Spivak Notes, Problems, and Solutions

Jack Ceroni

October 2020

Contents

1 Introduction

The goal of this set of notes is to solve the most challenging problems in Spivak, and write up the solutions in a clean and concise way. I apologize in advance for any possible mistakes, or instances in which I may skip over certain important points.

2 Chapter 3

Problem 3.17. Prove that if $f(x+y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$, where $f(x) \neq 0$, then $f(x) = x$ for all x.

Proof. We go through the steps of the proof, as organized in Spivak:

1. Clearly, we will have $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$, so either $f(1) = 0$ or $f(1) = 1$. If we assume that $f(1) = 0$, then this would imply that $f(n) = 0$ for all n (we can prove this by induction, assuming that $f(n) = 0$, and noting that $f(n+1) = f(n) + f(1) = 0$). This is a contradiction to our initial assumption, so $f(1) = 1$.

2. First, we note that:

$$f(0) = f(0+0) = f(0) + f(0) \;\Rightarrow\; f(0) = 0$$

   Next, we note that $f(n) = n$, for natural n. We prove this by induction, first assuming that $f(n) = n$, then noting that $f(n+1) = f(n) + f(1) = n+1$. We then note that $f(-n) = n - n + f(-n) = -n + f(n) + f(-n) = -n + f(0) = -n$. Thus, f is the identity for all integers.

   Now, we can see that:

$$f\left(\frac{1}{b}\right) = \frac{b}{b} \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(b) \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(1) = \frac{1}{b}$$

   Thus,

$$f\left(\frac{a}{b}\right) = f(a) \cdot f\left(\frac{1}{b}\right) = \frac{a}{b}$$

3. Assume that $x > 0$. It then follows that $\sqrt{x}$ is well-defined and greater than 0. We then have:

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = f(\sqrt{x})^2$$

   we know that for any real number r, we have $r^2 \geq 0$, so $f(x) \geq 0$. Assume that $f(x) = 0$. Since $x > 0$, this would imply that:

$$f(1) = f\left(\frac{x}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right) = 0$$

   a clear contradiction. Thus, $f(x) > 0$.

4. If $x > y$, then we know that $x - y > 0$, so it follows from previous result that:

$$f(x-y) > 0 \;\Rightarrow\; f(x) - f(y) > 0 \;\Rightarrow\; f(x) > f(y)$$

5. Assume that there exists some x such that $x < f(x)$. Since there exists a rational number between any two reals, it follows that we have:

$$x < \frac{a}{b} < f(x)$$

   for some a/b. From the previous result, we then get $f(x) < f(a/b)$, a clear contradiction to the right-most inequality above. Similarly, if we assume that $f(x) < x$, we will have:

$$f(x) < \frac{a}{b} < x$$

   so $f(a/b) < f(x)$, another contradiction. It follows that $f(x) = x$, and we have proved the proposition.

□

Problem 3.20B. If a function satisfies:

$$f(y) - f(x) \leq (x-y)^2$$

for all $, y \in \mathbb{R}$, then $f(x) = c$ for some c and all x

Part B is the interesting part of this problem, so I skipped writing out Part A

Proof. Assume that there exist distinct x and y such that $f(x) \neq f(y)$. Without loss of generality, let $f(x) < f(y)$. It follows that:

$$f(y) - f(x) \leq (y-x)^2$$

Consider what happens when we split up the interval from x to y into n "chunks". We let:

$$z_j = \Big(1 - \frac{j}{n}\Big)x + \frac{j}{n}y$$

so we get $z_0 = x$ and $z_n = y$. Clearly the distance between z_i and z_{j-1} is given by:

$$z_j - z_{j-1} = \Big(1 - \frac{j}{n}\Big)x + \frac{j}{n}y - \Big(1 - \frac{j-1}{n}\Big)x - \frac{j-1}{n}y = \frac{y-x}{n}$$

