Diagram chasing via generalized elements in Abelian categories

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I. Introduction

The main goal of this short note is to explain a very useful technique for performing "elementwise diagram chases" in potentially non-concrete Abelian categories by means of "generalized elements". When tasked with performing some kind of diagram chase while operating in the category of Abelian groups, or perhaps the category of R-modules for some ring R, we are usually able to construct/prove properties of maps by considering where they send particular *elements* of the group/module in question. In arbitrary Abelian categories, our underlying objects are not always sets, and our arrows are not always set maps, therefore speaking of an "element" of an object in an arbitrary Abelian category doesn't make sense naively. However, with some work, we can show that it *is* possible to come up with a notion of a *generalized element* of an object in an Abelian category, and moreover, these generalized objects behave in many ways similarly to the underlying elements of an object in some concrete, Abelian category (like Abelian groups or R-modules).

This treatment follows *Categories for the Working Mathematician*, by Mac Lane.

II. Generalized elements

Let us first prove a technical lemma:

Lemma II.1. Given a pullback square in an Abelian category $\mathcal{C}$, if the bottom edge f is epi, then the top edge f' is epi. Also, the kernel k of f is $k = g' \circ k'$, where g' is the left edge of the square and k' is the kernel of f'. All together, this forms a diagram of the form

$$\begin{tikzcd}
 & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
A \arrow[ur, "k'"] \arrow[r, "k"'] & B \arrow[r, "f"'] & C
\end{tikzcd}$$

Proof. A pullback in an Abelian category can be constructed by means of products and equalizers. In particular, we claim that if $B \oplus D$ is a biproduct with projections p_1, p_2 (which exists in our Abelian category), then $\mathrm{Ker}(f \circ p_1 - g \circ p_2)$, which also exists in the Abelian category, is a pullback. Letting S be the object of the kernel and m be the monic into $B \oplus D$, we have left-exact sequence

$$0 \longrightarrow S \xrightarrow{m} B \oplus D \xrightarrow{f \circ p_1 - g \circ p_2} C$$

We let $g' = p_1 \circ m$ and $f' = p_2 \circ m$. Of course, $(f \circ p_1 - g \circ p_2) \circ m = 0$, so $f \circ g' = g \circ f'$. In addition, given some S' and maps q_1, q_2 projecting onto B and D, where $f \circ q_1 = g \circ q_2$, so $f \circ q_1 - g \circ q_2 = 0$, then by the the universal property of the kernel, there is unique $j : S' \to S$ which makes the combined pullback diagram commute, implying that S is, in fact, a pullback. It is also easy to see that $f \circ p_1 - g \circ p_2$ is a cokernel of m.

Thus, we can assume without loss of generality that the f' and g' in the diagram of the lemma are of the form in the previous paragraph (as any pullback will be isomorphic in the proper sense to a pullback of the above form). In addition, if f is epi, note that $f \circ p_1 - g \circ p_2$ is epi, as if $h \circ (f \circ p_1 - g \circ p_2) = 0$, then using injection $i_1 : B \to B \oplus D$, we have

$$0 = h \circ (f \circ p_1 - g \circ p_2) \circ i_1 = h \circ f \tag{1}$$

implying $h = 0$, as f is assumed to be epi. From here, suppose $u \circ f' = u \circ p_2 \circ m = 0$. It follows by the universal property that there is a unique map $r : C \to X$ where for $(f \circ p_1 - g \circ p_2) : B \oplus D \to C$, the defining map of the cokernel (an epi), we have

$$r \circ (f \circ p_1 - g \circ p_2) = u \circ p_2 \tag{2}$$

so composing on the right with i_1 gives us $r \circ f = 0$, and since f is epi, $r = 0$, so $u \circ p_2 = 0$. Then, since p_2 is epi, $u = 0$, and f' is epi as desired.

