Advanced Putnam Problems 1

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Proposition 1 (Problem B11)**.** *Let T_n be the collection of non-empty subset of $\{1, \ ..., \ n\}$, where each subset contains no consecutive integers. For each $S \in T_n$, let P_S be the square of the product of all the elements of S. The sum of all P_S for $S \in T_n$ is given by $(n+1)! - 1$.*

Proof. Clearly, any subset of the set $\{1, \ ..., \ n+1\}$ containing $n+1$ will either be $\{n+1\}$, or be of the form $\{a_1, \ ..., \ a_k, \ n+1\}$ for $a_i \in \{1, \ ..., \ n-1\}$. Thus, we can form all sets S in T_{n+1} containing $n+1$ by simply "appending" $n+1$ to each $S \in T_{n-1}$. Let P_n be the sum of all P_S for $S \in T_n$. It follows that $(n+1)^2 P_{n-1}$ is the sum of all squares of product of such subsets.

We must also include all sets in T_n, which have square/product P_n. In addition, we include $\{n+1\}$, the square of which's product is $(n+1)^2$. Thus:

$$P_{n+1} = (n+1)^2 P_{n-1} + P_n + (n+1)^2$$

It is easy to check that the cases of $n = 1$ and $n = 2$ hold true for the above proposition. We assume the case of all natural numbers up to and including n for $n \geq 3$ (Strong induction) and prove $n+1$:

$$\begin{aligned} P_{n+1} &= (n+1)^2 P_{n-1} + P_n + (n+1)^2 = (n+1)^2(n! - 1) + (n+1)! + (n+1)^2 - 1 \\ &= (n+1)^2 n! + (n+1)! - 1 = (n+1+1)(n+1)! - 1 = (n+2)! - 1 \end{aligned}$$

This completes the proof. □

Proposition 2 (Problem A1)**.** *Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k - 1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.*

Proof. Consider the n-th derivative of $\frac{1}{x^k-1}$, given by $\frac{P_n(x)}{x^k-1}$. We take the derivative again, giving us:

$$\frac{d}{dx} \frac{P_n(x)}{(x^k-1)^{n+1}} = \frac{P_n'(x)}{(x^k-1)^{n+1}} - \frac{(n+1)kx^{k-1}P_n(x)}{(x^k-1)^{n+2}} = \frac{P_{n+1}(x)}{(x^k-1)^{n+2}}$$

We then multiply both sides by $(x^k-1)^{n+2}$ and changing the indices giving us:

$$P_n(x) = P_{n-1}'(x)(x^k - 1) - nkx^{k-1}P_{n-1}(x)$$

We are interested in $P_n(1)$, so we get:

$$P_n(1) = -nkP_{n-1}(1)$$

We assert that $P_n(1) = (-1)^n k^n n!$. This is clearly true in the case of $n = 1$, as $P_0(x) = 1$. We assume this is true for n and prove $n+1$:

$$P_{n+1}(1) = (-1)(n+1)kP_n(1) = (-1)(n+1)k(-1)^n k^n n! = (-1)^{n+1}k^{n+1}(n+1)!$$

Thus, we have proved this statement by induction. □