Spivak Notes, Problems, and Solutions

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Contents

1 Introduction

The goal of this set of notes is to solve the most challenging problems in Spivak, and write up the solutions in a clean and concise way. I apologize in advance for any possible mistakes, or instances in which I may skip over certain important points.

2 Chapter 3

Problem 3.17. Prove that if $f(x+y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$, where $f(x) \neq 0$, then $f(x) = x$ for all x.

Proof. We go through the steps of the proof, as organized in Spivak:

1. Clearly, we will have $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$, so either $f(1) = 0$ or $f(1) = 1$. If we assume that $f(1) = 0$, then this would imply that $f(n) = 0$ for all n (we can prove this by induction, assuming that $f(n) = 0$, and noting that $f(n+1) = f(n) + f(1) = 0$). This is a contradiction to our initial assumption, so $f(1) = 1$.

2. First, we note that:

$$f(0) = f(0+0) = f(0) + f(0) \;\Rightarrow\; f(0) = 0$$

   Next, we note that $f(n) = n$, for natural n. We prove this by induction, first assuming that $f(n) = n$, then noting that $f(n+1) = f(n) + f(1) = n + 1$. We then note that $f(-n) = n - n + f(-n) = -n + f(n) + f(-n) = -n + f(0) = -n$. Thus, f is the identity for all integers.

   Now, we can see that:

$$f\left(\frac{1}{b}\right) = \frac{b}{b} \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(b) \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(1) = \frac{1}{b}$$

   Thus,

$$f\left(\frac{a}{b}\right) = f(a) \cdot f\left(\frac{1}{b}\right) = \frac{a}{b}$$

3. Assume that $x > 0$. It then follows that $\sqrt{x}$ is well-defined and greater than 0. We then have:

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = f(\sqrt{x})^2$$

   we know that for any real number r, we have $r^2 \geq 0$, so $f(x) \geq 0$. Assume that $f(x) = 0$. Since $x > 0$, this would imply that:

$$f(1) = f\left(\frac{x}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right) = 0$$

   a clear contradiction. Thus, $f(x) > 0$.

4. If $x > y$, then we know that $x - y > 0$, so it follows from previous result that:

$$f(x-y) > 0 \;\Rightarrow\; f(x) - f(y) > 0 \;\Rightarrow\; f(x) > f(y)$$

5. Assume that there exists some x such that $x < f(x)$. Since there exists a rational number between any two reals, it follows that we have:

$$x < \frac{a}{b} < f(x)$$

   for some a/b. From the previous result, we then get $f(x) < f(a/b)$, a clear contradiction to the right-most inequality above. Similarly, if we assume that $f(x) < x$, we will have:

$$f(x) < \frac{a}{b} < x$$

   so $f(a/b) < f(x)$, another contradiction. It follows that $f(x) = x$, and we have proved the proposition.

□

Problem 3.20B. If a function satisfies:

$$f(y) - f(x) \leq (x-y)^2$$

for all $, y \in \mathbb{R}$, then $f(x) = c$ for some c and all x

Part B is the interesting part of this problem, so I skipped writing out Part A

Proof. Assume that there exist distinct x and y such that $f(x) \neq f(y)$. Without loss of generality, let $f(x) < f(y)$. It follows that:

$$f(y) - f(x) \leq (y-x)^2$$

Consider what happens when we split up the interval from x to y into n "chunks". We let:

$$z_j = \Big(1 - \frac{j}{n}\Big)x + \frac{j}{n}y$$

so we get $z_0 = x$ and $z_n = y$. Clearly the distance between z_i and z_{j-1} is given by:

$$z_j - z_{j-1} = \Big(1 - \frac{j}{n}\Big)x + \frac{j}{n}y - \Big(1 - \frac{j-1}{n}\Big)x - \frac{j-1}{n}y = \frac{y-x}{n}$$

