Review
Fuzzy Sets

Fuzzy Sets Operators

- Zadeh
  - $c(a) = 1-a$
  - $i(a,b) = \min\{a,b\}$
  - $u(a,b) = \max\{a,b\}$
- Lukasiewicz
  - $c(a) = 1-a$
  - $i(a,b) = \max\{a+b-1,0\}$
  - $u(a,b) = \min\{a+b,1\}$

Fuzzy Sets Properties

	Properties	Crisp sets	Zadeh	Lukasiewicz
(1)	Idempotence laws	Valid	Valid	Invalid
(2)	$A \cup \bar{A} = X$ and $A \cap \bar{A} = \emptyset$	Valid	Invalid	Valid
(3)	Distributivity laws	Valid	Valid	Invalid
(4)	De Morgan laws	valid	Valid	Valid

Fuzzy Sets Properties

- The idempotence properties (1) are only satisfied by Zadeh operators.

Fuzzy Sets Properties

- Any intersection and union satisfying De Morgan properties (4) are called dual with respect to the complement.

- The dual of an intersection operator is a union operator.
- The dual of a union operator is an intersection operator.

Fuzzy Sets Properties

- Any intersection and union defined on nondegenerate fuzzy sets that satisfy properties (2) of excluded middle and contradiction do not satisfy properties (3) of distributivity.
- Any intersection and union that satisfy properties (2) and (3) are defined on degenerate fuzzy sets (i.e. crisp values).

Fuzzy Partition

- If C is a fuzzy set, then $P = \{A_1, A_2, \dots, A_n\}$ is a fuzzy partition of C if
  - $C = A_1 \cup A_2 \cup \cdots \cup A_n$
  - $A_i \cap A_j = \emptyset$ for any i and j distinct
- The following statement is true if and only if the operators are Lukasievicz:
  - P is a fuzzy partition on C if and only if

$$A_1(x) + A_2(x) + \cdots + A_n(x) = C(x), \forall x \in X$$

Part 5.

Fuzzy Relations

Relations

Relations: A representation of association, interaction, or connection between elements of sets.

A crisp relation on two sets:

$$R: A \times B \rightarrow \{0,1\}$$

A crisp relation on three sets:

$$R: A \times B \times C \rightarrow \{0,1\}$$

A crisp relation on a set itself:

$$R: A \times A \rightarrow \{0,1\}$$

Example of Relations

$R: A \times A \rightarrow \{0,1\}$

R = "There is direct flight"

	Taipei	Kaohsiung	Hsinchu	Hualian
Taipei	0	1	0	1
Kaohsiung	1	0	0	1
Hsinchu	0	0	0	0
Hualian	1	1	0	0

Example of Relations

$$R: A \times B \times C \times D \to \{0,1\}$$

A: set of all students at the university

B: set of all courses at the university

C: set of all semesters

D: grades: {0,1,2,...,99,100}

Do you sense relational database?

Tuples

A tuple is a combination of one element from each set that forms the relation.

For example:

$\langle$Big John, Fall 1994, Fuzzy Set Theory, 80$\rangle$

is a tuple in our last relation.

A crisp relation can be defined as a subset of the product set of all the "base" sets.

Fuzzy Relations

How about a fuzzy subset of the product set?
This defines a fuzzy relation.

A fuzzy relation on two sets:

$$R: A \times B \rightarrow [0,1]$$

A fuzzy relation on three sets:

$$R: A \times B \times C \rightarrow [0,1]$$

A fuzzy relation on a set itself:

$$R: A \times A \rightarrow [0,1]$$

Example of Fuzzy Relations

$R: A \times B \times C \rightarrow [0,1]$

A: the four seasons

B: { cold, hot }

C: { wet, dry }

A relation:

Tuples	Membership
$\langle$spring, cold, wet$\rangle$	0.6
$\langle$summer, cold, wet$\rangle$	0.1
$\langle$winter, cold, dry$\rangle$	0.5
$\langle$fall, hot, wet$\rangle$	0.4
⋮	⋮

Example of Fuzzy Relations

$R: A \times A \rightarrow [0,1]$ A: students in this class

A relation:

The degree of whether two students are "good friends" ...

Projection

This is the process of getting a relation

$$R: A \times B \rightarrow [0,1]$$

from a relation of more "base" sets

$$R: A \times B \times C \rightarrow [0,1]$$

This is like the "projection", say, from a 3-D space to a 2-D space

Projection

$\mathcal{X}$: the original family of base sets

$\mathcal{Y}$: a subset of $\mathcal{X}$

For example, $\mathcal{X} = \{A,B,C\}$ and $\mathcal{Y} = \{A,B\}$

The projection $(R{\downarrow}\mathcal{Y})$ for a given tuple $\mathbf{y}$ in $A \times B$ is given by the maximum membership of all tuples $\mathbf{x}$ in $A \times B \times C$ that contain $\mathbf{y}$.

Example of Projection

$R: A \times B \times C \rightarrow [0,1]$

A: the four seasons

B: { cold, hot }

C: { wet, dry }

A relation:

	spring	summer	fall	winter
cold, wet	0.6	0.1	0.3	0.8
cold, dry	0.4	0.2	0.5	0.5
hot, wet	0.5	0.6	0.4	0.1
hot, dry	0.3	0.8	0.6	0.1

Projection on $A \times B$:

	spring	summer	fall	winter
cold				
hot				

Cylindric Extension

This is the process of getting a relation

$$R: A \times B \times C \rightarrow [0,1]$$

from a relation of less "base" sets

$$R: A \times B \rightarrow [0,1]$$

Since there is no information regarding the "additional" base set, all its should be considered equal.

Cylindric Extension

$\mathcal{Y}$: the original family of base sets

$\mathcal{X}$: the bigger family that contains $\mathcal{Y}$ and other sets

For example, $\mathcal{Y} = \{A, B\}$ and $\mathcal{X} = \{A, B, C\}$

The cylindric extension ($R\uparrow\mathcal{X}-\mathcal{Y}$) for a given tuple $\mathbf{x}$ in $A\times B\times C$ is given by the membership of the $\mathbf{y}$ in $A\times B$ that is contained in $\mathbf{x}$.

Example of Cylindric Extension

$R: A \times B \rightarrow [0,1]$

A: the four seasons

B: { cold, hot }

C: { wet, dry }

A relation:

	spring	summer	fall	winter
cold	0.5	0.1	0.4	0.7
hot	0.4	0.8	0.6	0.2

Cylindric Extension to $A \times B \times C$:

	spring	summer	fall	winter
cold, wet				
cold, dry				
hot, wet				
hot, dry				

Cylindric Closure

This is the "intersection" of multiple cylindric extensions from different family of sets.

For example:

A relation on $\mathbf{A \times B}$:

	spring	summer	fall	winter
cold	0.5	0.1	0.4	0.7
hot	0.4	0.8	0.6	0.2

A relation on $\mathbf{A \times C}$:

	spring	summer	fall	winter
wet	0.7	0.3	0.2	0.5
dry	0.2	0.6	0.6	0.3

Example of Cylindric Closure

(use "minimum" for intersection)

	spring	summer	fall	winter
cold, wet				
cold, dry				
hot, wet				
hot, dry				

Composition of Fuzzy Relations

This is the process of getting a relation

$$R: A \times C \to [0,1]$$

from two other relations

$$R_1: A \times B \to [0,1] \quad \text{and} \quad R_2: B \times C \to [0,1]$$

$$(R_1 \circ R_2)(\langle x,z \rangle) = \bigvee_{y \in B} [R_1(\langle x,y \rangle) \wedge R_2(\langle y,z \rangle)]$$

for all $x \in A$ and $z \in C$

This is called the min-max composition. It's common but not the only way.

