Axler Algebra Notes, Problems and Solutions

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1 Section 3B

Problem 3.12. Suppose that V is finite-dimensional and $T \in \mathcal{L}(V,\ W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and range $T = \{Tu : u \in U\}$.

Proof. Let us consider a basis B of null T. We then choose some basis B' of V, which, by rank-nullity theorem, will have cardinality greater than or equal to B. We use B to extend B' to a basis C of V (which we can do, as each B' is linearly independent).

Let $U = \text{span}(C - B')$ (linear combinations of the elements in the new basis that are not in B'). We assert that this is the U that satisfies these conditions.

Firstly, it is clear that U and null T contain the zero vector. In addition, if there were some non-zero vector v in U and null T, this would imply that there exist coefficients such that:

$$v = a_1u_1 + \ \cdots \ + a_nu_n = b_1v_1 + \ \cdots \ + b_mv_m$$

where $u_i \in U$ and $v_i \in B'$. We know that $U \cup B'$ forms a basis for V, so the above equation implies that:

$$a_ju_j = b_1v_1 + \ \cdots \ + b_mv_m - a_1u_1 + \ \cdots \ + a_{j-1}v_{j-1} - a_{j+1}v_{j+1} + \ \cdots \ + a_nu_n$$

where we know that at least one a_i (namely a_j) is non-zero, and at least one b_i is non-zero to conclude that the existence of v violates the linear independence of $U \cup B'$.

Clearly, $\{Tu : u \in U\} \subset \text{range } T$. In addition, we pick some $T(x) \in \text{range } T$. We have:

$$x = a_1u_1 + \ \cdots \ + a_nu_n + b_1v_1 + \ \cdots \ + b_mv_m$$

as $U \cup B'$ is a basis for V. We then get:

$$T(x) = T(a_1u_1 + \ \cdots \ + a_nu_n) + T(b_1v_1 + \ \cdots \ + b_mv_m) = T(a_1u_1 + \ \cdots \ + a_nu_n) = T(u)$$

where $u \in U$. Thus, range $T \subset \{Tu : u \in U\}$. We have inclusion both ways, so $\{Tu : u \in U\} =$ range T. This completes the proof.

□

Problem 3.19. Suppose that V and W are finite dimensional and U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V,\ W)$ such that null$T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. First, assume that exists such a T. From rank-nullity theorem, we have:

$$\dim V = \dim \text{range } T + \dim \text{null} T = \dim \text{range } T + \dim U \leq \dim W + \dim U$$

which clearly implies that $\dim U \geq \dim V - \dim W$. Conversely, assume that $\dim U \geq \dim V - \dim W$. Consider the basis $u_1,\ ...,\ u_n$ of U. We extend this to a basis for V by adding vectors $v_1,\ ...,\ v_m$.

We define T to be the map that takes each u_k to 0. We define a basis for W, which we label $w_1,\ ...,\ w_r$. We know that $\dim W \geq \dim V - \dim U$, which is equal to the number of vectors v_k. Thus, we are able to assign each v_k to some vector w_s of the basis for W.

We have assigned values to each basis vector of V, which means that T is linear. In addition, it is clear that null$T = U$.

□

Problem 3.26. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}),\ \mathcal{P}(\mathbb{R}))$ is such that deg $Dp = (\text{deg } p) - 1$ for every non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$. Prove that D is surjective.

Proof. Consider some $p \in \mathcal{P}(\mathbb{R})$ such that the degree of p is n. Consider the subset $\{x^{n+1},\ x^n,\ ...,\ x\}$ of $\mathcal{P}(\mathbb{R})$. We map each of these terms under D to get the set $B = \{D(x^{n+1},\ D(x^n),\ ...,\ D(x)\}$.

The k-th elements of this list will be a polynomial of degree $n+1-k$. It is easy to check that such a list is linearly independent: we complete the redundancy-removal procedure, starting at $D(x)$, noting that for each $D(x^k)$, we cannot write $D(x^k)$ as a sum of the polynomials $\{D(x^{k-1},\ ...,\ D(x)\}$ as $D(x^k)$ contains a term of degree $n+1-k$, which none of the other elements posses.

