Spivak Problem Set 0.1

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Quick disclaimer: I am using the third edition of Spivak (not the fourth), so there is a non-zero probability that the question assigned by Professor Meinrenken are different than the ones solved here (due to question numbering changing between editions).

1 Chapter 1

1.1 Problem 3

1. $\frac{a}{b} =$
2. $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = (db)^{-1}(ad + cb) = \frac{ad + cb}{db}$
3. $(ab)^{-1} = e \cdot e \cdot (ab)^{-1} = (aa^{-1}) \cdot (bb^{-1}) \cdot (ab)^{-1} = a \cdot a^{-1} \cdot b \cdot b^{-1} \cdot (ab)^{-1} = (a^{-1}b^{-1}) \cdot (ab) \cdot (ab)^{-1} = a^{-1}b^{-1}$
4. $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = a \cdot c \cdot d^{-1} \cdot b^{-1} = (ac) \cdot (db)^{-1} = \frac{ac}{db}$
5. $\frac{a/b}{c/d} = \frac{a}{b} \cdot \left(\frac{c}{d}\right)^{-1} = \frac{a}{b} \cdot (cd^{-1})^{-1} = \frac{a}{b} \cdot \left(c^{-1}(d^{-1})^{-1}\right) = \frac{a}{b} \cdot (c^{-1} \cdot (d^{-1})^{-1} \cdot e) = \frac{a}{c} \cdot (c^{-1} \cdot (d^{-1})^{-1} \cdot (dd^{-1})) = \frac{a}{b} \cdot (c^{-1}d) = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

1.2 Problem 8

Lemma 1. *If we assume that modified definition of an order relation (see Problem 8 in Spivak), then the Trichotomy law holds.*

Proof. Consider the pair of numbers 0 and a. By the first axiom, it follows that either $a = 0$, $a > 0$ (which implies that $a \in P$, by definition of P), or, $a < 0$. Using the third axiom, we add $-a$ to both sides to get: $a + (-a) = 0 < -a$. Thus, in this case, it follows that $-a \in P$. This proves the Trichotomy law. □

Lemma 2. *The modified definition of an order relation implies that if $a,\ b \in P$, then $a + b \in P$.*

Proof. Assume that $a,\ b \in P$. It follows that $0 < a$ and $0 < b$. By the third axiom, we have: $0 + a = a < b + a$. Thus, we have:

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$$0 < a < b + a$$

By the second axiom, we have $0 < a + b$. Thus, $a + b \in P$. $\square$

Lemma 3. *The modified definition of an order relation implies that if $a, b \in P$, then $a \cdot b \in P$.*

Proof. Assume that $a, b \in P$. We thus have $0 < a$ and $0 < b$. It follows from the fourth axiom that $0 \cdot b = 0 < a \cdot b$. Thus, $a \cdot b \in P$. $\square$

1.3 Problem 13

Proposition 1. *For two numbers x and y, we have:*

$$max(x, y) = \frac{x + y + |y - x|}{2}$$

$$min(x, y) = \frac{x + y - |y - x|}{2}$$

Proof. Let us pick two numbers, x and y. Assume, without loss of generality, that $x < y$. Consider the quantity:

$$\max(x, y) = \frac{x + y + |y - x|}{2}$$

Since $x < y$, it follows that $0 < y - x$, and we have $|y - x| = y - x$:

$$\max(x, y) = \frac{x + y + y - x}{2} = \frac{2y}{2} = y$$

which is in fact the larger of the pair x, y. We can also consider the quantity:

$$\min(x, y) = \frac{x + y - |y - x|}{2} = \frac{x + y - y + x}{2} = x$$

which is in fact the smaller of the pair x, y. Thus, the proof is complete. $\square$

Derive expressions for $\max(x, y, z)$ and $\min(x, y, z)$.
Since $\max(x, y, z) =$

2 Chapter 28

2.1 Problem 5

Lemma 4. *For any field, we have:*

$$\underbrace{(e + \cdots + e)}_{m\ times} \cdot \underbrace{(e + \cdots + e)}_{n\ times} = \underbrace{(e + \cdots + e)}_{mn\ times}$$

for all natural numbers n and m.