Example

This is a possible application of fuzzy relations in approximate reasoning:

A: a set of patients

B: a set of symptons

C: a set of possible diagnoses

Input from tests, etc.: $R_1 : A \times B \rightarrow [0,1]$

Knowledge base: $R_2 : B \times C \rightarrow [0,1]$

We can then use $(R_1 \circ R_2)$ to generate possible diagnoses for these patients.

Part 6.

Fuzzy Logic

Binary Logic Functions

$$f: \{0,1\}^N \rightarrow \{0,1\}$$

How many functions are there if $N = 2$?

Multi-Valued Logic Functions

$$f: \{0, 1/2, 1\}^N \to \{0, 1/2, 1\}$$

For example, when we need to represent "not sure".

Lukasiewice's definitions:

$$(a \text{ AND } b) = \min(a,b)$$
$$(a \text{ OR } b) = \max(a,b)$$
$$(\text{NOT } a) = 1 - a$$
$$(a \Rightarrow b) = \min(1, 1-a+b)$$

Finer divisions of truth values:

$$f: T^N \to T, \quad T = \{ 0, 1/n, 2/n, \ldots , (n-1)/n, 1 \}$$

Fuzzy Propositions

A basic proposition: **P: V is A.**

Examples:

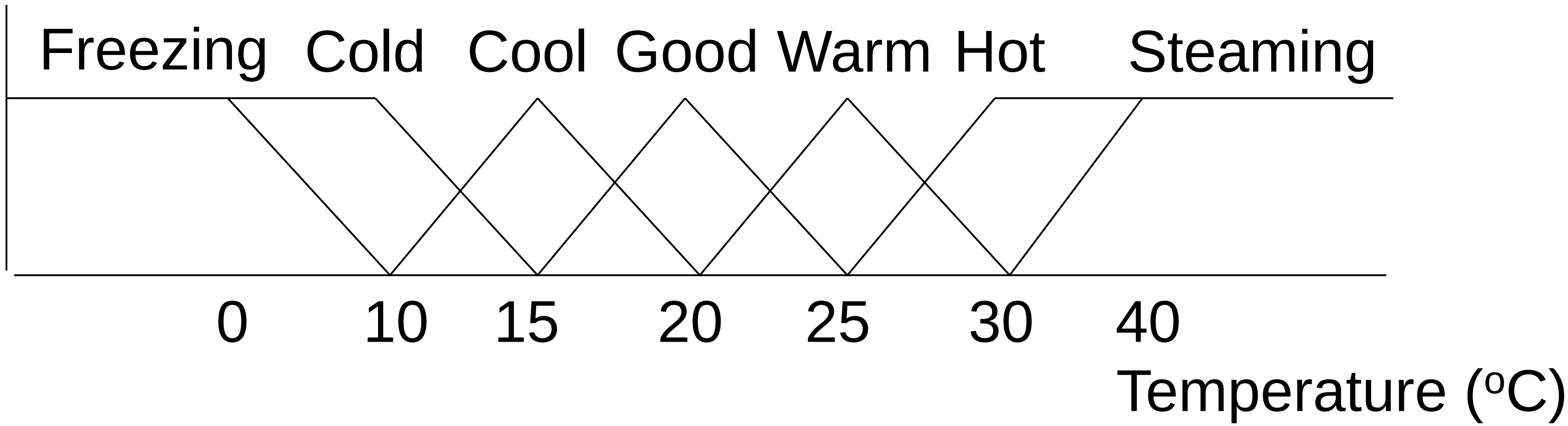
"38 degrees is HOT"

"This afternoon is HOT"

"200cm is TALL"

"Big John is TALL"

Linguistic Variables



Linguistic Hedges

Modifiers made to fuzzy variables

Examples:

HOT $\Rightarrow$ VERY HOT

WARM $\Rightarrow$ MORE-OR-LESS WARM

Common methods:

"VERY": take the square

"MORE-OR-LESS": take the square root

Combining Propositions

Conjunctive: **P: (V is A) AND (W is B)**.

- Uses intersection operator
- Forms a fuzzy relation

Disjunctive: **P: (V is A) OR (W is B)**.

- Uses union operator
- Forms a fuzzy relation

Implicative Propositions

P: (V is A) $\Rightarrow$ (W is B).

Uses fuzzy implication operator

Binary implication operators:

- (NOT a) OR b
- (NOT a) OR (a AND b)
- ((NOT a) AND (NOT b)) OR b

Implication Operators

- (NOT a) OR b
  $\Rightarrow$ u(c(a), b)

- (NOT a) OR (a AND b)
  $\Rightarrow$ u(c(a), i(a,b))

- ((NOT a) AND (NOT b)) OR b
  $\Rightarrow$ u(i(c(a),c(b)), b)

Logically equivalent for binary logic,
but not for fuzzy logic.

Implication Operators

Use

$(a \text{ OR } b) = \min(1, a+b)$
$(\text{NOT } a) = 1 - a$
$(a \Rightarrow b) = (\text{NOT } a) \text{ OR } b$

$\rightarrow$ $(a \Rightarrow b) = \min(1, 1-a+b)$ Lukasiewicz's implication

Other examples:

$(a \Rightarrow b) = \min(1-a, b)$ Kleene-Dienes implication
$(a \Rightarrow b) = (1-a+ab)$ Reichenbach implication

Implication Operators

- $\max\{\ x \in \{0,1\} \mid (a \text{ AND } x) \leq b\ \}$
  $\Rightarrow \sup\{\ x \in [0,1] \mid i(a,x) \leq b\ \}$

Examples:

Godel implication:

$$i(a,b) = \min(a,b) \quad \rightarrow (a \Rightarrow b) = \begin{cases} 1 & \text{when } a \leq b \\ b & \text{when } a > b \end{cases}$$

$$i(a,b) = ab \quad \rightarrow (a \Rightarrow b) =$$

$$i(a,b) = \max(0,\ a+b-1) \quad \rightarrow (a \Rightarrow b) =$$

Common Properties

$$a \leq b \ \rightarrow \ \mathscr{I}(a,x) \geq \mathscr{I}(b,x)$$

$$a \leq b \ \rightarrow \ \mathscr{I}(x,a) \leq \mathscr{I}(x,b)$$

$$\mathscr{I}(0,a) = 1$$

$$\mathscr{I}(1,b) = b$$

$$\mathscr{I}(a,a) = 1$$

$$\mathscr{I}(a, \mathscr{I}(b,x)) = \mathscr{I}(b, \mathscr{I}(a,x))$$

$$\mathscr{I}(a,b) = 1 \ \Leftrightarrow \ a \leq b$$

$$\mathscr{I}(a,b) = \mathscr{I}(c(b), c(a))$$

$\mathscr{I}(a,b)$ is continuous

Inference

Inference: Derivation of new results from given facts and rules.

