

Axler Algebra Notes, Problems and Solutions

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1 Section 3B

Problem 3.12. Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and $\text{range } T = \{Tu : u \in U\}$.

Proof. Let us consider a basis B of $\text{null } T$. We then choose some basis B' of V , which, by rank-nullity theorem, will have cardinality greater than or equal to B . We use B to extend B' to a basis C of V (which we can do, as each B' is linearly independent).

Let $U = \text{span}(C - B')$ (linear combinations of the elements in the new basis that are not in B'). We assert that this is the U that satisfies these conditions.

Firstly, it is clear that U and $\text{null } T$ contain the zero vector. In addition, if there were some non-zero vector v in U and $\text{null } T$, this would imply that there exist coefficients such that:

$$v = a_1u_1 + \cdots + a_nu_n = b_1v_1 + \cdots + b_mv_m$$

where $u_i \in U$ and $v_i \in B'$. We know that $U \cup B'$ forms a basis for V , so the above equation implies that:

$$a_ju_j = b_1v_1 + \cdots + b_mv_m - a_1u_1 - \cdots - a_{j-1}u_{j-1} - a_{j+1}u_{j+1} - \cdots - a_nu_n$$

where we know that at least one a_i (namely a_j) is non-zero, and at least one b_i is non-zero to conclude that the existence of v violates the linear independence of $U \cup B'$.

Clearly, $\{Tu : u \in U\} \subset \text{range } T$. In addition, we pick some $T(x) \in \text{range } T$. We have:

$$x = a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_mv_m$$

as $U \cup B'$ is a basis for V . We then get:

$$T(x) = T(a_1u_1 + \cdots + a_nu_n) + T(b_1v_1 + \cdots + b_mv_m) = T(a_1u_1 + \cdots + a_nu_n) = T(u)$$

where $u \in U$. Thus, $\text{range } T \subset \{Tu : u \in U\}$. We have inclusion both ways, so $\{Tu : u \in U\} = \text{range } T$. This completes the proof. □

Problem 3.19. Suppose that V and W are finite dimensional and U is a subspace of V . Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. First, assume that exists such a T . From rank-nullity theorem, we have:

$$\dim V = \dim \text{range } T + \dim \text{null } T = \dim \text{range } T + \dim U \leq \dim W + \dim U$$

which clearly implies that $\dim U \geq \dim V - \dim W$. Conversely, assume that $\dim U \geq \dim V - \dim W$. Consider the basis u_1, \dots, u_n of U . We extend this to a basis for V by adding vectors v_1, \dots, v_m .

We define T to be the map that takes each u_k to 0. We define a basis for W , which we label w_1, \dots, w_r . We know that $\dim W \geq \dim V - \dim U$, which is equal to the number of vectors v_k . Thus, we are able to assign each v_k to some vector w_s of the basis for W .

We have assigned values to each basis vector of V , which means that T is linear. In addition, it is clear that $\text{null } T = U$. □

Problem 3.26. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$. Prove that D is surjective.

Proof. Consider some $p \in \mathcal{P}(\mathbb{R})$ such that the degree of p is n . Consider the subset $\{x^{n+1}, x^n, \dots, x\}$ of $\mathcal{P}(\mathbb{R})$. We map each of these terms under D to get the set $B = \{D(x^{n+1}), D(x^n), \dots, D(x)\}$.

The k -th elements of this list will be a polynomial of degree $n + 1 - k$. It is easy to check that such a list is linearly independent: we complete the redundancy-removal procedure, starting at $D(x)$, noting that for each $D(x^k)$, we cannot write $D(x^k)$ as a sum of the polynomials $\{D(x^{k-1}), \dots, D(x)\}$ as $D(x^k)$ contains a term of degree $n + 1 - k$, which none of the other elements possess.

It follows that the elements of B are linearly independent. Let us consider the subspace $V_n \subset \mathcal{P}(\mathbb{R})$ of all polynomials of degree n . Clearly, such a space will have degree $n + 1$. It is also clear that each element of B is in V_n . Thus, B is a linearly independent list of length $n + 1$ contained in V_n . It follows that B is a basis for V_n .

Thus, for the p that we considered initially, we can write:

$$p = c_1 D(x) + \dots + c_{n+1} D(x^{n+1}) = D(c_1 x + \dots + c_{n+1} x^{n+1})$$

Therefore, p can be written as the image of some element of $\mathcal{P}(\mathbb{R})$ and the map D is surjective. \square

Problem 3.29. Suppose $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose that $u \in V$ is not in $\text{null } \phi$. Prove that:

$$V = \text{null } \phi \oplus \{au : a \in \mathbb{F}\}$$

Proof. In the case that ϕ is the trivial map, the null space of ϕ is all V and the theorem is proved.