Proof. Pick ome arbitrary natural number m. We proceed by induction. Clearly, the lemma is true in the case of $n = 1$. Let us assume the case of n. Consider the case of $n + 1$. We have:

$$\underbrace{(e + \cdots + e)}_{m \text{ times}} \cdot \underbrace{(e + \cdots + e)}_{n + 1 \text{ times}} = \underbrace{(e + \cdots + e)}_{m \text{ times}} \cdot \left[\underbrace{(e + \cdots + e)}_{n \text{ times}} + e\right]$$

Now, we use the distributive property of fields and the definition of the identity, along with the assumtpion that the lemma holds true in the case of n to get:

$$\Rightarrow \left[\underbrace{(e + \cdots + e)}_{mn \text{ times}} + e \cdot \underbrace{(e + \cdots + e)}_{m \text{ times}}\right] = \left[\underbrace{(e + \cdots + e)}_{mn \text{ times}} + \underbrace{(e + \cdots + e)}_{m \text{ times}}\right] = \underbrace{(e + \cdots + e)}_{m(n + 1) \text{ times}}$$

So the lemma is proved. $\square$

Theorem 1. *If in some field F we have:*

$$\underbrace{e + \cdots + e}_{n \text{ times}} = 0$$

then the smallest n for which this is true is prime.

Proof. Assume that n isn't prime. It follows that we can write n as a product of at least two whole numbers less than n and greater than 1. Thus, $n = ab$. By the previous lemma, we have:

$$\underbrace{e + \cdots + e}_{n \text{ times}} = \underbrace{(e + \cdots + e)}_{a \text{ times}} \cdot \underbrace{(e + \cdots + e)}_{b \text{ times}} = 0$$

In a field, we know that $a \cdot 0 = a$, as 0 is the element of the field such that $a + 0 = a$. We then have (by distribution) that $(a \cdot a) + (a \cdot 0) = (a \cdot a) = (a \cdot a) + 0$. By left cancellation, we have $a \cdot 0 = 0$. Assume that both the right-hand sums of e (for a and b) are non-zero. It follows that they have inverses. Let us denote the two sums by A and B. It follows that:

$$e = A^{-1}AB^{-1}B = (A^{-1}B^{-1}) \cdot (AB) = (A^{-1}B^{-1}) \cdot 0 = 0$$

which is a contradiction to the definition of a field, as the additive and multiplicative identities must be different. Thus, at least one of these sums is equal to 0 it follows that either a or b is a whole number less than n such that:

$$\underbrace{e + \cdots + e}_{a \text{ or } b \text{ times}} = 0$$

which is a contradiction. Thus, n must be prime.

□

2.2 Problem 6

Lemma 5. *For some field F with a finite number of elements, there exist distinct natural numbers m and n such that:*

$$\underbrace{e + \cdots + e}_{m \textit{ times}} = \underbrace{e + \cdots + e}_{n \textit{ times}}$$

Proof. Let $|F| = k$ be the cardinality of the set defining the field (which we know is some finite natural number, k). Let:

$$E(n) = \underbrace{e + \cdots + e}_{n \text{ times}}$$

Now, consider the set $\{E(1),\ E(2),\ ...,\ E(k),\ E(k + 1)\}$. It follows that there must exist two elements of this set that are equal, or else we would have a subset of F that contains $k + 1$ **distinct** elements, a clear contradiction. Hence, there exist m and n such that:

$$\underbrace{e + \cdots + e}_{m \text{ times}} = \underbrace{e + \cdots + e}_{n \text{ times}}$$

□

Theorem 2. *In a field F with a finite number of elements, there exists some natural number r such that:*

$$\underbrace{e + \cdots + e}_{r \textit{ times}} = 0$$

Proof. By the previous lemma, we know there exist m and n such that $E(m) = E(n)$. Without loss of generality, let $n < m$ (the two numbers are distinct, so one is larger than the other). We have:

$$0 + \underbrace{e + \cdots + e}_{n \text{ times}} = \underbrace{e + \cdots + e}_{m \text{ times}} = \underbrace{e + \cdots + e}_{m - n \text{ times}} + \underbrace{e + \cdots + e}_{n \text{ times}}$$

So by right cancellation, we have:

$$\underbrace{e + \cdots + e}_{m - n \text{ times}} = 0$$

It follows that $r = m - n$ and the theorem is proved. □