

The Interfacial Dynamics of Bursting Bubbles

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1 Introduction

Bubbles within liquids are incredibly common, occurring in both a wide spectrum of natural and industrial processes [1, 2, 3, 4, 5, 6, 7, 8]. These occurrences can range from water and carbon-dioxide bubbles in magma providing driving forces for an eruption [1] to carbon-dioxide bubbles in champagne enhancing the evaporation of volatile organic compounds dispersed in the liquid phase. When a bubble rises to the surface of a liquid, it may burst, scattering many droplets. These droplets are an important process of transport exchange across the liquid gas interface[?], they are considered the main source of sea spray aerosols and impact air pollution[] as well as the transmission of infectious diseases[]. Bubbles formed due to the breaking of waves contribute to the transfer of heat, mass and other contaminants between the oceans and the atmosphere. The efficiency of this transfer is governed by the initial size and speed of the ejected drops.

There are two Phenomena that occur when a bubble bursts that produce aerosols, the first being when the thin film separating the bubble from the atmosphere is ruptured. The atomisation of the film can produce several hundred droplets of around a micrometer in diameter. Due to the length scales of the rupture being order $O(100\text{nm})$, we are unable to describe this problem with continuum mechanics. In fact, van der Waals forces or electrostatic repulsion must be considered, both of which are long-range intermolecular forces.

After the film rupture, the remaining cavity collapses causing a jet to form. This jet eventually breaches into one or more droplets. The key difference between these droplets and the droplets formed when the film ruptures is that they are much larger, order $O(100\mu\text{m})$ for a typical bubble with a diameter of a millimeter. They are also ejected vertically with a typical ejection velocity of order $O(1\text{ms}^{-1})$.

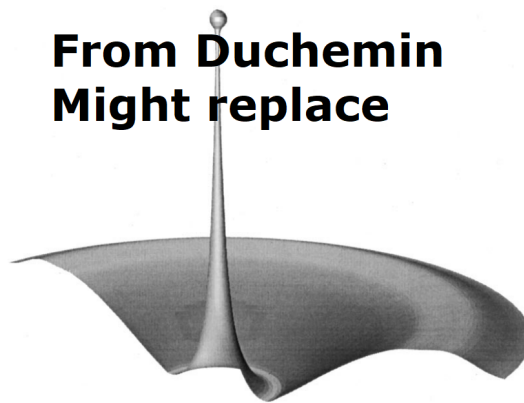


Figure 1: An image of jet formation due to a burst bubble

Literature review goes here:

just some notes:

$$\begin{aligned}
We &= \frac{\rho v^2 L}{\sigma} \\
La &= \frac{\sigma \rho L}{\mu^2} \\
Re &= \sqrt{La We} \\
Bo &= \frac{\Delta \rho g L^2}{\sigma} \\
Oh &= (La)^{-0.5}
\end{aligned}$$

2 Initial conditions

We need to determine the initial shape of a bubble floating at a gas-liquid interface before it's film ruptures. This derivation is based on the work done in [9]. We are considering the case where the fluid is stationary, i.e. $\mathbf{u} = \mathbf{0}$, meaning on the boundaries we only need to consider the Young-Laplace equation:

$$\Delta p = -\gamma \nabla \cdot \hat{n} \quad (1)$$

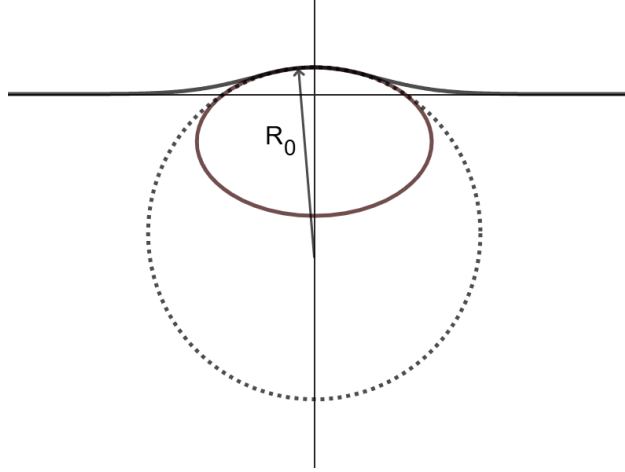


Figure 2: A bubble floating at a surface

We can divide the surface of a bubble into three regions: (A) the submerged portion of the interface, (B) the thin film over the top of the bubble separating the two gas regions, (C) the regions of the interface trailing away from the border of the thin film. We neglect the thickness of the film means we can ignore the weight of the film. Adding that the difference between the gas pressures inside and out of the bubble is uniform, the thin film

