Oceland: A conceptual model for ocean-land-atmosphere interactions based on water balance equations

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ABSTRACT: Enter the text of your abstract here.

1. Introduction

Explanation for proofreading: Red text means comments and questions. Blue and orange text is used when different versions of a sentence/paragraph are proposed.

2. Model description

Somewhere I would like to put a basic statement of our modelling objective, i.e. modelling water fluxes and their partitioning between land and ocean. Maybe this could be stated here as an introductory sentence to the model description. In this case, read the blue text below. Alternatively, it could be said in the end of the introduction. Or, yet another alternative: I could eliminate the heading "Design goals" and just have the text below directly after the heading "Model description". Then, the orange version would apply.

a. Design goals

Conceptual models do not try to explain natural processes in an exact, quantitative manner. Rather, they aim at helping us understand the dominant physical relationships that give rise to a certain natural phenomenon. These dominant factors often get modulated and thereby obscured by a plethora of other processes acting simultaneously in the real world and are therefore difficult to disentangle in observations or simulations with sophisticated climate models. Conceptual models can provide clarity at the expense of realism and with the danger of missing out on relevant physical processes. The successful development of a conceptual model is therefore an iterative process that begins with the most basic assumptions and ends when "the model is only as elaborate as it needs to be to capture the essence of a particular source of complexity, but is no more elaborate than this", as Held (2005) puts it. Version 1: It is our hope that the model proposed in this study meets this balance and that the assumptions and choices that were made in the model development process become clear. Version 2: The complexity we address in this work is the partitioning of precipitation and other water fluxes between land and ocean and its particular source might be the fundamental properties of the two surface types and how they interact with each other and the atmosphere to constrain the exchange of water.

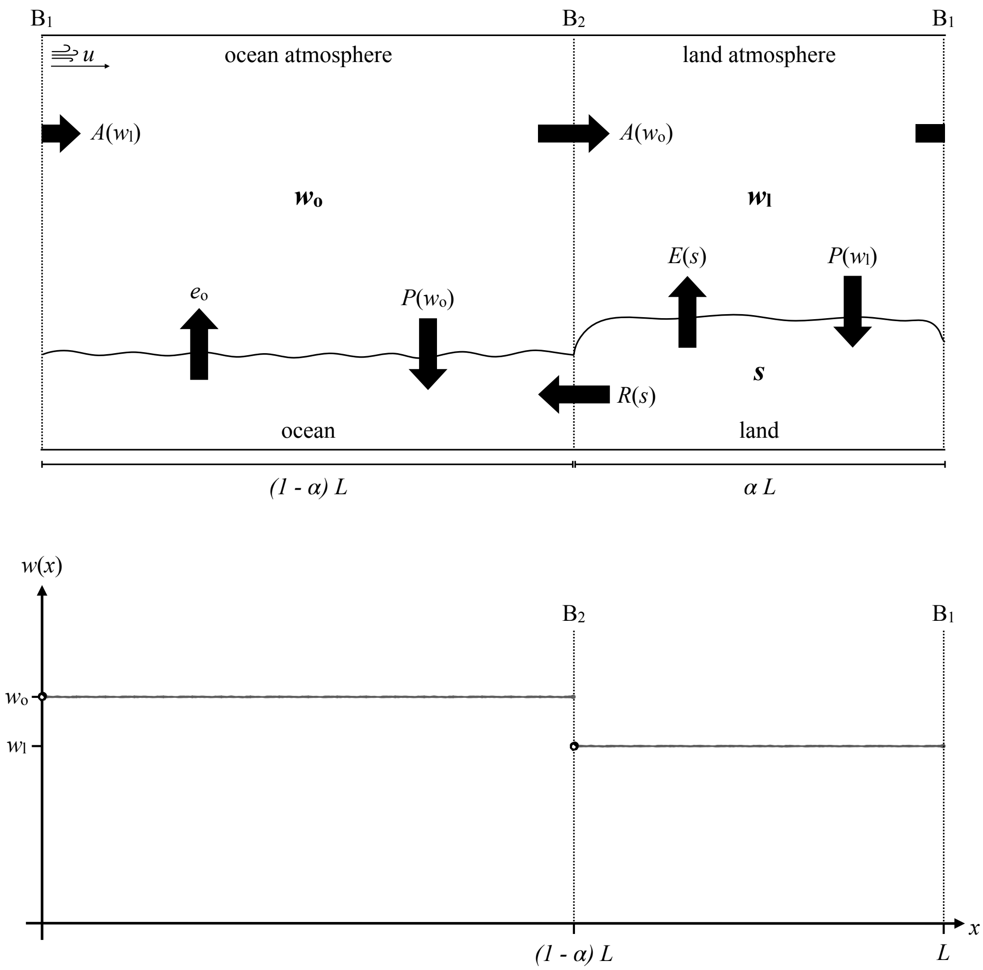

FIG. 1. Closed model sketch and water vapor pass distribution.

b. Closed model setup

We propose a 2D box model as sketched in the top panel of Figure 1, which consists of an ocean domain, denoted by subscript 'o', and a land domain, denoted by subscript 'l'. Actually, how many dimensions do we have? We don't resolve neither the x- nor y-direction explicitly. In this sense, we would have only the time-dimension, but we aren't even interested in time evolution as such but only the equilibrium. On the other hand, we implicitly assume the existence of spatial dimensions x and y by placing the boxes in a specific way. And the model extent in x-direction is even explicitly in the equations with L... Each of the two domains contains a ground box at the bottom and an atmospheric box aloft. However, this spatial arrangement is only relevant in so far as we are interested in the water fluxes across the boundaries connecting any two boxes. We assume that all fluxes can be expressed as functions of the mean moisture states of the model boxes so that an explicit dependence on the spatial variables (x, y) is obsolete. This choice trades some realism for the ease of working with ordinary differential equations (ODEs) instead of partial differential equations (PDEs).