Finally, let $k : A \to B$ be the kernel of f. By the universal property of the pullback, we get unique map $k' : A \to S$,

$$\begin{tikzcd}
A \arrow[dr, "k'" description] \arrow[drr, "0" description] \arrow[ddr, "k"'] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

To show that $k' : A \to S$ is the kernel of f', pick some map $j : R \to S$ such that $f' \circ j = 0$. We have map $g' \circ j : R \to B$ where $f \circ g' \circ j = g' \circ f \circ j = 0$, so there is unique $\ell : R \to A$ with

$$\begin{tikzcd}
& R \arrow[ddl, "g' \circ j"'] \arrow[d, "\ell"] \arrow[ddr, "0"] & \\
& A \arrow[dl, "k"] \arrow[dr, "0"'] & \\
B \arrow[rr, "f"'] & & C
\end{tikzcd}$$

so in particular, $g' \circ j = g' \circ k' \circ \ell$. In addition, we have $f' \circ j = 0 = f' \circ k' \circ \ell$ from the first diagram. Thus, putting j and $k' \circ \ell$ as the dashed arrow make the following diagram commute:

$$\begin{tikzcd}
R \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

so by uniqueness of the pullback, we must have $j = k' \circ \ell$. Thus, we have induced ℓ making the following diagram commute:

$$\begin{tikzcd}
& R \arrow[ddl, "j"'] \arrow[d, "\ell"] \arrow[ddr, "0"] & \\
& A \arrow[dl, "k'"] \arrow[dr, "0"'] & \\
S \arrow[rr, "f'"'] & & D
\end{tikzcd}$$

Moreover, ℓ is unique as if we replaced it with ℓ', then such an ℓ' would make the original kernel diagram where ℓ appeared commute as well, and by uniqueness of this diagram, $\ell' = \ell$. Thus, by definition, $k' : A \to S$ is the kernel of $f' : S \to D$, and the proof is complete. $\square$

There is also a dual result which we can prove through similar means (for this reason, we omit the proof).

Lemma II.2. Given a pushout square in an Abelian category $\mathcal{C}$, if the top edge g is epi, then the bottom edge g' is epi.

We will now use these technical lemmas to introduce machinery which will make our lives much easier, going forward.

Definition II.1 (Elements)**.** Let $\mathcal{C}$ be an Abelian category, let C be an object. We call an arrow $x : X \to C$ (with target C) a *member* of C, and denote it by $x \in C$. We say that two members x and y of C are equivalent if there are epis u and v such that $x \circ u = y \circ v$. One can easily see that this relation is symmetric and reflexive.

Claim II.1. The "equivalence of elements" relation defined above is transitive, hence an equivalence relation.

Proof. Suppose y and z are equivalent, so $y \circ v' = z \circ w$. We can combine together this commutative sqaure with $x \circ u = y \circ v$ to obtain the composite diagram

$$\begin{tikzcd}
\bullet \arrow[r, "v'"] \arrow[d, "w"'] & \bullet \arrow[d, "y"] & \bullet \arrow[l, "v"'] \arrow[d, "u"] \\
\bullet \arrow[r, "z"'] & a & \bullet \arrow[l, "x"]
\end{tikzcd}$$

We can then pullback in the top pair of arrows to obtain

$$\begin{tikzcd}
& \bullet \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\bullet \arrow[r, "v'"] \arrow[d, "w"'] & \bullet \arrow[d, "y"] & \bullet \arrow[l, "v"'] \arrow[d, "u"] \\
\bullet \arrow[r, "z"'] & a & \bullet \arrow[l, "x"]
\end{tikzcd}$$

where we note that now, $z \circ (w \circ p) = x \circ (u \circ q)$. Since both v and v' are epis, it follows from the technical lemma that p and q are epis. Moreover, w and u are epis, so the compositions $w \circ p$ and $u \circ q$ are epis. Thus, z is equivalent to x, and we have transitivity, as desired. When x and equivalent to y, we use the notation $x \sim y$, going forward. $\square$

Definition II.2. A *generalized element* (or just an *element*) of object C in Abelian category $\mathcal{C}$ is an equivalence class of members $x \in C$ under the equivalence relation $\sim$ defined above. The generalized element to which x belongs is denoted by $[x]$. We also use the notation $[x] \in C$ to denote a generalized element in C.