can be treated as a spherical surface of radius R_0 . The force balance equations at any point on these surfaces are,

$$(\rho - \rho')gz = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{4}{R_0}\right) \quad (2)$$

for region (A),

$$p = p_0 + \frac{4\gamma}{R_0} \quad (3)$$

for region (B),

$$(\rho - \rho')gz = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad (4)$$

for region (C) where γ represents surface tension of the liquid, g is acceleration due to gravity, ρ and ρ' are the densities of the liquid and gas respectively, p and p_0 are the gas pressure inside and outside the bubble respectively, g is acceleration due to gravity and z is the height above the still gas liquid interface in the far field. R_1 and R_2 are the principle radii of curvature of the surface at a point.

In order to calculate what R_1 and R_2 are we need to use some differential geometry. We assume the bubble can be described by a surfaces of revolution. Let Γ be a curve that lies in the plane (x, z) and let its equation have the equation $z = f(x)$. We then denote Φ as the surface defined by a rotation of Γ around the z -axis or axis of rotation. This surface can then be written in the form:

$$z = f(\sqrt{x^2 + y^2}) = f(r) \quad r = \sqrt{x^2 + y^2} \quad (5)$$

Then using the formulas derived in [10] we get

$$E = 1 + \left(\frac{x}{r}f'\right)^2, \quad G = 1 + \left(\frac{y}{r}f'\right)^2, \quad F = \frac{xy}{r^2}(f')^2, \quad EG - F^2 = 1 + (f')^2, \quad (6)$$

for the coefficients of the first fundamental form given by

$$I(\vec{\lambda}) = E(\lambda_1)^2 + 2F\lambda_1\lambda_2 + G(\lambda_2)^2 \quad (7)$$

where λ is some tangent vector. We can exploit the fact that since our surface is radially symmetric, we only need to find these geometric characteristics on some meridian of the surface, say $y=0$:

$$E = 1 + (f')^2, \quad G = 1, \quad F = 0. \quad (8)$$

Again using the formulas in [10], we obtain

$$L = \frac{f''}{\sqrt{1 + (f')^2}}, \quad M = 0 \quad N = \frac{f'}{x\sqrt{1 + (f')^2}} \quad (9)$$

for the coefficients of the second fundamental form given by

$$II(\vec{\lambda}, \vec{\mu}) = L\lambda_1\mu_1 + M(\lambda_1\mu_2 + \lambda_2\mu_1) + N\lambda_2\mu_2. \quad (10)$$

We then obtain the principle curvatures κ_1 and κ_2 satisfy:

$$(EG - F^2)\kappa^2 - (EN + GL - 2MF)\kappa + LN - M^2 = 0. \quad (11)$$

Substituting in our coefficients for the first and second fundamental forms, solving for κ and noting radius of curvature is one over the curvature we obtain:

$$\frac{1}{R_1} = \kappa_1 = \frac{L}{E} = \frac{f''}{(1 + (f')^2)^{\frac{3}{2}}}, \quad \frac{1}{R_2} = \kappa_2 = \frac{N}{G} = \frac{f'}{x\sqrt{1 + (f')^2}} \quad (12)$$

Notice that R_1 is exactly the radius of curvature of the meridian curve Γ . We now introduce the variable ϕ defined to be the angle at which the tangent to Γ makes with the x-axis. We can calculate ϕ using simple trigonometry to be $\phi = \arctan(f'(x))$. We then consider:

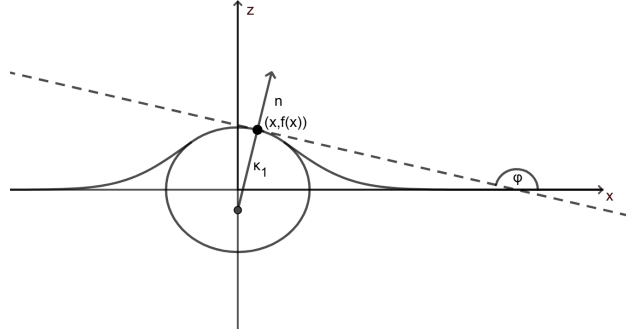


Figure 3: A bubble floating at a surface

$$f'(x) = \tan \phi = \frac{\sin \phi}{\cos \phi} \quad (13)$$

$$= \sin \phi \sec \phi \quad (14)$$

$$= \sin \phi \sqrt{1 + \tan^2 \phi} \quad (15)$$

$$\frac{f'(x)}{\sqrt{1 + f'(x)^2}} = \sin \phi \quad (16)$$

$$\frac{d}{dx} \frac{f'(x)}{\sqrt{1 + f'(x)^2}} = \frac{d}{dx} \sin \phi \quad (17)$$

$$\frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}} = \frac{d(\sin \phi)}{dx} \quad (18)$$

$$\frac{1}{R_1} = \frac{d(\sin \phi)}{dx} \quad (19)$$

Allowing us to obtain R_1 in terms of ϕ . reparametrising the curve in terms of z , i.e. considering the curve given by $(g(z), z)$ where $g(z) = f^{-1}(z)$, we can find another expression for R_1 . using the facts,

$$f'(x) = \frac{1}{g'(f(x))}, \quad f''(x) = \frac{-g''(f(x))}{g'(f(x))^3} \quad (20)$$

We can write the curvature in terms of z :

$$\frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}} = \frac{-\frac{g''(f(x))}{g'(f(x))^3}}{(1 + \frac{1}{g'(f(x))})^{\frac{3}{2}}} = \frac{-g''(f(x))}{(g'(f(x))^2 + 1)^{\frac{3}{2}}} = \frac{-g''(z)}{(g'(z) + 1)^{\frac{3}{2}}} \quad (21)$$

Now we can consider,

$$f'(x) = \frac{1}{g'(f(x))} = \tan \phi = \frac{\sin \phi}{\cos \phi} \quad (22)$$

$$g'(z) = \cos \phi \csc \phi \quad (23)$$

$$= \sin \phi \sqrt{1 + \cot^2 \phi} \quad (24)$$

$$\frac{g'(z)}{\sqrt{1 + g'(z)^2}} = \cos \phi \quad (25)$$

$$\frac{d}{dz} \frac{f'(z)}{\sqrt{1 + f'(z)^2}} = \frac{d}{dz} \cos \phi \quad (26)$$

$$\frac{g''(z)}{(1 + g'(z)^2)^{\frac{3}{2}}} = \frac{d(\cos \phi)}{dz} \quad (27)$$

$$\frac{1}{R_1} = -\frac{d(\cos \phi)}{dz} \quad (28)$$

We can also write R_2 in terms of ϕ by considering the normal to the curve at a point $(x, f(x))$. An equation for this normal is given by

$$(\bar{x} - x) + f'(x)(\bar{z} - f(x)) = 0 \quad (29)$$

where (\bar{x}, \bar{z}) are the coordinates of points on the straight line. We can then find the intersection of this line with the z -axis, $(0, \frac{x+ff'}{f'})$. Finding the distance R between this point and (x, f) gives

$$R = \sqrt{x^2 + (\frac{x+ff'}{f'} - f)^2} = \frac{x\sqrt{1+(f')^2}}{|f'|} = R_2. \quad (30)$$