It follows that the elements of B are linearly independent. Let us consider the subspace $V_n \subset \mathcal{P}(\mathbb{R})$ of all polynomials of degree n. Clearly, such a space will have degree $n+1$. It is also clear that each element of B is in V_n. Thus, B is a linearly independent list of length $n+1$ contained in V_n. It follows that B is a basis for V_n.

Thus, for the p that we considered initially, we can write:

$$p = c_1 D(x) + \ \cdots\ + c_{n+1} D(x^{n+1}) = D(c_1 x + \ \cdots\ c_{n+1} x^{n+1})$$

Therefore, p can be written asd the image of some element of $\mathcal{P}(\mathbb{R})$ and the map D is surjective. □

Problem 3.29. Suppose $\phi \in \mathcal{L}(V,\ \mathbb{F})$. Suppose that $u \in V$ is not in null ϕ. Prove that:

$$V = \text{null } \phi \oplus \{au\ :\ a \in \mathbb{F}\}$$

Proof. In the case that ϕ is the trivial map, the null space of ϕ is all V and the theorem is proved.

In the case that ϕ is not the trivial map, we know from rank-nullity theorem that:

$$\dim\ V = \dim\ \text{null } \phi + \dim\ \text{range } \phi$$

However, it is clear that range $\phi = \mathbb{F}$, so $\dim\ \text{range } \phi = \dim\ \mathbb{F} = 1$. This implies that:

$$\dim\ V - \dim\ \text{null } \phi = 1$$

Now, we know that given some V, and a subspace U of V, there exists some U' such that $V = U \oplus U'$. We let $U = \text{null } \phi$. Since the sum of these subspaces is direct, it follows that:

$$\dim\ V = \dim\ \text{null } \phi + \dim\ U' \ \Rightarrow\ \dim U' = \dim\ V - \dim\ \text{null } \phi = 1$$

where we used the equation above. Thus, U' must be a one-dimensional subspace. All one dimensional subspaces of some vector space V are all multiples of a single vector, u. In addition, since the sum of U' and the null space is direct, this vector cannot be in null ϕ. Therefore:

$$U' = \{au\ :\ a \in \mathbb{F}\}$$

and:

$$V = \text{null } \phi \oplus \{au\ :\ a \in \mathbb{F}\}$$

for some $u \in V$.

Now, the last thing we have to show is that U' can be multiples of **any** vector not in the null-space (not just u). Given some $v \in V$, we will have, from above:

$$v = n + au$$

for some n in the null space. Given some w also not in the null space, we choose c such that $a\phi(u) - c\phi(w) = 0$, which we can do as we know that both $\phi(u)$ and $\phi(w)$ are non-zero. Thus:

$$n + au = (n + au - cw) + cw = m + cw$$

where m is in the null space. We prove inclusion the other way in a similar fashion, implying that:

$$\text{null } \phi \oplus = \{au \ : \ a \in \mathbb{F}\} = \text{null } \phi \oplus = \{aw \ : \ a \in \mathbb{F}\}$$

Therefore, we are able to conclude that:

$$V = \text{null } \phi \oplus \{au \ : \ a \in \mathbb{F}\}$$

for **any** u not in the null space.

$\square$

Problem 3.30. Suppose ϕ_1 and ϕ_2 are linear maps from V to $\mathbb{F}$ that have the same null space. Show that there exists some $c \in \mathbb{F}$ such that $\phi_1 = c\phi_2$.