The mean moisture states of the model boxes represent their water content. For atmospheric boxes, we use the mean integrated water vapour pass w in mm, and for the land box the unitless mean relative soil moisture saturation s to describe the state. As the ocean is considered fully saturated at all times, the influence of the ocean state on fluxes is constant in time and can be prescribed in the form of a parameter. This means that the full information on the moisture state of the model at any given moment in time t is given by the set of state variables $\{w_o(t), w_l(t), s(t)\}$.

We limit the modelled water exchange between boxes to the following four flux types, denoted by arrows in Fig. 1: Evapotranspiration E from ground to atmosphere, precipitation P from atmosphere to ground, advection A between the atmospheric boxes and runoff R from land to ocean. Expressions for these fluxes as functions of the state variables are provided in Section d.

It is important to note that we assume the model to have closed boundaries at the top of the atmosphere and the bottom of the ground boxes, while periodic boundary conditions are used in horizontal direction. This is, the model topologically resembles the walls of a cylinder and the right boundary of the land domain connects to the left boundary of the ocean domain. A constant mean background wind is introduced to facilitate advection and gives the atmospheric moisture transport a fixed directionality. Motivated by a net easterly wind in the Tropics, ...?

Lastly, the relative size of the ocean and land domain is set by the land fraction parameter α. The spatial extent of the land in x-direction is given by αL, where L denotes the full model length. Conversely, the ocean has a horizontal extent of $(1-\alpha)L$.

c. Water balance equations

To a good approximation, the total amount of water is conserved within the tropical band. If we further assume that the mean water holding capacity of the atmosphere does not vary significantly over time, we can apply these properties of the tropics to our model and formulate a set of coupled differential equations that describe the rate of change of the water content in each of our model boxes. Maybe, we don't want to make a clear reference to the Tropics at this point. In this case, this paragraph can be reformulated in a more neutral way, where water conservation and constant water holding capacity are just general assumptions. As we assume the moisture state of the ocean to be constant in time, the number of equations reduces by one and we are left with the following expressions for the changes in soil moisture saturation and land and ocean mean water vapour passes:

$$\frac{ds}{dt} = \frac{1}{nz_\mathrm{r}}\left[P(w_\mathrm{l}) - R(s,w_\mathrm{l}) - E(s)\right] \tag{1}$$

$$\frac{dw_\mathrm{l}}{dt} = E(s) - P(w_\mathrm{l}) + A_\mathrm{l}(w_\mathrm{l}, w_\mathrm{o}) \tag{2}$$

$$\frac{dw_\mathrm{o}}{dt} = e_\mathrm{o} - P(w_\mathrm{o}) + A_\mathrm{o}(w_\mathrm{l}, w_\mathrm{o}). \tag{3}$$

The water transfer terms P, R, E, e_o, A_l and A_o are expressed as mean fluxes in mm/day. The advection terms A_l and A_o refer to *net* advection rate into the land and ocean atmosphere, respectively, and are positive for a net moisture import and negative for net moisture export. Ocean evaporation rate e_o in mm/day, dimensionless soil porosity n and hydrologically active soil depth z_r in mm are constant parameters of the system.

d. Parametrizations

While the conservation of water is a rather fundamental condition, there are no simple, fundamental laws governing the water fluxes between the model boxes. Instead, we need to draw inspiration from existing literature that provides empirical relationships between the flux quantities and our model state variables.

Bretherton et al. (2004) provide such an empirical parametrization for oceanic, tropical precipitation rate in mm/day as a function of the mean water vapor pass,

$$P(w) = \exp\left[a\left(\frac{w}{w_\mathrm{sat}} - b\right)\right]. \tag{4}$$

The parametrization introduces three parameters, two empirical dimensionless parameters $a \approx 15.6$ and $b \approx 0.6$ and the saturated water vapor pass w_sat in mm. Lacking a corresponding expression for tropical land regions, we will make the explicit assumption that the oceanic precipitation formulation can also be applied to land atmospheres. This assumption has major implications for the results presented in Section 4 as will be discussed in greater detail later. It might be worth it to spend a day or two looking at ERA5 data over(tropical) land and check if we see at least a similar relationship between water vapor pass and precipitation. In the end, this assumption is critical to our conclusion about PR<1 in the closed model. If there is absolutely no correlation between P and w over land, then we would have to think a lot harder about how to sell this. If I was the reviewer, I would probably pick on this and ask if we did a sanity check before making this assumption.. Furthermore, the same saturation water vapor pass is assumed over land and over ocean, implying similar energetic conditions across the entire model domain.

Runoff gets parametrized as the fraction R_f of precipitation that does not infiltrate the soil but returns to the ocean in the form of surface or sub-surface currents. This approach was, for instance, used in Rodriguez-Iturbe et al. (1991). The runoff fraction,

$$R_\mathrm{f}(s) = \epsilon s^r, \tag{5}$$

contains two empirical dimensionless parameters $\epsilon \approx 1$ and $r \approx 2$. It tells us that runoff intensifies as the soil moistens. The complete expression for the runoff rate reads

$$R(s, w_\mathrm{l}) = R_\mathrm{f}(s) P(w_\mathrm{l}), \tag{6}$$

but it proves to be convenient to combine precipitation and runoff in Eqn. (1) to $P(w_\mathrm{l}) - R(s, w_\mathrm{l}) = P(w_\mathrm{l})\Phi(s)$, where we introduced the infiltration function $\Phi(s) = 1 - R_\mathrm{f} = 1 - \epsilon s^r$. Note, that this parametrisation assumes that runoff discharge happens uniformly across the land domain and that its water does not participate in any secondary processes that could moisten the soil.