We can then use trigonometry to calculate R in terms of ϕ :

$$R = \frac{x}{\cos(\frac{\pi}{2} - \phi)} = \frac{x}{\sin(\phi)}. \quad (31)$$

Combining these all together we obtain

$$\frac{1}{R_1} = \frac{d(\sin(\phi))}{dx}, \quad (32)$$

$$\frac{1}{R_2} = \frac{\sin(\phi)}{x}, \quad (33)$$

$$\frac{dz}{dx} = \tan(\phi). \quad (34)$$

We now introduce a capillary constant a^2 having dimension of square length and defined

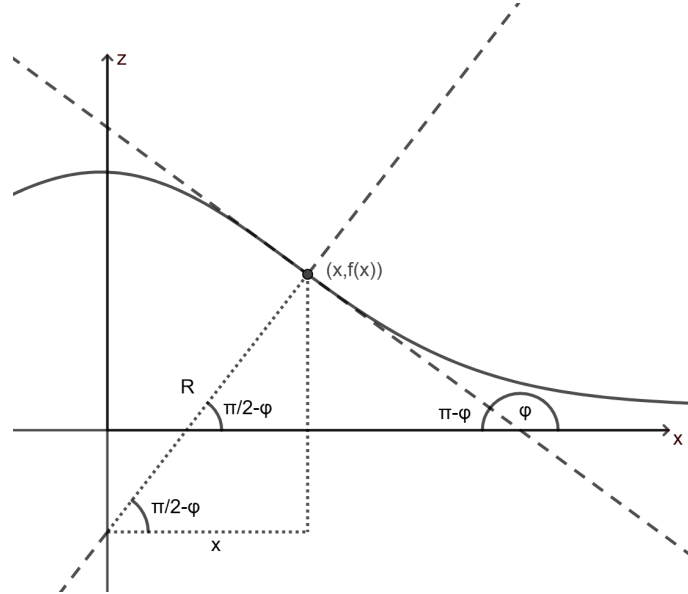


Figure 4: A bubble floating at a surface

by the equation

$$\frac{(\rho - \rho')g}{\gamma} = \frac{2}{a^2} \quad (35)$$

transform x, z, R_0 , etc. to dimensionless quantities $\bar{x}, \bar{z}, \bar{R}_0$, etc. by

$$x = a\bar{x}, \quad z = a\bar{z}, \quad R_0 = a\bar{R}_0, \quad \text{etc.} \quad (36)$$

We then consider two new transformed coordinates:

$$\bar{z}_1 = \bar{z} + \bar{h}, \quad \bar{z}_2 = \bar{z} + \bar{l} \quad (37)$$

where \bar{h} and \bar{l} are chosen such that \bar{z}_1 has its origin at the deepest point of the bubble for portion (A) and \bar{z}_2 has its origin at the centre of the sphere for portion (B). For portion (A), instead of 2, we obtain

$$\left. \begin{aligned} \frac{d\bar{z}_1}{d\bar{x}} &= \tan \phi \\ \frac{d(\sin \phi)}{d\bar{x}} &= 2\bar{z}_1 - \frac{\sin \phi}{\bar{x}} + \theta \end{aligned} \right\} \quad (38)$$

or,

$$\left. \begin{aligned} \frac{d\bar{x}}{d\bar{z}_1} &= \cot \phi \\ \frac{d(\cos \phi)}{d\bar{z}_1} &= -2\bar{z}_1 + \frac{\sin \phi}{\bar{x}} - \theta \end{aligned} \right\} \quad (39)$$

where,

$$\theta = \frac{4}{\bar{R}_0} - 2\bar{h} \quad (40)$$

and the equations 38 and 39 are used for $\tan \phi < 1$ and $\tan \phi > 1$ respectively. Similarly for portion (C), equation 4 becomes,

$$\left. \begin{aligned} \frac{d\bar{z}}{d\bar{x}} &= \tan \phi \\ \frac{d(\sin \phi)}{d\bar{x}} &= 2\bar{z} - \frac{\sin \phi}{\bar{x}} \end{aligned} \right\} \quad (41)$$

and equation 3 for portion (B) becomes

$$\bar{z}_2 = \sqrt{\bar{R}_0^2 + \bar{x}^2} \quad (42)$$

3 Conclusion

In summary, this has been a very successful project.

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