Proof. Using the previous result, we can write V as the sum:

$$V = \text{null } \phi_1 \ \oplus \ \{au : u \in \mathbb{F}\} = \text{null } \phi_2 \ \oplus \ \{au : u \in \mathbb{F}\}$$

Let us pick some $v \in V$. We will have $v = n + au$ where n is in the null-space of both maps. We will have:

$$\phi_1(v) = \phi_1(n + au) = \phi_1(n) + a\phi_1(u)$$

We then choose some c such that $\phi_1(u) = c\phi_2(u)$, which we can do as both $\phi_1(u)$ and $\phi_2(u)$ are non-zero. In addition, we will have: $\phi_1(n) = \phi_2(n) = 0$, as both maps have the same null-space. We note that $c\phi_2(n) = \phi_2(n)$. Thus, we will have:

$$\phi_1(n) + a\phi_1(u) = c\phi_2(n) + c\phi_2(au) = c\phi_2(n + au) = c\phi_2(v)$$

This completes the proof.

$\square$

2 Section 3C

Problem 3.6. Suppose that V and W are finite-dimensional and $T \in \mathcal{L}(V, \ W)$. Prove that if $\dim \text{range } T = 1$ if and only if there exists a basis of V and a basis of W such that with respect to these bases, all entires of $\mathcal{M}(T)$ are equal to 1.

Proof. Clearly, if there are bases of V and W such that $\mathcal{M}(T)$ has ones in all entries, then each basis vector in the chosen basis will get mapped to the sum of all the chosen basis vectors of W, which we call w. It follows that range $T = \text{span}(w)$, implying that the dimension of the range of T is 1.

Conversely, assume that $\dim \text{range } T = 1$. From rank-nullity theorem, it follows that $\dim \text{null} T = n - 1$, where n is the dimension of V. Since the dimension of the range is 1. There must exist some vector v of V such that $T(v) = w$, where $w \neq \mathbf{0}$. We choose a basis $w_1, \ ..., \ w_m$ of W, which means that:

$$w = a_1 w_1 + \ \cdots \ + a_m w_m$$

We let the set $\{a_1 w_1, \ ..., \ a_m w_m\}$ be a basis for W, and denote the k-th element of the basis w'_k. Now, consider some basis $v_1, \ ..., \ v_{n-1}$ for the null space of T. The set of vectors $\{v + v_0, \ v + v_1, \ ..., \ v + v_{n-1}\}$ (where $v_0 = \mathbf{0}$) will clearly be a basis for V, as each vector in the n-element set is linearly independent. We denote the $k + 1$-th element of this basis v'_k.

Now, consider T acting upon some basis vector:

$$T(v'_k) = T(v) + T(v_0) = w = w'_1 + \ \cdots \ + w'_m$$

So in the primed bases, each element of $\mathcal{M}(T)$ is 1, by definition.

$\square$

3 Section 3D

Problem 3.17. Suppose V is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{\mathbf{0}\}$ or $\mathcal{E} = \mathcal{L}(V)$.

Proof. Clearly, $\mathcal{E}$ can be the trivial subspace.

Now, consider what happens when we assume that there is some non-zero $T \in \mathcal{E}$. It follows that there must exist some $v \in V$ such that $T(v) = w_1$, where w_1 is non-zero. Extending w_1 to a basis for V, we get the set $w_1, \ ..., \ w_n$.

We let S_1^k be the map that takes w_k to v and all other basis elements to 0. We let S_2^k be the map that takes w_1 to w_k, and all other basis elements to 0. It follows that the map TS_1^k takes w_k to w_1, and all other basis vectors to 0, and is in $\mathcal{E}$. We can then conclude that $S_2^r TS_1^k$ is also in $\mathcal{E}$, and is the map that takes w_k to w_r, and all other basis elements to 0.

Clearly, any map from V to V can be written as a linear combination of maps of the form $S_2^r TS_1^k$. Since $\mathcal{E}$ is a subspace, all such linear combinations are in $\mathcal{E}$. This implies that $\mathcal{E} = \mathcal{L}(V)$.

It follows that $\mathcal{E}$ is either trivial, or the whole space $\mathcal{L}(V)$. □

4 Section 3E

Problem 3.18. Suppose that $T \in \mathcal{L}(V, \ W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, \ W)$ such that $T = S \circ \pi$ if and only if $U \subset \text{null } T$.

