Problem set 1

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1 Problem 0

Problem 1.1. Prove that if $|\cdot|$ is non-Archimedean on k, and $|x| \neq |y|$, then $|x+y| = \max(|x|, |y|)$.

Proof. We know that $|x+y| \leq \max(|x|, |y|)$. In addition, since $|x| \neq |y|$, we have $|y| < |x|$ or $|y| < |x|$. Suppose without loss of generality the former case. Then we have

$$|y| < |x| = |x+y-y| \leq \max(|x+y|, |y|) = |x+y| \tag{1}$$

where the final equality follows from the fact that if $\max(|x+y|, |y|) = |y|$, then we would have $|x| \leq |y|$, a contradiction. Thus, $\max(|x|, |y|) \leq |x+y|$, and we have equality. $\square$

Problem 1.2.

Problem 1.3.

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Problem 2.1. Prove that $|\cdot|$ is non-Archimedean if and only if $|1 + \cdots + 1| \leq 1$ for every finite sum of 1s.

Proof. Let $n = 1 + \cdots + 1$ repeated n times. Note that if $|\cdot|$ is non-Archimedean, then $|n| = |n - 1 + 1| \leq \max(|n-1|, 1)$, so the result follows via induction. Conversely, if $|n| \leq 1$ for all n, then note that if $|z| < 1$, we have

$$|1+z|^N = |(1+z)^N| = \left| \sum_{k=0}^{N} \binom{N}{k} z^k \right| \leq \sum_{k=0}^{N} \left| \binom{N}{k} \right| |z|^k \leq \sum_{k=0}^{N} |z|^k \leq \frac{1}{1-|z|} \tag{2}$$

for any N. In particular, $|1+z|^N$ is bounded as we take $N \to \infty$, so it must be the case that $|1+z| \leq 1$. Therefore, $|1+z| \leq \max(|z|, 1)$. It follows immediately that if $|x| \neq |y|$, so $|x| < |y|$ or $|y| < |x|$ (without loss of generality, assume the first case), then $\frac{|x|}{|y|} < 1$, and we have

$$|x+y| = |y| \left| 1 + \frac{x}{y} \right| \leq |y| = \max(|x|, |y|) \tag{3}$$

Therefore, the only remaining case is when $|x| = |y|$. Equivalently, suppose we have $|z| = 1$ and $|1+z| = \alpha > 1$. Then $|(1+z)^N| = \alpha^N$ grows exponentially, but as before

$$|(1+z)^N| \leq \sum_{k=0}^{N} |z|^k = \frac{N(N+1)}{2} \tag{4}$$

which grows polynomially in N. Hence, we have a contradiction: α^N must eventually exceed this bound. It follows that we must have $|1+z| \leq 1$ as well, which completes the proof. ☐

Remark 2.1. As an immediate corollary, it follows that if field F has positive characteristic p, so $p = 0$, then if $|\cdot|$ is some absolute value, we note that $|n|$ will be bounded for any integer n: it will be the maximum of $|1|, |2|, \ldots, |p-1|$. It follows that we must have $|n| \leq 1$ for all n, otherwise we could make $|n^K| = |n|^K$ arbitrarily large. If we have a finite field of characteristic p, pick some n, and note that n^K will eventually return to n for some large enough $K > 1$, so $|n| = |n|^K$, which means $|n|^{K-1} = 1$ (in the case $n \neq 0$). Thus, $|n| = 1$, and the absolute value is trivial.

Problem 2.2 (Ostrowski's theorem)**.** Prove that every non-trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime $p \leq \infty$.

Proof. Recall that the p-adic absolute value on $\mathbb{Q}$ is given by taking $|x|_p = p^{-v_p(x)}$, where $v_p(x)$ is the multiplicity of a prime factor in x. If norm $|\cdot|$ is non-trivial, then we will have $|n| \neq 0, 1$ for some $n \neq 0$. We can factor n into primes, so we must have $|p| \neq 0, 1$ for some prime p. Of course, $|p|_p = p^{-1}$, so we let $\alpha = -\frac{\log(p)}{\log(|p|)}$ (well-defined as $|p| \neq 0, 1$) and note that $|p|^\alpha = |p|_p$. Immediately, we have $|p^k|^\alpha = |p^k|_p$ for all powers of p. For any other prime $q \neq p$, we have $|q|_p = 1$, so we must show that $|q| = 1$ for all such q. We can write, for any n,

$$\frac{1}{p} = \frac{k}{q^n} + r \tag{5}$$

for some $k \in \mathbb{Z}$ less than q^n, and remainder r which has magnitude less than $\frac{1}{q^n}$. Note that we will never have $r = 0$, as we would have $q^n = pk$, but in the prime factorization of the right-hand side, the exponent of q is at most $n-1$. It then follows that

$$\left|\frac{1}{p}\right| = \left|\frac{k}{q^n} + r\right| \leq \frac{|k|}{|q|^n} + |r| \tag{6}$$

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Problem 2.3.

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