It follows that:

$$f(z_j) - f(z_{j-1}) \leq (z_j - z_{j-1})^2 = \frac{(y-x)^2}{n^2}$$

Now comes the crucial step. Notice that

$$\sum_{j=1}^{n} \big(f(z_j) - f(z_{j-1})\big) = f(z_n) - f(z_0) = f(y) - f(x)$$

as the rest of the terms cancel. Thus, we will have:

$$\sum_{j=1}^{n} \big(z_j - z_{j-1}\big) \leq \sum_{j=1}^{n} \frac{(y-x)^2}{n^2} \;\Rightarrow\; f(y) - f(x) \leq n \cdot \frac{(y-x)^2}{n^2} = \frac{(y-x)^2}{n}$$

for all possible values of n. Since we have assume $f(y) \neq f(x)$, it follows that $f(y) - f(x)$ is a positive real number, and that:

$$\epsilon = \frac{f(y) - f(x)}{(y-x)^2}$$

is a positive real as well. Thus, this implies that there exists a real number ϵ such that for any positive integer n:

$$\epsilon \leq \frac{1}{n}$$

But this clearly contradicts the Archimedean property of the real numbers. We have derived a contradiction, so it follows that for any x and y, $f(x) = f(y)$. Thus, the function f is constant.

□

3 Chpater 5

Lemma 1 (Uniqueness of Limits)**.** *The limit of a function is unique: If a function f approaches ℓ_1 as x approaches a, and f approaches ℓ_2 as x approaches a, then $\ell_1 = \ell_2$.*

Proof. Suppose the the function f approaches ℓ_1 and ℓ_2. It follows that given some $\epsilon > 0$, we can choose δ_1 and δ_2 such that:

$$|x-a| < \delta_1 \;\Rightarrow\; |f(x) - \ell_1| < \epsilon$$
$$|x-a| < \delta_2 \;\Rightarrow\; |f(x) - \ell_2| < \epsilon$$

Assume that $\ell_1 \neq \ell_2$, so $|\ell_1 - \ell_2| > 0$. Let us then pick $\epsilon = \frac{|\ell_1 - \ell_2|}{2}$. We can then pick δ_1 and δ_2 corresponding to this ϵ. We then let $\delta = \min(\delta_1,\ \delta_2)$ so:

$$|x-a| < \delta \;\Rightarrow\; |f(x) - \ell_1| < \;\epsilon \qquad \text{and} \qquad |f(x) - \ell_2| < \epsilon$$

It then follows that:

$$|x-a| < \delta \;\Rightarrow\; |f(x) - \ell_1| + |f(x) - \ell_2| < 2\epsilon = |\ell_1 - \ell_2|$$

We know that there exists some x_0 such that $|x_0 - a| < \delta$, which implies that:

$$|\ell_1 - \ell_2| \leq |f(x_0) - \ell_1| + |f(x_0) - \ell_2| < |\ell_1 - \ell_2|$$

a clear contradiction. It follows that ℓ_1 must equal ℓ_2.

□

Lemma 2 (Sums of Limits)**.** *If* $\lim_{x\to a} f(x) = m$ *and* $\lim_{x\to a} g(x) = \ell$*, then* $\lim_{x\to a}(f+g)(x) = m + \ell$*.*

Proof. Let us pick some $\epsilon > 0$. We will have, for $\epsilon/2$:

$$|x-a| < \delta_1 \;\Rightarrow\; |f(x) - m| < \epsilon/2$$
$$|x-a| < \delta_2 \;\Rightarrow\; |g(x) - \ell| < \epsilon/2$$

we choose $\delta = \min(\delta_1,\ \delta_2)$, giving us:

$$|x-a| < \delta \;\Rightarrow\; |f(x) - m| + |g(x) - \ell| < \epsilon$$

Then, given x such that $|x-a| < \delta$, we have:

$$|f(x) + g(x) - (m+\ell)| \leq |f(x) - m| + |g(x) - \ell| < \epsilon$$

Thus, given ϵ, we can choose a δ. It follows by definition that $\lim_{x\to a}(f+g)(x) = m + \ell$.

□

Problem 5.20. If $f(x) = x$ for rational x and $f(x) = -x$ for irrational x, show that $\lim_{x\to a} f(x)$ does not exist for $a \neq 0$.