Rules of inference:

modus ponens: $(a \text{ AND } (a \Rightarrow b)) \Rightarrow b$

modus tollens: $((\text{NOT } b) \text{ AND } (a \Rightarrow b)) \Rightarrow (\text{NOT } a)$

syllosium: $((a \Rightarrow b) \text{ AND } (b \Rightarrow c)) \Rightarrow (a \Rightarrow c)$

Composition of Fuzzy Relations

Getting $R: A \times C \to [0,1]$

from $R_1: A \times B \to [0,1]$ and $R_2: B \times C \to [0,1]$

$$R(x,z) = (R_1 \circ R_2)(x,z) = \bigvee_{y \in B} [R_1(x,y) \wedge R_2(y,z)]$$

for all $x \in A$ and $z \in C$

Example:

A: patients B: symptoms C: diagnoses

R_1: patient symptoms R_2: knowledge base

We can then use $(R_1 \circ R_2)$ to generate possible diagnoses for these patients.

Inference and Relations

Assume A has only one element so we drop the variable x:

$$R(z) = (R_1 \circ R_2)(z) = \bigvee_{y \in B} [R_1(y) \wedge R_2(y,z)] \qquad \text{for all } z \in C$$

This is like modus ponens:

$$(\underset{\substack{\uparrow \\ R_1}}{a} \text{ AND } (\underset{\substack{\uparrow \\ R_2}}{a \Rightarrow b})) \Rightarrow \underset{\substack{\uparrow \\ R}}{b}$$

Example

P: (V is A) $\Rightarrow$ (W is B).

A represents "fun days" of a week:

$$A = 0.4/\text{Thu} + 0.8/\text{Fri} + 1/\text{Sat} + 0.6/\text{Sun}$$

B represents "happy" defined on "degrees of happiness" = {1,2,3,4}:

$$B = 0.5/3 + 1/4$$

The fuzzy relation representing P, using implication operator min(1,1–a+b):

A / B	Mon	Tue	Wed	Thu	Fri	Sat	Sun
1							
2							
3							
4							

Example

Do inference for the following inputs:

- V = { Sat }
- V = { Thu }
- V = A
- V = NOT A (use Zadeh's complement)
- V = MORE-OR-LESS A (use square root)

Correlation-Min

The idea is to implement

$$\mathbf{P: (V \text{ is } A) \Rightarrow (W \text{ is } B).}$$

as

$$\mathbf{(V \text{ is } A) \text{ AND } (W \text{ is } B)}$$

using "min" for "AND" operation.

Convenient, and actually used in practice, but not a true implication operator.

The fuzzy relation representing P, using correlation-min:

A B	Mon	Tue	Wed	Thu	Fri	Sat	Sun
1							
2							
3							
4							

Example

Do inference for the following inputs:

- V = { Sat }
- V = { Thu }
- V = A
- V = NOT A (use Zadeh's complement)
- V = MORE-OR-LESS A (use square root)

Binary Relations

It is also possible to have a binary relation for "translating" this rule:

$$\mathbf{P: (V\ is\ A) \Rightarrow (W\ is\ B)}.$$

For example:

$$\mathbf{R(x,y)} = \begin{cases} 0 & \text{if } A(x) > B(y) \\ 1 & \text{if } A(x) \le B(y) \end{cases}$$

The fuzzy relation representing P:

A / B	Mon	Tue	Wed	Thu	Fri	Sat	Sun
1							
2							
3							
4							

Example

Do inference for the following inputs:

- V = A
- V = MORE-OR-LESS A (use square root)
- V = NOT A (use Zadeh's complement)

Preservation of function relations

Multi-Antecedent Rules

Conjunctive ("AND"):

$$\textbf{P: } (\mathbf{V_1 \text{ is } A_1 \text{ AND } V_2 \text{ is } A_2}) \Rightarrow (\mathbf{W \text{ is } B})$$

- First create a relation between A_1 and A_2 first (using an intersection operator).
- Then create a new relation (using an implication operator) between B and this relation.

Example

If Day is Fun and Weather is Good Then Happy.

Weather is defined on {Sunny, Cloudy, Rainy}
Good(Weather) = 1/Sunny + 0.5/Cloudy + 0/Rainy

The fuzzy relation representing the antecedents:

A_1 / A_2	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Sunny							
Cloudy							
Rainy							

Use "min" as intersection

Example

The combined (3-dimensional) relation R_{rule} (use Lukasiewice's implication operator):

y (Mood) = "1"

A_1 / A_2	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Sunny							
Cloudy							
Rainy							

y (Mood) = "2"

A_1 / A_2	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Sunny							
Cloudy							
Rainy							

y (Mood) = "3"

A_1 / A_2	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Sunny							
Cloudy							
Rainy							

y (Mood) = "4"

A_1 / A_2	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Sunny							
Cloudy							
Rainy							

*For simplicity, "Mood" replaces "Degree of Happiness"

Example

For input, get the relation R_{input} between V_1 and V_2 using the same intersection operator.

Inference:

$$W(y) = \bigvee_{x_1, x_2} [R_{input}(x_1,x_2) \wedge R_{rule}(x_1,x_2,y)]$$

Do inference for the following inputs:

- V_1 = "Sunday" AND V_2 = "Cloudy"
- $V_1 = A_1$ AND $V_2 = A_2$

Multi-Antecedent Rules

Disjunctive ("OR"):

$$\mathbf{P: (V_1 \text{ is } A_1 \;\; OR \;\; V_2 \text{ is } A_2) \Rightarrow (W \text{ is } B)}$$

Is this equivalent to the following?

$$\mathbf{P: (P_1 \;\; OR \;\; P_2)}$$
$$\mathbf{P_1: (V_1 \text{ is } A_1) \Rightarrow (W \text{ is } B)}$$
$$\mathbf{P_2: (V_2 \text{ is } A_2) \Rightarrow (W \text{ is } B)}$$

Test with

- "max" as OR and correlation-min for implication
- "max" as OR and Lukasiewicz's implication

Multiple Rules

$$P_j\text{: }(V_{j1} \text{ is } A_{j1} \text{ AND } \ldots \text{ AND } V_{jn} \text{ is } A_{jn}) \Rightarrow (W_j \text{ is } B_j)$$

Common: Conjunctive Normal Form

Combination of outputs for variable W:

- Take all rules with (W is B_j)
- (Common) Get the "max" or "sum" of the outputs ($W(y)$) of these rules

Example

If Day is Fun and Weather is Good Then Mood is Happy.

If Day is Fun and Weather is Bad Then Mood is Unhappy.

$$\text{Fun(Days)} = 0.4/\text{Thu} + 0.8/\text{Fri} + 1/\text{Sat} + 0.6/\text{Sun}$$

$$\text{Good(Weather)} = 1/\text{Sunny} + 0.5/\text{Cloudy}$$

$$\text{Bad(Weather)} = 1/\text{Rainy} + 0.4/\text{Cloudy}$$

$$\text{Happy(Mood)} = 1/4 + 0.5/3$$

$$\text{Unhappy(Mood)} = 1/1 + 0.5/2$$

Try to do inference for

- a Cloudy Saturday
- a Sunny Thursday

Why Correlation-Min?