It follows that:

$$f(z_j) - f(z_{j-1}) \leq (z_j - z_{j-1})^2 = \frac{(y-x)^2}{n^2}$$

Now comes the crucial step. Notice that

$$\sum_{j=1}^{n} \big(f(z_j) - f(z_{j-1})\big) = f(z_n) - f(z_0) = f(y) - f(x)$$

as the rest of the terms cancel. Thus, we will have:

$$\sum_{j=1}^{n} \big(z_j - z_{j-1}\big) \leq \sum_{j=1}^{n} \frac{(y-x)^2}{n^2} \;\Rightarrow\; f(y) - f(x) \leq n \cdot \frac{(y-x)^2}{n^2} = \frac{(y-x)^2}{n}$$

for all possible values of n. Since we have assume $f(y) \neq f(x)$, it follows that $f(y) - f(x)$ is a positive real number, and that:

$$\epsilon = \frac{f(y) - f(x)}{(y-x)^2}$$

is a positive real as well. Thus, this implies that there exists a real number ϵ such that for any positive integer n:

$$\epsilon \leq \frac{1}{n}$$

But this clearly contradicts the Archimedean property of the real numbers. We have derived a contradiction, so it follows that for any x and y, $f(x) = f(y)$. Thus, the function f is constant.

□

3 Chpater 5

Lemma 1 (Uniqueness of Limits)**.** *The limit of a function is unique: If a function f approaches ℓ_1 as x approaches a, and f approaches ℓ_2 as x approaches a, then $\ell_1 = \ell_2$.*

Proof. Suppose the the function f approaches ℓ_1 and ℓ_2. It follows that given some $\epsilon > 0$, we can choose δ_1 and δ_2 such that:

$$|x - a| < \delta_1 \;\Rightarrow\; |f(x) - \ell_1| < \epsilon$$
$$|x - a| < \delta_2 \;\Rightarrow\; |f(x) - \ell_2| < \epsilon$$

Assume that $\ell_1 \neq \ell_2$, so $|\ell_1 - \ell_2| > 0$. Let us then pick $\epsilon = \frac{|\ell_1 - \ell_2|}{2}$. We can then pick δ_1 and δ_2 corresponding to this ϵ. We then let $\delta = \min(\delta_1,\ \delta_2)$ so:

$$|x - a| < \delta \ \Rightarrow\ |f(x) - \ell_1| < \ \epsilon \qquad \text{and} \qquad |f(x) - \ell_2| < \epsilon$$

It then follows that:

$$|x - a| < \delta \ \Rightarrow\ |f(x) - \ell_1| + |f(x) - \ell_2| < 2\epsilon = |\ell_1 - \ell_2|$$

We know that there exists some x_0 such that $|x_0 - a| < \delta$, which implies that:

$$|\ell_1 - \ell_2| \leq |f(x_0) - \ell_1| + |f(x_0) - \ell_2| < |\ell_1 - \ell_2|$$

a clear contradiction. It follows that ℓ_1 must equal ℓ_2.

□

Lemma 2 (Sums of Limits)**.** *If* $\lim_{x\to a} f(x) = m$ *and* $\lim_{x\to a} g(x) = \ell$*, then* $\lim_{x\to a}(f + g)(x) = m + \ell$.

Proof. Let us pick some $\epsilon > 0$. We will have, for $\epsilon/2$:

$$|x - a| < \delta_1 \ \Rightarrow\ |f(x) - m| < \epsilon/2$$
$$|x - a| < \delta_2 \ \Rightarrow\ |g(x) - \ell| < \epsilon/2$$

we choose $\delta = \min(\delta_1,\ \delta_2)$, giving us:

$$|x - a| < \delta \ \Rightarrow\ |f(x) - m| + |g(x) - \ell| < \epsilon$$

Then, given x such that $|x - a| < \delta$, we have:

$$|f(x) + g(x) - (m + \ell)| \leq |f(x) - m| + |g(x) - \ell| < \epsilon$$

Thus, given ϵ, we can choose a δ. It follows by definition that $\lim_{x\to a}(f + g)(x) = m + \ell$.

□

Problem 5.20. If $f(x) = x$ for rational x and $f(x) = -x$ for irrational x, show that $\lim_{x\to a} f(x)$ does not exist for $a \neq 0$.

Proof. Assume that there exists some non-zero a such that:

$$\lim_{x\to a} f(x) = L$$

It follows that for any $\epsilon > 0$, we can choose a δ such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$. We begin by considering the case when $a > 0$. We let $\epsilon = a$ and assume that we can choose a δ such that:

$$|x - a| < \delta \ \Rightarrow\ |f(x) - L| < a$$

Now, since there exists a rational and an irrational number between any two reals, we pick rational r and irrational i from the interval $(a,\ a + \delta)$. We will then have:

$$|r - a| < \delta \ \Rightarrow\ |r - L| < a$$
$$|i - a| < \delta \ \Rightarrow\ |-i - L| = |i + L| < a$$

So we will have:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2a$$

But this is a contradiction, as $a < i,\ r$, so $2a < i + r$. Thus, we can choose no such $\delta > 0$, and the limit does not exist.

