# Spivak Problems and Solutions

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## 1 Introduction

The goal of this set of notes is to solve the most challenging problems in Spivak, and write up the solutions in a clean and concise way. I apologize in advance for any possible mistakes, or instances in which I may skip over certain important points.

# 2 Chapter 3

**Problem 3.17.** Prove that if $f(x+y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$, where $f(x) \neq 0$, then $f(x) = x$ for all $x$.

*Proof.* We go through the steps of the proof, as organized in Spivak:

1. Clearly, we will have $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$, so either $f(1) = 0$ or $f(1) = 1$. If we assume that $f(1) = 0$, then this would imply that $f(n) = 0$ for all $n$ (we can prove this by induction, assuming that $f(n) = 0$, and noting that $f(n+1) = f(n) + f(1) = 0$). This is a contradiction to our initial assumption, so $f(1) = 1$.

2. First, we note that:

$$f(0) = f(0+0) = f(0) + f(0) \;\Rightarrow\; f(0) = 0$$

Next, we note that $f(n) = n$, for natural $n$. We prove this by induction, first assuming that $f(n) = n$, then noting that $f(n+1) = f(n) + f(1) = n+1$. We then note that $f(-n) = n - n + f(-n) = -n + f(n) + f(-n) = -n + f(0) = -n$. Thus, $f$ is the identity for all integers.

Now, we can see that:

$$f\left(\frac{1}{b}\right) = \frac{b}{b} \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(b) \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(1) = \frac{1}{b}$$

Thus,

$$f\left(\frac{a}{b}\right) = f(a) \cdot f\left(\frac{1}{b}\right) = \frac{a}{b}$$

3. Assume that $x > 0$. It then follows that $\sqrt{x}$ is well-defined and greater than 0. We then have:

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = f(\sqrt{x})^2$$

we know that for any real number $r$, we have $r^2 \geq 0$, so $f(x) \geq 0$. Assume that $f(x) = 0$. Since $x > 0$, this would imply that:

$$f(1) = f\left(\frac{x}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right) = 0$$

a clear contradiction. Thus, $f(x) > 0$.

4. If $x > y$, then we know that $x - y > 0$, so it follows from previous result that:

$$f(x-y) > 0 \;\Rightarrow\; f(x) - f(y) > 0 \;\Rightarrow\; f(x) > f(y)$$

5. Assume that there exists some $x$ such that $x < f(x)$. Since there exists a rational number between any two reals, it follows that we have:

$$x < \frac{a}{b} < f(x)$$

for some $a/b$. From the previous result, we then get $f(x) < f(a/b)$, a clear contradiction to the right-most inequality above. Similarly, if we assume that $f(x) < x$, we will have:

$$f(x) < \frac{a}{b} < x$$

so $f(a/b) < f(x)$, another contradiction. It follows that $f(x) = x$, and we have proved the proposition.

$\square$

**Problem 3.20B.** If a function satisfies:

$$f(y) - f(x) \leq (x-y)^2$$

for all $, y \in \mathbb{R}$, then $f(x) = c$ for some $c$ and all $x$

*Part B is the interesting part of this problem, so I skipped writing out Part A*

*Proof.* Assume that there exist distinct $x$ and $y$ such that $f(x) \neq f(y)$. Without loss of generality, let $f(x) < f(y)$. It follows that:

$$f(y) - f(x) \leq (y-x)^2$$

Consider what happens when we split up the interval from $x$ to $y$ into $n$ "chunks". We let:

$$z_j = \left(1 - \frac{j}{n}\right)x + \frac{j}{n}y$$

so we get $z_0 = x$ and $z_n = y$. Clearly the distance between $z_i$ and $z_{j-1}$ is given by:

$$z_j - z_{j-1} = \left(1 - \frac{j}{n}\right)x + \frac{j}{n}y - \left(1 - \frac{j-1}{n}\right)x - \frac{j-1}{n}y = \frac{y-x}{n}$$

It follows that:

$$f(z_j) - f(z_{j-1}) \leq (z_j - z_{j-1})^2 = \frac{(y-x)^2}{n^2}$$

Now comes the crucial step. Notice that

$$\sum_{j=1}^{n} \big(f(z_j) - f(z_{j-1})\big) = f(z_n) - f(z_0) = f(y) - f(x)$$

as the rest of the terms cancel. Thus, we will have:

$$\sum_{j=1}^{n} \left(z_j - z_{j-1}\right) \leq \sum_{j=1}^{n} \frac{(y-x)^2}{n^2} \;\Rightarrow\; f(y) - f(x) \leq n \cdot \frac{(y-x)^2}{n^2} = \frac{(y-x)^2}{n}$$

for all possible values of $n$. Since we have assume $f(y) \neq f(x)$, it follows that $f(y) - f(x)$ is a positive real number, and that:

$$\epsilon = \frac{f(y) - f(x)}{(y-x)^2}$$

is a positive real as well. Thus, this implies that there exists a real number $\epsilon$ such that for any positive integer $n$:

$$\epsilon \leq \frac{1}{n}$$

But this clearly contradicts the Archimedean property of the real numbers. We have derived a contradiction, so it follows that for any $x$ and $y$, $f(x) = f(y)$. Thus, the function $f$ is constant.

$\square$