Proof. Assume that there exists some non-zero a such that:

$$\lim_{x\to a} f(x) = L$$

It follows that for any $\epsilon > 0$, we can choose a δ such that if $|x-a| < \delta$, then $|f(x) - L| < \epsilon$. We begin by considering the case when $a > 0$. We let $\epsilon = a$ and assume that we can choose a δ such that:

$$|x-a| < \delta \;\Rightarrow\; |f(x) - L| < a$$

Now, since there exists a rational and an irrational number between any two reals, we pick rational r and irrational i from the interval $(a,\ a+\delta)$. We will then have:

$$|r-a| < \delta \;\Rightarrow\; |r - L| < a$$
$$|i-a| < \delta \;\Rightarrow\; |-i - L| = |i + L| < a$$

So we will have:

$$|(r-L) + (i+L)| = |r + i| \leq |r - L| + |i + L| < 2a$$

But this is a contradiction, as $a < i,\ r$, so $2a < i + r$. Thus, we can choose no such $\delta > 0$, and the limit does not exist.

Now, assume that $a < 0$. We let $\epsilon = |a|$ and assume that we can choose a corresponding δ. We then choose rational and irrational $r, i \in (a - \delta, a)$. Similar to above, we get:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2|a|$$

But this is a contradiction, as $i, r < a$, so $i + r < 2a$, which implies that $|i + r| > 2|a|$ (as a, i, and r are negative). Thus, we can choose no such δ, and the limit does not exist.

We conclude that the limit does not exist for any $a > 0$, and any $a < 0$, making $a = 0$ the only point at which the limit exists. □

Problem 5.12. Suppose that $f(x) \leq g(x)$ for all x. Prove that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, assuming the limits exist.

Proof. We let the first limit be denoted by ℓ_f and the second by ℓ_g. Assume that $\ell_f > \ell_g$. It follows that $\ell_f - \ell_g$ is a positive real number, so we let $\epsilon = (\ell_f - \ell_g)/2$. Now, by definition of the limits, we can choose δ_1 and δ_2 such that:

$$\begin{aligned} 0 < |x - a| < \delta_1 &\ \Rightarrow\ |f(x) - \ell_f| < (\ell_f - \ell_g)/2 \\ 0 < |x - a| < \delta_2 &\ \Rightarrow\ |g(x) - \ell_g| < (\ell_f - \ell_g)/2 \end{aligned}$$

.
We let $\delta = \min\{\delta_1,\ \delta_2\}$. We then have:

$$0 < |x - a| < \delta\ \Rightarrow\ |\ell_f - f(x) + g(x) - \ell_g| \leq |f(x) - \ell_f| + |g(x) - \ell_g| < \ell_f - \ell_g$$

So we have:

$$|(g(x) - f(x)) + (\ell_f - \ell_g)| < \ell_f - \ell_g$$

but since $f(x) \leq g(x)$ and $\ell_g < \ell_f$, both numbers in the brackets will be greater than or equal to 0, so:

$$g(x) - f(x) + \ell_f - \ell_g < \ell_f - \ell_g$$

which is a contradiction. Thus, $\ell_f \leq \ell_g$ and the proof is complete.

□

Problem 5.23. Let f be a function with the following property: if g is a function for which $\lim_{x\to 0} g(x)$ does not exists, then $\lim_{x\to 0}[f(x) \cdot g(x)]$ also does not exist. Prove that f has this property if and only if $\lim_{x\to 0} f(x)$ exists.

Proof. We start by considering the case where $\lim_{x\to 0} f(x)$ exists and is equal to $m \neq 0$. Let $g(x)$ be a function such that $\lim_{x\to 0} g(x)$ does not exist. Assume that $\lim_{x\to 0}[f(x) \cdot g(x)]$ exists, so it is equal to some real ℓ. We then have:

$$\frac{\lim_{x\to 0}[f(x) \cdot g(x)]}{\lim_{x\to 0} f(x)} = \lim_{x\to 0} \Big(\frac{f(x) \cdot g(x)}{f(x)}$$

□

Problem 5.24. Suppose that A_n is, for each natural n, some finite set of numbers of $[0,\ 1]$, and that A_n and A_m are disjoint if $n \neq m$. Define f as follows:

$$f = \begin{cases} 1/n & x \in A_n \\ 0 & x \notin A_n \forall n \in \mathbb{N} \end{cases}$$

Prove that $\lim_{x\to a} f(x) = 0$ for any $a \in [0,\ 1]$.