The qualitative dependence of evapotranspiration (ET) on soil moisture saturation is long-known and was first introduced by Budyko (1956). ET is close to zero for soil moisture saturation values below the permanent wilting point, $s < s_\mathrm{pwp}$, increases approximately linearly in a transition range between the permanent wilting point and a critical value close to the field capacity, $s_\mathrm{pwp} < s < s_\mathrm{fc}$ and reaches a plateau for higher s-values, $s > s_\mathrm{fc}$, where evapotranspiration is nearly constant. Is it ok to write it like this or should I rather explain that it is no longer water-limited beyond s_fc? ...Because technically, a higher temperature could lead to higher evapotranspiration. We just don't model this energy-relationship. For computational convenience, we parametrize this sometimes piecewise defined behaviour by the following smooth equivalent:

$$E(s) = \frac{E_{\mathrm{p}}}{2}\left[\tanh\left(10\left(s - \frac{s_{\mathrm{pwp}} + s_{\mathrm{fc}}}{2}\right)\right) + 1\right]. \tag{7}$$

The potential evapotranspiration E_{p} signifies the value of the plateau beyond s_{fc}. This parametrization implies that the entire land box is either covered by a single vegetation type or that a combination of vegetation types can be modelled by means of an effective mean value of s_{pwp}, s_{fc} and E_{p}. Furthermore, the model does not consider energy conservation, so that an even higher evapotranspiration beyond E_{p} due to an enhanced radiative energy input is precluded by design.

It remains to find expressions for the *mean net* advection rates into the land and ocean atmospheres, hereafter mean land/ocean advection rates. The net total advection flux into a given box is the difference between the moisture entering and leaving the box per unit time. It can be written as the windward boundary water vapour pass times wind speed minus the analogous term at the leeward boundary of the box,

$$A_{\mathrm{tot}} = (w_{\mathrm{in}} - w_{\mathrm{out}})u. \tag{8}$$

The sketch in the bottom panel of Figure 1 illustrates the assumed water vapour pass distribution across the full model domain. Since we only have two boxes and periodic boundary conditions, the total net advection rate A_{tot} into the land and ocean atmospheres are identical in magnitude but with opposite signs. If the ocean has a net advective outflux, then the land atmosphere gains this moisture as net advective influx. By applying Eqn. (8) to the w-distribution in Figure [***] for the land and ocean atmosphere boxes, respectively, and translating the obtained total net advection fluxes into *mean* advection rates per unit land/ocean length, we obtain

$$A_{\mathrm{l}} = \frac{(w_{\mathrm{o}} - w_{\mathrm{l}})u}{\alpha L} \tag{9}$$

and

$$A_{\mathrm{o}} = -\frac{(w_{\mathrm{o}} - w_{\mathrm{l}})u}{(1-\alpha)L}, \tag{10}$$

where α and L are the land fraction and full model length, respectively, as introduced in Section b.

TABLE 1. Parameter ranges for closed model Monte Carlo simulations with uniform sampling.

Parameter	Minimum	Maximum	Range choice motivated by
s_{pwp}	0.2	0.54	Hagemann and Stacke (2015)
s_{fc}	0.5	0.84	Hagemann and Stacke (2015)
e_{p} [mm/day]	4.1	4.5	Rodriguez-Iturbe et al. (1991)
nZr [mm]	90.0	110.0	Rodriguez-Iturbe et al. (1991)
e_{o} [mm/day]	2.8	3.2	C. Hohenegger, private communications
ϵ	0.9	1.1	Rodriguez-Iturbe et al. (1991)
r	2.0	2.0	fixed due to computational method, Rodriguez-Iturbe et al. (1991)
a	11.4	15.6	Bretherton et al. (2004)
b	0.522	0.603	Bretherton et al. (2004)
w_{sat} [mm]	65.0	80.0	Bretherton et al. (2004)
α	0.0	1.0	full possible range
u [m/s]	1.0	10.0	needs more research/thoughts
L [km]	1000.0	40000.0	needs more research/thoughts
$\tau = u/L$ [day^{-1}]	0.00216	0.864	computed from extreme u and L

With these parametrizations, the model has a total of 14 free parameters which we can reduce to 12 if we treat nz_{r} in mm as one combined parameter and $\tau = u/L$ in day^{-1} as a characteristic rate of atmospheric transport. Table 1 provides sensible ranges for the 12 parameters. These ranges are used to constrain the precipitation ratio across the parameter space and test the sensitivity of the model results to parameter variations.

3. Evaluation methods

In this study, our primary interest is directed at equilibrium states of the model and the magnitude and ratio of different flux quantities that result from the equilibrium values of $\{s, w_{\text{l}}, w_{\text{o}}\}$. Moreover, we want to know how sensitive the ratio between land and ocean precipitation rates is to a variation of the model parameters. In the following, we outline the analysis methods that are employed to address these questions.

a. Equilibrium states

Mathematically, equilibrium states are fixed points of the system of coupled ODEs presented in Eqns. (1) to (3), i.e. the set of state variables $\{s, w_{\text{l}}, w_{\text{o}}\}$ for which all three time-derivatives are zero. Due to the nonlinear complexity of the equations, no analytical solution exists and we need to resort to numerical methods. A number of numerical algorithms exists to find fixed points of systems of ODEs. Our code is written in `Julia` (Bezanson et al. (2012)) and we use the `JuliaDynamics.jl` library (Datseris (2018)) to find all roots of the model equations along with the information whether each root represents a stable or unstable fixed point of the system. The advantage of this approach over other solution strategies such as, for example, sufficiently long time evolution, is the independence of the result on initial conditions and that we are guaranteed to find all equilibrium states within a defined range of possible state variable values. We define this range as $s \times w_l \times w_o = [0.0, 1.0] \times [0.0, w_{sat}] \times [0.0, w_{sat}]$. It turns out that our model has only one equilibrium state for any given set of parameters. With the equilibrium values for s, w_l and w_o and corresponding parameter choices at hand, we can compute all fluxes and flux ratios of interest using the parametrizations introduced in Section 2.d.

b. Scanning the parameter space

Adopting an agnostic view on the plausibility of each combination of parameter values from the ranges given in Tab. 1, we are confronted with a uniformly sampled, 12-dimensional parameter space. To answer the question of how sensitive the equilibrium state is to a variation of the model parameters, we need to scan the full parameter space. In order to minimize computational costs and systematic biases when solving for "all" possible combinations of parameter values, we sample the parameter space randomly by running n model simulations, each of which is composed of three steps:

1. Compute a random set of parameter values, assuming the parameter ranges from Tab. 1 with a uniform distribution of values within each range.
2. Find the fixed point for the system using the random parameter set from step 1.
3. Compute flux quantities from the obtained equilibrium state and store them together with the corresponding parameter values in a dataset.

c. Scatter plot analysis

Having obtained a sufficiently large dataset from scanning the parameter space, the sensitivity of a computed quantity Q such as the precipitation ratio to a given parameter p can be visually evaluated with scatter plots. With p on one axis and Q on the other, each of the n simulations can be represented by one data point in the scatter plot. A random distribution of data points across the entire p-range indicates insensitivity of Q to a variation in p. In this case, the choice of a certain p-value has no predictive power for the value of Q. In contrast, a scatter plot distribution where data points cluster in a non-uniform way points to a stronger sensitivity. Various degrees of sensitivity exist, ranging from a slight trend of Q across the range of p-values with considerable scatter for weak sensitivity to a clear, nearly functional dependency $Q(p)$ with narrow scatter range in the case of strong sensitivity. In either case, the influence of all other parameter variations combined determines the spread of the data points around some mean value of Q at any given value of p.