Now, assume that $a < 0$. We let $\epsilon = |a|$ and assume that we can choose a corresponding δ. We then choose rational and irrational $r, i \in (a - \delta, a)$. Similar to above, we get:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2|a|$$

But this is a contradiction, as $i, r < a$, so $i + r < 2a$, which implies that $|i + r| > 2|a|$ (as a, i, and r are negative). Thus, we can choose no such δ, and the limit does not exist.

We conclude that the limit does not exist for any $a > 0$, and any $a < 0$, making $a = 0$ the only point at which the limit exists. □

Problem 5.12. Suppose that $f(x) \leq g(x)$ for all x. Prove that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, assuming the limits exist.

Proof. We let the first limit be denoted by ℓ_f and the second by ℓ_g. Assume that $\ell_f > \ell_g$. It follows that $\ell_f - \ell_g$ is a positive real number, so we let $\epsilon = (\ell_f - \ell_g)/2$. Now, by definition of the limits, we can choose δ_1 and δ_2 such that:

$$0 < |x - a| < \delta_1 \;\Rightarrow\; |f(x) - \ell_f| < (\ell_f - \ell_g)/2$$
$$0 < |x - a| < \delta_2 \;\Rightarrow\; |g(x) - \ell_g| < (\ell_f - \ell_g)/2$$

.
We let $\delta = \min\{\delta_1,\ \delta_2\}$. We then have:

$$0 < |x - a| < \delta \;\Rightarrow\; |\ell_f - f(x) + g(x) - \ell_g| \leq |f(x) - \ell_f| + |g(x) - \ell_g| < \ell_f - \ell_g$$

So we have:

$$|(g(x) - f(x)) + (\ell_f - \ell_g)| < \ell_f - \ell_g$$

but since $f(x) \leq g(x)$ and $\ell_g < \ell_f$, both numbers in the brackets will be greater than or equal to 0, so:

$$g(x) - f(x) + \ell_f - \ell_g < \ell_f - \ell_g$$

which is a contradiction. Thus, $\ell_f \leq \ell_g$ and the proof is complete.

□

Problem 5.23. Let f be a function with the following property: if g is a function for which $\lim_{x\to 0} g(x)$ does not exists, then $\lim_{x\to 0}[f(x) \cdot g(x)]$ also does not exist. Prove that f has this property if and only if $\lim_{x\to 0} f(x)$ exists.

Proof. We start by considering the case where $\lim_{x\to 0} f(x)$ exists and is equal to $m \neq 0$. Let $g(x)$ be a function such that $\lim_{x\to 0} g(x)$ does not exist. Assume that $\lim_{x\to 0}[f(x) \cdot g(x)]$ exists, so it is equal to some real ℓ. We then have:

$$\frac{\lim_{x\to 0}[f(x) \cdot g(x)]}{\lim_{x\to 0} f(x)} = \lim_{x\to 0} \left(\frac{f(x) \cdot g(x)}{f(x)}\right.$$

□

Problem 5.24. Suppose that A_n is, for each natural n, some finite set of numbers of $[0,\ 1]$, and that A_n and A_m are disjoint if $n \neq m$. Define f as follows:

$$f = \begin{cases} 1/n & x \in A_n \\ 0 & x \notin A_n \forall n \in \mathbb{N} \end{cases}$$

Prove that $\lim_{x\to a} f(x) = 0$ for any $a \in [0,\ 1]$.

Proof. Let us pick some $a \in [0,\ 1]$ and some $\epsilon > 0$. By the Archimedean property, we can pick some natural n such that $1/n < \epsilon$. Since each A_m contains only a finite number of elements, it follows that the union of the collection of set $\{A_1,\ ...,\ A_{n-1}\}$ also contains a finite number of elements.

By definition of f, this implies that there are a finite number of $x \in [0,\ 1]$ such that $1/n < f(x)$. We denote the set of such x by X. Then, we let:

$$\delta = \min\{|x - a| \mid x \in X - \{a\}\}$$

where the minimum of the set is well-defined, as X contains a finite number of elements. It then follows that given some y such that $0 < |y - a| < \delta$, y cannot possibly be in X, so it must be true that $f(x) \leq 1/n < \epsilon$.

□