

# Spivak Problems and Solutions

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## 1 Introduction

The goal of this set of notes is to solve the most challenging problems in Spivak, and write up the solutions in a clean and concise way. I apologize in advance for any possible mistakes, or instances in which I may skip over certain important points.

## 2 Chapter 3

**Problem 3.17.** Prove that if  $f(x + y) = f(x) + f(y)$  and  $f(x \cdot y) = f(x) \cdot f(y)$ , where  $f(x) \neq 0$ , then  $f(x) = x$  for all  $x$ .

*Proof.* We go through the steps of the proof, as organized in Spivak:

1. Clearly, we will have  $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$ , so either  $f(1) = 0$  or  $f(1) = 1$ . If we assume that  $f(1) = 0$ , then this would imply that  $f(n) = 0$  for all  $n$  (we can prove this by induction, assuming that  $f(n) = 0$ , and noting that  $f(n + 1) = f(n) + f(1) = 0$ ). This is a contradiction to our initial assumption, so  $f(1) = 1$ .
2. First, we note that:

$$f(0) = f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0$$

Next, we note that  $f(n) = n$ , for natural  $n$ . We prove this by induction, first assuming that  $f(n) = n$ , then noting that  $f(n + 1) = f(n) + f(1) = n + 1$ . We then note that  $f(-n) = n - n + f(-n) = -n + f(n) + f(-n) = -n + f(0) = -n$ . Thus,  $f$  is the identity for all integers.

Now, we can see that:

$$f\left(\frac{1}{b}\right) = \frac{b}{b} \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(b) \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(1) = \frac{1}{b}$$

Thus,

$$f\left(\frac{a}{b}\right) = f(a) \cdot f\left(\frac{1}{b}\right) = \frac{a}{b}$$

3. Assume that  $x > 0$ . It then follows that  $\sqrt{x}$  is well-defined and greater than 0. We then have:

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = f(\sqrt{x})^2$$

we know that for any real number  $r$ , we have  $r^2 \geq 0$ , so  $f(x) \geq 0$ . Assume that  $f(x) = 0$ . Since  $x > 0$ , this would imply that:

$$f(1) = f\left(\frac{x}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right) = 0$$

a clear contradiction. Thus,  $f(x) > 0$ .

4. If  $x > y$ , then we know that  $x - y > 0$ , so it follows from previous result that:

$$f(x - y) > 0 \Rightarrow f(x) - f(y) > 0 \Rightarrow f(x) > f(y)$$

5. Assume that there exists some  $x$  such that  $x < f(x)$ . Since there exists a rational number between any two reals, it follows that we have:

$$x < \frac{a}{b} < f(x)$$

for some  $a/b$ . From the previous result, we then get  $f(x) < f(a/b)$ , a clear contradiction to the right-most inequality above. Similarly, if we assume that  $f(x) < x$ , we will have:

$$f(x) < \frac{a}{b} < x$$

so  $f(a/b) < f(x)$ , another contradiction. It follows that  $f(x) = x$ , and we have proved the proposition.

□

**Problem 3.20B.** If a function satisfies:

$$f(y) - f(x) \leq (x - y)^2$$

for all  $x, y \in \mathbb{R}$ , then  $f(x) = c$  for some  $c$  and all  $x$

*Part B is the interesting part of this problem, so I skipped writing out Part A*

*Proof.* Assume that there exist distinct  $x$  and  $y$  such that  $f(x) \neq f(y)$ . Without loss of generality, let  $f(x) < f(y)$ . It follows that:

$$f(y) - f(x) \leq (y - x)^2$$

Consider what happens when we split up the interval from  $x$  to  $y$  into  $n$  “chunks”. We let:

$$z_j = \left(1 - \frac{j}{n}\right)x + \frac{j}{n}y$$

so we get  $z_0 = x$  and  $z_n = y$ . Clearly the distance between  $z_i$  and  $z_{j-1}$  is given by:

$$z_j - z_{j-1} = \left(1 - \frac{j}{n}\right)x + \frac{j}{n}y - \left(1 - \frac{j-1}{n}\right)x - \frac{j-1}{n}y = \frac{y-x}{n}$$

It follows that:

$$f(z_j) - f(z_{j-1}) \leq (z_j - z_{j-1})^2 = \frac{(y-x)^2}{n^2}$$

Now comes the crucial step. Notice that

$$\sum_{j=1}^n (f(z_j) - f(z_{j-1})) = f(z_n) - f(z_0) = f(y) - f(x)$$

as the rest of the terms cancel. Thus, we will have:

$$\sum_{j=1}^n (z_j - z_{j-1}) \leq \sum_{j=1}^n \frac{(y-x)^2}{n^2} \Rightarrow f(y) - f(x) \leq n \cdot \frac{(y-x)^2}{n^2} = \frac{(y-x)^2}{n}$$

for all possible values of  $n$ . Since we have assume  $f(y) \neq f(x)$ , it follows that  $f(y) - f(x)$  is a positive real number, and that:

$$\epsilon = \frac{f(y) - f(x)}{(y-x)^2}$$

is a positive real as well. Thus, this implies that there exists a real number  $\epsilon$  such that for any positive integer  $n$ :

$$\epsilon \leq \frac{1}{n}$$

But this clearly contradicts the Archimedean property of the real numbers. We have derived a contradiction, so it follows that for any  $x$  and  $y$ ,  $f(x) = f(y)$ . Thus, the function  $f$  is constant. □

### 3 Chpater 5

**Problem 5.20.** If  $f(x) = x$  for rational  $x$  and  $f(x) = -x$  for irrational  $x$ , show that  $\lim_{x \rightarrow a} f(x)$  does not exist for  $a \neq 0$ .

*Proof.* Assume that there exists some non-zero  $a$  such that:

$$\lim_{x \rightarrow a} f(x) = L$$

It follows that for any  $\epsilon > 0$ , we can choose a  $\delta$  such that if  $|x - a| < \delta$ , then  $|f(x) - L| < \epsilon$ . We begin by considering the case when  $a > 0$ . We let  $\epsilon = a$  and assume that we can choose a  $\delta$  such that:

$$|x - a| < \delta \Rightarrow |f(x) - L| < a$$

Now, since there exists a rational and an irrational number between any two reals, we pick rational  $r$  and irrational  $i$  from the interval  $(a, a + \delta)$ . We will then have:

$$\begin{aligned} |r - a| < \delta &\Rightarrow |r - L| < a \\ |i - a| < \delta &\Rightarrow |-i - L| = |i + L| < a \end{aligned}$$

So we will have:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2a$$

But this is a contradiction, as  $a < i, r$ , so  $2a < i + r$ . Thus, we can choose no such  $\delta > 0$ , and the limit does not exist.

Now, assume that  $a < 0$ . We let  $\epsilon = |a|$  and assume that we can choose a corresponding  $\delta$ . We then choose rational and irrational  $r, i \in (a - \delta, a)$ . Similar to above, we get:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2|a|$$

But this is a contradiction, as  $i, r < a$ , so  $i + r < 2a$ , which implies that  $|i + r| > 2|a|$  (as  $a, i$ , and  $r$  are negative). Thus, we can choose no such  $\delta$ , and the limit does not exist.

We conclude that the limit does not exist for any  $a > 0$ , and any  $a < 0$ , making  $a = 0$  the only point at which the limit exists.  $\square$