Troical Moduli Spaces

Count Me In – Algebra, Geometry and Topology
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Abstract

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Introduction

1 Metric Graphs

Definition 1.1 (Metric graph)**.** A *metric graph* Γ is defined to be a graph together with a length function, i.e. $\Gamma = (G, \ell)$ where G is a graph with edge set $E(G)$ and vertex set $V(G)$ and $\ell \in \mathbb{R}_{>0}^{E(G)}$ a distance function.

Definition 1.2 (Weighted Graph)**.** A *weighted graph* (G, w) is a connected graph G together with a weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$. Isomorphisms between graphs need to preserve weight.

Definition 1.3 (Stable Graph)**.** A *stable graph* is graph (G, w) such that $2w(v) - 2 + val(v) > 0$, $\forall v \in V(G)$. Note that if we have an unweighted graph, this is equivalent to $w(v) = 0, \forall v \in V(G)$, so the unweighted graph is stable if $val(v) - 2 > 0$, $\forall v \in V(G)$.

In this report we consider stable, connected graphs. We will denote the weight of a vertex by a cross on that vertex with an integer beside it denoting the weight. A cross with no number denotes a weight of 1.

Definition 1.4 (Genus)**.** The *genus* of a graph G, denoted g, is defined as $g = |E(G)| - |V(G)| + f$ where f denotes the number of components of the graph. Note all the graphs we consider are connected, so $f = 1$. The genus of a weighted graph (G, w) is defined as $g(G, w) = |E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} w(v)$.

Another way to consider the genus is as the number of independent cycles a metric graph has.

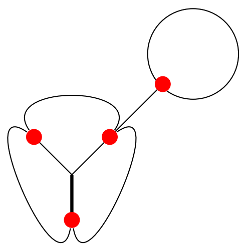


Figure 1: A metric graph of genus 4

Definition 1.5 (Tropical Divisor)**.** A *(tropical) divisor* on Γ is a function $D : \Gamma \to \mathbb{Z}$ such that D has finite support. Equivalently, $D(q) \neq 0$ for at most finitely many $q \in \Gamma$.

Definition 1.6 (Degree)**.** The *degree*, d, of a divisor D is defined by $d := \sum_{p \in \Gamma} D(p)$

Definition 1.7 (Divisor group)**.** $Div(\Gamma)$ is the set of all divisors and forms a group under point wise addition. $Div^d(\Gamma)$ is the set of all divisors of degree d. If $d = 0$, the $Div^d(\Gamma)$ is a group, otherwise it is a torsor of $Div^0(\Gamma)$.

Definition 1.8 (Effective Divisor)**.** A divisor $D \in Div(\Gamma)$ is called *effective* if it is non-negative at all points on the graph, so for all $p \in \Gamma$, $D(p) \geq 0$.

Definition 1.9 (Partial Ordering of Dividers)**.** There exists a partial ordering of dividers, where is $E, D \in Div(\Gamma)$, $D \geq E$ if, for all $p \in \Gamma$, $D(p) \geq E(p)$. Hence, for any effective divisor E, $E \geq 0$.

Definition 1.10 (Principal Divisor)**.** Let $\phi : \Gamma \to \mathbb{R}$ be a continuous piece-wise linear function with integer slopes. Then ϕ induces a divisor, denoted $div(\phi)$, which is defined by $div(\phi)(p) =$ the sum of the incoming slopes of ϕ at p for all $p \in \Gamma$. A divisor that is induced by such a ϕ is called a *principle divisor*. $Prin(\Gamma)$ is the set of all principal divisors on Γ.

Note that all principal divisors have degree 0.

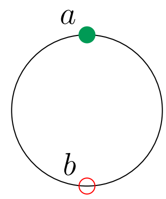


Figure 2: A divisor of degree 0 which is not a principal divisor

Above we can see a divisor D of degree 0 which is not principal. Set all lengths between chips to 1. Suppose $\exists \phi$ such that $div(\phi) = D$. Then at a, $\phi(a) = \phi_0 \in \mathbb{R}$. Let the gradient of ϕ from b to a on the left edge be c, this implies the gradient coming out of a on the right is $c - 1$. From these two gradients we find that $\phi(b) = \phi_0 - c = \phi_0 + c - 1$ which implies $c = \frac{1}{2}$. But the gradients of ϕ must be integers, so we have a contradiction.

Definition 1.11 (Linearly Equivalent)**.** Two divisors $D, E \in Div(\Gamma)$ are said to be *linearly equivalent*, denoted $D \simeq E$ if $D - E = div(\phi)$ for some principal divisor $div(\phi)$. A divisor D is principal if and only if $D \simeq div(\phi)$ for some function ϕ as defined above.

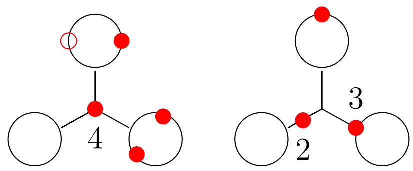


Figure 3: Examples of Linearly Equivalent Divisors

Definition 1.12 (Linear System of D)**.** The *linear system* of a divisor $D \in Div(\Gamma)$, denoted $|D| := \{E \in Div(\Gamma) | E \simeq D, E \geq 0\}$.

Definition 1.13 (Cycle)**.** A *cycle* of a graph is a finite or infinite sequences of edges, which join as a sequence of vertices, in which no edge is repeated, the first vertex is also the last vertex, and there are no other vertex repetitions.

To make linearly equivalent divisors, we can consider the total momentum of chips moving along each cycle of the graph. If the total momentum of chips on each cycle is conserved, then the divisor made before moving the chips and the divisor made after are equivalent.

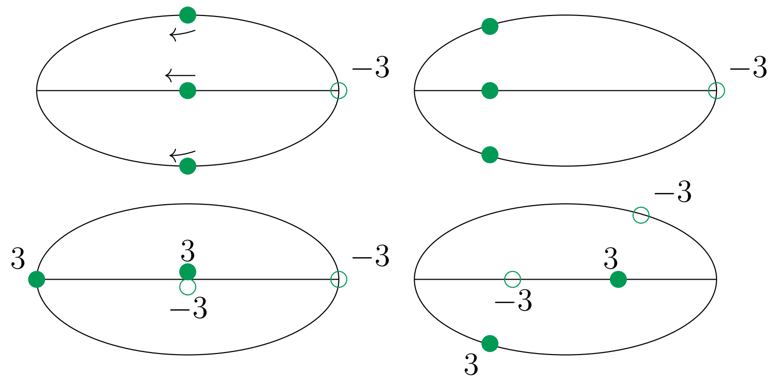


Figure 4: Preserving momentum in independent cycles

To prove that preserving momentum creates linearly equivalent graphs, we can first consider the simplest case: Consider a cycle, Γ, with a divisor D such that $D(a) = a_1$ and $D(b) = b_1$ where a and b are points on the cycle and $a_1, b_1 \in \mathbb{Z}$. We can then create a new divisor, D', by moving chips at a and b a distance of s and t to points a' and b' respectively. Without loss of generality, We can choose the clockwise direction as positive and assume s is positive. In order to preserve momentum, s and t must satisfy $a_1 s + b_1 t = 0$. We will also assume s and $|t|$ are less than the length of the cycle. Now consider the divisor $D - D'$, if D and D' are linearly equivalent then there exists a ϕ such that $div(\phi) = D - D'$. Define A as the region of length s from a to a'. Define B as the region of length t from b to b'. We can then define the gradients of ϕ in the clockwise direction by

$$\text{Gradient of } \phi \text{ at x} = \begin{cases} -a_1 & \text{if } x \in A \setminus B \\ -\operatorname{sign}(t)b_1 & \text{if } x \in B \setminus A \\ -a_1 - \operatorname{sign}(t)b_1 & \text{if } x \in A \cap B \\ 0 & \text{if } x \in \Gamma \setminus (A \cup B) \end{cases} \tag{1.1}$$

We can then check ϕ is continuous by checking the total increase of ϕ round the cycle is equal to the total decrease of ϕ round the cycle. Let u be the length of $A \cap B$. The contribution from $A \setminus B$ is $-\operatorname{sign}(s)a_1(|s| - u)$, the contribution from $B \setminus A$ is $-\operatorname{sign}(t)b_1(|t| - u)$, the contribution from $A \cap B$ is $(-\operatorname{sign}(s)a_1 - \operatorname{sign}(t)b_1)u$, and the contribution from $\Gamma \setminus (A \cup B)$ is 0. Therefore the total change in ϕ around one clockwise loop is $(-\operatorname{sign}(s)a_1 - \operatorname{sign}(t)b_1)u - \operatorname{sign}(s)a_1(|s| - u) - \operatorname{sign}(t)b_1(|t| - u) = -\operatorname{sign}(s)a_1|s| - \operatorname{sign}(t)b_1|t| = -(a_1 s + b_1 t) = 0$. Therefore we have a continuous function. Now we need to check that $D - D' = div(\phi)$. For $x \in \Gamma \setminus a, a', b, b'$ we clearly have $div(\phi)(x) = 0$. Consider $div(\phi)(b)$, if $b \notin A$, then the sum of incoming slopes at b is $(-1)^2 \operatorname{sign}(t)^2 b_1 + \operatorname{sign}(t)\dot{0} = b_1$. If $b \in A$, then the sum of incoming slopes at b is $\operatorname{sign}(t)(a_1 + \operatorname{sign}(t)b_1) - \operatorname{sign}(t)a_1 = b_1$. Similarly we can check $div(\phi)(b') = -b_1$, $div(\phi)(a) = a_1$, $div(\phi)(a') = -a_1$. Therefore we have shown D and D' are linearly equivalent. If we are moving more than two sets of chips, we can always decompose it into a sequence of moving two chips at a time. If we want to move a chip more than a cycle we can repeatedly find linear equivalences of distances less than the cycle distance. If a chip is on multiple loops, we can find a ϕ as we did above for both loops and just match the constant areas where they are the loops overlap. Therefore, if momentum is preserved along all loops, the graphs are linearly equivalent.

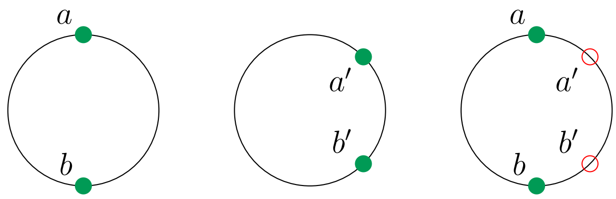


Figure 5: A divisor of degree 0 which is not a principal divisor

2 Moduli Spaces

A moduli space is a geometric object that classifies objects we are interested in by their paramaters, so every point in a moduli space is an object we are interested in.

2.1 Moduli space of stable tropical curves of genus 1

$\mathcal{M}_1^{trop}$ is the moduli space of stable tropical curves of genus 1.

Figure 6: Metric graphs of genus $g = 1$

All stable tropical curves of genus 1 are just single cycles of varying lengths, so the only parameter is the length of the cycle, $\ell \geq 0$ with $\ell = 0$ being the trivial graph. So the moduli space of all such tropical curves is $\mathbb{R}_{\geq 0}$. We include 0 so that the moduli space is compact (every open cover has a finite subcover).

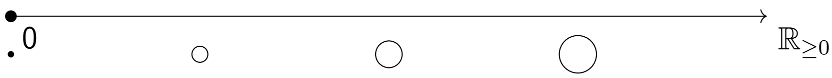


Figure 7: Tropical Moduli Space for Genus $g = 1$

2.2 Moduli Space of Tropical Curves of genus 2

For genus two metric graphs there are two types, dumbbell and theta graphs. We consider graphs with a total edge length of 1, so $a + b + c = 1$.

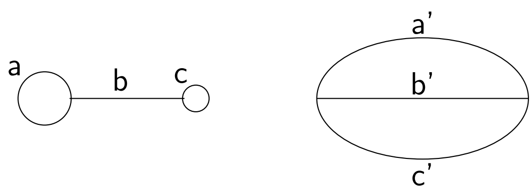


Figure 8: Left: A dumbbell graph. Right: A Theta graph

Considering first the dumbell graph, there are three parameters, the lengths of a, b, c, where $(a, b, c) \in \mathbb{R}^3_{\geq 0}$ and $a + b + c = 1$. This forms a triangle in $\mathbb{R}^3$.

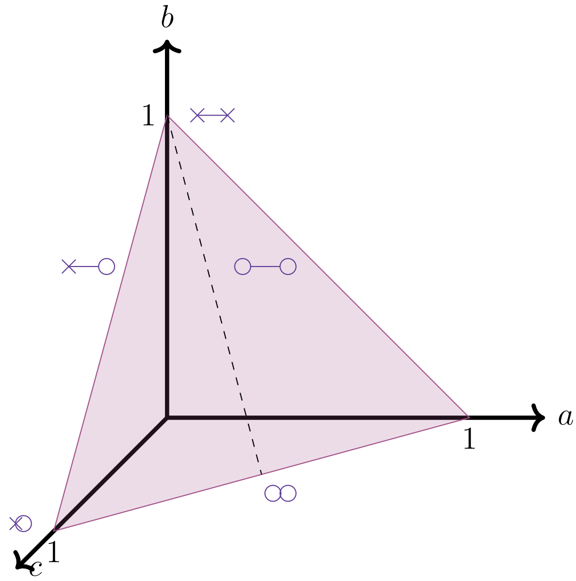


Figure 9: Moduli Space for genus 2 dumbbell graphs with total edge length one

However we can see that there is symmetry between the a and c co-ordinates, for example the graphs given by $(10, 4, 7)$ and $(7, 4, 10)$ are isomorphic. So we only need half of the triangle - see dotted line. So the moduli space of stable genus 2 dumbell graphs with total length 1, $\mathcal{M}_2^{trop,dumbbell} = \{(a, b, c) \in \mathbb{R}^3_{\geq 0} | a + b + c = 1, a \leq c\}$. Next, consider what happens at $b = 0$. As the line connecting the two cycles shrinks to a point, the graph becomes:

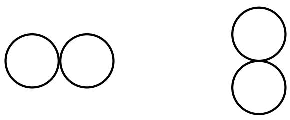


Figure 10: Shrinking b and b' to 0 on a dumbbell and theta graph respectively

Where b has been reduced to a vertex with zero length where edges a and c meet. This is clearly the same as if a theta graph had the length of b' reduced to 0. Thus, along the line $b = 0$ these two graphs are equivalent.

Note that when we shrink an edge to zero and it reduces the genus of the graph, for example, in the dumbbell case, if we reduced one of the loop edges to 0, then we need to add a weight of 1 to the point remaining on the graph which the reduced edge goes to, so that $g = v - e + 1 + \sum_{p \in \Gamma} w_p$ where w_p is the weight at $p \in \Gamma$.

The moduli space for theta graphs is also determined by the lengths of the three edges $(a', b', c') \in \mathbb{R}^3_{\geq 0}$, $a' + b' + c' = 1$. Here we have $3!$ permutations between the edges, as all three edges a, b, c are equivalent in this graph, hence their lengths can be permuted as we wish.

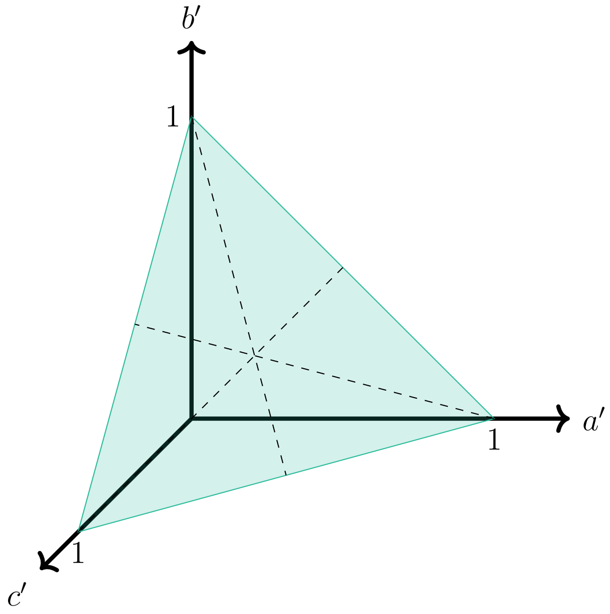


Figure 11: Moduli space for genus 2 theta graphs

So if we can permute all three edge lengths, we have $\mathcal{M}_2^{trop,theta} = \{(a', b', c') \in \mathbb{R}^3_{\geq 0} | a' + b' + c' = 1, b' \leq a' \leq c'\}$.

Now we glue these spaces together along the line $b = b' = 0$. We consider all other possible edges and find that they produce different graphs.

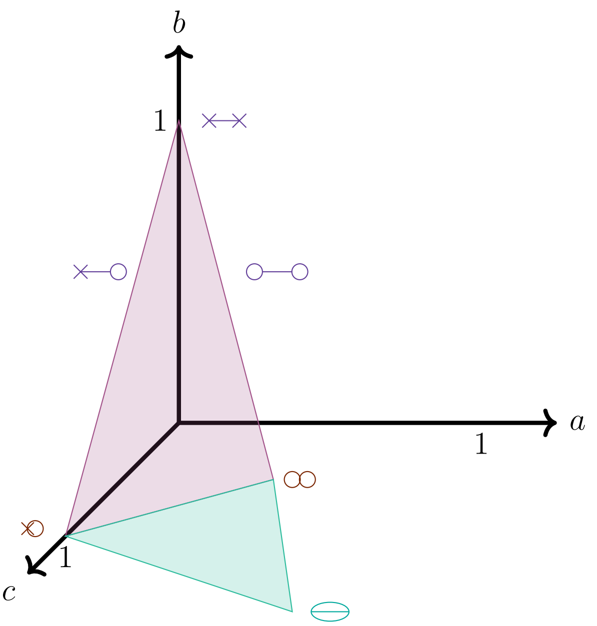


Figure 12: Moduli space for genus 2 metric graphs of total edge length 1, glued together along line $b = 0$

This is just one slice of the moduli space of genus 2 metric graphs with all total edge lengths. The line where the purple and green triangles meet is the line $b = 0$, the bottom-most green line corresponds to the line $a = b$, the last line of the green triangle corresponds to $a = c$. The left most line of the purple triangle corresponds to $a = 0$ and the rightmost line of the purple triangle to $a = c$. Hence at the top point, $a = c = 0$.