Given some arrow $f : C \to D$, note that if $x \in C$ is a member, then $f \circ x \in D$. Moreover, if $x \sim y$ in C, then $f \circ x \sim f \circ y$ in D, so the arrow f is a well-defined map from the generalized elements of C, to the generalized elements of D, $f([x]) = [f \circ x]$. Because we are working in an Abelian category, note that every object has a zero element, the equivalence class of the zero map $0 \to C$ and member x has a negative $-x$, so we denote $-[x] = [-x]$ (if $x \sim y$, then it is easy to check that $-x \sim -y$). Note, however, that we generally cannot perform arithmetic on generalized elements (i.e. $x \simeq x'$ and $y \sim y'$ does not imply that $x + y \sim x' + y'$), because the epis that we precompose these elements with may differ.

Now, let us prove the main theorem which characterizes elements

Theorem II.1. If $\mathcal{C}$ is an Abelian category, then the following hold:

1. Arrow $f : C \to D$ is monic if and only if, for all elements $[x] \in C$, $f([x]) = [0]$ implies $[x] = [0]$.

2. Moreover, $f : C \to D$ is monic if and only if, for all $[x], [x'] \in C$, $f([x]) = f([x'])$ implies $[x] = [x']$.

3. Arrow $g : C \to D$ is epi if and only if for all $[z] \in D$, there exists $[y] \in C$ such that $g([y]) = [z]$.

4. Arrow $h : C \to D$ is the zero arrow if and only if $h([x]) = [0]$ for all $[x] \in C$.

5. Let $f : C \to D$ be an arrow, let $j : K \to C$ be a kernel of f, and let $i : F \to D$ be an image. Then for every $[x] \in C$ such that $f([x]) = [0]$, there is unique $[k] \in K$ such that $[x] = j([k])$. Since $f(j([k])) = [0]$ for all $[k]$, it follows that such $[x]$ and the $[k]$ are in bijective correspondence.

6. Additionally, elements of the form $f([x])$ and $i([z])$ for $[z] \in F$ are in bijective correspondence.

7. If the sequence $B \to C \to D$ with arrows f and g is exact, then $g \circ f = 0$ and for all $[y] \in C$ with $g([y]) = [0]$, there exists $[x] \in B$ with $f([x]) = [y]$. The converse is also true.

Proof. To begin, assume the condition that $f([x]) = [f \circ x] = [0]$ implies $[x] = [0]$. Then if $f \circ x = 0$, then $x \circ v = 0$ for some epi v, so $x = 0$, and f is monic. On the other hand, if f is monic, and $f([x]) = [0]$, then there is epi u such that $f \circ x \circ u = 0$, so $x \circ u = 0$, so $[x] = [0]$. In addition, the condition "$f([x]) = f([x'])$ implies $[x] = [x']$" means that if $f \circ x = 0 = f \circ 0$, so $f([x]) = f([0])$, then $[x] = [0]$, so $x \circ v = 0$, so $x = 0$. In addition, if f is monic, then $f \circ x \circ u = f \circ x' \circ v$ implies that $x \circ u = x' \circ v$, so $[x] = [x']$.

If there is $[y]$ with $g([y]) = [z]$ for each $[z]$, then note that if $f \circ g = 0$, we can pick $[y] \in C$ such that $g([y]) = [\mathrm{id}]$, so that $g \circ y \circ u = v$ for some epi v. Then we get

$$0 = f \circ g \circ y \circ u = f \circ u \tag{3}$$

and since u is an epi, $f = 0$. Thus, g is an epi. Conversely, if g is epi, then note that for given z, we have pullback

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "y"'] & \bullet \arrow[d, "z"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

so $g \circ y = z \circ u$, where u is epi because g is (from the technical lemma). Thus, $g([y]) = [z]$. Moving on to the fourth point, if $h([x]) = [0]$ for all x, then $h \circ x \circ u = 0$ for all x, so since u is epi, $h \circ x = 0$ for all x, so setting $x = \mathrm{id}$, $h = 0$. On the other hand, if h is the zero arrow, clearly $h([x]) = [0]$ for all $[x]$.