4. Closed model results

The results presented in this section are based on the data of 10000 simulations of the closed model (CM) which randomly sampled the parameter space as explained in Section 3.b, each yielding the equilibrium solution for a unique point in the parameter space provided in Table 1. The obtained dataset will henceforth be referred to as "CM data". The section is organised in two parts. First, we discuss basic features of the model and their implications for the partitioning of precipitation between land and ocean. Second, we examine to which parameters the precipitation ratio is most sensitive and which physical arguments explain the individual relationships.

a. Basic model behaviour

From a first visual inspection, it is clear that the equilibrium states and resulting equilibrium mean water fluxes P_l, P_o, E_l, R, A_l and A_o, show a strong dependence on the choice of land fraction α. It is therefore instructive to discuss basic features of the model output with the help of scatter plots of the water fluxes over α. These plots are provided in Figure 2. Similar figures that show the dependence on other parameters are provided in appendix [***]. Note, that the ocean advection rate A_o has a negative value in all runs and is therefore multiplied by -1 in order to obtain the absolute values which are more easily compared to the other fluxes.

All mean fluxes are functions of the equilibrium solutions to Eqns. (1) - (3) and therefore depend implicitly on the choice of parameter values. Expressions for the fluxes as explicit, analytical functions of α or other parameters are cumbersome to find or may not exist. We therefore explain the observed features with qualitative, physical arguments rather than with mathematical rigor.

To begin with, Figure 2 shows that all equilibrium mean fluxes lie in the range $[0, e_{\mathrm{o}}]$, with $e_{\mathrm{o}} \approx 3$ mm/day. With the exception of $-A_{\mathrm{o}}$, maximum values are attained for $\alpha \to 0$ and the fluxes decrease monotonically but in nonlinear ways with increasing land fraction. To understand these general features and draw first conclusions for the partitioning of precipitation between land and ocean, two observations about the mean land advection rate A_{l} (bottom left panel) and runoff rate R (middle right panel) are key:

1. Land advection is positive (ocean advection is negative) for all equilibrium states.

2. The mean land advection rate is identical to mean runoff rate, i.e. $A_{\mathrm{l}} = R$.

The first observation implies a clear directionality of the atmospheric water transport for the system in equilibrium. Moisture is supplied *by* the ocean atmosphere *to* the land atmosphere. This directionality of advection sets the upper limit of the precipitation ratio in the following way: From Eq. (9) follows that a positive mean land advection rate requires the ocean atmosphere to be moister than the land atmosphere, i.e. $w_{\mathrm{o}} > w_{\mathrm{l}}$. As we assume that the same parametrization holds for precipitation over ocean and land, it further follows that $P_{\mathrm{o}} > P_{\mathrm{l}}$ and, consequently,

$$PR = \frac{P_{\mathrm{l}}}{P_{\mathrm{o}}} < 1. \tag{11}$$

Should a discussion of possible pathways for $PR > 1$ come already here or later?

The second observation helps to explain why the unidirectionality of moisture transport from ocean to land exists. In order to sustain a constant, nonzero equilibrium soil moisture value, $s > 0$, land precipitation needs to balance the water loss of the soil through evapotranspiration and runoff. While the amount of precipitation that is turned into evapotranspiration resides in a self-sustaining recycling loop between land atmosphere and soil, runoff is irretrievably lost to the ocean. Its share in the precipitation balance needs to be supplied to the land atmosphere in the form of advection,

$$P_{\mathrm{l}} = E_{\mathrm{l}} + R = E_{\mathrm{l}} + A_{\mathrm{l}}. \tag{12}$$

If we imagine a system without advection, e.g. because the land and ocean atmosphere were separated by an impenetrable barrier, runoff would continuously reduce the soil moisture saturation and with it the evapotranspiration and precipitation fluxes. Eventually, the system would attain the trivial equilibrium solution $\{s = 0, w_l = 0\}$. On the ocean side of this hypothetical system, equilibrium conditions would be rather moist with w_o such that $P_o(w_o) = e_o$. We conclude that it is the fundamental property of land to lose water in the form of runoff that requires an atmospheric moisture flux from ocean to land for any nontrivial equilibrium solution.

Based on these insights, we can also understand why no individual water flux can exceed the value of the ocean evaporation. Ocean precipitation needs to be smaller than e_o because some of the evaporated water gets advected by the land atmosphere and is no longer available for precipitation. Over land, we already established that $P_l < P_o$ with the consequence that $P_l < e_o$. The land precipitation is partitioned into E_l and R so that each of these two fluxes must be smaller than e_o. Land advection is constrained by $A_l = P_l - E_l < e_o$ and ocean advection is limited to $A_o \lesssim e_o$ as the ocean atmosphere can only export as much moisture as it receives from the ocean surface minus the amount that precipitates. At the same time, A_o cannot attain e_o as the basic requirement for advection is $w_o > w_l > 0$ which comes along with nonzero ocean precipitation.