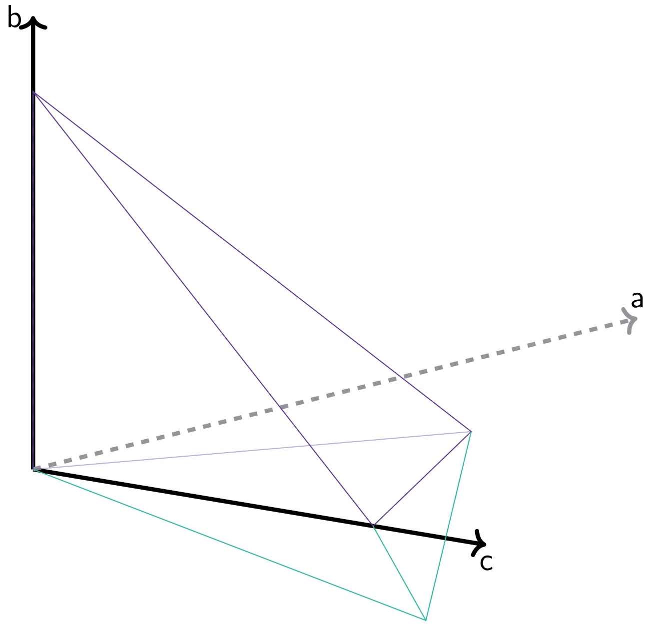


Figure 13: Sketch of moduli-space for all genus 2 metric graphs with slice taken out

In this representation we can see that as total side length tends to 0 all graphs tend to the trivial graph, and that the total moduli space is a space in $\mathbb{R}^3$ made by gluing together two cones along the place defined by $b = 0$.

3 Theta Characteristics

Definition 3.1 (Canonical Divisor)**.** Let Γ be a metric graph. The *canonical divisor*, K_Γ, of Γ is defined at every point $p \in \Gamma$ as $K_\Gamma(p) = val(p) - 2$ where $val(p)$ is the number of half-edges adjacent to p.

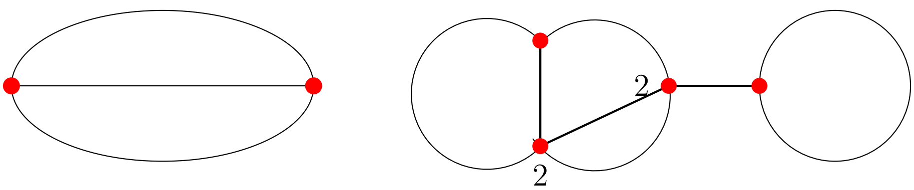


Figure 14: Examples of Canonical Divisors

Definition 3.2 (Theta Characteristic)**.** On a metric graph Γ, a *theta characteristic* is a linear equivalence class of divisors D such that $2D$ is linearly equivalent to the canonical divisor on Γ. A theta characteristic is called *effective* if one can choose D to be an effective divisor.

Lemma 3.3. *Let Γ be a metric graph of genus g. Then Γ has 2^g theta characteristics, of which $2^g - 1$ are effective and the remaining 1 is not effective.*

Jensen and Len (2018) provide an algorithm for finding effective and non-effective theta characteristics on a given graph Γ.

4 Moduli Spaces of Metric Graphs with Theta Characteristics

Consider $g = 2$. Then the canonical divisors of the theta graph and dumbbell graph are:

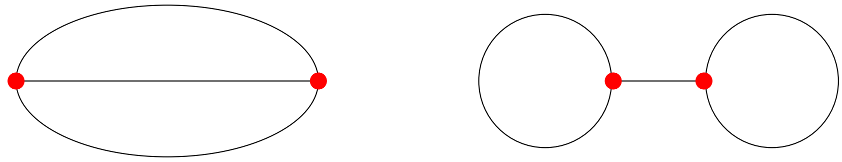


Figure 15: Canonical divisors of genus 2 metric graphs

Now their theta characteristics are:

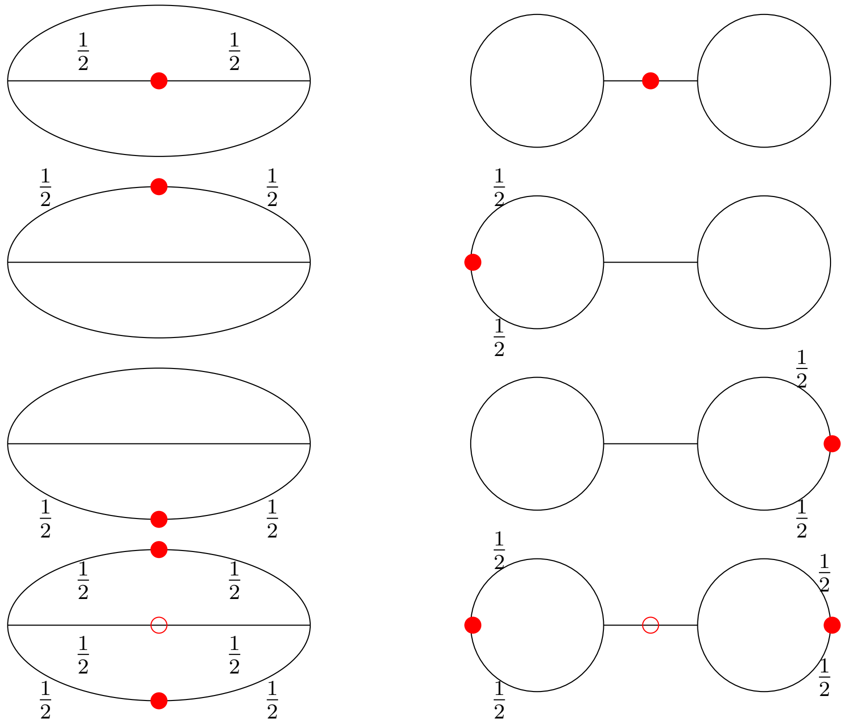


Figure 16: Theta Characteristics of genus 2 metric graphs

Recall that these are representatives of linear equivalence classes of divisors. In cases where distances have been marked, that is the proportion of the edge length that the chip must be along, in cases where distances have not been marked the chip can be anywhere along that edge, and all such placements are linearly equivalent and hence part of the same theta characteristic. We are only interested in a whole class of theta characteristics at a time. Also recall that we are only constructing the metric spaces of effective theta characteristics, so we can ignore the 4th one in the above diagram on both sides. Also recall that we set the total edge length of the graph to 1. Now let us consider the moduli space of genus 2 metric graphs with theta characteristics. Considering the dumbbell graphs first, and first looking at the moduli space for each effective theta characteristic separately:

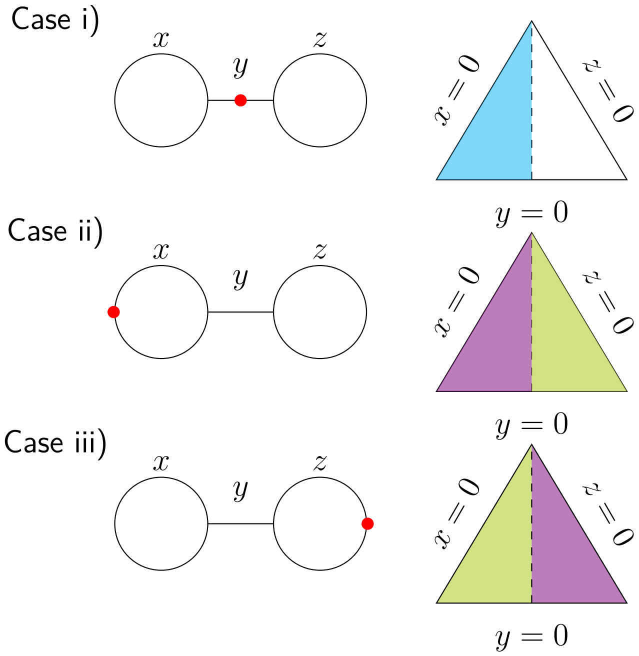


Figure 17: Moduli spaces of dumbbell theta characteristics

In case i) we can see there is a symmetry between the lengths of edges x and z, so we can consider, for example, only graphs where $x \leq z$, the blue shaded portion of the triangle. In cases ii) and iii) we can more easily consider the symmetry between them if we consider x in case ii) and z in case iii) as 'the edge with the chip' and z in case ii) and x in case iii) as 'the edge with no chip'. This shows us that the triangles highlighted in green and purple are equivalent to each-other. Putting these together yields:

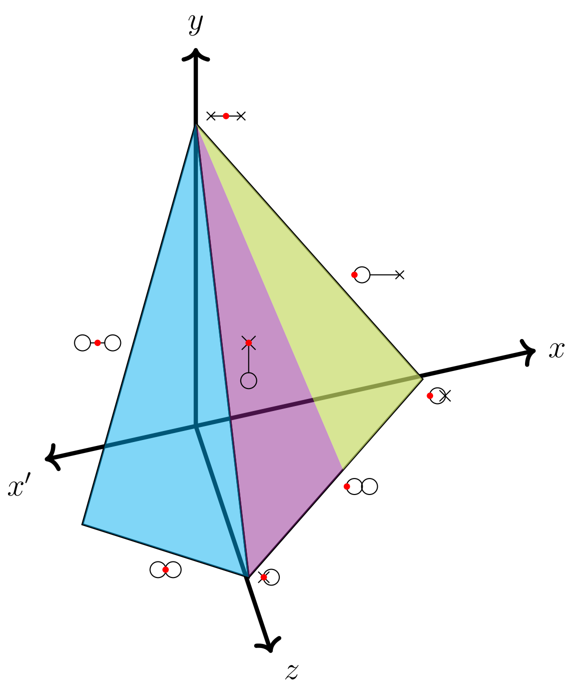


Figure 18: Moduli space of genus 2 dumbbell graphs with theta characteristics

In 18 we consider x to be 'the edge with the node' and z to be 'the edge without the node' from cases ii) and iii). Note the similarity in the points $x = 1$ and $z = 1$. In the $x = 1$ case this is

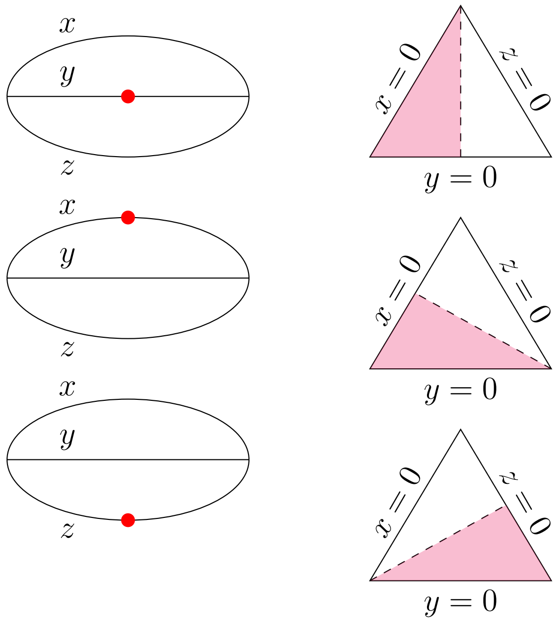


Figure 19: Individual moduli spaces of theta genus 2 graphs with theta characteristics

because the edge with the chip on it has length 1, and all others have length 0, and in the $z=1$ the edge wihout the chip on it has length 1 and all others have length 0 - but this means the chip moves to the remaining edge in the picture - so these two points alone should potentially be glued together. All other points on the $y=0$ line are unique, as the edge lengths of the edges with and without chips are different.

Now consider the same process for theta graphs of genus 2 with effective theta characteristics and total edge length 1.

In 19 we see that there is symmetry between the two edges which do not have a chip on them in each case - represented by the dotted lines on the triangles. In fact one half of all these triangles are equivalent to each other, so the total moduli space is half an equalateral triangle, all we need to know is how long the edge with the chip, and then of the remaining length, what proportion of it is devoted to the longest of the two non-chip edges. As the edges can be permuted, it doesn't matter which edge is the chip edge. From there, everything is determined. These shapes in some sense glue onto the bottom to the shape for dumbbell graphs (though not with $y=0$).

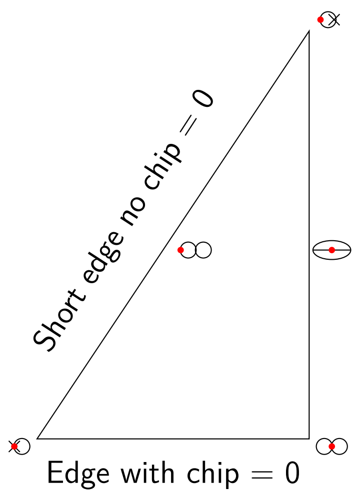


Figure 20: Moduli Space of genus 2 theta graphs with theta characteristics

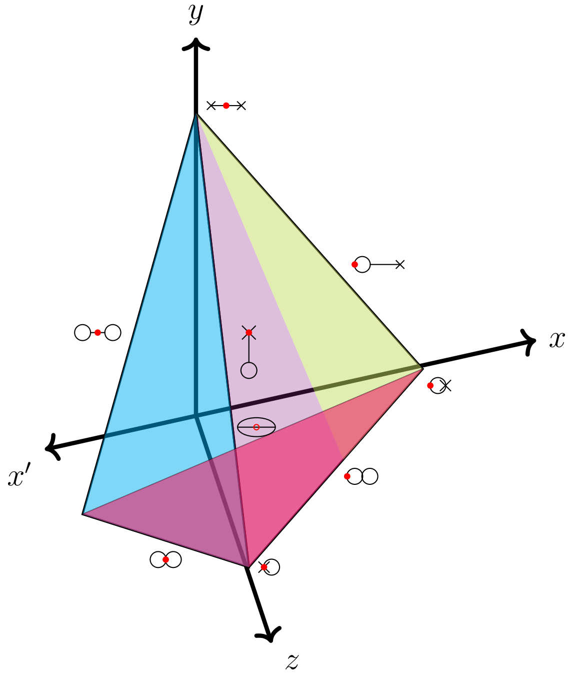


Figure 21: Moduli space of genus 2 graphs with theta characteristics

A better way to draw this might be from the other side:

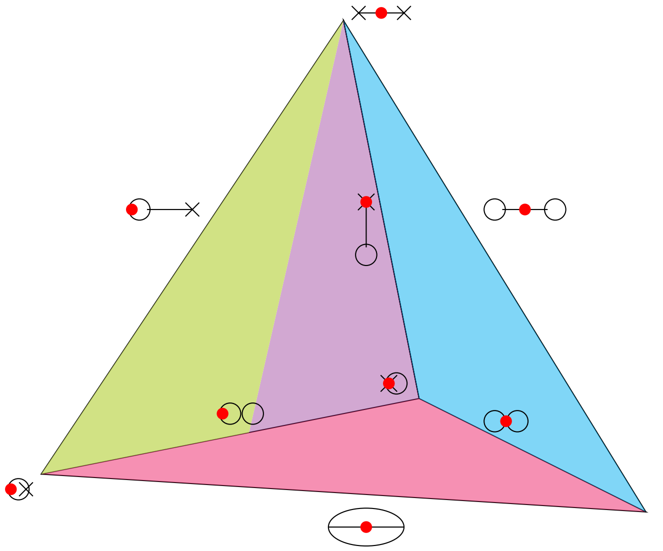


Figure 22: Another view of combined moduli space

Thus now consider the non-effective theta characteristics of genus $g = 2$.

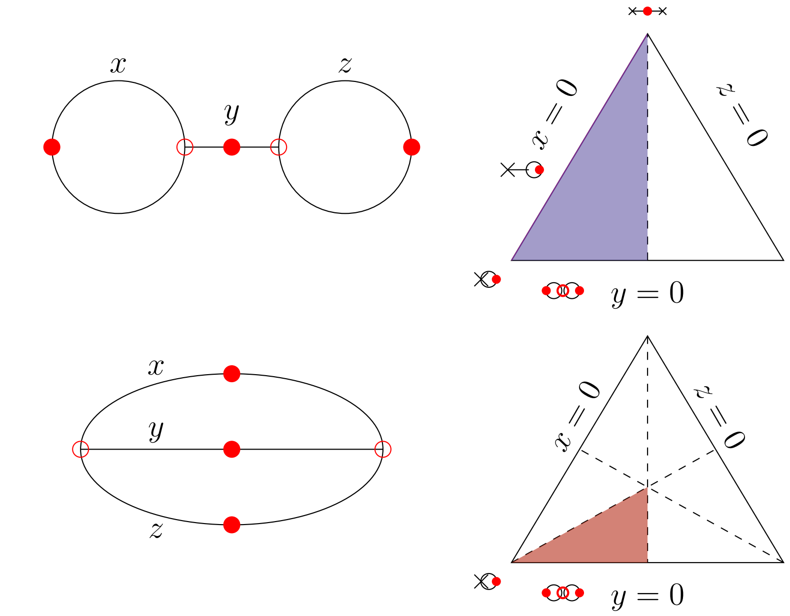


Figure 23: Genus 2 non-effective theta characteristics moduli spaces

We can see, as expected, that 23 is a copy of the moduli space of genus 2 graphs without theta characteristics, as seen in 12, but also that along the edge $x = 0$ on the dumbbell moduli space the theta characteristic is effective, as the anti-chips have been cancelled out (also in the bottom right corner, $y = 0$ and $x = 0$ of the theta graph moduli space). Therefore in these places we expect this moduli space to be glued onto the moduli space of effective theta characteristics on genus 2 graphs. Note that in this diagram we revert back to the original representatives of the theta characteristics.