In the case that ϕ is not the trivial map, we know from rank-nullity theorem that:

$$\dim V = \dim \text{null } \phi + \dim \text{range } \phi$$

However, it is clear that $\text{range } \phi = \mathbb{F}$, so $\dim \text{range } \phi = \dim \mathbb{F} = 1$. This implies that:

$$\dim V - \dim \text{null } \phi = 1$$

Now, we know that given some V , and a subspace U of V , there exists some U' such that $V = U \oplus U'$. We let $U = \text{null } \phi$. Since the sum of these subspaces is direct, it follows that:

$$\dim V = \dim \text{null } \phi + \dim U' \Rightarrow \dim U' = \dim V - \dim \text{null } \phi = 1$$

where we used the equation above. Thus, U' must be a one-dimensional subspace. All one-dimensional subspaces of some vector space V are all multiples of a single vector, u . In addition, since the sum of U' and the null space is direct, this vector cannot be in $\text{null } \phi$. Therefore:

$$U' = \{au : a \in \mathbb{F}\}$$

and:

$$V = \text{null } \phi \oplus \{au : a \in \mathbb{F}\}$$

for some $u \in V$.

Now, the last thing we have to show is that U' can be multiples of **any** vector not in the null-space (not just u). Given some $v \in V$, we will have, from above:

$$v = n + au$$

for some n in the null space. Given some w also not in the null space, we choose c such that $a\phi(u) - c\phi(w) = 0$, which we can do as we know that both $\phi(u)$ and $\phi(w)$ are non-zero. Thus:

$$n + au = (n + au - cw) + cw = m + cw$$

where m is in the null space. We prove inclusion the other way in a similar fashion, implying that:

$$\text{null } \phi \oplus = \{au : a \in \mathbb{F}\} = \text{null } \phi \oplus = \{aw : a \in \mathbb{F}\}$$

Therefore, we are able to conclude that:

$$V = \text{null } \phi \oplus \{au : a \in \mathbb{F}\}$$

for **any** u not in the null space. □

Problem 3.30. Suppose ϕ_1 and ϕ_2 are linear maps from V to \mathbb{F} that have the same null space. Show that there exists some $c \in \mathbb{F}$ such that $\phi_1 = c\phi_2$.

Proof. Using the previous result, we can write V as the sum:

$$V = \text{null } \phi_1 \oplus \{au : u \in \mathbb{F}\} = \text{null } \phi_2 \oplus \{au : u \in \mathbb{F}\}$$

Let us pick some $v \in V$. We will have $v = n + au$ where n is in the null-space of both maps. We will have:

$$\phi_1(v) = \phi_1(n + au) = \phi_1(n) + a\phi_1(u)$$

We then choose some c such that $\phi_1(u) = c\phi_2(u)$, which we can do as both $\phi_1(u)$ and $\phi_2(u)$ are non-zero. In addition, we will have: $\phi_1(n) = \phi_2(n) = 0$, as both maps have the same null-space. We note that $c\phi_2(n) = \phi_2(n)$. Thus, we will have:

$$\phi_1(n) + a\phi_1(u) = c\phi_2(n) + c\phi_2(au) = c\phi_2(n + au) = c\phi_2(v)$$

This completes the proof. □

2 Section 3C

Problem 3.6. Suppose that V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that if $\dim \text{range } T = 1$ if and only if there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are equal to 1.

Proof. Clearly, if there are bases of V and W such that $\mathcal{M}(T)$ has ones in all entries, then each basis vector in the chosen basis will get mapped to the sum of all the chosen basis vectors of W , which we call w . It follows that $\text{range } T = \text{span}(w)$, implying that the dimension of the range of T is 1.

Conversely, assume that $\dim \text{range } T = 1$. From rank-nullity theorem, it follows that $\dim \text{null } T = n - 1$, where n is the dimension of V . Since the dimension of the range is 1. There must exist some vector v of V such that $T(v) = w$, where $w \neq \mathbf{0}$. We choose a basis w_1, \dots, w_m of W , which means that:

$$w = a_1w_1 + \dots + a_mw_m$$

We let the set $\{a_1w_1, \dots, a_mw_m\}$ be a basis for W , and denote the k -th element of the basis w'_k . Now, consider some basis v_1, \dots, v_{n-1} for the null space of T . The set of vectors $\{v+v_0, v+v_1, \dots, v+v_{n-1}\}$ (where $v_0 = \mathbf{0}$) will clearly be a basis for V , as each vector in the n -element set is linearly independent. We denote the $k+1$ -th element of this basis v'_k .

Now, consider T acting upon some basis vector:

$$T(v'_k) = T(v) + T(v_0) = w = w'_1 + \dots + w'_m$$

So in the primed bases, each element of $\mathcal{M}(T)$ is 1, by definition. □

3 Section 3D

Problem 3.17. Suppose V is finite-dimensional and \mathcal{E} is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{\mathbf{0}\}$ or $\mathcal{E} = \mathcal{L}(V)$.