As we increase the land fraction from $\alpha = 0$ to $\alpha = 1$, all fluxes except A_o decrease in magnitude. The dependence of our system on α enters our model equations through the mean advection rates A_l and A_o. The same amount of exchanged water per unit time $(w_o - w_l)u$, that solely depends on the atmospheric moisture contents and wind speed, translates to an amount per time and unit length for the ocean with factor $1/((1-\alpha)L)$ and for land with factor $1/(\alpha L)$. Combining Eqns. (2) and (3) under the equilibrium assumption of vanishing time derivatives, we can formulate the equilibrium condition,

$$e_o = P_o(w_o) + \frac{\alpha}{1-\alpha} \underbrace{[P_l(w_l) - E_l(s)]}_{R(s)}. \tag{13}$$

Equation (13) tells us, that the constant ocean evaporation rate needs to balance the sum of ocean precipitation rate and the difference between land precipitation and ET that is multiplied by an α-dependent term. This term, $\alpha/(1-\alpha)$, goes to zero for $\alpha \to 0$ and increases monotonically until it diverges to infinity for $\alpha \to 1$. As α increases, the equilibrium state needs to adjust by either decreasing w_o, w_l or s. However, due to the coupling between all three state variables, a decrease in only one of the state variables does not result in a new equilibrium state. Instead, all state variables have to decrease together so that the new equilibrium state is dryer in all boxes (except the ocean).

We can also understand this more intuitively: Despite the constant mean evaporation rate of the ocean, the total moisture input from the ocean is reduced as the land surface increases and the ocean surface shrinks. A lesser amount of water is available to sustain the soil moisture value of a larger land box. Consequently, starting from a rather moist state when the ocean and its total water input into the atmosphere are large, the entire system undergoes drying with increasing land fraction. This process terminates when the entire model domain is covered by land ($\alpha = 1$). Just before this point, when α is close to 1, a tiny ocean atmosphere exports almost the entire moisture that gets evaporated from the ocean surface, $(1-\alpha)L\,|A_\mathrm{o}| \lesssim e_\mathrm{o}$, but this amount is just sufficient to keep the large land atmosphere at a moisture value $w_\mathrm{l} \gtrsim 0$ so that the resulting land precipitation stabilises the land at a very small soil moisture value $s \gtrsim 0$.

Do you think I should say anything about the specific shape of the relationships in Fig. 2? I don't have very satisfying, easy physical reasons to explain them. It seems to me like the shapes are simply a result of the interplay of the different nonlinear parametrisations that we use. I could probably reason about them in a similar style as the discussion about the upper limit e_o for the fluxes, i.e. by showing an equation and then discussing what needs to happen if we increase α by a bit in different α regimes. But this seems very boring and pointless to me.

b. Parameter sensitivity of PR

Building on the preceding general description of the model behavior, we now draw our attention to the sensitivity of the precipitation ratio with respect to a variation of different model parameters. Three parameters stand out in having a particularly strong impact on PR: Land fraction α, atmospheric moisture transport parameter τ and permanent wilting point s_pwp. We discuss the underlying relationships using the same CM data as before.

Land fraction α: Figure 3 shows a scatter plot of PR values over α. Despite considerable spread in PR, we can see that $PR \to 1$ for both limits, $\alpha \to 0$ and $\alpha \to 1$. This reflects very similar moisture conditions in the two atmospheres when α is extreme. Knowing that $w_\mathrm{o} > w_\mathrm{l}$ for all equilibrium states, it follows that PR will only decrease if $\Delta w = w_\mathrm{o} - w_\mathrm{l}$ increases. As has been discussed in the preceding section, the system's equilibrium states for a tiny land domain are relatively moist. For $\alpha \to 0$, a large Δw cannot be sustained since the resulting advection amount $\Delta w u$ would translate to a large land advection rate, $\Delta w u/(\alpha L)$, that would immediately moisten