Proof. Let us pick some $a \in [0,\ 1]$ and some $\epsilon > 0$. By the Archimedean property, we can pick some natural n such that $1/n < \epsilon$. Since each A_m contains only a finite number of elements, it follows that the union of the collection of set $\{A_1,\ ...,\ A_{n-1}\}$ also contains a finite number of elements.

By definition of f, this implies that there are a finite number of $x \in [0,\ 1]$ such that $1/n < f(x)$. We denote the set of such x by X. Then, we let:

$$\delta = \min\{|x-a| \mid x \in X - \{a\}\}$$

where the minimum of the set is well-defined, as X contains a finite number of elements. It then follows that given some y such that $0 < |y-a| < \delta$, y cannot possibly be in X, so it must be true that $f(x) \leq 1/n < \epsilon$. □

4 Chapter 7

Problem 7. Suppose that ϕ is continuous and $\lim_{x\to\infty} \phi(x)/x^n = 0 = \lim_{x\to-\infty} \phi(x)/x^n$.

Prove that if n is odd, then there is a number x such that $x^n + \phi(x) = 0$

Proof. Since ϕ and x^n are both continuous, the function $g(x) = x^n + \phi(x)$ is also continuous. We also note that:

$$\lim_{x\to\pm\infty} \frac{g(x)}{x^n} = \lim_{x\to\pm\infty} \big(1 + \frac{\phi(x)}{x^n}\big) = 1$$

So it follows that:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{g(x)}{x^n} x^n = \lim_{x\to\infty} x^n \lim_{x\to\infty} \frac{g(x)}{x^n} = \lim_{x\to\infty} x^n = \infty$$

and similarly:

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{g(x)}{x^n} x^n = -\infty$$

Thus, it follows that there must exist C such that $f(C) < 0$ and D such that $f(D) > 0$. Thus, we apply intermediate value theorem to the interval $[C,\ D]$, to conclude there must exist some x such that $g(x) = x^n + \phi(x) = 0$. This completes the proof. □

Prove that if n is even, then there is a number y such that $y^n + \phi(y) \leq x^n + \phi(x)$ for all x.

Proof. Again, we let $g(x) = x^n + \phi(x)$ and note that the limit as $g(x)/x^n$ approaches $\pm\infty$ is 1. From this, we proceed in a similar fashion to the previous proof, noting that the limit as $g(x)$ approaches $\pm\infty$ is ∞.

Thus, we can choose some A such that $x > A > 0 \ \Rightarrow\ g(x) > g(0)$, and some B such that $x < B < 0 \ \Rightarrow$ $g(x) > g(0)$. We then apply the extreme value theorem to the interval $[A,\ B]$ to conclude that this interval has a minimum, which occurs at the point $x = y$. We note that all points not in this interval are greater than $g(0)$, which is itself in the interval, so y must be the global minimum of the function. By definition:

$$y^n + \phi(y) \leq x^n + \phi(x)$$

for all x and the proof is complete. □

Problem 10.19a. Suppose that f is continuous on $[0,\ 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x such that $f(x) = f(x + 1/n)$.

Proof. Consider the continuous function $g(x) = f(x) - f(x + 1/n)$. Suppose that $g(x) \neq 0$ for all x. The intermediate value theorem implies that $g(x)$ must then either be positive for all x, or negative for all x, as if it took on both positive and negative values, it would have to take on the value 0.

This would then imply that $f(x) > f(x+1/n)$ or $f(x) < f(x+1/n)$ for all x. In either case, we would have $f(0) > f(1/n)$ or $f(1/n) < f(0)$. It is then easy to prove inductively that for all k from 1 to n, we will have

$f(0) > f(k/n)$ or $f(k/n) < f(0)$, which gives us $f(0) > f(1)$ or $f(1) < f(0)$, both clear contradictions.

Thus, there must exist a value of x for which $g(x) = 0$, implying that $f(x) = f(x + 1/n)$. $\square$

Problem 10.19b.