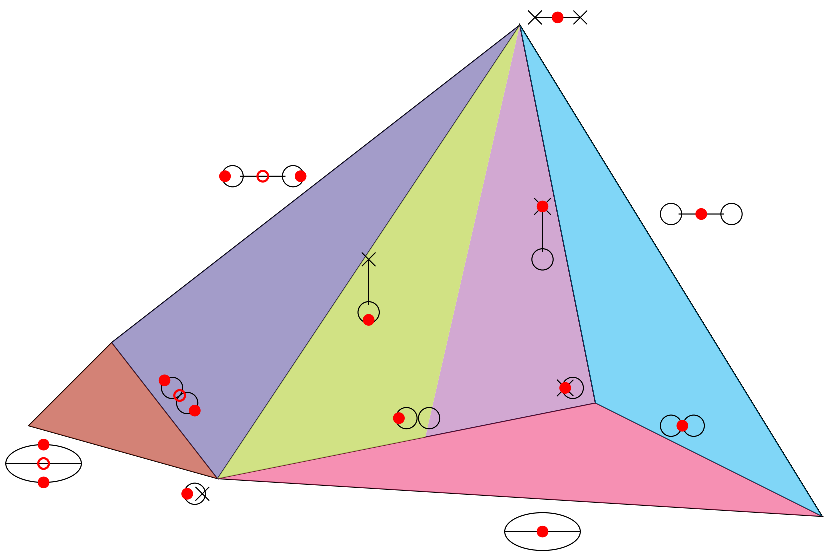


Figure 24: Moduli space with non-effective theta characteristics

4.1 Theta characteristics Isomorphic to Cyclic Subgraphs

Consider now a different representation of the theta characteristics given in Caporaso, Melo, and Pacini (2023). Consider the vector spaces over $\mathbb{F}_2$ respectively spanned by E, denoted $\mathcal{E}_G$ and spanned by V, denoted $\mathcal{V}_G$. Then consider the map $\delta : \mathcal{E}_G \to \mathcal{V}_G$ given, for all subsets $S \subset E$ by: $\delta(\sum_{e \in S} e) = \sum_{e \in S}(u_e + v_e)$ where u_e and v_e are the endpoints of edge e and addition is 'modulo 2' on the RHS. Then denote the kernel of this map $\mathcal{C}_G$ and call its elements 'cyclic subgraphs of G' or 'cyclic'. It is well known that $\mathcal{C}_G$ is generated by the cycles of G - that is connected subgraphs of G whose vertices all have valency 2. Also define $deg_P(v)$ to be the number of edges in some subset $P \subset E$ incident to the vertex v and p_e to be the midpoint of the edge e for all $e \in E$.

Now, for all $P \in \mathcal{C}_G$, define the divisor T_P as follows:

$$T_P = \sum_{v \in V} \Big(\frac{deg_P(v)}{2} - 1 + w(v) \Big) + \sum_{e \in E \setminus P} p_e. \tag{4.1}$$

Caporaso, Melo, and Pacini (2023) shows that these divisors T_P all represent unique theta characteristics and hence there is a bijection between elements of $\mathcal{C}_G$ and theta characteristics on G. Hence we will frequently identify theta characteristics by the cyclic subgraph P which can be used to generate them in this way.

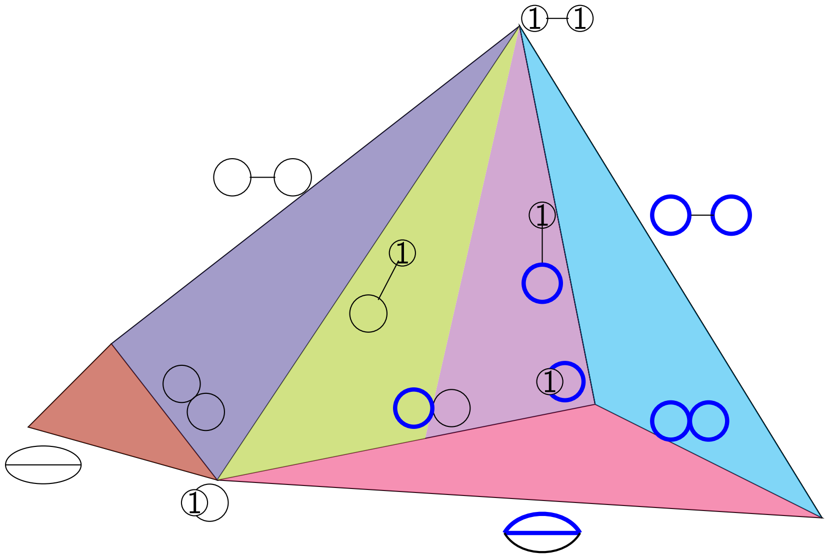


Figure 25: Cyclic Subgraphs P represented in blue

4.2 Connectedness

We want to show that the moduli space of metric graphs with theta characteristics and edge lengths summing to one is connected. First let us examine what an edge collapse looks like on a metric space with a theta characteristic defined by a cycle subgraph P. Recall that all theta characteristics on Γ may be equivalently represented by the cyclic subgraph $P \in \mathcal{C}_G$ which is used to define them. We want to show that edge contractions of edges on $(\mathbf{G}, P)$ and a graph $\mathbf{G}$ with chips on it corresponding to $[T_P]$ lead to the same theta characteristic.

Lemma 4.1. *Consider a metric graph $\mathbf{G}$ with cyclic subgraph $P \in \mathcal{C}_G$, and the metric subgraph $\mathbf{G}$ with the divisor $T_P = \sum_{v \in V(G)} \Big(\frac{val_P(v)}{2} - 1 + \omega(v) \Big) + \sum_{e \in E \setminus P} p_e$ on it. Define $G_1 = (\mathbf{G}, P)$ and $G_2 = \mathbf{G} \cup T_P$. Then given an edge collapse $\gamma : G \to G/e$ for some $e \in E(G)$, $\gamma(G, P) = (G', P')$ the theta characteristic on G' given by $[T_{P'}]$ is equivalent to performing reducing the length of e to 0 and keeping the chips in G_2 all at the same points.*

Proof. Suppose the edge e is a loop. We will show that the theta characteristic generated by T_P when the edge e is contracted to 0 agrees with theta characteristic generated after we edge contract e, ie when there is $+1$ weight at the vertex v incident to e.

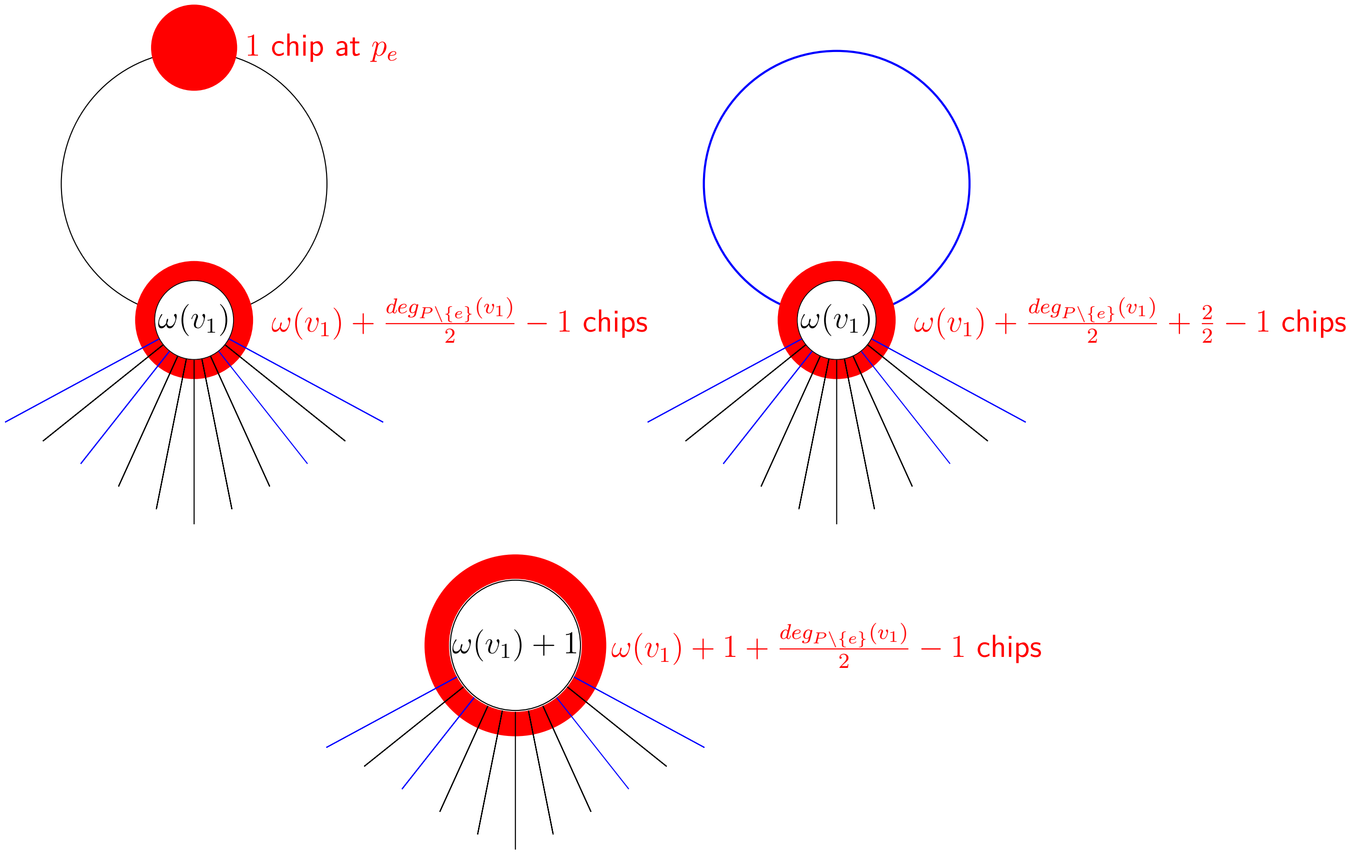


Figure 26: How theta characteristic changes by edge contraction of a loop

First consider a loop e when $e \notin P$. Construct the theta characteristic generated by the cycle P on G, then let the length of e tend to 0. There will be 1 chip placed at p_e - the midpoint of e, and $\frac{val_P(v_1)}{2} - 1 + \omega(v_1)$ placed at v_1 the vertex incident to e.

If we consider reducing the edge of e to 0 with the chips still on it, as $\ell_e \to 0$, $\frac{\ell_e}{2} = d(p_e, v_1) \to 0$ so the midpoint of edge e gets closer and closer to v_1 until $p_e = v_1$, so the chip from the middle of loop e is now at v_1, so there are $\frac{val_P(v_1)}{2} - 1 + \omega(v_1) + 1$ chips at v_1.

Now consider the case where we first perform edge contraction γ to edge contract e and then calculate the placement of chips in $T_{P'}$.

If we perform an edge contraction of e, then we will increase the weight of the vertex v by 1 and the edge $e \notin E(G')$. Hence the number of chips placed at v_1 will be $\frac{val_P(v_1)}{2} - 1 + \omega(v_1) + 1$ which agrees exactly with the place where we placed chips before edge contraction.

All other chips will either remain where they were (for chips on vertices), as neither the weight nor valency of any other vertex has been altered, or in the case of mid-edge chips they will move by $\frac{\ell_e}{2(|E(G)|-1)}$ when the edge contraction happens to account for the lengthening of all remaining edges to account for the length of e going to 0.

If $e \in P$, then placing chips before edge contraction we have no chips on e, as there are no weights on e (they would make it not actually a loop but a cycle of edges between weights) and $e \in P$ so by definition of T_p there is no mid-edge weight. Now divide the edges in P incident at v_1 into two distinct subsets, $P = e \cup (P \setminus \{e\})$ then as these are a disjoint partition of P, $val_P(v_1) = val_e(v_1) + val_{P\setminus\{e\}}(v_1)$.

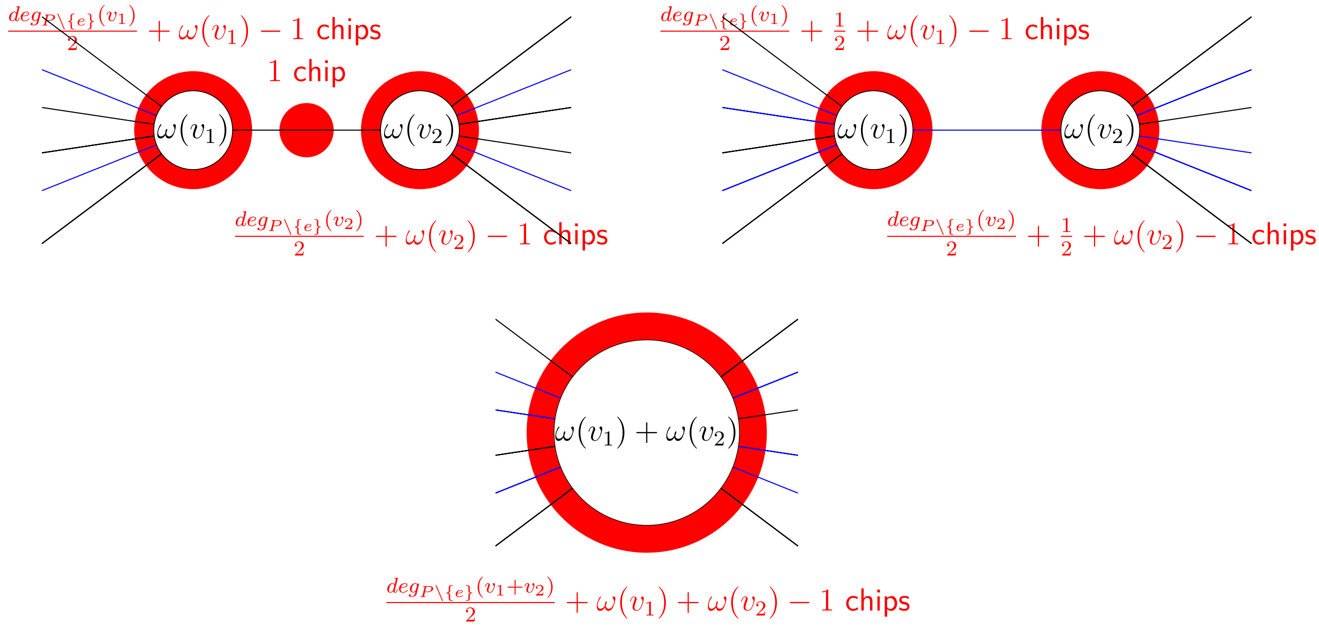


Figure 27: How theta characteristic changes by edge contraction of a non-loop

Consider an edge collapse of an edge e which is not a loop, so e is incident to two distinct vertices say v_1 and v_2. By definition of edge contraction, if we edge contract e then we will produce a new vertex $v_1 + v_2$ such that all other edges incident at either v_1 or v_2 are still incident at $v_1 + v_2$ (indeed any other edge that was previously incident at both will become a loop at $v_1 + v_2$) and $\omega(v_1 + v_2) = \omega(v_1) + \omega(v_2)$. So $val(v_1 + v_2) = val(v_1) + val(v_2) - 2$ where the -2 comes from the fact that $v_1 + v_2$ is no longer incident to either end of e and

$$val_P(v_1 + v_2) = val_{P\setminus\{e\}}(v_1) + val_{P\setminus\{e\}}(v_2) - 2\delta_{P,e}$$

where $\delta_{P,e} = 1$ if $e \in P$ and $\delta_{P,e} = 0$ otherwise.

If $e \in P$, considering the theta characteristic defined by T_P then there will be no chips anywhere along e, and there will be

$$\frac{val_{P\setminus\{e\}}(v_1) + 1}{2} - 1 + \omega(v_1)$$

chips on v_1 and

$$\frac{val_{P\setminus\{e\}}(v_2) + 1}{2} - 1 + \omega(v_2)$$

chips on v_2.

Then letting ℓ_e tend to 0, $d(v_1, v_2)$ tends to 0 until they become the new vertex $v_1 + v_2$ which will have

$$\frac{val_{P\setminus\{e\}}(v_1) + 1}{2} - 1 + \omega(v_1) + \frac{val_{P\setminus\{e\}}(v_1) + 1}{2} - 1 + \omega(v_1) = \frac{val_{P\setminus\{e\}}(v_1 + v_2)}{2} - 1 + \omega(v_1 + v_2)$$

.

If we first edge contract e to make a new vertex $v_1 + v_2$, then there will be no chips on e and the number of chips on the new vertex will be

$$\frac{val_{P\setminus\{e\}}(v_1 + v_2)}{2} - 1 + \omega(v_1 + v_2)$$

as before.