Proof. Clearly, \mathcal{E} can be the trivial subspace.

Now, consider what happens when we assume that there is some non-zero $T \in \mathcal{E}$. It follows that there must exist some $v \in V$ such that $T(v) = w_1$, where w_1 is non-zero. Extending w_1 to a basis for V , we get the set w_1, \dots, w_n .

We let S_1^k be the map that takes w_k to v and all other basis elements to 0. We let S_2^k be the map that takes w_1 to w_k , and all other basis elements to 0. It follows that the map TS_1^k takes w_k to w_1 , and all other basis vectors to 0, and is in \mathcal{E} . We can then conclude that $S_2^r TS_1^k$ is also in \mathcal{E} , and is the map that takes w_k to w_r , and all other basis elements to 0.

Clearly, any map from V to V can be written as a linear combination of maps of the form $S_2^r TS_1^k$. Since \mathcal{E} is a subspace, all such linear combinations are in \mathcal{E} . This implies that $\mathcal{E} = \mathcal{L}(V)$.

It follows that \mathcal{E} is either trivial, or the whole space $\mathcal{L}(V)$. □

4 Section 3E

Problem 3.18. Suppose that $T \in \mathcal{L}(V, W)$ and U is a subspace of V . Let π denote the quotient map from V onto V/U . Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subset \text{null } T$.

Proof. Assume that there exists S such that $T = S \circ \pi$. Let us pick some $u \in U$. We note that $Tu = (S \circ \pi)(u) = S([u]) = S([0]) = 0$, so $U \subset \text{null } T$.

Assume that $U \subset \text{null } T$. Since U is a subspace of the null space, it follows that for $u \in U$, we have $T(u) = 0$. Thus, given w and v in V such that $\pi(w) = \pi(v)$, we can notice that $w - v \in U$, by definition of the quotient space, so

$$T(w - v) = T(w) - T(v) = 0 \Rightarrow T(w) = T(v)$$

Thus, we define S to be the map that takes $[v]$ in the quotient space to $T(v)$ in W . Such a map is well defined as if $[v] = [w]$, then $S([w]) = T(w) = T(v) = S([v])$. Clearly, such a map is linear, as:

$$S([w] + [v]) = S([w + v]) = T(w + v) = T(w) + T(v) = S([w]) + S([v])$$

and:

$$\lambda S([w]) = \lambda T(w) = T(\lambda w) = S([\lambda w]) = S(\lambda[w])$$

and the proof is complete. □

5 Section 3F

Proposition 1. Let U^0 be the annihilator of U as a subspace of V . It follows that:

$$\dim U^0 + \dim U = \dim V$$

Proof. We attempt to prove this in the language of linear functionals.

We know that V' is the space of functionals from V to \mathbb{F} . We know that U is a subspace of V , so it follows that we can choose a basis v_1, \dots, v_n of U , then extend it to a basis for V by adding vector

v_{n+1}, \dots, v_m .

Using this basis, we can define the dual basis on V' of the elements $\phi_i(v_k)$ for v_k in the basis of V .

We define a linear map $T : V' \rightarrow V'$ which takes the basis element ϕ_i to itself if $1 \leq i \leq n$ (so the corresponding v_i is in U), and to 0 otherwise.

We assert that $\text{null } T = U^0$. Let us pick some $\phi \in \text{null } T$. We will have:

$$T(a_1\phi_1 + \dots + a_m\phi_m) = a_1\phi_1 + \dots + a_n\phi_n = 0$$

Since each element of the dual basis is linearly independent, all a_k must be 0, thus, ϕ is a linear combination of the ϕ_k basis elements for $k \geq n+1$. It follows that $\phi(u) = 0$ for all $u \in U$, as u is a linear combination of exclusively the basis elements v_k from $k = 1$ to $k = n$. Thus, ϕ is in U^0 .

Now, if $\phi \in U^0$, it follows that $\phi(u) = 0$ for all $u \in U$, so we will have:

$$T(\phi) = a_1T(\phi_1) + \dots + a_mT(\phi_m) = a_1\phi_1 + \dots + a_n\phi_n$$

Now, given some v_k for k between 1 and n , we will have:

$$(a_1\phi_1 + \dots + a_n\phi_n)(v_k) = a_k\phi_k(v_k) = a_k = 0$$

so each a_k is equal to 0, implying that $T(\phi)$ is the zero map, so ϕ is in the null space. Thus, $U^0 = \text{null } T$.

Finally, using the fundamental theorem of linear maps:

$$\dim V' = \dim \text{range}(T) + \dim \text{null}(T) = \dim U' + \dim U^0$$

But we know that $\dim V' = \dim V$ and $\dim U' = \dim U$, so:

$$\dim V = \dim U + \dim U^0$$

□