P: (V_1 is A_1 AND V_2 is A_2) $\Rightarrow$ (W is B)

$$W(y) = \bigvee_{x_1,x_2} \{ i(V_1(x_1),V_2(x_2)) \wedge [1 \wedge (1 - i(A_1(x_1),A_2(x_2)) + B(y))] \}$$

using Lukasiewice's implication

Much simpler with correlation-min for implication:

$$W(y) = \bigvee_{x_1,x_2} [(V_1(x_1) \wedge V_2(x_2) \wedge A_1(x_1) \wedge A_2(x_2) \wedge B(y))]$$

Example

If Day is Fun and Weather is Good Then Mood is Happy.

If Day is Fun and Weather is Bad Then Mood is Unhappy.

$$\text{Fun(Days)} = 0.4/\text{Thu} + 0.8/\text{Fri} + 1/\text{Sat} + 0.6/\text{Sun}$$

$$\text{Good(Weather)} = 1/\text{Sunny} + 0.5/\text{Cloudy}$$

$$\text{Bad(Weather)} = 1/\text{Rainy} + 0.4/\text{Cloudy}$$

$$\text{Happy(Mood)} = 1/4 + 0.5/3$$

$$\text{Unhappy(Mood)} = 1/1 + 0.5/2$$

Try to do inference for

- a Cloudy Saturday
- a Sunny Thursday

Why Correlation-Min?

P: (V_1 is A_1 AND V_2 is A_2) $\Rightarrow$ (W is B)

$$W(y) = \bigvee_{x_1, x_2} \{ i(V_1(x_1), V_2(x_2)) \wedge [1 \wedge (1 - i(A_1(x_1), A_2(x_2)) + B(y))] \}$$

using Lukasiewice's implication

Much simpler with correlation-min for implication:

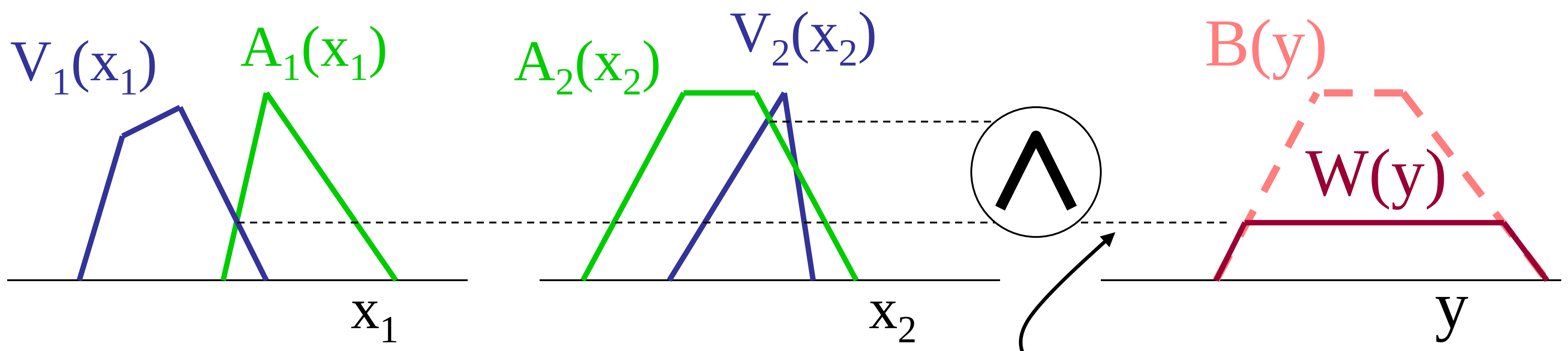
$$W(y) = \bigvee_{x_1, x_2} [(V_1(x_1) \wedge V_2(x_2) \wedge A_1(x_1) \wedge A_2(x_2) \wedge B(y)]$$

$$= \left\{ \bigvee_{x_1, x_2} [(V_1(x_1) \wedge V_2(x_2) \wedge A_1(x_1) \wedge A_2(x_2)] \right\} \wedge B(y)$$

Graphical Representation

$$W(y) = \left\{ \bigvee_{x_1,x_2} [(V_1(x_1) \wedge V_2(x_2) \wedge A_1(x_1) \wedge A_2(x_2))] \right\} \wedge B(y)$$

$$= \left\{ \bigvee_{x_1} [(V_1(x_1) \wedge A_1(x_1))] \right\} \bigwedge \left\{ \bigvee_{x_2} [V_2(x_2) \wedge A_2(x_2)] \right\} \bigwedge B(y)$$



Firing strength of rule

Firing of Multiple Rules

Assume the same input and output domains for all rules. For example, all rules have the form:

IF (Day) AND (Weather) THEN (Mood)

P_1: (V_1 is A_{11} AND V_2 is A_{12}) $\Rightarrow$ (W is B_1)

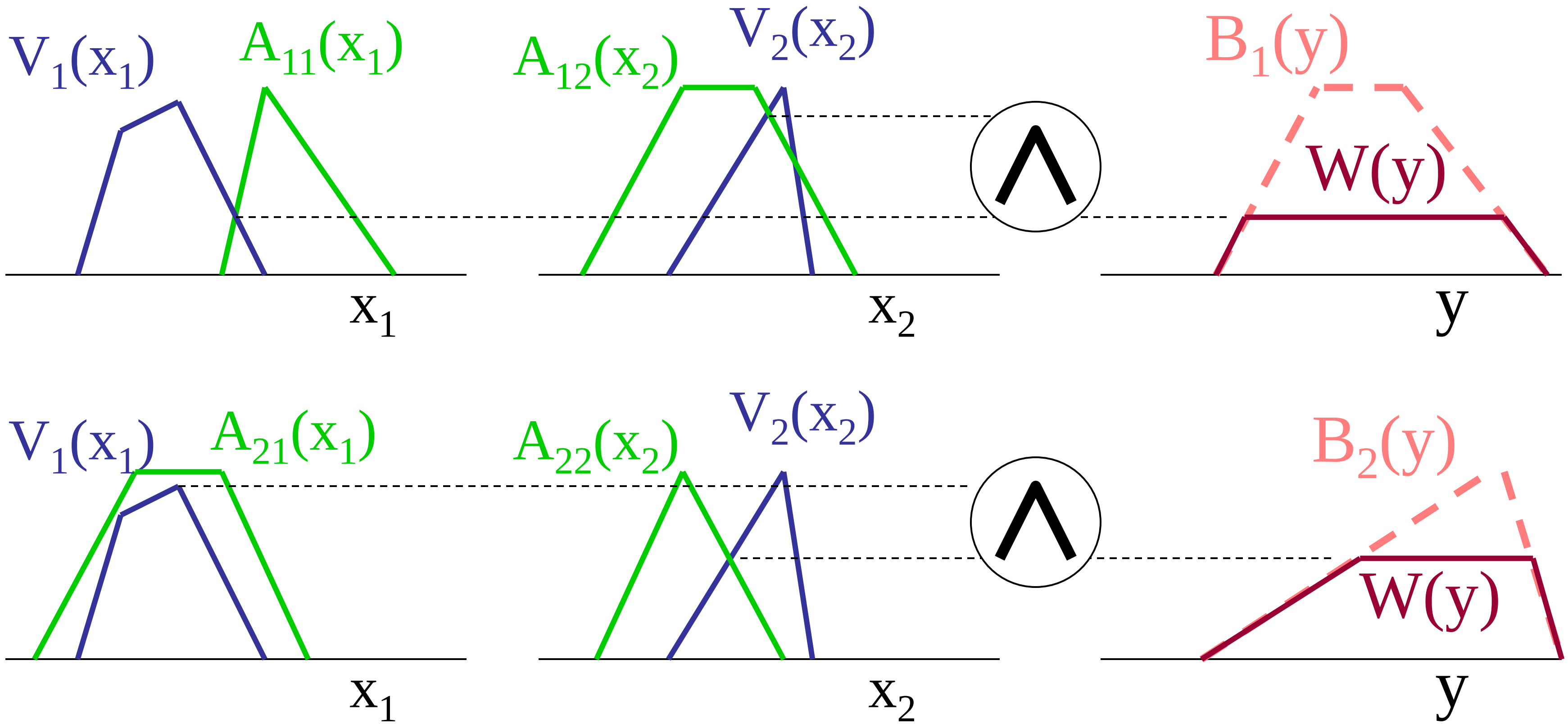
P_2: (V_1 is A_{21} AND V_2 is A_{22}) $\Rightarrow$ (W is B_2)