If $e \notin P$ then there will be one chip on the midpoint of e, p_e and the number of chips on v_1

will be

$$\frac{val_{P\setminus\{e\}}(v_1)}{2} - 1 + \omega(v_1)$$

and on v_2 will be

$$\frac{val_{P\setminus\{e\}}(v_2)}{2} - 1 + \omega(v_2)$$

, so as we let $\ell_e \to 0$, $d(p_e, v_1) \to 0$ and $d(v_1, v_2) \to 0$, so the chips on all these points all tend to chips on $v_1 + v_2$, which will have

$$\frac{val_{P\setminus\{e\}}(v_1)}{2} - 1 + \omega(v_1) + \frac{val_{P\setminus\{e\}}(v_1)}{2} - 1 + \omega(v_1) + 1$$

chips, which is

$$\frac{val_{P\setminus\{e\}}(v_1 + v_2)}{2} - 1 + \omega(v_1 + v_2)$$

chips as required.

As the edge contraction of any subset of edges $F \subset E$ can be thought of as a finite series of edge contractions, hence by repeating this proof we can see that any series of edge contractions preserves the theta characteristic generated by T_P, whether the edges contracted are in P or not, or a mixture of in P and not. □

Thus we can just consider the cyclic subgraph P when considering a graph with a theta characteristic, and any edge contractions of (G, P) will be correctly represented as changes to the theta characteristic, in particular:

Corollary 4.1.1. *Given a cycle $F \in \mathcal{C}_G$ and edge contractions of each edge of P to a single weight, the theta characteristic generated after the those contractions will agree with the one generated before it, regardless of how many edges of F are in P.*

Proof. If there are at least two edges in the cycle F, choose 1 and edge contract. By above the resultant theta characteristic will be consistent with the theta characteristic made before edge contraction. Repeat until only edge left in the cycle F. Then we have shown that edge contracting this loop also leads to a consistent theta characteristic. □

Theorem 4.2. *The moduli space of metric graphs of any genus g with theta characteristics is connected.*

A quick sketch of the proof is given below:

1. Assume the moduli space of metric graphs of genus $g - 1$ with theta characteristics, T_{g-1}^{trop}, is connected.

2. Take two metric graphs of genus g with theta characteristics, say G_1 and G_2. Edge contract both to obtain metric graphs with theta characteristics of genus g with one more weight added - denote them G_1' and G_2'.

3. Discard the additional weights on both graphs, to obtain metric graphs of genus f with theta characteristics - denote these by G_1^0 and G_2^0. Find a path between G_1^0 and G_2^0 in the connected moduli space T_{g-1}^{trop}.

4. The path found above is equivalent to smoothly deforming G_1^0 into G_2^0. Now apply those same deformations / transformations to G_1'. This will result in a metric graph of genus g with theta characteristic that "looks" like G_2' but with the weight in a different place.

5. Smoothly move the weight on this resultant metric graph with theta characteristic, so that we obtain G_2'.

6. We have found a path between G_1 and G_2, so the moduli space is connected.

Lemma 4.3. *From any graph G with $g \geq 3$ and $b_1(G) \geq 1$ we can perform a finite series of edge contractions to find a graph G' of genus g with $b_1(G') = b_1(G)$ if G is not pure or $b_1(G') = b_1(G) - 1$ if G is pure which satisfies WR.*

Note that the restrictions of this lemma still cover most stable graphs with genus $g \geq 2$ with total edge length 1 (or indeed any non-zero total edge length) as we can't have a single vertex with no edges (no edge length) or a tree (not stable), a graph of first Betti number 0. A single pure cycle with no weights on it (a genus 1 graph) can't be reduced with this formula, nor can a single cycle with a single weight on it. For our proof by induction we will start with a proof for $g = 2$, so neither case is a concern.

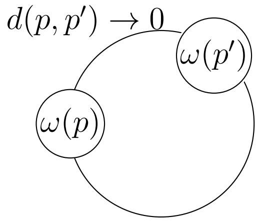


Figure 28: How to edge contract $\mathbf{G}$ to satisfy the weight requirement (WR)

Proof.

Case (i): If G is not pure then there exists some point $p \in \Gamma$ such that $\omega(p) \geq 1$.

Case (i.a): If $val(p) + \omega(p) \geq 4$ we are done. Else $val(p) = 1$ or $val(p) = 2$.

Case (i.b): If there are no points $p \in \Gamma$ where $val(p) \geq 3$ then we must have a graph G that is a single cycle. Thus there is another point $p' \in \Gamma$ where $\omega(p') \geq 1$, by the requirements of the Lemma. Choose the point p' such that $d(p, p')$ is minimal and $\omega(p') \geq 1$ then edge contract/collapse the edge between p and p'. The new vertex $p + p'$ will have $val(p + p') = 2$ and $\omega(p + p') \geq 2$, so this will satisfy WR.

Case (i.c): As our graph is stable any vertices must have $val(v) \geq 3$, and as there are no weights on points with valency greater than or equal to 3 hence we have that there must be at least one vertex in the graph, and this vertex must have valency greater than or equal to 3 where at least 3 edges incident to v are either half-edges of loops or lead to other vertices of valency greater than or equal to 3 and there must be a finite path of edges between p and v. Take the shortest such path and edge contract each edge on it.

Case (ii): If the graph G is pure, i.e. has no weights on it, then $b_1(G) \geq 3$ then there are at least 3 independent cycles of the graph G, and all vertices have valency greater than or equal to 3 as the graph is stable. If all cycles meet at one vertex that vertex must have valency $2g > 3$ and we can contract any cycle to result in a weight at that vertex.

If there is at least one cycle that does not meet at the common vertex, which has valency $val(v) \geq 3$, then edge contract that cycle to get a weight at a point p. Again we are in case (i.c). $\square$

Let $\boldsymbol{G}$ be a weighted graph of genus g satisfying:

There is at least one vertex with at least one weight and valency greater than or equal to 3 (valency plus weight greater than or equal to 4.) (WR) (write about why we want this condition to be true.) If v is such a vertex, we define the *weight reduction of* $\boldsymbol{G}$ as the weighted graph $\boldsymbol{G}^-$ obtained by removing one weight from G at v. Note that $\boldsymbol{G}^-$ has genus $g-1$ and that if P is a cyclic subgraph of $\boldsymbol{G}$, than it is still a cyclic subgraph of $\boldsymbol{G}^-$ (in fact the underlying discrete graphs of $\boldsymbol{G}$ and $\boldsymbol{G}^-$ are the same, i.e. $G = G^-$). If

$$\boldsymbol{\psi} = ((\boldsymbol{G}, [P]), (l_G(e) : e \in E(G))) \in \mathcal{T}_g^{trop}$$

then we define

$$\boldsymbol{\psi}^- := ((\boldsymbol{G}^-, [P]), (l_G(e) : e \in E(G))) \in \mathcal{T}_{g-1}^{trop}$$

.

Oppositely, for a weighted graph $\boldsymbol{G}$ with vertex v, we define the *weight increment of* $\boldsymbol{G}$ *at* v as the weighted graph $\boldsymbol{G}^{+(v)}$ obtained by adding one weight to $\boldsymbol{G}$ at vertex v. Similarly, if

$$\boldsymbol{\psi} = ((\boldsymbol{G}, [P]), (l_G(e) : e \in E(G))) \in \mathcal{T}_g^{trop}$$

then we define

$$\boldsymbol{\psi}^{+(v)} := ((\boldsymbol{G}^{+(v)}, [P]), (l_G(e) : e \in E(G))) \in \mathcal{T}_{g+1}^{trop}$$

.

We finish the discussion on this notation with an observation regarding edge collapses. Let $(\boldsymbol{G}, P), \boldsymbol{H}, Q) \in \mathcal{C}_g$ such that there is an edge collapse $(\boldsymbol{G}, P) \to (\boldsymbol{H}, Q)$ (with underlying function $\phi : X(G) \to X(H)$) and let $u \in V(G)$ and $v \in V(H)$. Note that $X(\boldsymbol{G}) = X(\boldsymbol{G}^{+(u)}$, $r_{\boldsymbol{G}} = r_{\boldsymbol{G}^{+(u)}}$ and $s_{\boldsymbol{G}} = s_{\boldsymbol{G}^{+(u)}}$ and similarly for $\boldsymbol{H}$ and $\boldsymbol{H}^{+(v)}$. So $\phi : X(\boldsymbol{G}^{+(u)}) \to X(\boldsymbol{H}^{+(v)})$ satisfies all of the conditions to be a morphism, except possibly:

$$\forall w \in V(\boldsymbol{H}^{+(v)}) \text{ with } S_w = \phi^{-1}(w) \text{ and } \bar{S}_w = (S_w, r_G \restriction_{S_w}, s_G \restriction_{S_w}) \text{ we have}$$

$$g\left((\bar{S}_w, \omega_{\boldsymbol{G}^{+(u)}} \restriction_{S_w})\right) = \omega_{\boldsymbol{H}^{+(v)}}(w).$$

Particularly, we need to check

$$\text{for } S_v = \phi^{-1}(v) \text{ and } \bar{S}_v = (S_v, r_G \restriction_{S_v}, s_G \restriction_{S_v}) \text{ we have}$$
$$g\left((\bar{S}_v, \omega_{\boldsymbol{G}^{+(u)}} \restriction_{S_v})\right) = \omega_{\boldsymbol{H}^{+(v)}}(v) = \omega_{\boldsymbol{H}}(v) + 1.$$

We have that $\omega_{\boldsymbol{H}}(v) = g\left((\bar{S}_v, \omega_{\boldsymbol{G}} \restriction_{S_v})\right)$ since ϕ is a morphism from $\boldsymbol{G}$ to $\boldsymbol{H}$. And

$$g\left((\bar{S}_v, \omega_{\boldsymbol{G}} \restriction_{S_v})\right) + 1 = |E(\bar{S}_v)| - |V(\bar{S}_v)| + 1 + \sum_{w \in V(\bar{S}_v)} \omega_G(w) + 1 = g\left((\bar{S}_v, w_{\boldsymbol{G}^{+(u)}} \restriction_{S_v})\right)$$

if and only if the sum of the weights on $(\bar{S}_v, w_{\boldsymbol{G}} \restriction_{S_v})$ and $(\bar{S}_v, w_{\boldsymbol{G}^{+(u)}} \restriction_{S_v})$ difer by 1, which is if and only if $u \in S_v = \phi^{-1}(v)$. That is, ϕ is an edge collapse morphism $\boldsymbol{G}^{+(u)} \to \boldsymbol{H}^{+(v)}$ if and only if u is in the pre-image of v under ϕ.

Lemma 4.4. *Let $(\boldsymbol{G}, P)$, $(\boldsymbol{H}, Q) \in \mathcal{C}_g$ such that there is an edge collapse $(\boldsymbol{G}, P) \to (\boldsymbol{H}, Q)$ (with underlying function $\phi : X(G) \to X(H)$) and let $u \in V(G)$ and $v \in V(H)$. Then ϕ is an edge collapse morphism $\boldsymbol{G}^{+(u)} \to \boldsymbol{H}^{+(v)}$ if and only if u is in the pre-image of v under ϕ.*

Let $(\boldsymbol{G}, [P])$, $(\boldsymbol{H}, [Q]) \in [\mathcal{C}_g]$. If $(\boldsymbol{G}, [P]) \leq (\boldsymbol{H}, [Q])$ we say that $(\boldsymbol{G}, [P])$ is obtained by *edge contraction* of $(\boldsymbol{H}, [Q])$ or that $(\boldsymbol{G}, [P])$ is an *edge contraction* of $(\boldsymbol{H}, [Q])$. We also say that $(\boldsymbol{H}, [Q])$ is obtained by *edge expansion* of $(\boldsymbol{H}, [Q])$ or that $(\boldsymbol{H}, [Q])$ is an *edge expansion* of $(\boldsymbol{G}, [P])$.

Let $(\boldsymbol{G}, [P]) \in [\mathcal{C}_g]$. Let $V((\boldsymbol{G}, [P])) = V(G)$ and $E((\boldsymbol{G}, [P])) = E(G)$.

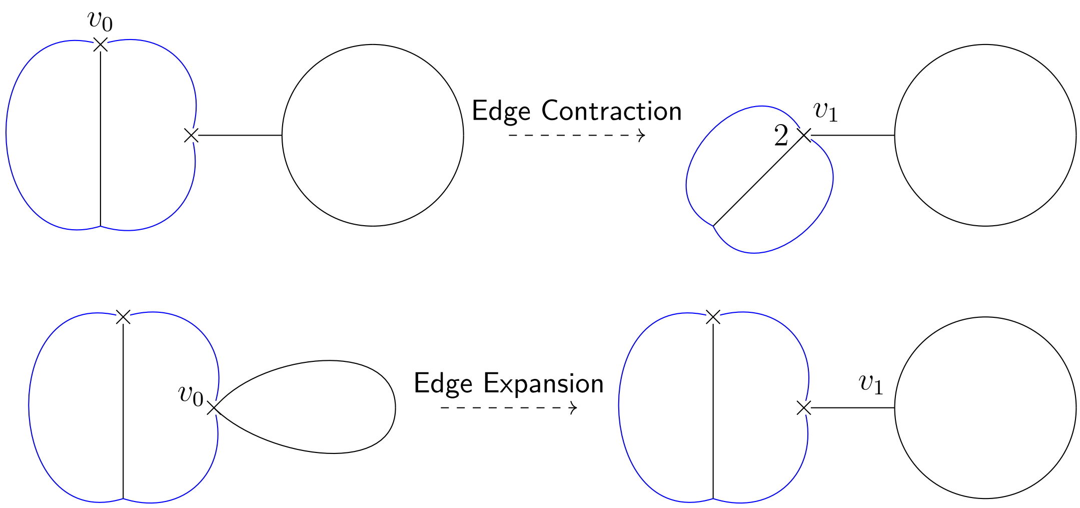
Lemma 4.5. *Let $\boldsymbol{\psi} = ((\boldsymbol{G}, [P]), (l_G(e) : e \in E(G)))$ and $\boldsymbol{\chi} = ((\boldsymbol{H}, [Q]), (l_H(e) : e \in E(H)))$ be elements of $\mathcal{T}_g^{trop}$ such that $\boldsymbol{G}$ and $\boldsymbol{H}$ satisfy (WR). If there is a connected path between $\boldsymbol{\psi}^-$ and $\boldsymbol{\chi}^-$ in $\mathcal{T}_{g-1}^{trop}$ then there exists a connected path between $\boldsymbol{\psi}$ and $\boldsymbol{\chi}'$ in $\mathcal{T}_g^{trop}$, where $\boldsymbol{\chi}'$ is identical to $\boldsymbol{\chi}$ except possibly with one weight in a different place on its underlying weighted graph.*

Proof. Let $f : [0,1] \to \mathcal{T}^{trop}_{g-1}$ be the continuous function representing the path between $\boldsymbol{\psi}^-$ and $\boldsymbol{\chi}^-$. Let $f_0 : [0,1] \to [\mathcal{C}_g]$ and, for each $(\boldsymbol{K}, [R]) \in [\mathcal{C}_g]$ and $e \in E(K)$, let $f_e^{(\boldsymbol{K},[R])} : [0,1] \to \mathbb{R}_{>0}$ such that

$$f(t) = (f_0(t), (f_e^{f_0(t)}(t) : e \in E(f_0(t))).$$

We would like to show that this path can be lifted into the space $\mathcal{T}^{trop}_g$. First we'll demonstrate that the path this function takes through $[\mathcal{C}_{g-1}]$ corresponds to a connected path through $[\mathcal{C}_g]$ from $(\boldsymbol{G}, [P])$ to $(\boldsymbol{H'}, [Q])$, where $\boldsymbol{H'} \in \mathcal{G}_g$ is identical to $\boldsymbol{H}$ except possibly with one weight in a different place.

Let $s_1 \in [0,1]$ be the smallest such that $f_0(0) \neq f(s_1)$. Now, for each $i \geq 2$, let $s_i \in [0,1]$ be the smallest such that $s_{i-1} < s_i$ and $f_0(s_{i-1}) \neq f_0(s_i)$. Note that we can assume this sequence is finite, because $[\mathcal{C}_{g-1}]$ is finite and so there exists $n \in \mathbb{N}$ such that $f_0(s_n) = (\boldsymbol{H}^-, [Q])$. Since each cell in $\mathcal{T}^{trop}_{g-1}$ is connected (prove that each cell in $\mathcal{T}^{trop}_g$ is connected for any g) we could always adjust the path so that between s_n and 1 it stays in $(\boldsymbol{H}^-, [Q])$ and moves straight to $\boldsymbol{\chi}^-$. Let v_0 be the vertex of $\boldsymbol{G}$ containing the weight which was removed to obtain $\boldsymbol{G}^-$ (recall that $G^- = G$, so this is also a vertex of $\boldsymbol{G}^-$). Finally let $(\boldsymbol{K_i}, [R_i]) = f_0(s_i)$ for each i. We know that $(\boldsymbol{K_1}, [R_1])$ will be the result of either an edge contraction or edge expansion of $(\boldsymbol{G}^-, [P])$. If it is the result of an edge contraction, than let v_1 be the vertex on $(\boldsymbol{K_1}, [R_1])$ that v_0 is mapped to by the edge contraction - that is, for the edge contraction morphism $\gamma : (\boldsymbol{G}^-, [P]) \to (\boldsymbol{K_1}, [R_1])$, let $v_1 = \gamma(v_0)$. Otherwise, let v_1 be any of the vertices (at most two) in the pre-image of v_0 under the edge contraction map from $(\boldsymbol{K_1}, [R_1])$ to $(\boldsymbol{G}^-, [P])$ - that is, for the edge contraction morphism $\gamma : (\boldsymbol{K_1}, [R_1]) \to (\boldsymbol{G}^-, [P])$, let $v_1 \in \gamma^-1(v_0) \cap V(K_1)$. We now define the vertex v_i of $(\boldsymbol{K_i}, [R_i])$ iteratively, as either the image or a vertex in the pre-image of v_{i-1} under the edge contraction map that links $(\boldsymbol{K_{i-1}}, [R_{i-1}])$ and $(\boldsymbol{K_i}, [R_i])$.



