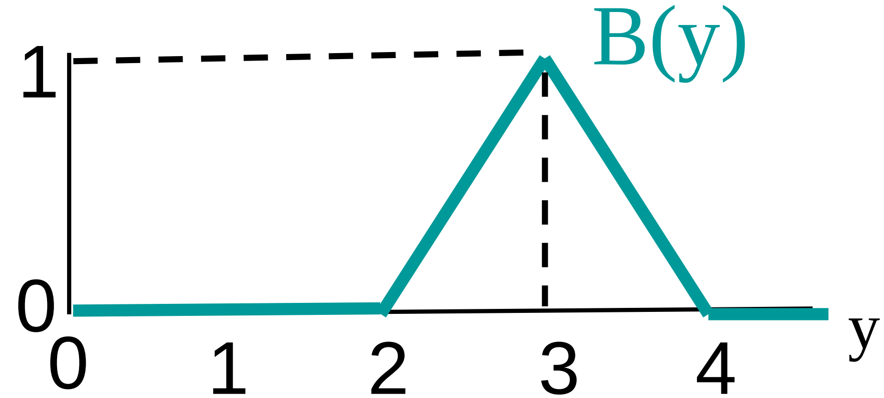
$$C(z) = \bigvee_{x+y=z} [A(x) \wedge B(y)]$$

Example

Fuzzy number arithmetic with Extension Principle.



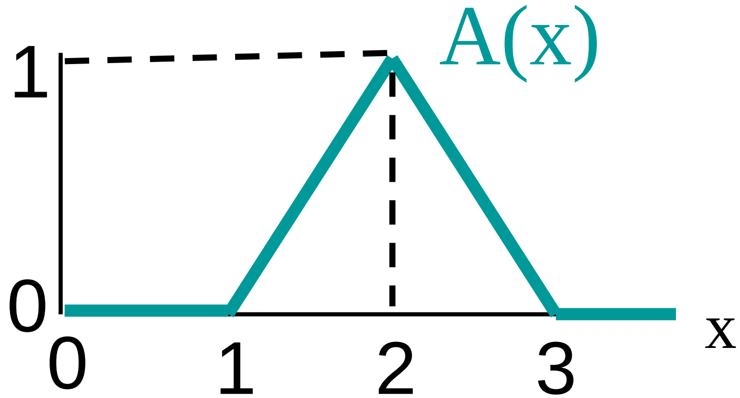
+



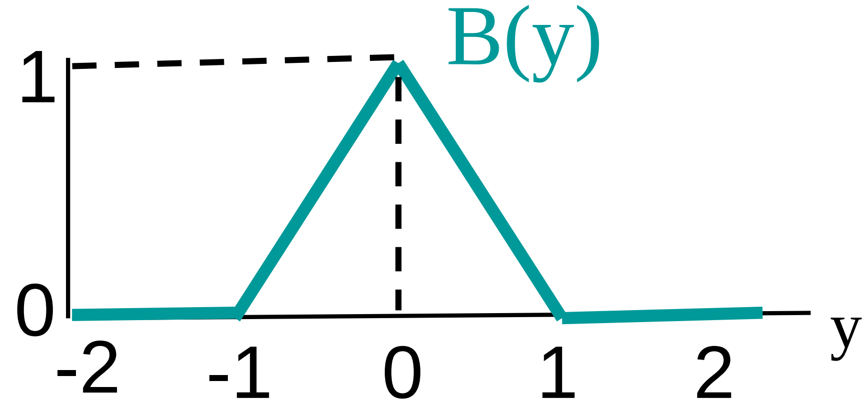
$$(A+B)(z) = \bigvee_{x+y=z} [A(x) \wedge B(y)]$$

Example

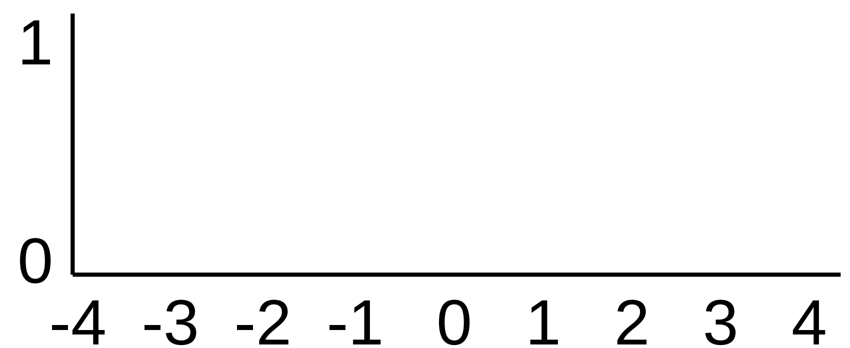
Fuzzy number arithmetic with Extension Principle.



*



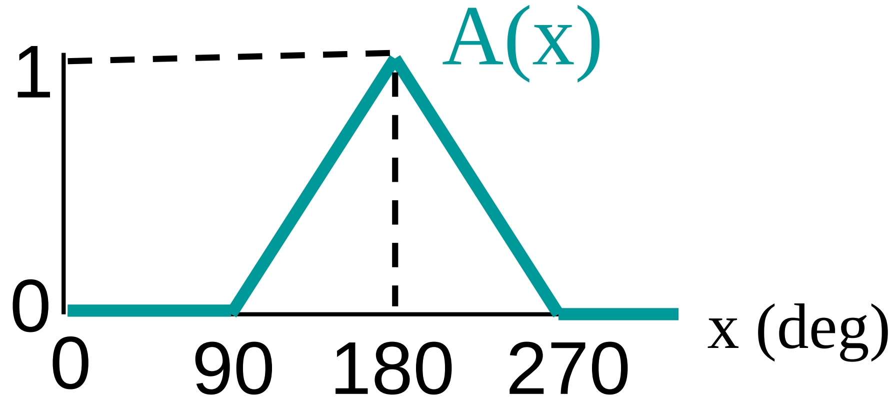
$$(A*B)(z) = \bigvee_{xy=z} [A(x) \wedge B(y)]$$



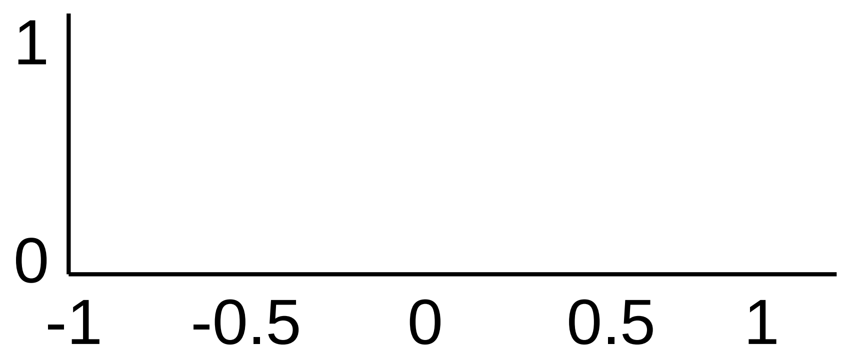
Example

How about other functions?

$$B = \sin(A)$$



$$B(y) = \bigvee_{\sin(x)=y} A(x)$$



About Extension Principle

- Very versatile for extending known operations and functions to fuzzy sets – not limited to numbers.
- Can be easily used for functions with more variables.
- Difficult to implement and computationally expensive.
- Specialized methods are usually necessary in practice.