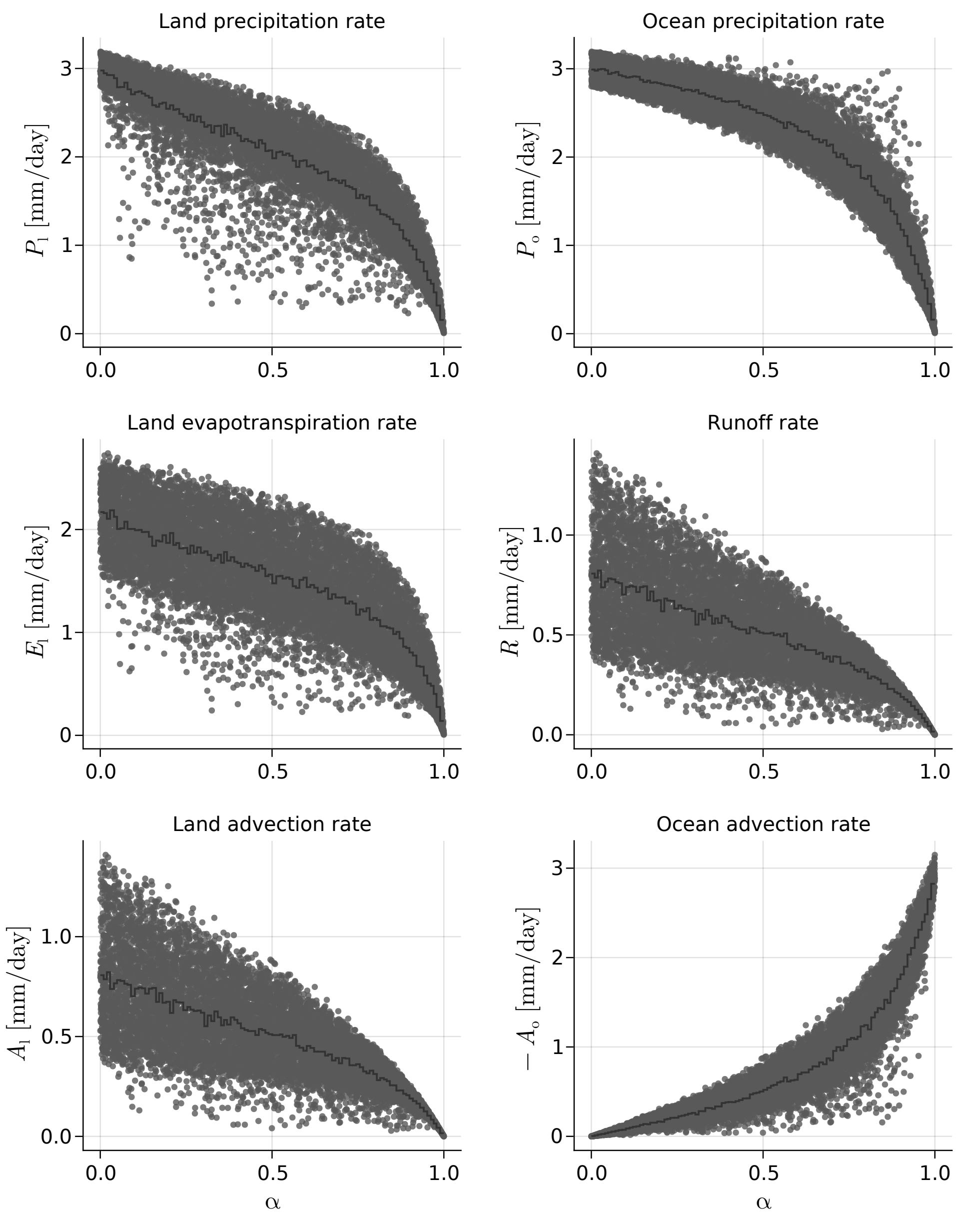

FIG. 2. Mean water fluxes computed from the equilibrium states of 10000 closed model runs with randomly sampled parameter values and plotted over land fraction α. The dark grey line shows the mean values of bins of 100 consecutive α-values. The negative ocean advection rate A_o reflects a net transport of water out of the ocean and into the land atmosphere. Multiplication by -1 simplifies the comparison of its magnitude with the other flux quantities.

the land atmosphere and assimilate w_o and w_l. On the other end of the range, when the ocean is tiny, i.e. $\alpha \to 1$, large moisture differences are likewise impossible: This time, Δw is limited by the total amount of water that enters the system through the ocean surface. The ocean atmosphere cannot export more water than it receives. Therefore, the total amount of evaporated water sets the upper limit for advection, $\Delta w\, u < (1-\alpha) L\, e_o$. This amount decreases with increasing α, so that Δw needs to decrease with it. Moreover, Δw needs to stay below this limit since the ocean atmosphere has to stay moister than the land atmosphere to facilitate advection in the first place.

Along the mid-α range, PR decreases until it reaches a minimum beyond which the ratio increases again. This behaviour is somewhat concealed by the large spread in PR for intermediate land fractions but is both visible in the means of bins of 100 consecutive α values (dark grey line) and in graphs for which all parameters except α were kept fixed (not shown). A mathematically rigorous analysis of $PR(\alpha)$ in this range and, in particular, the location of the minimum is difficult due to the lack of an analytical expression for the relationship between precipitation ratio and land fraction. We can write,

$$PR(\alpha) = \frac{P_l(\alpha)}{P_o(\alpha)} = \frac{E_l(s) + \frac{(w_o - w_l)u}{\alpha L}}{e_o - \frac{(w_o - w_l)u}{(1-\alpha)L}}, \tag{14}$$

but we may not overlook the fact that our state variables are implicit functions of α, too, i.e. $s(\alpha)$, $w_o(\alpha)$ and $w_l(\alpha)$. Even though we don't know the analytical form of these state variable dependencies, Eqn. (3) gives a useful indication of why the precipitation ratio should decrease for small but increasing α and why it should increase again as α approaches one. This indication lies in the factors $f = 1/\alpha$ and $g = 1/(1-\alpha)$ in the land and ocean advection rates, respectively. Assuming that the system resides in an equilibrium state for some α close to zero, a small increase in α would lead to a rather strong drop in the land advection rate (strong negative slope of f at low α) compared to the rather mild increase in the magnitude of ocean advection (weakly positive slope of g at low α)...

I stopped here because I wondered if it makes sense to explain the shape of $PR(\alpha)$ in such great detail. Maybe all this could be described in a much simpler way by starting from total moisture input rather than mean rates. The argument would go something like this: increasing land = generally less water available to the circulation in the system. Consequently, the moisture state as a whole must become drier, i.e. all state variables decrease but at different rates. Land precip (and with it w_l) decrease both trough a reduction of E_l and a rather sharp drop in A_l due to factor

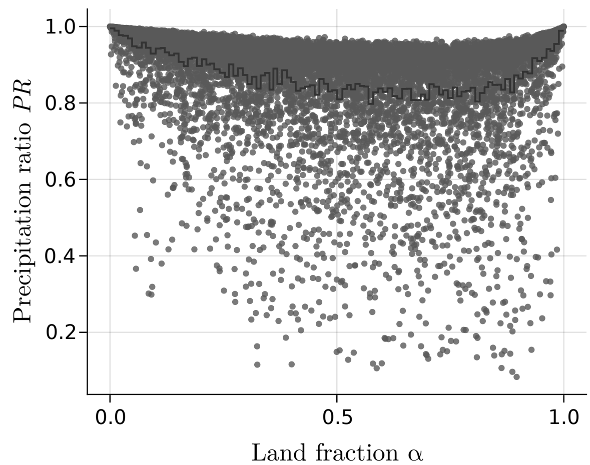

FIG. 3. Smile plot

f. Ocean precip only decreases by slight increase of $-A_o$. For large α the system is already in a rather dry state. E_l decreases only slightly with decreasing s and impact of f is less strong. For ocean precip, the opposite is true. Here, g plays a stronger role now and increases the ocean advection rate strongly. In the end, the interplay of the different nonlinear parametrisations make the behaviour of PR asymmetric around $\alpha = 0.5$ and hard to understand in detail.

