Baby Rudin Chapter 1 Notes

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September 2020

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1 Motivation for Writing Notes

The reason I'm writing these notes is to better understand certain aspects of Rudin's book on real analysis. By writing things in my own words and going through concepts that I don't fully understand in a systemic fashion, I'm confident my understanding of the material will be much deeper by the time I'm done reading this book.

2 Introduction

Theorem 1. *The number $\sqrt{2}$ is irational.*

Proof. By definition, $\sqrt{2}$ is the number p that satisfies the equation:

$$p^2 = 2$$

Assume that p is rational, so it can be written as an irreducable quotient of integers $p = m/n$. It follows that:

$$m^2 = 2n^2$$

This implies that m^2 is even, so m must be even (an odd times and odd is equal to an odd). Let us write m as $2a$. We thus have:

$$4a^2 = 2n^2 \Rightarrow n^2 = 2a^2$$

it follows that n is also even. Let us write n as $2b$. Thus, we have:

$$p = \frac{m}{n} = \frac{2a}{2b} = \frac{a}{b}$$

Thus, m and n do not form a irreducable fraction, contrary to our initial assumption. This implies that p cannot be written as such a fraction and is thus irrational. ∎

Remark 1. *There is a decent bit to be said about how Rudin went about choosing the formula used to demonstrate that the sets of rationals with $p^2 < 2$ and $p^2 > 2$ have no greatest and smallest element. However, I'll probably circle back and fill this section in when I have two to draw some graphs to really build visual intuition.*

2.1 The Real Field

Theorem 2. *If $x,\ y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that:*

$$nx > y$$

Remark 2. *The idea behind this proof is to derive a contradiction by showing that some subset of the reals with an upper bound has no least upper bound, as we are only working with that assumption.*

Proof. Assume that there existed some pair (x, y) such that there existed no positive integer n such that $nx > y$. It follows that for any integer n:

$$nx \ \leq \ y$$

It follows that y is an upper bound of the set of all nx, for integer n. By definition of $\mathbb{R}$, this set has a least upper bound. Let m be this least-upper bound:

$$nx \ \leq \ m$$

Consider the number $m - x$. There must be some $nx > m - x$, or else $m - x$ would be a lower upper bound than m oon the set of all nx. It follows that $(n + 1)x > m$. Since $n + 1$ is an integer, this is a contradiction. Thus, no such m exists. It follows that there must exist some n such that $nx > y$, for the pair x and y. ■

Theorem 3. *If $x \in \mathbb{R}$ and $y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.*

Remark 3. *This proof seemed significantly more difficult than the previous one, so I will simply be making remarks on Rudin's proof rather than attempt it myself. The general idea behind this proof is that we wish to scale both x and y by some integer n, until they are far enough apart such that there is an integer m between them. Dividing by n then gives m/n between x and y.*

The real question is how we go about finding an n that allows for this. Well, since $x - y > 0$, we can choose some n such that $n(x - y) > 1$. If $x - y > 1$, then n can just be one and if $1 > x - y$, then we apply the previous theorem. Conceptually, it makes sense that we should be able to find an integer m in this range. If we're given a number line and mark two points that are separated by a distance greater than 1, that interval should overlap with an integer.

To find this integer, We make use of the fact that there is some $m_1 > nx$ (by the previous theorem). Similarly, we can prove there is an integer below nx by noting that there is some $m_2 > -nx \ \Rightarrow \ -m_2 < nx$. Since nx is located between two integers, it follows that it must be between two succesive integers, m and $m - 1$.

Now, we just need to show that $m < ny$. This follows from the fact that the distance between nx and ny is greater than 1:

$$nx < m < nx + 1 < ny$$

So there is an m between nx and ny. It follows that:

$$x < \frac{m}{n} < y$$

Remark 4. *For the moment, we will not discuss the "power-uniqueness" proof.*

3 Basic "Topology"

The reason I put topology in quotes is because we are technically only studying topology on the real line and products of the real line.

3.1 Finite and Infinite Sets

Theorem 4. *Every infinite subset of a countable set is countable.*

Remark 5. *Rudin presents a non-rigorous version of the proof of this fact. The "adult" version requires the well-ordering principle of the integers and the principle of recursive definition.*

Theorem 5. *A set is open if and only if its complement is closed.*

Proof. Consider the set U in X. Let U be open. It follows that for each $x \in U$, there exists some $N_r(x)$ such that $x \in N_r(x) \subset U$. Consider the complement of U, which we denote by $X - U$. Assume that there is a limit point y of $X - U$ that is in U. Thus, there must exist some neighbourhood around x that is contained in U, which contradicts the fact that y is a limit point. Thus, $X - U$ must contain all of its limit points and is closed.

Conversely, assume $X - U$ is closed, and thus contains all of its limit points. Assume there is some $x \in U$ that is contained in no neighbourhood that is a subset of U. It follows that each neighbourhood of x interssects $X - U$, making it a limit point. This is a contradiction, so each point of U must be an interior point, making U open. ∎