Proof. Assume that there exists S such that $T = S \circ \pi$. Let us pick some $u \in U$. We note that $Tu = (S \circ \pi)(u) = S([u]) = S([0]) = 0$, so $U \subset \text{null } T$.

Assume that $U \subset \text{null } T$. Since U is a subspace of the null space, it follows that for $u \in U$, we have $T(u) = 0$. Thus, given w and v in V such that $\pi(w) = \pi(v)$, we can notice that $w - v \in U$, by definition of the quotient space, so

$$T(w - v) = T(w) - T(v) = 0 \ \Rightarrow T(w) = T(v)$$

Thus, we define S to be the map that takes $[v]$ in the quotient space to $T(v)$ in W. Such a map is well defined as if $[v] = [w]$, then $S([w]) = T(w) = T(v) = S([v])$. Clearly, such a map is linear, as:

$$S([w] + [v]) = S([w + v]) = T(w + v) = T(w) + T(v) = S([w]) + S([v])$$

and:

$$\lambda S([w]) = \lambda T(w) = T(\lambda w) = S([\lambda w]) = S(\lambda [w])$$

and the proof is complete.

□

5 Section 3F

Proposition 1. *Let U^0 be the annihiltor of U as a subspace of V. It follows that:*

$$\dim U^0 + \dim U = \dim V$$

Proof. We attempt to prove this in the language of linear functionals.

We know that V' is the space of functionals from V to $\mathbb{F}$. We know that U is a subspace of V, so it follows that we can choose a basis $v_1, \ ..., \ v_n$ of U, then extend it to a basis for V by adding vector

$v_{n+1}, \ ..., \ v_m$.

Using this basis, we can define the dual basis on V' of the elements $\phi_i(v_k)$ for v_k in the basis of V.

We define a linear map $T : V' \ \rightarrow \ V'$ which takes the basis element ϕ_i to itself if $1 \leq i \leq n$ (so the corresponding v_i is in U), and to 0 otherwise.

We assert that null $T = U^0$. Let us pick some $\phi \in$ null T. We will have:

$$T(a_1\phi_1 + \ \cdots \ + a_m\phi_m) = a_1\phi_1 + \ \cdots \ + a_n\phi_n = 0$$

Since each element of the dual basis is linearly independent, all a_k must be 0, thus, ϕ is a linear combination of the ϕ_k basis elements for $k \geq n+1$. It follows that $\phi(u) = 0$ for all $u \in U$, as u is a linear combination of exclusively the basis elements v_k from $k = 1$ to $k = n$. Thus, ϕ is in U^0.

Now, if $\phi \in U^0$, it follows that $\phi(u) = 0$ for all $u \in U$, so we will have:

$$T(\phi) = a_1T(\phi_1) + \ \cdots \ a_mT(\phi_m) = a_1\phi_1 + \ \cdots \ + a_n\phi_n$$

Now, given some v_k for k between 1 and n, we will have:

$$(a_1\phi_1 + \ \cdots \ + a_n\phi_n)(v_k) = a_k\phi_k(v_k) = a_k = 0$$

so each a_k is equal to 0, implying that $T(\phi)$ is the zero map, so ϕ is in the null space. Thus, $U^0 =$ null T.

Finally, using the fundmanetal theorem of linear maps:

$$\dim V' = \dim \ \text{range}(T) + \dim \ \text{null}(T) = \dim \ U' + \dim U^0$$

But we know that $\dim V' = \dim V$ and $\dim U' = \dim U$, so:

$$\dim V = \dim U + \dim U^0$$

□

6 Section 5A

Proposition 2. *Given a set of m distinct eigenvalues $\lambda_1, \ ..., \ \lambda_m$, along with a set of corresponding eigenvectors $V = \{v_1, \ ..., \ v_m\}$, the set V is linearly independent.*