Atmospheric rate of transport τ: The ratio between mean horizontal wind speed and spatial extent of the model, $\tau = u/L$, is a measure for the efficiency with which moisture is transported across the model atmosphere. Its inverse value, τ^{-1}, corresponds to the time that an air parcel would need to travel across the full domain length L. In the advection terms of Eqn. (2) and (3), τ appears as the rate at which moisture is moved across the boundaries between the two atmospheric boxes. It has therefore major implications for the ability of advection to assimilate the moisture conditions over ocean and land. A very small value of τ, i.e. a low rate of transport, corresponds to a combination of large domain size and low wind speed while a small domain and strong wind result in a very large value of τ. Assuming a fixed land fraction α, a larger moisture difference Δw is needed to move the same total amount of water across a box boundaries when the rate of transport is small, compared to when it is large. Except for the special cases of extreme land fractions, $\alpha \to \{0, 1\}$, where α enforces very similar moisture conditions over land and ocean, it is primarily τ that sets the moisture difference which is needed to attain the equilibrium state. This dominant role is illustrated in Figure 4 which shows the scatter plot of precipitation ratio over τ.

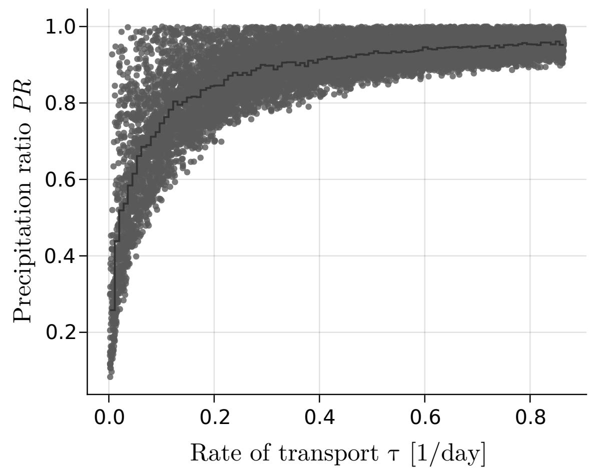

FIG. 4. τ-dependence

While we already assessed that α sets the overall upper limit of PR, Fig. 4 shows that τ sets the overall lower bound. It explains the large spread for PR values in the mid-α range in Figure 3, where the efficiency of atmospheric moisture transport is particularly important. Only high rates of transport enable the system to attain an equilibrium state with rather similar moisture conditions over land and ocean. For instance, if $\tau > 0.4\,\mathrm{day}^{-1}$, then PR stays above 0.8 regardless of the choice of values for other parameters. Note, that τ combines the information about both wind and spatial extent of the model. If one fixes one of the two, e.g. $L = 40000\,\mathrm{km}$ to simulate the full Tropics along the equator, the physically sensible range of τ is limited. For example, in order to obtain a rate of transport larger than $0.4\,\mathrm{day}^{-1}$, such a large L would require a minimum wind speed of 185 m/s, a value that lies beyond the highest wind speed ever measured on Earth. More realistic mean wind speed values for such a large domain could lie around 5 to 10 m/s (Needs to be checked! Maybe by looking at ERA5 data?) with corresponding rates of transport, $\tau \approx 0.01 - 0.02$. At these low values of τ, the spread of PR values is considerable which means that also other parameters have a substantial influence on the attained equilibrium state.

Permanent wilting point $s_{\mathbf{pwp}}$: It takes work to extract water from the soil and the drier the soil, the more work is needed to facilitate evapotranspiration. Regardsless of whether the land surface is bare or covered with vegetation, s_{pwp} is a characteristic property of the soil type which denotes the relative soil moisture saturation value below which practically no water can be extracted. The left panel of Figure 5 shows the parametrization function of evapotranspiration, $E_l(s)$, for different choices of the permanent wilting point. For instance, $s_{\mathrm{pwp}} = 0.3$ might correspond to ***** and $s_{\mathrm{pwp}} = 0.5$ to ***** (Hagemann and Stacke (2015)) The asterixes are place holders for soil types which I have to look up when I am back in the office. In the evapotranspiration graphs, s_{pwp} determines the soil moisture value at which the curve transitions from $E_{\mathrm{l}} \approx 0$ to the regime of steeply increasing E_{l}. Since the field capacity s_{fc} lies $\Delta s = 0.3$ higher than s_{pwp} for all relevant soil types, a change in s_{pwp} merely shifts the evapotranspiration graph along the s-direction, while its shape remains unchanged.

Figure 6 shows a negative trend of the precipitation ratio with increasing s_{pwp} for the performed model runs. The impact of soil type on the precipitation ratio is weaker than, for example, the impact of τ but it is nonetheless clearly visible and s_{pwp} represents the third most sensitive model parameter. To understand the dependence of PR on s_{pwp}, it is convenient to think of a system in equilibrium for some permanent wilting point, e.g. $s_{\mathrm{pwp}} = 0.3$. The mean equilibrium soil moisture value in the CM data for $s_{\mathrm{pwp}} = 0.3$ is $s = 0.43$. This initial state of the model is displayed as a blue dot in Figure 5. An abrupt increase of s_{pwp} to $s_{\mathrm{pwp}} = 0.4$ leads to a significant drop of E_{l} as illustrated by the first red arrow connecting the blue and green dot in the left panel of Fig. 5. The green dot represents a temporary state where the model is not in equilibrium because the state variables have not yet adapted to the new situation. At this point, the soil receives the same amount of precipitation but loses less water through evapotranspiration. As a result, the soil moistens. As time progresses, the system attains a new equilibrium state at a higher s value which is marked by the orange dot. This moistening of the soil is shown in the right panel of Fig. 5, where the equilibrium s values of the CM data are plotted over the corresponding values of s_{pwp}. However, as s increases, runoff and land advection rate increase, too. Assuming that $\tau/(\alpha L)$ is kept fixed, Δw has to increase to facilitate the increase of advection. The water that is supplied to the land atmosphere as advection is taken from the ocean atmosphere, where w_{o} decreases as a consequence. Hence, an increase in advection is only possible, if w_{l} decreases more strongly than w_{o}. The increase in R combined with a decrease in P_{l} is the reason why the new equilibrium state for $s_{\mathrm{pwp}} = 0.4$ will have a moister soil but a lower evapotranspiration rate than the initial state for $s_{\mathrm{pwp}} = 0.3$. The fact that w_{l} must decrease more strongly than w_{o} in the adaptation process is the reason why PR declines with increasing s_{pwp}. Actually, I am not so sure about this from a rigorously mathematical point of view. I tried to proof that a more strongly decreasing w_{l} compared