P_3: (V_1 is A_{31} AND V_2 is A_{32}) $\Rightarrow$ (W is B_3)

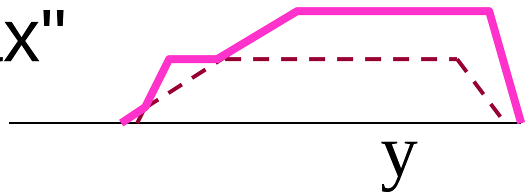
etc.

If Day is Fun and Weather is Good Then Mood is Happy.
If Day is Fun and Weather is Bad Then Mood is OK.
If Day is Work and Weather is Bad Then Mood is Unhappy.
etc.

Firing of Multiple Rules



Combined **$W(y)$** by "max"

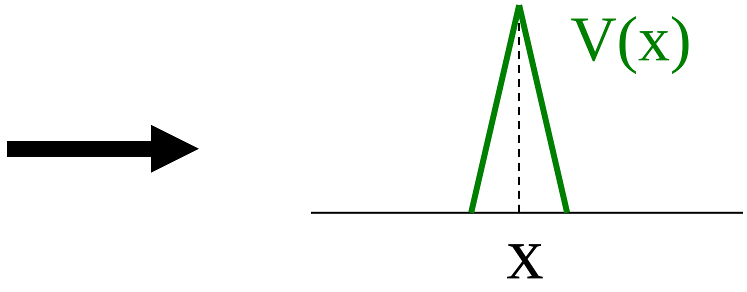


Fuzzification

What happens when the inputs are individual numbers, such as readings from instruments?

We can create a "narrow" fuzzy set around the number. For example, with a triangle:

x is the input



Does not work on discrete domains.

Singleton Excitation

This happens when inputs are individual elements.
Very common in fuzzy logic applications.

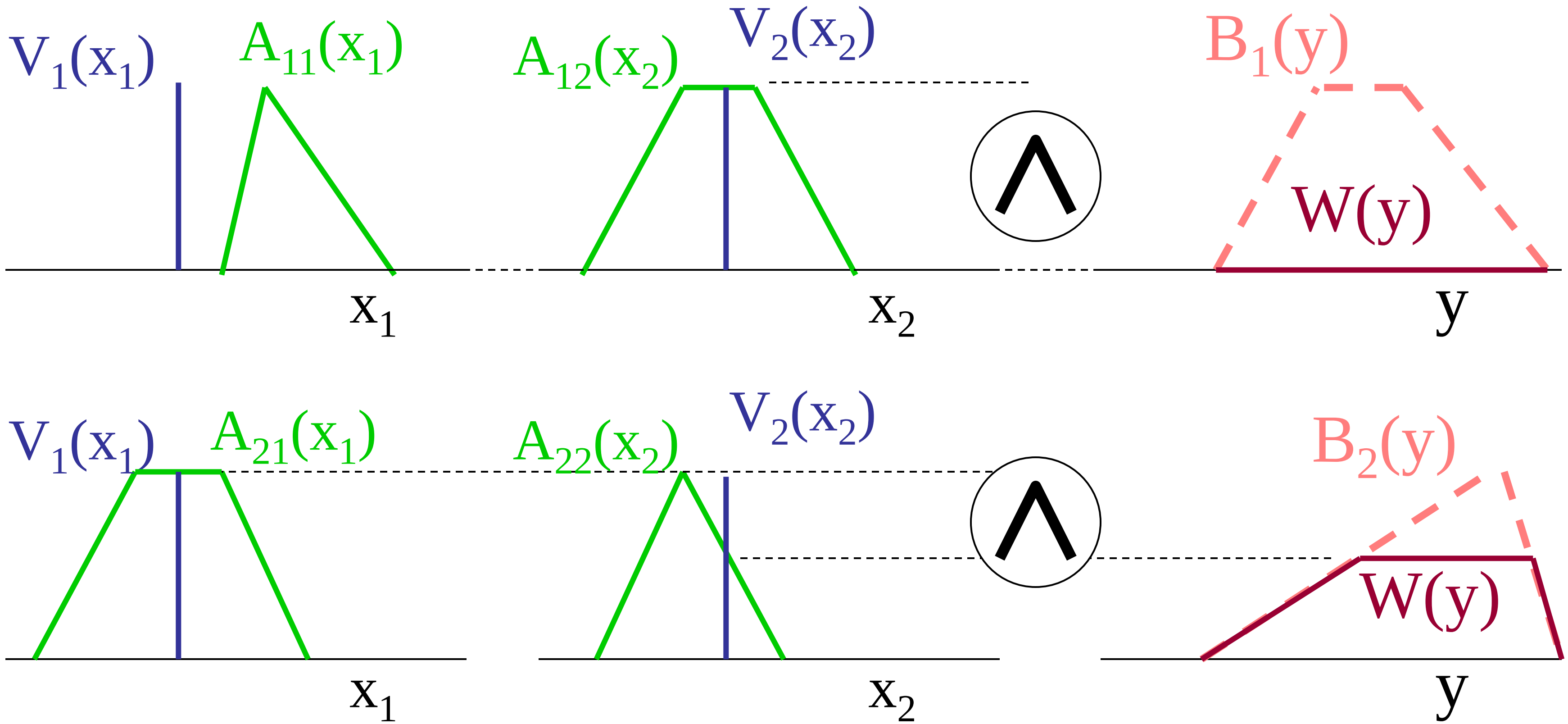
$$V_i(x_i) = \begin{cases} 1 & \text{when } x_i = x_i^* \\ 0 & \text{otherwise} \end{cases}$$

$$W(y) = \left\{ \bigvee_{x_1, x_2} [(V_1(x_1) \wedge V_2(x_2) \wedge A_1(x_1) \wedge A_2(x_2)] \right\} \wedge B(y)$$