Problem 10.20a. Prove that there does not exist a continuous function on $\mathbb{R}$ that takes on every value exactly twice.

Proof. Suppose that we have a function f such that it takes on every value exactly twice. It follows that given some value a, there exists exactly one other value b such that $f(a) = f(b)$. Assume without loss of generality that $a < b$.

It follows that we can apply the intermediate value theorem to the interval $[a,\ b]$, to conlude that either $f(a) < f(x)$ for all $x \in (a,\ b)$ or $f(x) < f(a)$ for all $x \in (a,\ b)$, as if the function changes sign, then there must be some $c \in (a,\ b)$ such that $f(a) = f(c) = f(b)$, a contradiction.

We can again assume without loss of generality that $f(x) < f(a)$ for all $x \in (a,\ b)$. Now, by extreme value theorem, there is a value $y \in (a,\ b)$ such that $f(y) \geq f(x)$ for all $x \in [a,\ b]$. Thus, we apply intermediate value theorem to the intervals $[a,\ y]$ and $[y,\ b]$ to find that every $c \in [f(a),\ f(y)]$, except $f(y)$, is equal to $f(x)$ for exactly two $x \in [a,\ b]$.

Since f takes on the same value exactly twice, it follows that for $x < a$ or $x > b$, we must have $f(x) < f(a)$, or else there would exist some value that the function takes on at least three times. But $f(a) \leq f(y)$, so it follows that $f(y)$ is only taken on by one value of x. This is a contradiction, so such a function cannot exist. $\square$

5 Chapter 10

Problem 10.27. Suppose f is differentiable at 0, and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g which is continuous at 0.

Proof. Since f is differentiable at 0, it follow that:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

is a well-defined real number. Thus, we define $g(x) = f(x)/x$ for all x other than 0 and $f'(0)$ when $x = 0$. Clearly, since f is differentiable, it is continuous, so:

$$\lim_{x \to a} g(x) = \lim_{x \to a} \frac{f(x)}{x} = \frac{f(a)}{a}$$

for non-zero a, and $f'(0)$ for $a = 0$. Thus, the function g is continuous at all a, and $f(x) = xg(x)$ for all x. This completes the proof. $\square$

Problem 10.28. Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$

Proof. Assume that this is possible. It would follow from differentiating that:

$$f'(x)g(x) + g'(x)f(x) = 1$$

where $f'(x)$ and $g'(x)$ are well defined. We then must have $f'(0)g(0) + g'(0)f(0) = 1$, which is impossible, as $f(0) = g(0) = 0$. Thus, we cannot find f and g such that $f(x)g(x) = x$ and the proof is complete. $\square$

6 Other Proofs

Proposition 1. *If $f : A \to \mathbb{R}$ is differentiable at a, and $g : B \to \mathbb{R}$ is differewntiable at $b = f(a)$, with $f(A) \subset B$. Then $g \circ f$ is differentiable at a with:*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. First, we define two functions that effectively act as the "derivatives" of f and g for x approaching a and $b = f(a)$ respectively:

$$\tilde{f}(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases} \tag{1}$$

$$\tilde{g}(x) = \begin{cases} \frac{g(x)-g(b)}{x-b} & \text{if } x \neq b \\ g'(b) & \text{if } x = b \end{cases} \tag{2}$$

Clearly, by definition of the limit and the derivative, these functions are continuous at a and b respectively.

We assert that:

$$\frac{g(f(x)) - g(f(a))}{x - a} = \tilde{g}(f(x))\tilde{f}(x)$$

for all $x \neq a$, which can easily be verified by checking the cases of $f(x) = f(a)$ and $f(x) \neq f(a)$. The main idea here is that we are showing the expression of which we are calculating the limit is equal to the product of expressions which one would take the limit of to get $g'(f(a))$ and $f'(a)$.

From here, it is just a matter of actually carrying out the limit:

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} \tilde{g}(f(x))\tilde{f}(x) = \lim_{x \to a} \tilde{g}(f(x)) \lim_{x \to a} \tilde{f}(x) = \tilde{g}(b)f'(a) = g'(f(a))f'(a)$$

and the proof is complete. $\square$

Proposition 2.