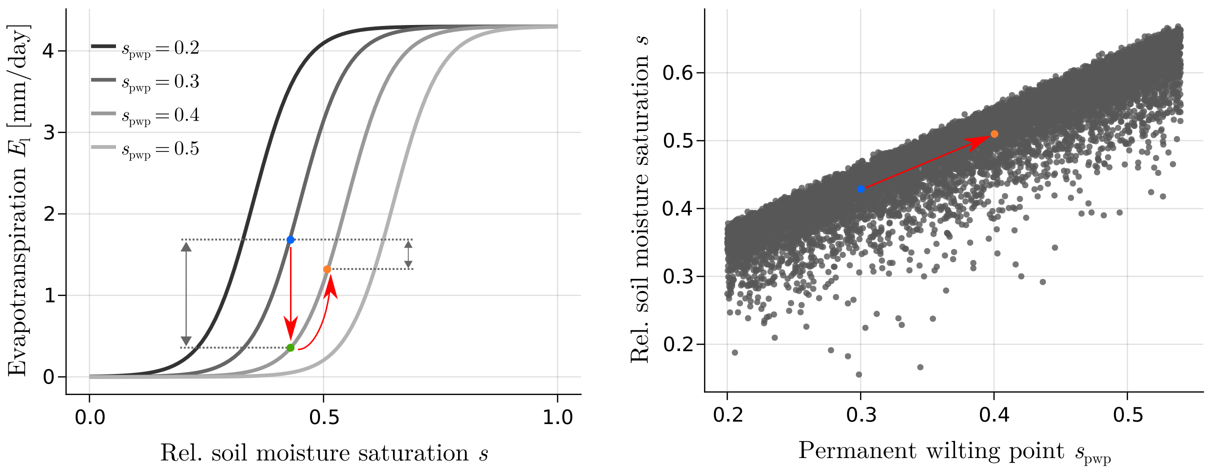

FIG. 5. Influence of an increase in s_{pwp} on the equilibrium state. Left: Higher values of s_{pwp} shift the graph of the E_l parametrization towards larger s. Right: Equilibrium values of the soil moisture saturation from CM data plotted over s_{pwp} values. In left panel, next to left black arrows will stand something like ΔE_{inst} for instantaneous ET-difference and next to the right black arrows $\Delta E_{\mathrm{final}}$ to denote the ET difference to the final, new equilibrium state.

to w_o also leads to a more strongly decreasing $P(w_l)$ compared to $P(w_o)$ but it is hard. In fact, one ends up with the condition,

$$\underbrace{\frac{P(w_o)}{P(w_l)}}_{>1} dw_o < dw_l,$$

which needs to be fulfilled for PR to always decrease with increasing Δw.

5. Open model formulation

The closed model discussed so far can be applied to any system for which the total net advection is zero. Such conditions might be met in the real world when we look at very large scales, e.g. global domains such as the tropical band. However, in the case of more local, small scale phenomena, the net advection might not be zero and the situation is better captured by an open model configuration, where moisture inflow at the windward model boundary is a model parameter and no constraints apply to the moisture outflow at the leeward boundary. In this model configuration, the modelled domain can have a net advection larger or smaller than zero. In the following, we present the formalism and analysis of an open model with two oceanic domains and an island inbetween them.

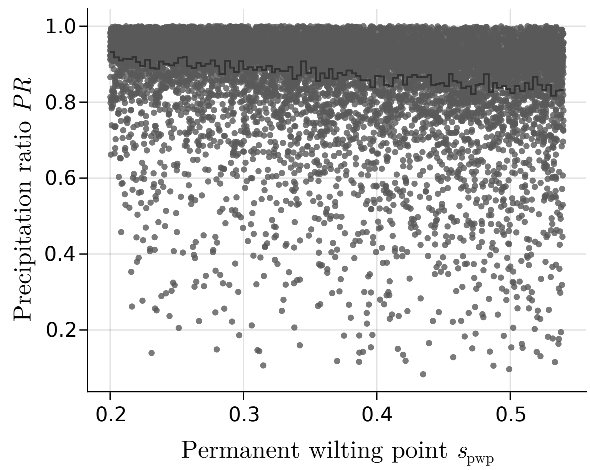

FIG. 6. s_{pwp}-dependence

a. Open model equations

The model equations for an open configuration are similar to the ones for the closed model. This time, four instead of three equations are needed as the system has now one more ocean domain. The meaning of the soil moisture variable s is unchanged, while a different notation is employed for the water content of the atmospheric boxes. The index $i = 1,2,3$ is used to denote the mean integrated water vapour pass w_i and net advection rate A_i of the first ocean atmosphere ($i = 1$), land atmosphere ($i = 2$) and second ocean atmosphere ($i = 3$), respectively. With this, the model equations read

$$\frac{ds}{dt} = \frac{1}{nz_{\mathrm{r}}}\left[P(w_2) - R(s,w_2) - E(s)\right] \tag{15}$$

$$\frac{dw_1}{dt} = e_{\mathrm{o}} - P(w_1) + A_1 \tag{16}$$

$$\frac{dw_2}{dt} = E(s) - P(w_2) + A_2 \tag{17}$$

$$\frac{dw_3}{dt} = e_{\mathrm{o}} - P(w_3) + A_3, \tag{18}$$

with

$$A_{\mathrm{i}} = \frac{(w_{\mathrm{i-1}} - w_{\mathrm{i}})u}{L_{\mathrm{i}}}. \tag{19}$$

Note, that a new parameter w_0 was introduced which denotes the boundary condition of the water vapor pass at the windward end of the model domain. It reflects the synoptic scale? conditions which the model is embedded in.

b. Open model results

- How the open model relaxes the condition that PR<1 (PR>1 only under certain conditions)
- The role of synoptic moisture conditions in the atmosphere
- Transforming the open model into the closed model

6. Discussion and summary

- Which conditions need to be met to end up with a precipitation ratio larger one?
- What are possible use cases for the models?
- What can the model(s) tell us and what not and why? (e.g. land distribution not representative for the Tropics)

Acknowledgments.

Data availability statement.

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