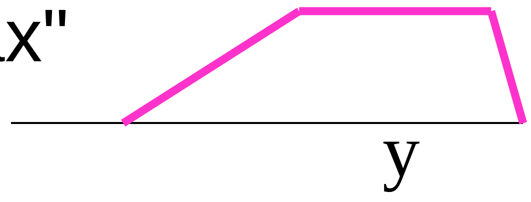
$$= \left\{ \bigvee_{x_1} [(V_1(x_1) \wedge A_1(x_1)] \right\} \bigwedge \left\{ \bigvee_{x_2} [V_2(x_2) \wedge A_2(x_2)] \right\} \bigwedge B(y)$$

$$= [A_1(x_1^*) \bigwedge A_2(x_2^*)] \bigwedge B(y)$$

Singleton Excitation

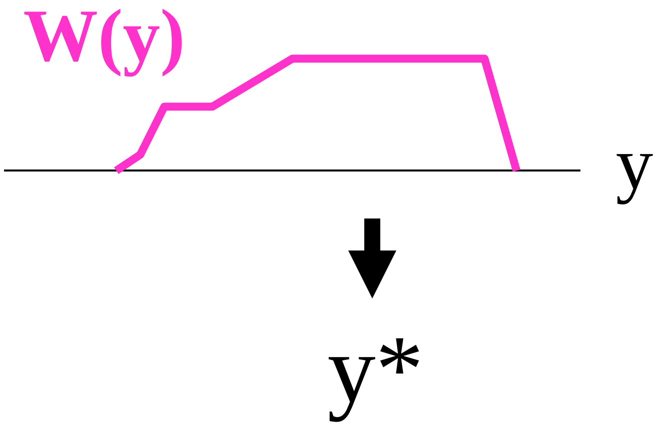


Combined **W(y)** by "max"



Defuzzification

The process of converting a fuzzy set to a single number so that we can do something about it.



Common methods:

- centroid
- maxima (or center of maxima)

This number y^* can be used for ranking, or as inputs to other devices, etc.

Grid of Rules

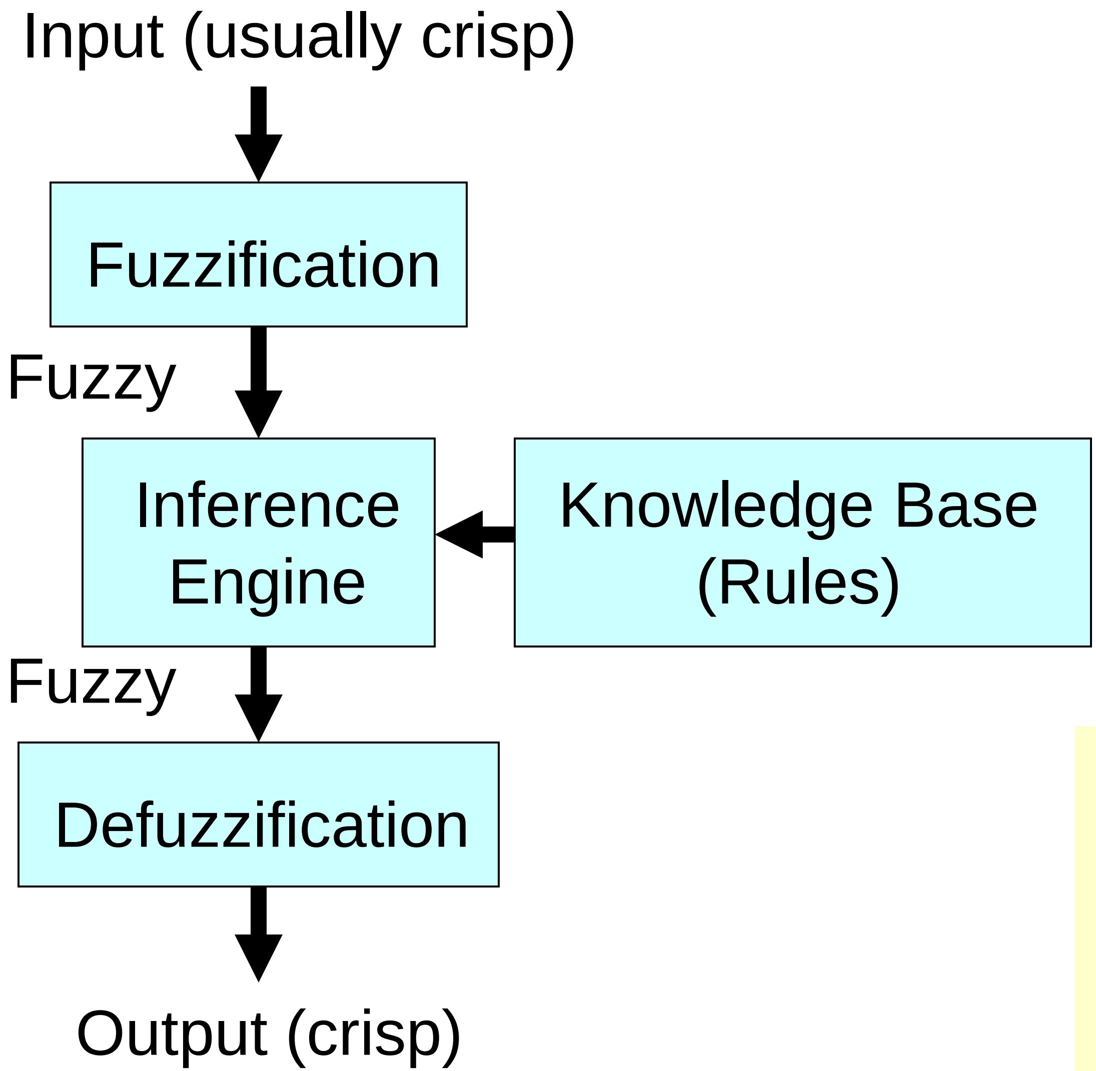
Common representation of a set of 2-antecedent rules with the same input and output domains.

For example: IF (Day) AND (Weather) THEN (Mood)

$$\mathbf{P: (V_1 \text{ is } A_1 \;\; AND \;\; V_2 \text{ is } A_2) \Rightarrow (W \text{ is } B)}$$