Recall that $[\mathcal{C}_g]$ is a poset with partial relation defined $(\boldsymbol{G}, [P]) \leq (\boldsymbol{G'}, [P'])$ if and only if there exists an edge collapse $\phi : \boldsymbol{G'} \to \boldsymbol{G}$ such that there exists $Q \in [P]$ and $Q' \in [P']$ such that $\phi(Q') = Q$. We want to show that for each i, if

$$(\boldsymbol{W_i}, [R_i]) \leq (\boldsymbol{W_{i+1}}, [R_{i+1}]) \quad \text{then} \quad (\boldsymbol{W_i}^{+(v_i)}, [R_i]) \leq (\boldsymbol{W}_{i+1}^{+(v_{i+1})}, [R_{i+1}]),$$

and if

$$(\boldsymbol{W_i}, [R_i]) \geq (\boldsymbol{W_{i+1}}, [R_{i+1}]) \quad \text{then} \quad (\boldsymbol{W_i}^{+(v_i)}, [R_i]) \geq (\boldsymbol{W}_{i+1}^{+(v_{i+1})}, [R_{i+1}]).$$

i.e. when $(\boldsymbol{W_{i+1}}, [R_{i+1}])$ is obtained via an edge expansion or contraction of $(\boldsymbol{W_i}, [R_i])$ than so to can $(\boldsymbol{W}_{i+1}^{+(v_{i+1})}, [R_{i+1}])$ from $(\boldsymbol{W_i}^{+(v_i)}, [R_i])$. This follows from the way we have defined v_i and v_{i+1} and Lemma 4.4.

So $\tilde{f}_0 : [0,1] \to [\mathcal{C}_g]$ defined $\tilde{f}_0(t) := (\boldsymbol{K_i}^{+(v_i)}, [R_i])$ when $s_i \leq t < s_{i+1}$ is the function for a connected path through $[\mathcal{C}_g]$ as we wanted.

It remains to find a connected path through each cell $\tilde{f}_0(t)$. First note that because $\boldsymbol{G}$ and $\boldsymbol{H}$ satisfy (WR) and from how we defined each $\boldsymbol{K_i}^{+(v_i)}$, we can be sure that the extra weight on each of the graphs $\boldsymbol{K_i}^{+(v_i)}$ are on vertices of $\boldsymbol{K_i}$ and so the edges of both are in one to one correspondence. Hence we can very directly lift the connected path in each cell $f_0(t)$ onto the corresponding cell $\tilde{f}_0(t)$. That is, for each $i \in \mathbb{N}$ and $e \in E(K_i)$, we define $\tilde{f}_e^{(\boldsymbol{K_i}^{+(v_i)}, [R_i])}(t) := f_e^{(\boldsymbol{K_i}, [R_i])}(t)$ and then let

$$\tilde{f}(t) = (\tilde{f}_0(t), \, (\tilde{f}_e^{\tilde{f}_0(t)}(t) : e \in E(\tilde{f}_0(t))).$$

This function represents the connected path we want.

$\square$

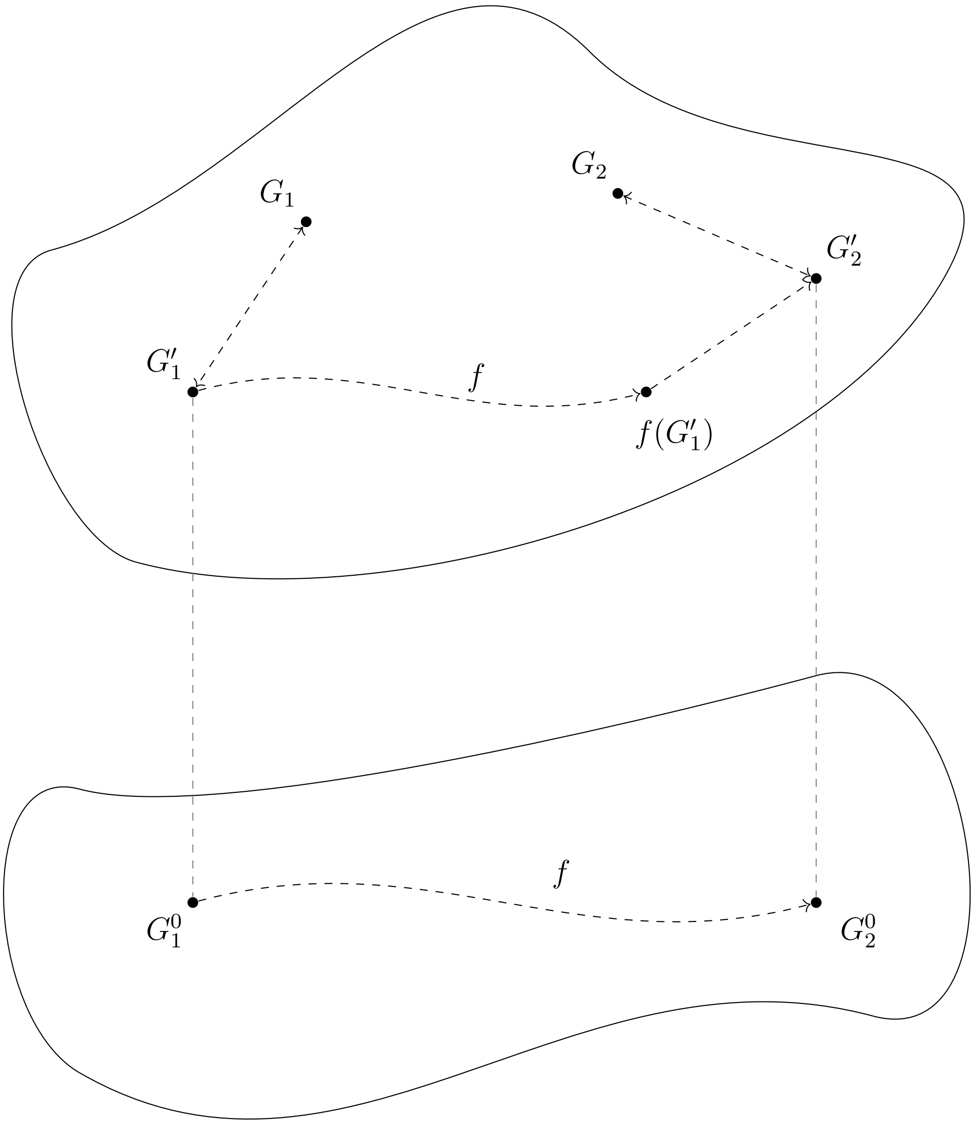
Lemma 4.6. *The subspace A of the $\mathscr{T}_g^{trop}$ moduli space where is is generated by placing a weight anywhere on some graph (Γ, P), where Γ is a tropical curve of genus $g-1$ and P is some theta characteristic, is connected. Equivalently, smoothly moving a weight around a graph corresponds to a smooth movement of a point in the moduli space.*

Proof. We know that if two graphs are connected if we can perform a series of edge contractions and expansions from one to another what we want to show is that using edge contractions and expansions we can manipulate the position of a weight to anywhere on a graph.

Case 1. First we consider 'moving' the weight along edges. This is quite simple. For any weight incident to two edges, we edge contract one edge and edge expand the other at the same rate. This has the effect of moving the weight along the edge.

Case 2. Now we consider when the weight is at a vertex. At this point, the valency of this point is at least 4 due to the presence of the weight. This means we can edge expand an edge of zero length connected to the vertex and the weight to push the weight onto an edge.

Using these two cases, we can smoothly transform the graph such that the weight can be moved to any point on the graph, meaning it is connected. $\square$



4.3 Connectedness in Codimension One

Our moduli space is made up of connected cells. Each cell is the moduli space of one combinatorial type of graph, (G, ω) together with one class of cyclic subgraphs $[P]$ on that combinatorial type. We will call this tuple $(\mathbf{G}, [P])$ the cell type. For a fixed g we wish to consider cells of with the maximum number of dimensions, which we will show is $3g-4$. We then show that any graph and cyclic subgraph (G, P) is a limit of a sequence of graphs in a maximal cell, so all graphs in our moduli space are in a maximal cell or the boundary of a maximal cell. We then show that any two maximal cells can be connected through a sequence of edge contractions and expansions of only one edge contraction and expansion at a time, so the space is connected in codimension one. This section is heavily inspired by the discussion in Section 3 of Brannetti, Melo, and Viviani (2011).

Lemma 4.7. *Cells of type $(G, 0, [P])$ for any $P \in \mathcal{C}_G$ where all vertices in G have valency 3 are of maximal dimension in the moduli space and their dimension is $3g-4$.*

Proof. From the definition of a stable graph, for any stable graph with cyclic subgraph $(\Gamma, \omega, [P])$, for all vertices $v \in V(G)$,

$$2\omega(v) - 3 + val(v) \geq 0.$$

So summing over all vertices $v \in V(G)$ we see that,

$$\begin{aligned} 2\sum_{v\in V(G)} \omega(v) - \sum_{v\in V(G)} 3 + \sum_{v\in V(G)} val(v) &\geq 0, \\ 2|\underline{\omega}| + 2|E(G)| &\geq 3|V(G)|, \end{aligned}$$

as each edge in the graph is composed of two half edges, each of which add one to the total valency of all vertices in the graph, so each edge adds two to the total valency, so sum of total valency is two times the number of edges.

Now we need to express number of edges in terms of the fixed genus g. Recall the formula for genus $g = |E| - |V| + 1 + \sum_{v\in V} \omega(v)$. Thus:

$$\begin{aligned} 2|\underline{\omega}| + 2|E(G)| &\geq 3|E(G)| + 3 + 3|\underline{\omega}| - 3g, \\ 3g - 3 - |\underline{\omega}| &\geq |E(G)|. \end{aligned} \tag{4.2}$$

What are the dimensions of one of our cells of a graph G underlying a tropical curve Γ with length function $\underline{\ell}$? If we parametrise G with a parameter representing the length each edge then we can see each G will be parametrised by $|E(G)|$ parameters. So we can associate to the cone $\sigma_{(G,P)} = \mathbb{R}_{>0}^{|E(G)|}$ where all edge lengths are positive integers (an edge length being 0 would change the underlying graph type and reduce the number of edges). However, we are looking at the slice of this cone for which the edge lengths sum to 1, so each cell of our moduli space will be associated with the simplex $\Delta_{(G,P} = \{\mathbb{R}_{>0}^{|E(G)|} \mid \sum_{e\in E(G)} \ell_e = 1\}$. You can see that $\sigma_{(G,P)}$ has dimension $|E(G)|$ and that this is equivalent to the sum of all edge lengths ranging over all non-negative real values. Thus by imposing the additional condition that edge lengths sum to 1 we have reduced the degrees of freedom and hence the dimension of this space by 1, so its dimension is $|E(G)| - 1$, so by 4.2, the $dim \leq 3g - 4 - |\underline{\omega}|$. So the maximal possible dimension of cells of underlying graphs of fixed genus g is $3g - 4$, and this occurs when the number of edges of the graph is $3g - 3$ and the graph has no weights on it $|\underline{\omega}| = 0$.

Clearly the graphs in our lemma are pure. Now we will show that if $val(v) = 3$ for all $v \in V(G)$, then the cells will be maximal dimensional. The number of half edges connected to a vertex equals the valency, so $\sum_{v\in V} val(v) = 2|E|$, and here $val(v) = 3 \forall v \in V$, so $2|E| = 3|V|$, so $|V| = \frac{2}{3}|E|$. Subbing into the formula for genus we have

$$\begin{aligned} g &= |E| - \frac{2}{3}|E| + 1, \\ g &= \frac{1}{3}|E| + 1, \\ 3g - 3 &= |E|, \end{aligned}$$

as required. So all such cells have maximal dimension $3g - 4$. $\square$

Lemma 4.8. *All weighted tropical curves $(\Gamma, \omega, [P])$ are in the closure of a cell of type $(G, 0, [P])$ for some G where $val(v) = 3$ for all $v \in V(G)$.*

Proof. The proof that any weighted tropical curve (Γ', ω') can be obtained via a finite series of edge contractions on a tropical curve (Γ, ω) where Γ is trivalent, $(val(v) = 3 \forall v \in V(G))$ is given in (Caporaso and Viviani, 2010) Appendix A.2 for example. We wish to extend it to graphs with a cyclic subgraph. Let $\gamma_i : (\Gamma, \omega, P) \to (\Gamma*, \omega*, P*)$ be an edge contraction from one tropical curve equipped with a cyclic subgraph to another. So if γ_i contracts edge e_i, $\Gamma* = \Gamma/e_i$ and $P* = P/e_i$. We know that, for any $\Gamma*$ of genus g there exists some trivalent, pure tropical curve

Γ of genus g such that there exists a finite series of edge contractions that take Γ to $\Gamma*$. Call these edge contractions $\gamma_1, \ldots, \gamma_n$, each edge contracting edge $e_1, \ldots, e_n$ respectively. Now we need to show that we can always choose $P \in \mathcal{C}_G$ such that $\gamma_n(\ldots(\gamma_1(P)) = P*$. Label all edges of $\Gamma*$ with $e_{n+1}, \ldots, e_{|E(G*)|+n}$ and then label the edges that correspond to those edges (will become those edges after our chosen series of edge contractions) with the same labels on Γ, then add labels for all remaining edges on Γ by what order they are edge contracted away, $e_1, \ldots, e_n$. First take all edges of $P*$ and add the corresponding edges to a set $S \subset E(\Gamma)$. If S already forms a cyclic subgraph on Γ then we are done, after edge contracting away all other edges it will still be a cyclic subgraph and will be exactly $P*$.

However, if S does not form a cyclic subgraph on Γ, that is, there exist at least two vertices of Γ such that $val_S(v) \equiv 1 \pmod 2$, then we need to add some edges in the set $\{e_1, \ldots e_n\}$ to it to make it a cyclic subgraph. As S forms a cyclic subgraph on $\Gamma*$ there must be some minimal set of edges $S' \subset \{e_1, \ldots, e_n\}$ which, when they are edge contracted, make S into a cyclic subgraph. Choose these edges in the minimal set and denote $S* = S \cup S'$. Note that, as S' is a minimal set it will not contain any loops, as these can always be removed from a cyclic subgraph without changing the fact that it is a cyclic subgraph, as they increase val_P of a single vertex by 2. Clearly after edge contractions $\gamma_1, \ldots, \gamma_n$, $S*$ will have become $P*$. Also $S*$ is a cyclic subgraph because S' was chosen in such a way that $S*$ would be a cyclic subgraph. Can we always chose S' in such a way? Starting from the final edge contraction of a non-loop edge, γ_{n-i}, which combines the vertices incident to edge e_{n-i}, say $v_{2n-2i-1}$ and v_{2n-2i}, and after the edge contraction combines them to a single vertex $\tilde{v}_{2n-2i}$. We know that S forms a cyclic subgraph on $\Gamma*$, so $val_S(\tilde{v}_{2n-2i} \equiv 0 \pmod 2)$. So in the pre-image, either both $v_{2n-2i-1}$ and v_{2n-2i} have even valency in S or neither do. If neither, add e_{n-i} to S', if both do then do not add e_{n-i} to S'.

For the next non-loop edge, consider the valency of the two vertices in the pre-image relative to $S* = S \cup S'$ for the edges we have already added to S'. Continue this process until all non-loop edges have been expanded. In this way $S*$ will always remain a cyclic subgraph, so P will be a cyclic subgraph on Γ.

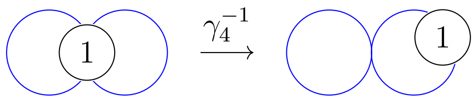


Figure 29: Whether to add the pre-image of an edge contraction to S'

In 29 after the pre-image of the edge contraction, $val_S(v_8) = 1$ and $val_S(v_7) = 3$, so we need to add e_4 to S'.

Note that this is not a tropical curve with cyclic subgraph on it, several different tropical curves with cyclic subcycles may be able to specialise to the same tropical curve with cyclic subcycle. However, as long as one trivalent, underlying graph exists such that its specialisation is the desired underlying graph and cyclic subcycle then there exists a sequence of tropical curves where, for all $0 < \epsilon < \frac{1}{|E(G)|}$ we set $\ell_{e_i} = \frac{\epsilon}{n}$ for all edges that were contracted to make the desired underlying graph and $\ell_{e_{n+j}} = \ell_{\tilde{e}_{n+j}} + \frac{\epsilon}{E(G)-n}$. Then letting $\epsilon \to 0$, this sequence of graphs in the maximal dimensional cell will tend to our desired non-(pure and trivalent) graph with cyclic subcycle, so it is in the boundary of at least one maximal dimensional cell.