Proof. We will prove this proposition by induction. Clearly, this will be true in the case of one eigenvalue, λ. Assume that it holds true given n eigenvalues. We prove it holds true for $n+1$.
Consider the set of eigenvalues $\{\lambda_1, \ ..., \ \lambda_{n+1}\}$ with corresponding eigenvectors $\{v_1, \ ..., \ v_{n+1}\}$. Assume that there is a non-trivial linear combination:

$$a_1v_1 + \ \cdots \ + a_nv_n + a_{n+1}v_{n+1} = 0$$

Note that since eigenvectors are non-zero, for this non-trivial linear combination to be 0, we must have at least two a_i not equal to 0 otherwise we would have $a_kv_k = 0$, for non-zero a_k, which can't be the case. It follows that at least one a_i with $1 \leq i \leq n$ is non-zero.
We define the linear operator $(T - \lambda_{n+1}I)$. We then have:

$$(T - \lambda_iI)(a_1v_1 + \ \cdots \ + a_nv_n + a_{n+1}v_{n+1}) = \sum_{k \neq n+1} a_k(\lambda_k - \lambda_{n+1})v_k = 0$$

But since all eigenvalues are unique, we must have $\lambda_k - \lambda_{n+1} \neq 0$. In addition, it least one a_i in this sum is non-zero. Thus, we have found a non-trivial linear combination of n eigenvectors that yields the zero vector, a contradiction to the inductive hypothesis.

It follows that the set $\{v_1, \ ..., \ v_{n+1}\}$ is linearly independent and the proof is complete.

□

Problem 5.28. Suppose V is finite-dimensional with $\dim V \geq 3$ and $T \in \mathcal{L}(V)$ is such that every 2-dimensional subspace of V is invariant under T. Prove that T is a scalar multiple of the identity operator.

Proof. Consider some $v \in V$. Since the dimension of V is greater than or equal to 3, we can also choose two other vectors, w and z that form a linearly independent set $\{v,\ w,\ z\}$. We consider the two-dimensional subspaces $A = \text{span}(v,\ w)$ and $B = \text{span}(v,\ z)$. We know that A is invariant, so it follows that $Tv = av + bw$, but we know that B is also invariant, so $Tv = cv + dz$. This implies that:

$$(c - a)v + dz - bw = 0$$

and since these vectors are linearly independent, we have $d = b = 0$, so it follows that v is sent to a multiple of itself.

Now, we pick linearly independent v and w in V such that $Tv = av$ and $Tw = bw$. We will have:

$$T(v + w) = c(v + w) = T(v) + T(w) = av + bw$$

so since v and w are linearly independent, it follows that $c = a = b$, so $Tv = cv$ and $Tw = cw$. Thus, T is a scalar multiple of the identity map and the proof is complete.

□

Problem 5.35. Suppose V is finite-dimnensional, $T \in \mathcal{L}(V)$, and U is invariant under T. Prove that each eigenvalue of T/U is an eigenvalue of T.

Proof. Clearly, if λ is an eigenvalue of T, then there is v such that $Tv = \lambda v$, so it follows that $Tv - \lambda v = 0$, implying that $Tv - \lambda v \in U$, so:

$$Tv + U = \lambda v + U \ \Rightarrow\ (T/U)(v + U) = \lambda(v + U)$$

so λ is an eigenvalue of T/U.

Conversely, let us assume that is $v + U$ such that $(T/U)(v + U) = T(v) + U = \lambda v + U$. This implies that $Tv - \lambda v \in U$.

To do this, it is enough to show that the map $T - \lambda I$ is not surjective.

□

7 Section 8

Proposition 3. *Given some operator $T : V \ \rightarrow\ V$ with V finite dimensional, there is a minimal polynomial $p(z)$ such that $p(T) = 0$. In other words, there exists a polynomial such that for all $q(z)$, with $q(T) = 0$, we have:*

$$q(z) = t(z)p(z)$$

for some polynomial $t(z)$

Proof. □

Proposition 4.