A_1 (Day) A_2 (Weather)	Fun	Work
Good	Happy	OK
Bad	OK	Unhappy

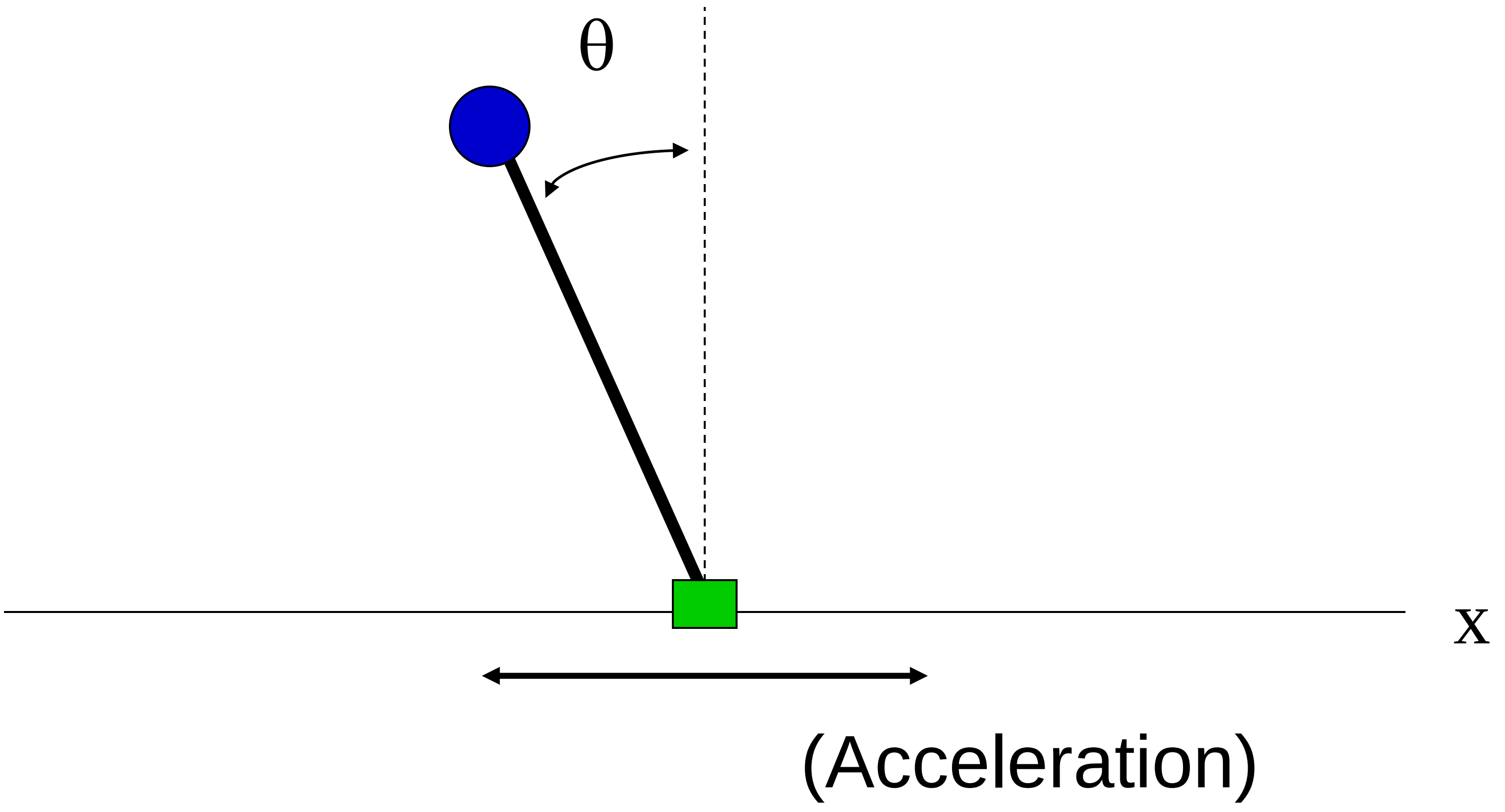
Fuzzy Inference System



Choices:
- How to fuzzify
- How to defuzzify
- Rules
- Inference

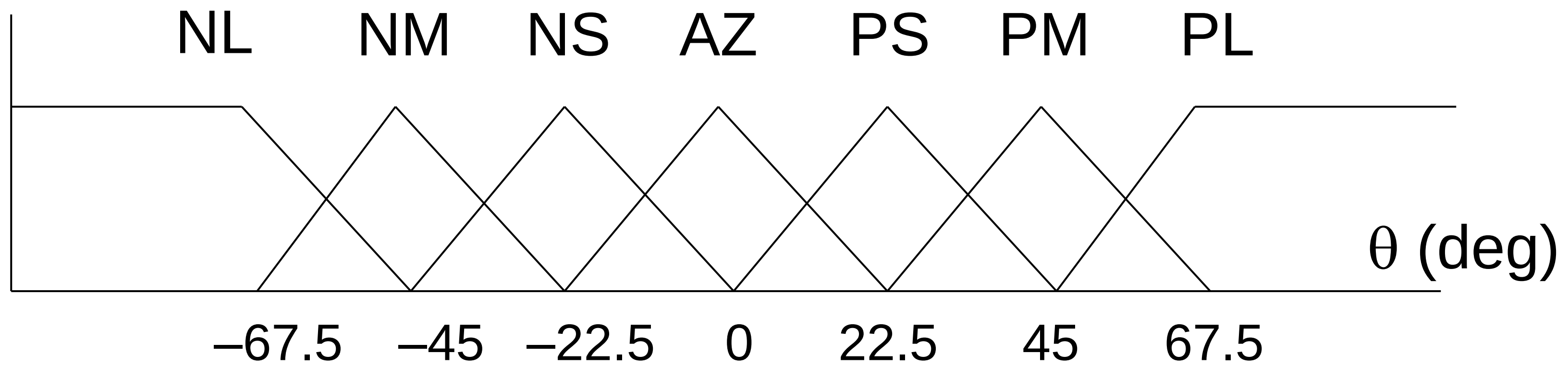
Inverse Pendulum Demo

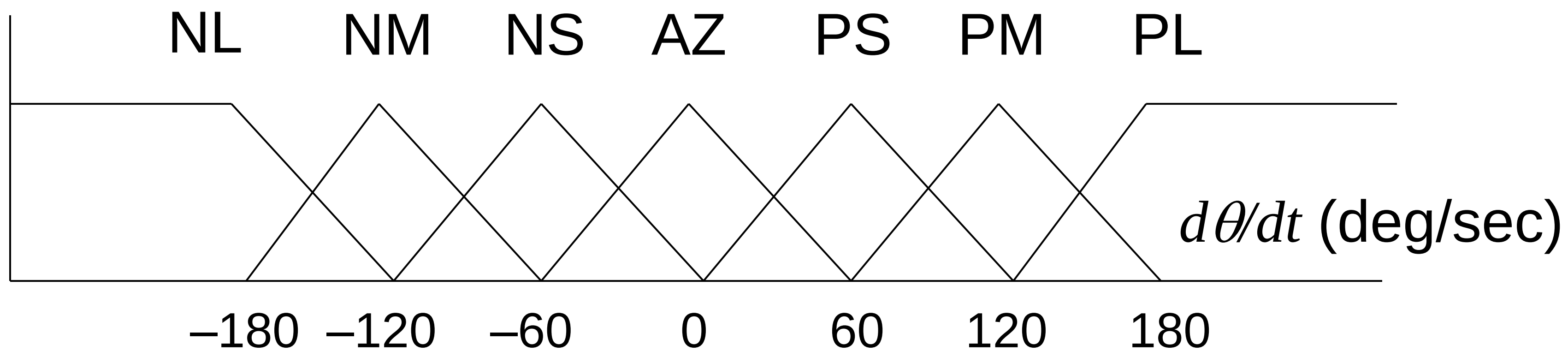
Inverse pendulum is a classical control problem.