This further shows that any underlying graph (G, P) which is not trivalent or not weightless (or both) cannot be a maximal dimensional cell, as it is reached via edge contraction and hence must have fewer edges and be represented by a lower dimensional cell. Thus the only maximal dimensional cells are those associated with graphs $(G, 0, [P])$ where G is trivalent. $\square$

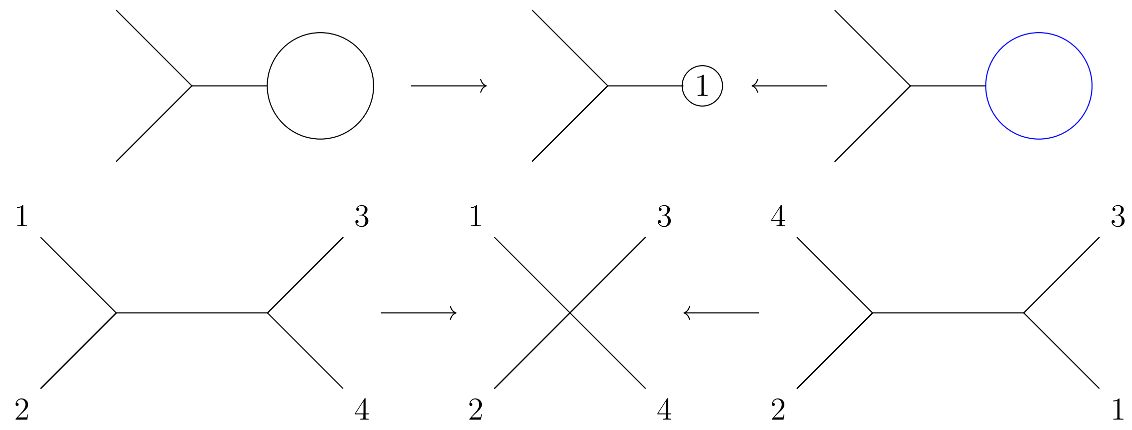


Figure 30: One Edge Contraction Possibilities for Codimension One, where 1-4 represent half-edges

Lemma 4.9. *All codimension one cells are formed by one edge contraction, so are formed by either an edge contraction of a loop to a vertex of weight 1 and valency 1 or by edge contraction of an edge to a single vertex of valency 4.*

Proof. Let $C(G, \omega, [P])$ be a codimension one cell of $\mathcal{T}_g^{trop}$. Then the dimension of $C(G, \omega, [P])$ is $3g-5$ and the underlying graph G has $3g-4$ edges. As we have seen that all graphs of genus g can be formed by edge contractions of weightless trivalent graphs, thus all graphs with $3g-4$ edges can be formed from a graph of $3g-3$ edges via one edge contraction. As all vertices in the graph with $3g-3$ edges are trivalent there are only two possible configurations of this edge contraction, either edge contracting an edge between two different vertices of valency 3, or edge contracting a loop which starts and ends at the same vertex. Case i will produce a vertex of valency 4 and weight 0, case ii will produce a vertex of valency 1 and weight 1. Whether the edge we contract is in the cyclic subgraph P or not is immaterial. As all vertices have valency 3 and all edges go between either one or two vertices these are the only two possible cases. □

Theorem 4.10. *The moduli space is connected in codimension one.*

Proof. The main tool of this proof is this twist operation described by, for example, Caporaso (2012) in section 2.4 and Tsukui (1996) in section 3. It has already been shown that any trivalent graph of genus g can be made into any other trivalent graph of genus g by a series of twists. We will show that these twists are identical to an edge contraction and expansion, show that they preserve theta characteristic cycles and finally that we can get from any maximal dimensional cell to any other by only passing through cells of one dimension lower.

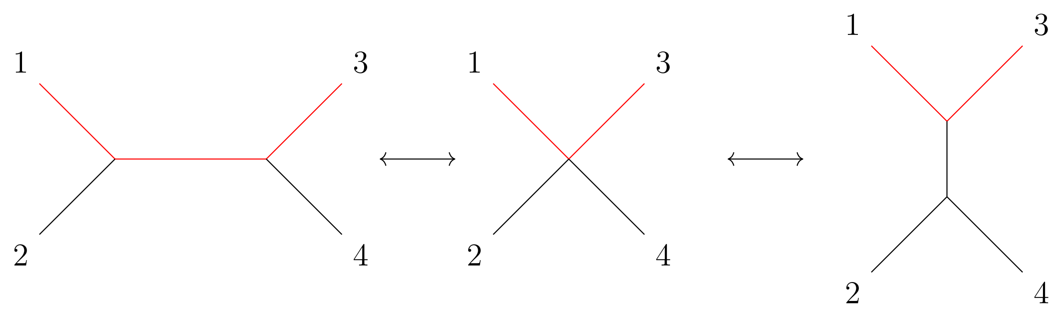


Figure 31: Example how twists are equivalent to and edge contraction and expansion

When edge contracting and expanding we can preserve any cycle, denoted by the red lines in Lemma 31, by choosing whether or not the new edge is in the cycle. Also, as we know any graph can be obtained through a series of twists, we can always create a "Lollipop" graph, consisting of only independent loops.

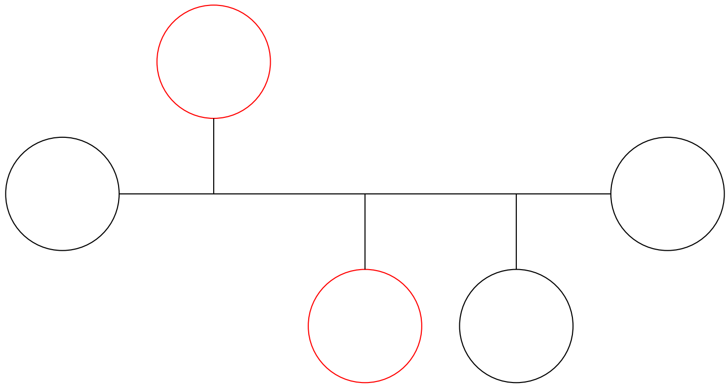


Figure 32: Lollipop graph of genus 5

From the "lollipop" graph, we can edge expand and expand any loop, as seen in Lemma 4.9, to change all cycles to be in the cyclic sub graph P. As any cell can be transformed into the "lollipop" graph and any "lollipop" graph can be transformed into having all it's cycles be in P, this means we can construct a path between any two cells, only using one edge contraction and expansion at a time. Therefore, from Lemma 4.8, we deduce the moduli space is connected in codimension one.

□

4.4 Simply Connected

Definition 4.11 (loop)**.** Let X be a metric space and $\phi : [0,1] \to X$. We say that ϕ is a *loop* if ϕ is continuous, $\phi(0) = \phi(1)$ and $\phi(s) \neq \phi(t)$ for all $0 < s < t < 1$.

Definition 4.12 (simply connected)**.** Let X be a metric space. We say that X is *simply connected* if for all $\phi : [0,1] \to X$ a loop, there exists a continuous function $\hat{\phi} : [0,1] \times [0,1] \to X$ such that

(i) $\hat{\phi}(0,t) = \phi(t)$ for all $t \in [0,1]$,

(ii) $t \mapsto \hat{\phi}(s,t)$ is a loop for all $s \in [0,1)$,

(iii) $\hat{\phi}(1,t) = \hat{\phi}(1,s)$ for all $s,t \in [0,]$.

Informally this mean that any loop in X can be smoothly shrunk to a point.

Examples of simply connected spaces include the euclidean plane and the surface of a sphere. The annulus however is not simply connected.

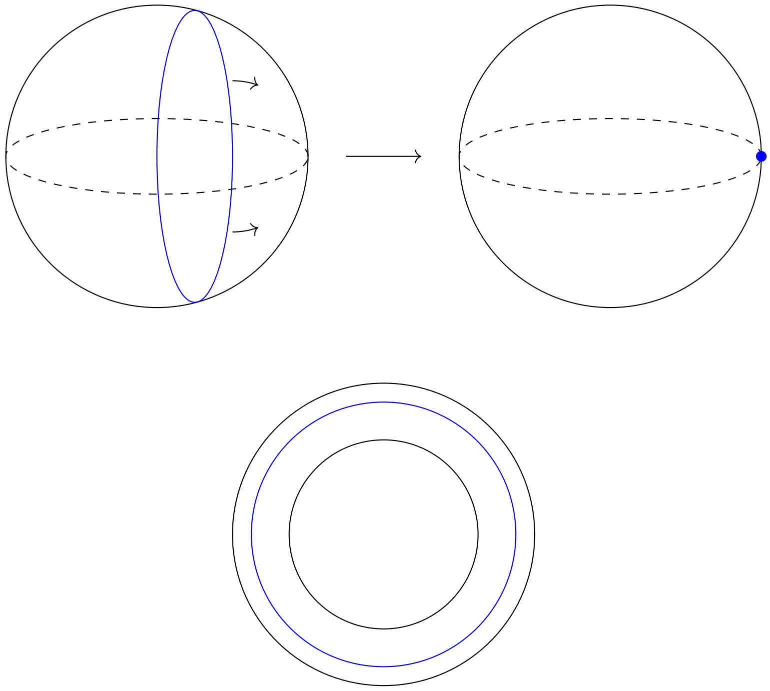
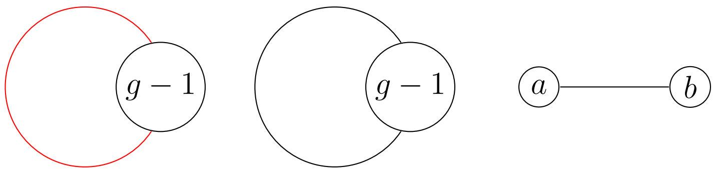


Figure 33: Examples of simply connected and not simply connected spaces.



where $a+b=g$ and $a,b>0$.

Figure 34: Zero dimensional points on the moduli space of genus g

Just as a note, for a combinatorial argument for how many distinct zero dimensional points there are for a fixed g, consider that the line with weights a,b can be rotated to the line with weights b,a, so choose $a \leq b$. Then the possible values of a are $a \in \{1,2,\ldots,\lfloor\frac{g}{2}\rfloor\}$, so there are $\lfloor\frac{g}{2}\rfloor$ possible values of a - each of which uniquely determine the value of b, so in total there are $\lfloor\frac{g}{2}\rfloor+2$ zero dimensional points in the moduli space of genus g.

Now we will define a 'snake' graph and cyclic subcycle $(\mathbf{G},P)$ of genus g. This is defined as a pure, trivalent graph made up a series of cycles of two edges, with single bridges between adjacent cycles, and two loops on the end, on which one ending loop must be in P and one must not be in P. Other cycles may be in P or not, so there are 2^{g-2} ways of selecting this snake. (and no such graph exists for $g=0$ or $g=1$.)

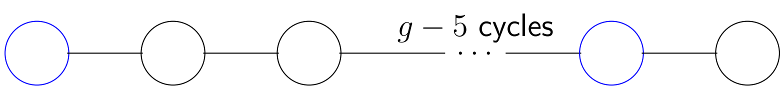


Figure 35: Examples of 'snake' graph and cyclic subgraph in blue

If we choose an edge in P, contract all other edges and increase the length of our chosen edge to 1 then we will have the first zero dimensional point. If we choose an edge in a cycle that is not in P and contract all others we will have the second type of zero dimensional points. If we choose the edge that is not a loop with a cycles to its left and b cycles to its right and edge contract all other edges then we have the third zero dimensional case, where $a+b=g$ and $a,b>0$. Thus all zero dimensional points in the moduli space are in the boundary of the cell defined for the snake graph.

We chose one such snake graph cell and call it the 'main cell'. As every type of graph with cyclic subgraph can be edge contracted to least one type of zero dimensional point, by choosing one edge and edge contracting all others to 0, hence the boundary of every cell in our moduli space must be connected to the main cell at at least one point - including the boundary of cells already in the boundary of other cells.

Also any cell whose boundary meets the main cell at more than one point will meet it for the entire one dimensional boundary defining a cell of a graph with two non-zero (non-contracted) edges between those two points. These facts will enable us to show that any loop that enters and leaves every cell it is by two different boundaries in fact must be continuously deformable to a loop in the main cell, which is itself simply connected by Lemma 4.17.

Any boundary that a loop passes through is itself a cell, and hence connected to the main cell. If the loop passes through multiple boundaries of the same cell, then all those boundaries must connect to the main cell, as must the one dimensional cells between them, so the loop may be deformed to that one dimensional cell in the boundary of the main cell, and thus into the main cell itself.

Lemma 4.13. *Every zero dimensional cell of the moduli space is in the boundary of the main cell.*

Proof. Zero dimensional cells in $\mathcal{T}_g^{trop}$ represent tropical curves will underlying graphs with no choice of edge lengths, so the underlying graphs must only have one edge of length 1. A single edge may be either a loop or a straight line.

In order to be of genus g, a loop, of first Betti number 1 ($b_1(G)=|E|-|V|+1=1-1+1=1$), must have $g-1$ weights on it, and for the loop to be one single edge all $g-1$ weights must be at the same vertex, loops with vertices placed at different points will be equivalent up to isomorphism. The loop may then be part of the cyclic subgraph P or not, so there are two cases, a single loop in P and a single loop not in P, each with a single vertex v where $\omega(v)=g-1$.

In order to be of genus g a non-loop edge, of first Betti number 0 ($b_1(G)=|E|-|V|+1=1-2+1=0$), must have g weights on it. The weights must be at the vertices at either end of the edge, as weights elsewhere would be mean we have two distinct edges, and so are no longer in the zero dimensional cell. Also, in order for the graph to be stable, as each vertex has valency 1, there must be at least one weight on each end. So let the weights on each end be $\omega(v_1),\omega(v_2)=a,b\in\mathbb{Z}$, where $a+b=g, a,b\geq 1$. As a line with weights a,b on v_1,v_2 and a line with weights b,a on v_1,v_2 are equivalent we can say without loss of generality that $a\leq b$. Thus there are $\lfloor\frac{g}{2}\rfloor$ different possible values of a,b. So there are $2+\lfloor\frac{g}{2}\rfloor$ zero-dimensional cells $\mathcal{T}_g^{trop}$.

Now we show all these zero-dimensional cells are in the boundary of the main cell, so that there is a series of edge contractions that transform a snake graph to any of these underlying graphs. For the loops, choose one edge in one of the cycles of the snake graph, either an edge in P or not, and then edge contract every other edge of the graph. This will involve edge contracting $g-1$ other cycles, to leave a loop with a single weight of $g-1$ on it.

For the non-loops, choose the edge of the graph with a cycles to one side and b cycles to the other, this edge will always exist, as $a,b\geq 1$ so we at least need the loops on the ends of the snake graph, and there are g cycles in the snake graph, so can always partition these cycles into a and b and choose the edge between these two sections. Then edge contract all other edges, this will give a weights on one side of the edge and b on the other, as required, as each cycle will contribute 1 weight when edge contracted.

□

For ease of reference we will label these underlying graphs in the following way: G_P^1 for the single loop in P, G_{PC}^1 for the single loop not in P (in P^C) and $G_{(a,b)}^0$ for the single non-loop edge with weights a, b on its vertices. Their cells will be denoted $C(G_P^1)$ for example, and all cells will be singletons.

Lemma 4.14. *Every cell in the moduli space (including cells in the boundary between other cells) has at least one zero dimensional point in its boundary. In fact every boundary of a cell/boundary between cells has a zero dimensional cell in its boundary.*

Proof. As we have seen in the proof of connectedness in codimension one, the cells of all graphs that are not pure and trivalent with a cyclic subgraph are in the boundary of at least one pure trivalent graph's cell. In fact the boundaries of the cells are formed by edge contraction of at least one edge of the underlying graph with cyclic subgraph. Thus every boundary of a cell represents the moduli space of some new type of underlying graph with cyclic subgraph, and each boundary may be stratified into the cells of different graphs corresponding to different edge contraction(s) of the underlying graph in our original cell. Consider a cell of an arbitrary graph of genus g with cyclic subgraph, $C(G, \omega, [P])$. Then choose any one edge of that graph, and edge contract all other edges to 0. If the edge chosen was in a cycle (any cycle) in the graph, then the result will be a loop, in P or not depending on whether initial loop selected was in P. If the edge selected is not in any cycle, then the resultant graph will be a non-loop edge with weights on each end $a + b = g$ and $a, b > 0$. So at least one of these points must be in the boundary for every cell in the moduli space. $\square$

Lemma 4.15. *Every pair of zero dimensional cells are in the boundary of at least one one-dimensional cell which 'connects' them.*

Proof. Any underlying graph G such that $|E(G)| = 2$ has a one dimensional cell represented by a line, $\{(\ell_{e_1}, \ell_{e_2}) \in \mathbb{R}^2_{>0} \mid \ell_{e_1}, \ell_{e_2} > 0, \ell_{e_1} + \ell_{e_2} = 1\}$. The boundary of this line are the points $(0, 1)$ and $(1, 0)$ found by taking ℓ_{e_1} and ℓ_{e_2} to 0 respectively, equivalent to performing an edge contraction on one of the two edges. Thus the boundary of any graph with two edges (with cyclic subgraphs) will be the cell(s) representing one or two graphs (with cyclic subgraphs) with one edge.