Now, if $f([x]) = [0]$, then $f \circ x \circ u = 0$ for some epi u, so $f \circ x = 0$. Thus, there is unique k pointing to K such that $j \circ k = x$, so $j([k]) = [x]$.

In addition, given some $f([x])$, we let $q : D \to Q$ be the cokernel map of f, and note that we have

$$q \circ f \circ x \circ u = 0 \tag{4}$$

so $q \circ (f \circ x) = 0$, so there is z pointing to F such that $i \circ z = f \circ x$, which means that $i([z]) = f([x])$. To see that this $[z]$ is unique, if we also had $i([z']) = f([x])$, then $i \circ z \circ u = i \circ z' \circ v$, so since i is monic, $z \circ u = z' \circ v$ and $[z] = [z']$. Conversely, given $i([z])$, note that via the universal property, we have induced map $\widetilde{f}$,

$$\begin{tikzcd} & C \arrow[ddl, "f"'] \arrow[d, "\widetilde{f}"] \arrow[ddr, "0"] & \\ & F \arrow[dl, "i"] \arrow[dr, "0"'] & \\ D \arrow[rr, "q"'] & & Q \end{tikzcd}$$

so that $f = i \circ \widetilde{f}$. We can prove that $\widetilde{f}$ is epi by noting that if $u \circ \widetilde{f} = 0$, we have pushout of $i : F \to D$ and $u : F \to \cdot$. Note that

$$u' \circ f = u' \circ i \circ \widetilde{f} = i' \circ u \circ \widetilde{f} = 0 \tag{5}$$

so we get an induced map (the dahsed line), by universal property of the cokernel, in the following diagram:

$$\begin{tikzcd}
& C \arrow[dl, "\widetilde{f}"'] \arrow[dr, "f"] & & \\
F \arrow[rr, "i"'] \arrow[d, "u"'] & & D \arrow[r, "q"'] \arrow[d, "u'"] & Q \arrow[dl, dashed, "\phi"] \\
\bullet \arrow[rr, "i'"'] & & \bullet &
\end{tikzcd}$$

It then follows that $i' \circ u = \phi \circ q \circ i = 0$. We know from the technical lemmas that i' is mono, so $u = 0$, which means $\widetilde{f}$ is epi. Therefore, for some $[z]$, we can choose $[x]$ such that $\widetilde{f}([x]) = [z]$, so $i([z]) = f([x])$, as desired.

Moving along, if $B \to C \to D$ is exact, we have $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Let $j : K \to C$ be the shared defining monic morphism. First note that if $g([y]) = 0$, then we can pick $[z] \in K$ such that $j([z]) = [y]$, and $[x]$ such that $f([x]) = j([z])$. Moreover, since every $f([x])$ is of the form $j([z])$, we have

$$(g \circ f)([x]) = (g \circ j)([z]) = [0] \tag{6}$$

for all $[x]$, so $g \circ f = 0$, as desired.

To prove the converse, take the kernel map $j : K \to C$ and note that $[j] \in C$ with $g([j]) = 0$, so $f([x]) = [j]$ for some $[x]$, so $f \circ x \circ u = j \circ v$ for epis u and v. Thus, if we let $q : C \to Q$ be the quotient of $\mathrm{Coker}(f)$, then $q \circ j \circ v = 0$, so $q \circ j = 0$, so we have unique arrow from K to F, the image of f, making the universal diagram commute. The fact that $g \circ f = 0$ implies that $g \circ i = 0$, as any $(g \circ i)([z])$ can be written as $(g \circ i)([x]) = [0]$, for any $[z]$. Thus, there is a unique arrow from F to K as well, making the universal diagram commute. Putting these diagrams together, it is easy to check that the composite diagram commutes, and uniqueness implies these arrows will be inverses of each other, so we have the desired exactness:

$$\begin{tikzcd}
& K \arrow[rr, dashed, shift left] \arrow[dl, "0"'] \arrow[dr, "j"] & & F \arrow[ll, dashed, shift left] \arrow[dl, "i"'] \arrow[dr, "0"] & \\
D & & C \arrow[ll, "g"] \arrow[rr, "q"'] & & Q
\end{tikzcd}$$

and the proof is finally complete. $\square$