Inverse Pendulum Demo

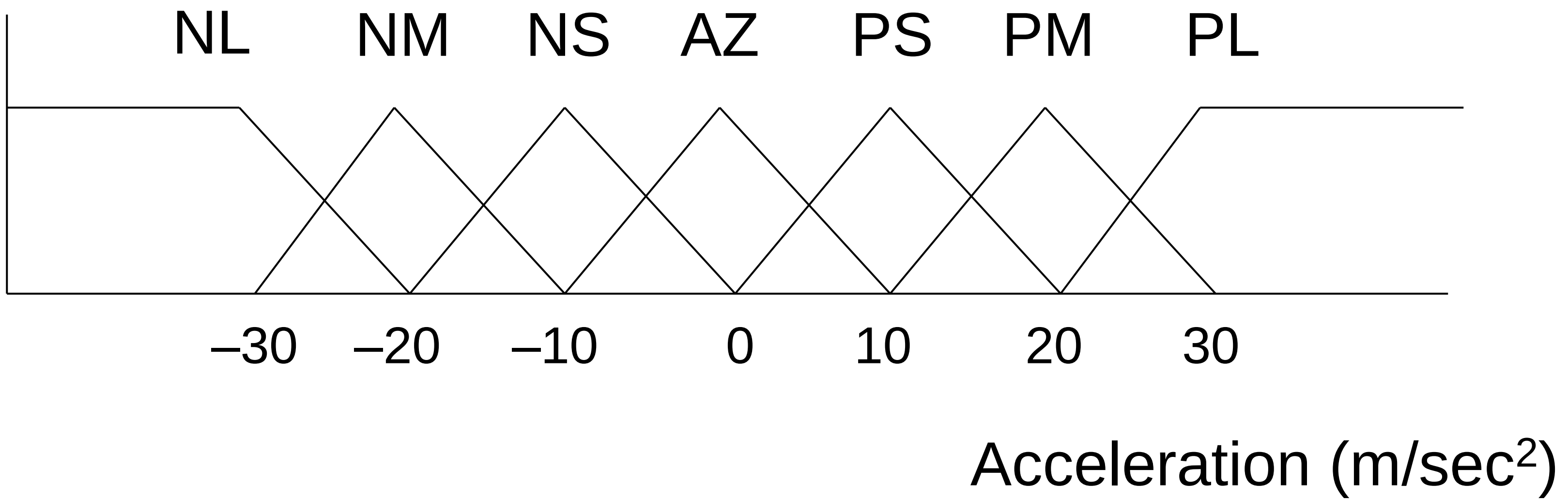
Variables (inputs):





Inverse Pendulum Demo

Variables (output):



Stick assumed to be 1-m long

Inverse Pendulum Demo

Rules: **IF (Angle) AND (Angle Change Rate) THEN (Acceleration)**

$d\theta/dt$ θ	NL	NM	NS	AZ	PS	PM	PL
NL	PL	PL	PL	PL	PM	AZ	AZ
NM	PL	PL	PL	PL	PM	AZ	AZ
NS	PM	PM	PM	PS	AZ	NM	NM
AZ	PM	PM	PS	AZ	NS	NM	NM
PS	PM	PM	AZ	NS	NM	NM	NM
PM	AZ	AZ	NM	NL	NL	NL	NL
PL	AZ	AZ	NM	NL	NL	NL	NL

TSK Model of Fuzzy Inference

The fuzzy inference method used so far is called the Mandani-Assilian (MA) model.

Takagi-Sugeno-Kung (TS or TSK) model uses functions of inputs, instead of fuzzy sets, as outputs:

$$\mathbf{P_j: (V_1 \text{ is } A_{j1} \text{ AND } \ldots \text{ AND } V_n \text{ is } A_{jn}) \Rightarrow (y \text{ is } \boldsymbol{f_j}(\vec{V}))}$$

Consider just singleton excitation: $\mathrm{V_i} \rightarrow \mathrm{x_i}$

$$\mathbf{P_j: (A_{j1}(x_1) \text{ AND } \ldots \text{ AND } A_{jn}(x_n)) \Rightarrow (y = \boldsymbol{f_j}(\vec{x}))}$$

TSK Model

Weighted mean for aggregation of multiple inputs.

$$y(\vec{x}) = \frac{\Sigma\, \alpha_j(x_j) f_j(x_j)}{\Sigma\, \alpha_j(x_j)}$$ similar to centroid

Firing strength:

$$\alpha_j(x_j) = A_{j1}(x_1) \wedge \ldots \wedge A_{jn}(x_n)$$

or

$$\alpha_j(x_j) = A_{j1}(x_1) \times \ldots \times A_{jn}(x_n)$$

TSK model is very popular with fuzzy controls and description of dynamic systems.

Fuzzy Propositions

Unconditional and unqualified propositions

- p: *V* is *F*
  - *V* is a variable with values from the universe V
  - *F* is a fuzzy set on V that represents a fuzzy predicate
- E.g.: John is tall
- T(p) = F(v) for any value v of variable *V*

Unconditional and qualified propositions

- p: *V* is *F* is *S*
  - *V* is a variable with values from the universe V
  - *F* is a fuzzy set on V that represents a fuzzy predicate
  - *S* is a fuzzy truth qualifier (e.g. fairly true, very false)
- E.g.: John is tall is fairly true
- T(p) = S(F(v)) for any value v of variable *V*

Conditional and unqualified propositions

- p: if *X* is *A* then *Y* is *B*
  - *X*, *Y* are variables with values in X, Y
  - *A*, *B* are fuzzy sets on X, Y
- May be viewed as <*X*,*Y*> is *R*
- T(p) = R(x,y) for each values *x*, *y* of *X*, *Y*
- T(p) = R(x,y) = $\mathcal{J}$(A(x),B(y))
  - $\mathcal{J}$ denotes the fuzzy implication operator

Conditional and qualified propositions

- p: if *X* is *A* then *Y* is *B* is *S*

- Truth value determined in a way similar to the cases above

Fuzzy quantifiers: absolute

- Absolute quantifiers are defined on **R**
  - about 10, more than 100, at least 5
- p: There are *Q i*'s in *I* such that *V*(*i*) is *F*
  - *I* is a set of individuals
  - *Q* is a fuzzy number on **R**
- T(p) = $\mathrm{Q}(\sum_{i \in I} F(V(i)))$

Fuzzy quantifiers: absolute

- p: There are *Q* *i*'s in *I* such that *V*1(*i*) is *F*1 and *V*2(*i*) is *F*2
  - *I* is a set of individuals
  - *Q* is a fuzzy number on **R**
- T(p) = $\mathrm{Q}(\sum_{i \in I} \min(F_1\left(V_1(i)\right), F_2(V_2(i))))$

Fuzzy quantifiers: relative

- Relative quantifiers are defined on [0,1]
  - Almost all, about half, most of
- p: Among *i*'s in *I* such that *V*1(*i*) is *F*1 there are *Q* *i*'s in *I* such that *V*2(*i*) is *F*2
  - *I* is a set of individuals
  - *Q* is a fuzzy number on [0,1]
- T(p) = $Q\left(\frac{\sum_{i\in I}\min\left(F_1\left(V_1(i)\right),F_2(V_2(i))\right)}{\sum_{i\in I}F_1\left(V_1(i)\right)}\right)$

Linguistic hedges

- Linguistic terms that modify other linguistic terms
  - very, more or less, fairly, extremely
- The fuzzy set that defines the hedge is called *modifier*
- E.g. the proposition “x is young [is true]”:
  - “x is very young is true”
  - “x is young is very true”
  - “x is very young is very true”