Now to show that for any pair of zero dimensional cells we select, such a one-dimensional cell does exit that has them both in its boundary. We may select both the loops in and not in P, then the graph with one vertex of valency 4 and weight $g - 2$ and two loops coming off that vertex with one in P and one not is an underlying graph with a cell of dimension one between them. We may select a loop and a non-loop with weights a and b. Then consider a graph with two vertices, v_1, v_2 where $val(v_1) = 1, \omega(v_1) = a$ and $val(v_2) = 3, \omega(v_2) = b - 1$, with one loop (either in P or not depending on the zero dimensional underlying graph selected), then this graph has two edges so its cell is of dimension one. (We may also have the cell where the underlying graph has $val(v_1) = 1, \omega(v_1) = a - 1$ and $val(v_2) = 3, \omega(v_2) = b$ - if $a \neq b$ then these will be two different one dimensional cells with our zero dimensional cells in their boundaries). We may select two different non-loops with different weights on their ends, say one edge with weigths a, b and one edge with weights c, d. Then there exists some $r \in \mathbb{Z}$, $-a < r < g - a = b$ such that $c = a + r$ and $d = b - r$.

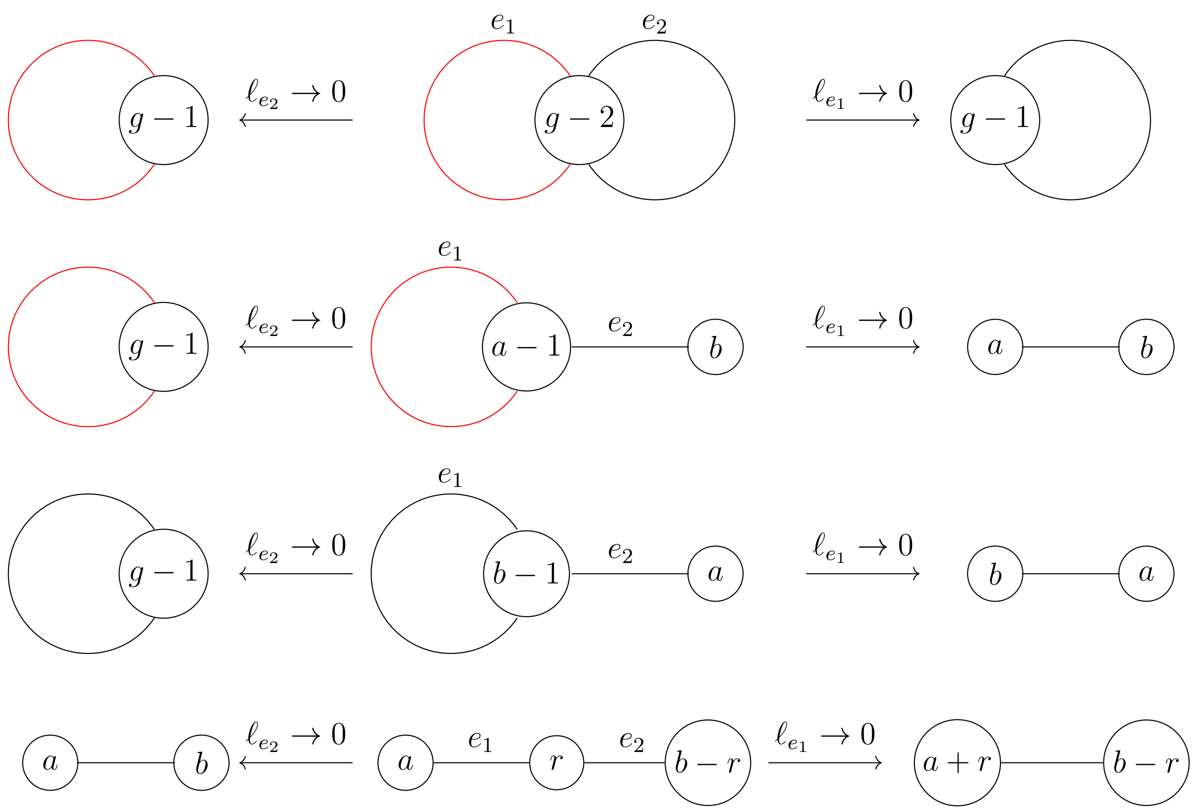


Figure 36: One Dimensional Cells which connect two zero dimensional cells

□

Note there are other one dimensional cells, which only have one zero dimensional cell in their boundary, such as a figure 8 graph where both loops are not in P, with $g-2$ weight on the central vertex.

For a one dimensional cell, denote the underlying graph $G^2_{P,PC}$ for the first case, $G^1_{P,(a,b)}$ for the second case, $G^1_{PC,(b,a)}$ for the third and $G^0_{(a,b),(a+r,b-r)}$ for the fourth case.

Lemma 4.16. *For every cell with two zero dimensional cells in its boundary will also have a one dimensional line connecting them in its boundary, and this one dimensional line will also be in the boundary of the main cell.*

For every cell with two different zero-dimensional cells contained in its boundary, there must be at least one edge in the underlying graph of the cell which forms that type of underlying graph when all other edges are edge contracted. We will choose two such of these edges (one of each type) and edge contract all other edges. The result will be a tropical curve with an underlying graph of with two edges. We want to show that all tropical curves in the cell with this underling graph are both in the closure of our original cell with two zero-dimensional points in its boundary, and in the closure of the main cell. Consider a cell of a weighted graph with cyclic subgraph $C(G, \omega, [P])$ where $C(G^1_P), C(G^1_{PC}) \in \overline{C(G, \omega, [P])}$. Then there exists $e_i, e_j \in E(G)$ such that when $\ell_{e_k} = 0 \forall k \neq i$ and $\ell_{e_i} = 1$, the underlying graph is of the type G^1_P and when $\ell_{e_k} = 0 \forall k \neq j$ and $\ell_{e_j} = 1$, the underlying graph is of the type G^1_{PC}.

Consider a point $\{([G^2_{P,PC}], \ell_{\tilde{e}_1}, \ell_{\tilde{e}_2}) \mid \ell_{\tilde{e}_1}, \ell_{\tilde{e}_2} > 0, \ell_{\tilde{e}_1} + \ell_{\tilde{e}_2} = 1\} \in C(G^2_{P,PC})$. Let $\epsilon > 0$ be sufficiently small, $\epsilon < \ell_{\tilde{e}_1}, \ell_{\tilde{e}_2}$. Then consider the graph of type $(G, \omega, [P])$ with edges of lengths, $\ell_{e_i} = \ell_{\tilde{e}_1} - \frac{\epsilon}{2}$, $\ell_{e_j} = \ell_{\tilde{e}_2} - \frac{\epsilon}{2}$ and $\ell_{e_k} = \frac{\epsilon}{|E(G)|-2}$. Then $\sum_{n=1}^{|E(G)|} \ell_{e_n} = \ell_{e_i} + \ell_{e_j} + \sum_{n=1, n\neq i,j}^{|E(G)|} \ell_{e_n} = \ell_{\tilde{e}_1} - \frac{\epsilon}{2} + \ell_{\tilde{e}_2} - \frac{\epsilon}{2} + \sum_{n=1}^{|E(G)|-2} \frac{\epsilon}{|E(G)|-2} = \ell_{\tilde{e}_1} - \frac{\epsilon}{2} + \ell_{\tilde{e}_2} - \frac{\epsilon}{2} + \epsilon = \ell_{\tilde{e}_1} + \ell_{\tilde{e}_2} = 1$. So this point is in the cell $C(G,\omega,[P])$ for all $0<\epsilon<\min \ell_{\tilde{e}_1},\ell_{\tilde{e}_2}$. Now letting $\epsilon \to 0$, we have a sequence of tropical curves in the cell which tend to the arbitrary point in $C(G^2_{P,PC})$ so $C(G^2_{P,PC}) \subset \overline{C(G,\omega,[P])}$. In fact a similar argument works for most other cases, the only remaining thing to do is check that a tropical curve of the correct underlying type can always be generated by edge contracting all but two edges.

In case 1) when edge contracting all edges except one in P and one not in P where both edges are in cycles this is the only type of underlying graph that may be produced, as there are no other graphs of genus g made up of precisely two loops and no other edges with one loop in P and one loop not.

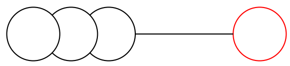


Figure 37: An example of an underlying graph and cyclic subgraph where only one possible placement on the loop on the vertices with weights is possible

In case 2) when edge contracting all edges except one in P and one non-cycle edge, which can't be in P as it isn't in any cycle, there are two possibilities, depending on whether the edge in P is to the side of the non-loop edge with a weights on it or b weights on it. It may be there is a choice of edges in P, so we can choose one either side of the non-loop edge, or it may be that all edges in P are on one side of the non-loop edge, for example if P is one cycle. Case 3) is very similar, but with cycles not in P instead. Both of these one dimensional cells $C(G^1_{P,(a,b)}), C(G^1_{P,(b,a)})$ may be in the closure of $C(G,\omega,[P])$ or only one of them, depending on whether we can choose an edge e_i on both sides of the bridge with weights a and b.

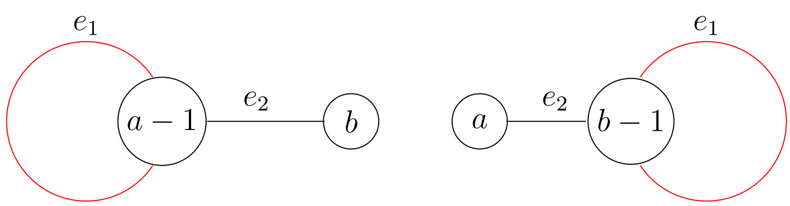


Figure 38: Two different possible one dimensional cells

In case 4), the only thing to note is that two non-cycle edges with different weights on each side after all other edges have been edge contracted must be different edges in the initial underlying graph. Thus if our graph has two non-loop edges e_1 and e_2 where e_1 has a cycles to one side and b cycles to the other, and e_2 has $a+r$ cycles to one side and $b-r$ cycles to the other, then edge contracting all other edges except e_1 and e_2 will create an underlying graph with no edges, a weights to one side, r weights in the middle and $b-r$ weights to the other.

Or if e_2 has $a-r$ cycles to one side and $b+r$ to the other, then edge contracting all other edges will give an underlying graph with two edges and $a-r$ weights to one side, r weights in the middle and b weights to the other side. By using a similar proof as above we can show that both dimension one cells corresponding to these graphs will be in the boundary of our original cell.

Now we need to show that these one dimensional cells will also be in the boundary of our main cell.

By definition of the snake graph there must be at least one edge in a cycle in P and at least one edge in a cycle not in P, so labelling two such edges e_i and e_j by similar argument to above, $C(G^2_{P,PC}) \subset \overline{C('main')}$.

For case 2) we want to show that both possible one dimensional cells $C(G_{P,(a,b)}), C(G_{P,(b,a)})$ are in the boundary of the main cell. In the case where $a=b=\frac{g}{2}$ these two cells are equivalent.This can only happen if g is even. In the snake graph there will be one edge with half the cycles to each side of it, label that one e_j and the cycle at the end that is in P as e_i. Then the one dimensional cell is in the boundary of the main cell again by the above. If $a \neq b$ then there are two edges in the snake graph with a cycles on one side and b on the other. Given that $a \leq b$, if we want the cycle e_1 to be on the side of the non-loop edge with $a-1$ weights, then take the edge that is closer to the ending loop in P, and if we want the cycle e_1 to be on the side of the non-loop edge with $b-1$ weights then take the edge that is further away from the ending loop in P. For case 3) do as in case 2), but considering distance to ending loop not in P instead. For case 4) start at one end of the underlying graph, and consider the edge after a cycles, then the edge after a further r cycles. If $a+b=g$ and $0<r<b$, then these will be two different edges both in the graph. Edge contract all other edges to 0, this will give a graph a weights on one vertex, r weights on the middle vertex and $b-r$ weights on the final vertex as required. Labelling these two edges e_1 and e_2 and preceding as above should show that the entire cell of this underlying graph is in the boundary of the main cell. The case of $a-r$ and $b+r$ is equivalent.

Let $\phi : S^1 \to \mathcal{T}_g^{trop}$ be some loop. We let $\phi_0 : S^1 \to [\mathcal{C}_g]$ and, for each $(\boldsymbol{K}, [R]) \in [\mathcal{C}_g]$ and $e \in E(K)$, let $\phi_e^{(\boldsymbol{K},[R])} : S^1 \to \mathbb{R}_{>0}$ be the maps such that

$$\phi(t) = (\phi_0(t), (\phi_e^{\phi_0(t)}(t) : e \in E(\phi_0(t))).$$

By a *loop deformation of* ϕ we mean a sequence of loops $\phi^s : S^1 \to \mathcal{T}_g^{trop}$ such that the map $\hat{\phi} : (s,t) \mapsto \phi^s(t)$ is continuous.

Let $\mathcal{C}_g^k$ be the set of cells in $\mathcal{T}_g^{trop}$ of effective dimension k. So $\mathcal{C}_g^{3g-4}$ is the set of all maximal dimensional cells in $\mathcal{T}_g^{trop}$.

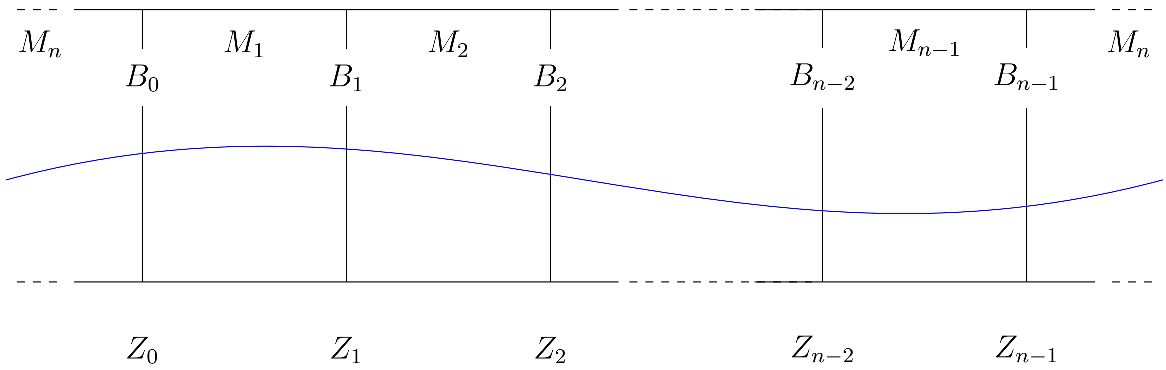
For the remainder of this section we consider loops in $\mathcal{T}_g^{trop}$ where $g \geq 3$.

Let $A_\phi = \{T \subseteq S^1 : \exists M \in \mathcal{C}_g^{3g-4}, T \text{ is a maximal arc such that } \phi_0(t) \leq M \ \forall t \in T\}$. A_ϕ is the set of maximal arcs of S^1 that map onto portions of the loop contained entirely in the closure of maximal dimensional cells. We can order A_ϕ by clockwise rotation. We pick one arc in A_ϕ and denote in by A_1 and then by A_i the remaining arcs in order of clockwise rotation. Let $M_1, \ldots, M_n$ denote the cells in $\mathcal{C}_g^{3g-4}$ whose closure contains all cells in $\phi_0(A_1)$, …, $\phi_0(A_n)$ respectively.

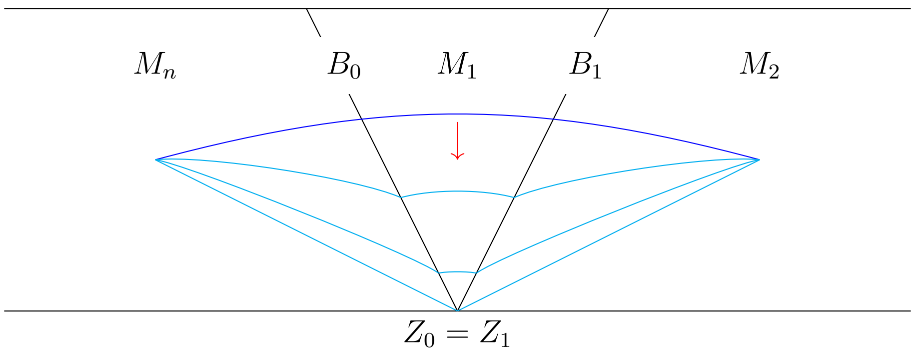
Now we pay particular attention to the M_i's where $\phi_0(\min(A_i)) = \phi_0(\max(A_i))$ (recall how we order points in S^1) - that is the portions of the loop that 'enter' and 'leave' the closure of the maximal dimensional cell via the same lower dimensional cell. We will call these arcs and associated portions of the loop *redundant*. We argue that we can deform the loop ϕ so as to remove these redundant portions. The deformation process is relatively simple, the only thing we need to be careful of here, is when equal M_i's contain different redundant portions of the loop. In this case we must be careful to first deform/shrink down the portions which in the process doesn't form a loop that intersects with our other portions of the loop. (meh)

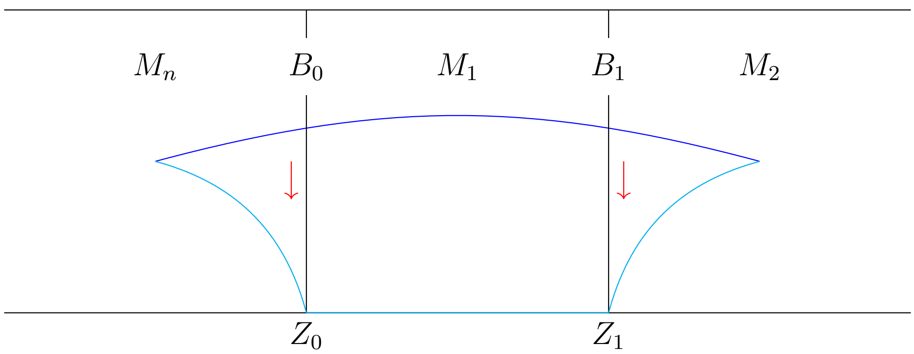
(portion of loop now contained in the boundary of another maximal dimensional cell.)

So any loop ϕ not contained in the closure of one maximal dimensional cell can be deformed to a loop θ which is either contained entirely in one maximal dimensional cell, or no arcs in A_θ are redundant - that is, the loop always leaves and exits the closure of any maximal dimensional cell via two different cells. In the first case we are done, by lemma **??**, so assume the latter. We pick one arc in A_θ and denote it by A_1 and then by A_i the remaining arcs in order of clockwise rotation. Let $M_1, \ldots, M_n$ denote the cells in $\mathcal{C}_g^{3g-4}$ whose closure contains all the cells in $\theta_0(A_1)$, …, $\theta_0(A_n)$ respectively. For $0 \leq i < n$, let $B_i = \phi(\min(A_{i+1}))$, the cell the portion of the loop $\phi(A_{i+1})$ enters M_{i+1} through. By lemma 4.14, we kind find a path contained in B_i and B_{i+1} that connects the point $\theta(\min(A_{i+1}))$ and $\theta(\max(A_{i+1}))$ to a zero dimensional cell Z_i and Z_{i+1} respectively. So we have this abstract picture below.



We first demonstrate what we'd like to be able to do for each M_i, and then discuss why we need to be more careful. First consider M_n, M_1 and M_2. Since B_0 and B_1 are also completely contained in the closure of M_n and M_2, we can deform the loop so that it now passes through Z_0 and Z_1, being careful to also shrink the portion of the loop $\theta(A_1)$ to Z_0 if $Z_0 = Z_1$ (this is to avoid a pinch point where the new loop intersects itself). Lemma 4.16 tells us that Z_0 and Z_1 are joined by a one dimensional cell contained in the closure of both the main cell and the maximal dimensional cell M_1. So if $Z_0 \neq Z_1$ then we can deform the portion of the loop between $\theta(\min(A))$ and $\theta(\max(A))$ to this line.

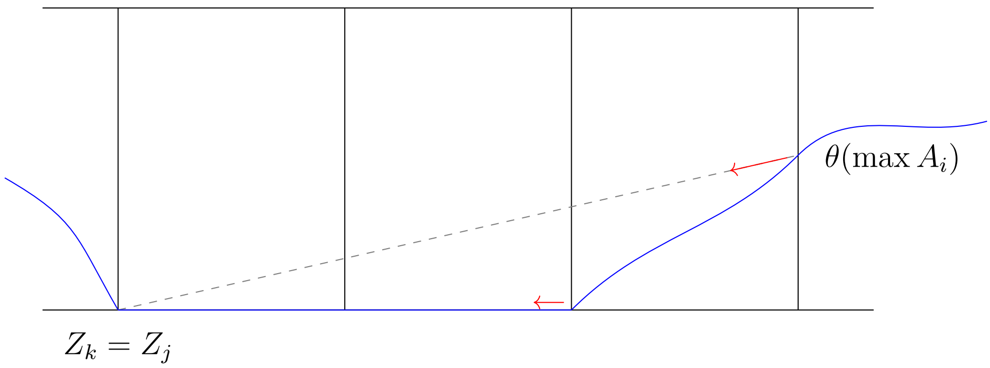




Ideally we'd like to be able to do this for each M_i so as to move the loop into the closure of the main cell, but unfortunately there is one hindrance. If multiple portions of the loop cross through the closure of the same maximal cell (that is, $M_i = M_j$ for some $i \neq j$) and the cells on the boundary they enter and exit from share only one or two zero dimensional cells, then our deformation above will result in a loop that intersects itself. We now present an algorithm which deals with this possibility.

First note that if multiple portions of the loop run through the same maximal dimensional cell, the order in which we deform each segment may matter. For example in the 2 dimensional case, we would need to first deform the portion down to the 1 dimensional line in the main cell that is 'below' any other portions. Also note that if $M_i = M_j$ portion $\theta(A_i)$ **must** be deformed before portion $\theta(A_j)$ in some maximal cell M_i, then it can't be the case that $\theta(A_{i+1})$ must be deformed before $\theta(A_{j+1})$ in the cell M_{i+1} ($i+1=1$ if $i=n$), where P' and Q' are the portions of the loop connected to P and Q respectively.

We perform the loop deformation process outlined above for each M_i, until we get to a j such that $M_j = M_i$ for some $i < j$. If $Z_i \neq Z_j$ and $Z_{i-1} \neq Z_j$ then we are fine and can deform as usual. If $Z_k = Z_j$ for some $k \in \{i, i+1\}$ then there must be a 1 dimensional cell joining $\theta(\max(A_i))$ to Z_k that is also contained in the boundary of M_i, M_j and M_{j+1}. Therefore we can deform the loop by taking the point $\theta(\max(A_i))$ and moving it along this 1 dimensional line to Z_k, while shrinking the portion of the new loop between Z_k and $\theta(\max(A_i))$ to Z_k.



This then gives us an algorithm to deform the loop into hte boundary of the main cell.

Lemma 4.17. *Any cell in $\mathscr{T}_g^{trop}$ is simply connected*

Proof. Consider any loop $\phi : S^1 \to \mu^0(\mathbf{G}, P)$ in a cell $(\mathbf{G}, [P])$. Define $\hat{\phi}_s : S^1 \times [0,1] \to \mu^0(\mathbf{G}, P)$ such that $\hat{\phi}(t, 0) = \phi(t)$. We can separate ϕ into it's components: $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_{|E(G)|}(t))$. we can then define $\hat{\phi}$ component-wise:

$$\hat{\phi}_i(t, s) = \frac{s}{|E|} + (1-s)\phi_i(t)$$

This function is certainly continuous as it is piece-wise linear. The function also satisfies $\hat{\phi}(t, 1) = (\frac{1}{|E|}, \ldots, \frac{1}{|E|})$. Now we just need to check $\hat{\phi}$ always maps to $\mu^0(\mathbf{G}, P)$.

$$\begin{aligned}
\sum_{i=1}^{|E|} \hat{\phi}_i(t, s) &= \sum_{i=1}^{|E|} \frac{s}{|E|} + (1-s)\phi_i(t) \\
&= s + (1-s)\sum_{i=1}^{|E|} \phi_i(t) \\
&= s + 1 - s \\
&= 1
\end{aligned}$$

as required $\square$

We can construct an algorithm to always deform any loop to a point. Assume a loop passes through more than one cell, otherwise we are done. There are a few cases to consider:

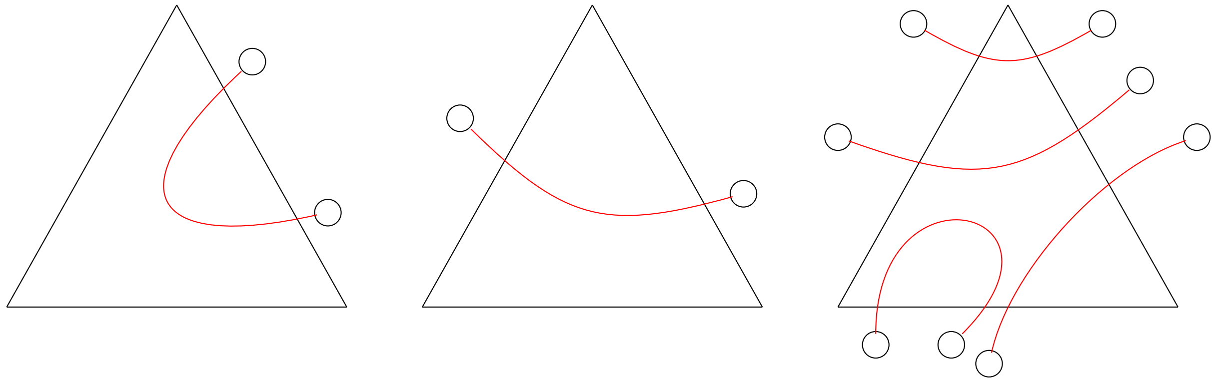


Figure 39: Cases of loop passing through cells

The two main cases are the case where the loop enters and exits a cell form the same boundary or the loop enters and exits a cell through a different boundary. The first case was discussed above and we can repeatedly deform the loop until this case no longer appears. The other case is a little trickier. We can also have a combination of the cases or multiple of these cases at the same time, however, the first case can always be removes and any instances of cells with multiple instances of the second case, we can simply treat each case seperatly and the proof works.

5 Glossary

This section is a glossary of rigorous definitions and notation Graph Edges Weighted Graph Valency Stability Genus Morphism Edge Collapse Tropical Curve Metric graph Metric graph Valency $\mathscr{C}_G$ Combinatorial type T_Γ^{trop} $T_{(G,P)}^{trop}$ $\mathscr{G}_g^0$ $\mathscr{C}_g$ $[\mathscr{C}_g]$ $\mathscr{T}_g^{trop}$ Cell type $[P]$

Glossary

$T_{(G,P)}^{trop}$ Let G be a graph and $P \in \mathscr{C}_G$. We define $T_{(G,P)}^{trop}$ as the set of isomorphism classes of θ-characteristis of type $[T_p]$ on all tropical curves $\Gamma = ((G), l)$ where the edge lengths sum to 1. 35

T_Γ^{trop} We denote the set of θ-characteristics on Γ as T_Γ^{trop}. Note $T_\Gamma^{trop} \cong \mathscr{C}_G$. 35

$[P]$ The cycle class is defined by $[P] = \{\tau \dot{P} : \tau \in \text{Aut}(\mathbf{G})$. 35

$[\mathscr{C}_g]$ $[\mathscr{C}_g] = \bigsqcup_{\mathbf{G} \in \mathscr{G}_g} \mathscr{C}_G/\text{Aut}(\mathbf{G}) = \{(\mathbf{G}, [P]) : \mathbf{G} \in \mathscr{G}_g, [P] \in \mathscr{C}_G/\text{Aut}(\mathbf{G})\}$. 35

$\mathscr{C}_G$ We denote the set of cyclic subgroups of G by $\mathscr{C}_G$. 35

$\mathscr{C}_g$ We define the disjoint union of $\mathscr{C}_g : \forall \mathbf{G} \in \mathscr{G}_g$ by $\mathscr{C}_g = \bigsqcup_{\mathbf{G} \in \mathscr{G}_g} \mathscr{C}_G = \{(\mathbf{G}, P) : \mathbf{G} \in \mathscr{G}_g, P \in \mathscr{C}_G\}$. This is a poset with partial ordering defined by $(\mathbf{G}, P) \leq (\mathbf{G}', P') \Leftrightarrow \exists$ an edge collapse $\phi : \mathbf{G}' \to \mathbf{G}$ such that $\phi(P') = P$. 35

$\mathscr{G}_g^0$ We define the poset of stable graphs of genus g partially ordered by edge collapses excluding the single point as $\mathscr{G}_g^0$. The partial ordering is defined by $\mathbf{G} \leq \mathbf{G}' \Leftrightarrow \exists$ an edge collapse $\phi : \mathbf{G}' \to \mathbf{G}$. 35

$\mathscr{T}_g^{trop}$ $\mathscr{T}_g^{trop} = \bigsqcup_{(\mathbf{G},[P]) \in [\mathscr{C}_g]} T_{(G,P)}^{trop} \cong \bigsqcup_{(\mathbf{G},[P]) \in [\mathscr{C}_g]} \mu^0(\mathbf{G}, P)/\text{Aut}(\mathbf{G}, P)$ where $\mu^0(\mathbf{G}, P) = \{(x) \in \mathbb{R}_{>0}^{|E(G)|} : \sum_{i=1}^{|E(G)|} = 1\}$. 35

Cell type The cell type of a moduli space is the tuple $(\mathbf{G}, [P])$ where $\mathbf{G}$ is a Weighted Graph and $[P]$ is a . 35

Combinatorial type We define the combinatorial type of Γ as $\mathbf{G} = (G, \omega)$. 35

Edge Collapse $\forall e \in E(G) : \exists$ a morphism $f : \mathbf{G} \to \mathbf{G}/e$ where $\mathbf{G}/e$ is a weighted graph obtained from $\mathbf{G}$ by collapsing e together with its two endpoints to a single vertex $[e] \in \mathbf{G}/e$. Any morphism can be described as a finite sequence of edge collapses followed by an isomporphism. . 35

edges The set of edges $E(G)$ is defined by $E(G) = H(G)/$ where the equivalence relation is given by $x\ y \Leftrightarrow x = s_G(y)$. 35

Genus the genus, g, of a graph is defined by $g(\mathbf{G}) = |E(G)| + |V(G)| - 2 + |\omega|$ where $|\omega| = \sum_{v \in V(G)} \omega(v)$. 35

Graph A graph, denoted G, is a finite set $X(G) = V(G) \sqcup H(G)$ together with two functions $s_G, r_G : X(G) \to X(G)$ satisfing $s_G^2 = \mathrm{Id}$, $r_G^2 = r_G$ and that

$$\{x \in X(G) | r_G(x) = x\} = \{x \in X(G) | s_G(x) = x\} = V(G)$$

Altogether $\bar{G} = (X(G), r_G, s_G)$ Informally, $V(G)$ is the set of vertices and $H(G)$ is the set of half edges. s_G sends half-edges to it's other half and r_G sends half-edges to it's incident vertex. 35

Metric graph Let W be the disjoint union of intervals of length $l(e)$ where e is not a loop and circles of circumference $l(e)$ where e is a loop. Let $d : W \times W \to \mathbb{R}_{\geq 0}$ defined by the length of the shorted path between points on $\mathbf{G}$, then $\mathbf{\Gamma} = (W, d)$ is a metric space. 35

Metric graph Valency We can define the valency function for all points $p \in W$ on $\mathbf{\Gamma} = (W, d)$

$$\text{val}_{\mathbf{Gamma}}(p) = \begin{cases} \text{val}(p) & \text{if } p \in V(G), \\ 2 & \text{otherwise} \end{cases}$$

. 35

Morphism A morphism $\mathbf{G} \to (G)'$ is a function $f : X(\mathbf{G}) \to x(\mathbf{G}')$ satisfying:

- $f \circ r_{\mathbf{G}} = r_{\mathbf{G}'} \circ f$
- $f \circ s_{\mathbf{G}} = s_{\mathbf{G}'} \circ f$
- $\forall h \in H(\mathbf{G}') : |f^{-1}(h)| = 1$
- each $v \in V((G)')$ determines a subset $S_v = f^{-1} \subset X(G)$ and $\mathbf{S}_v = (S_v, r|_{S_v}, s|_{S_v})$ is a graph that is connected and such that the genus of $(\mathbf{S_v}, \omega|_{\mathbf{S_v}}) = \omega(v)$

. 35

Stability A graph is stable if $\forall v \in V(G) : 2\omega(v) - 2 + val(v)$. 35

Tropical Curve A tropical curve is a weighted graph with a length assigned to each edge. Let $\mathbf{G}$ be a weighted graph and $\ell : E(G) \to \mathbb{R}_{>0}$ a function from the edges of G to the reals, effectively assigning each edge a length, then $\Gamma = (\mathbf{G}, \ell)$ is a tropical curve.. 35

Valency the Valency of a vertex, $val(v)$ is defined by $val(v) = |\{h \in H(G) : r_G(h) = v|$. 35

Weighted Graph A weighted graph, denoted $\mathbf{G}$, is defined by $\mathbf{G} = (G, \omega)$, where G is a graph and $\omega : V(G) \to \mathbb{Z}_{\geq 0}$ is a function from the verticies of the graph to the non-negative integers. 35

References

Brannetti, S., Melo, M., and Viviani, F. (2011). "On the tropical Torelli map". In: *Advances in Mathematics* 226.3, pp. 2546–2586. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2010.09.011. URL: https://www.sciencedirect.com/science/article/pii/S0001870810003348.

Caporaso, L. (2012). "Geometry of tropical moduli spaces and linkage of graphs". In: *Journal of Combinatorial Theory, Series A* 119.3, pp. 579–598. ISSN: 0097-3165. DOI: https://doi.org/10.1016/j.jcta.2011.11.011. URL: https://www.sciencedirect.com/science/article/pii/S0097316511001865.

Caporaso, L., Melo, M., and Pacini, M. (2023). "Theta-characteristics on tropical curves". In: *Journal of Pure and Applied Algebra* 227.5, p. 107307. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2022.107307. URL: https://www.sciencedirect.com/science/article/pii/S002240492200305X.

Caporaso, L. and Viviani, F. (May 2010). "Torelli theorem for graphs and tropical curves". In: *Duke Mathematical Journal* 153.1. DOI: 10.1215/00127094-2010-022. URL: https://doi.org/10.1215%2F00127094-2010-022.

Jensen, D. and Len, Y. (Jan. 2018). "Tropicalization of theta characteristics, double covers, and Prym varieties". In: *Selecta Mathematica* 24.2, pp. 1391–1410. DOI: 10.1007/s00029-017-0379-6.

Tsukui, Y. (1996). "Transformations of cubic graphs". In: *Journal of the Franklin Institute* 333.4, pp. 565–575. ISSN: 0016-0032. DOI: https://doi.org/10.1016/0016-0032(96)00015-4. URL: https://www.sciencedirect.com/science/article/pii/0016003296000154.

To Do

Appendices