Challenge Accepted, Matt

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Contents

1 Problem 1

Proposition 1. *The degree of* $(x - a_1) \cdots (x - a_n)$ *is* n*, for* $a_k \in \mathbb{F}$.

Proof. Clearly, this product does not have a non-zero term of the form ax^k, for $k > n$ (this can be formally demonstrated using induction). In addition, this polynomial has a term of the form x^n.

There exists no field in which $1 = 0$, so it follows that x^n is a non-zero term in the expansion. $\square$

Part 1

Assume that $p(x)$ does split over $\mathbb{R}$. We then must have $x^2 + 1 = (x - a)(x - b)$, as if there were any more terms in the product, the degree of the resulting polynomial would be greater than 2.

We then have:

$$x^2 + 1 = x^2 - (a + b)x + ab$$

so $a = -b$ and $ab = -a^2 = 1$, which implies that $a^2 = -1$. However, from the axioms of $\mathbb{R}$, the square of any $r \in \mathbb{R}$ must be positive, so this is a contradiction. It follows that $p(x)$ cannot be split over $\mathbb{R}$.

Part 2

Assume $q(x) = x^2 + x + 1$ does split over $\mathbb{F}_2$. By Proposition 1:

$$x^2 + x + 1 = (x - a)(x - b) = x^2 - (a + b)x + ab$$

So $-a - b = 1$ and $ab = 1$. The second equation implies that we must have $a = b = 1$, but $-1 + (-1) = 0 \neq 1$, so we have a contradiction. Thus, $q(x)$ does not split over $\mathbb{F}_2$.

Part 3

Let T be a map such that $p_T(x)$ splits. It follows that T can be put in Jordan form. In other words, there exists a basis such that the matrix of T with respect to this basis is in Jordan form.

There is no notion of similarity between linear maps, only matrices, so this question is not well-defined.

Part 4

Pick some matrix $\mathcal{A} \in M_n(\mathbb{C})$. We define a linear operator $A : \mathbb{F}^{n \times 1} \to \mathbb{F}^{n \times 1}$, over the field $\mathbb{C}$, such that $A(v) = \mathcal{A}v$ (clearly such a map is linear, by definition of matrix multiplication).

It is easy to see that if β is the standard basis on $\mathbb{F}^{n \times 1}$, then $\mathcal{A} = {}_{\beta}[A]_{\beta}$

We let p_A be the minimal polynomial of A. We know that any polynomial can be factored over the complex field, so it follows that p_A splits. Therefore, from Part 3, there exists a basis β' such that the matrix of A with respect to this basis is in Jordan form. In other words:

$${}_{\beta'}[A]_{\beta'} = {}_{\beta'}I_{\beta\beta}[A]_{\beta\beta}I_{\beta'} = P^{-1}\mathcal{A}P$$

is in Jordan form, where $P = {}_{\beta}I_{\beta'}$. Thus, by definition, $\mathcal{A}$ is similar to a matrix in Jordan form and the proof is complete.

Part 5

Proposition 2. *If* A *is an upper-diagonal matrix, then* A^n *is also upper-diagonal, for any natrual* n.

Proof. Clearly, this is true in the case of $n = 1$. Assume the case of n. For the case of $n + 1$, we note that:

$$A^{n+1}_{ij} = \sum_r A^n_{ir} A_{rj}$$

Assume that $i > j$ (these are entries below the upper diagonal). If $r \geq i$, then $r > j$, so $A_{rj} = 0$ and $A_{ij}^{n+1} = 0$. If $r < i$, then from the inductive hypothesis, $A_{ir}^n = 0$, so $A_{ij}^{n+1} = 0$. Thus, A^{n+1} is upper-diagonal and the proof by induction is complete. □

Proposition 3. *If A is upper-diagonal with diagonal entries A_{kk}, then for the diagonal entires of A^n (which we call A_{kk}^n), we have $A_{kk}^n = (A_{kk})^n$.*

Proof. Clearly, this is true in the case of $k = 1$. Assume the case of $k = n$. Consider the case of $k = n + 1$. The entries of the matrix A^{n+1} will be given by:

$$A_{ij}^{n+1} = \sum_r A_{ir}^n A_{rj}$$

So for $i = j$, we will have:

$$A_{ii}^{n+1} = \sum_r A_{ir}^n A_{ri}$$

Clearly A^n is upper-diagonal, so for $r < i$, we will have $A_{ir}^n = 0$. In the case of $r > i$, we have $A_{ri} = 0$. Thus, the only term in the sum that can be non-zero is $r = i$, By the inductive hypothesis:

$$\sum_r A_{ir}^n A_{ri} = A_{ii}^n A_{ii} = (A_{ii})^n A_{ii} = (A_{ii})^{n+1}$$

and the proof is complete. □

2 Problem 2

Part 1

We will have, from the defintition

$$e^U = \sum_{n=0}^{\infty} \frac{U^n}{n!}$$

where we define $U^0 = I$. The matrix sum in entry-wise, so if we let E_{rj} be the $(r,\ j)$-th element of e^U and U_{rj}^n be the $(r,\ j)$-th entry of U^n, then we will have:

$$E_{rj} = \sum_{n=0}^{\infty} \frac{U_{rj}^n}{n!}$$

We note that from Problem 5, we will have $U_{rr}^n = a_r^n$, so it follows that:

$$E_{rr} = \sum_{n=0}^{\infty} \frac{a_r^n}{n!} = e^{a_r}$$

by definition of the function e^x. This completes the proof.

Part 2

Proposition 4. *For some natrual n and some arbitrary A:*

$$(QAQ^{-1})^n = QA^nQ^{-1}$$

Proof. Clearly, this is true for $n = 1$. Assume the case of n. For $n + 1$, we have:

$$(QAQ^{-1})^{n+1} = (QAQ^{-1})(QAQ^{-1})^n = (QAQ^{-1})(QA^nQ^{-1}) = QA^{n+1}A^{-1}$$

and the proof by induction is complete. □

We will have:

$$\exp(QAA^{-1}) = \sum_{n=0}^{\infty} \frac{(QAQ^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{QA^nQ^{-1}}{n!} = Q\Big[\sum_{n=0}^{\infty} \frac{A^n}{n!}\Big]Q^{-1} = Qe^AQ^{-1}$$

Part 3

Proposition 5. $e^{x+y} = e^x e^y$

Proof.

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^j}{j!} \frac{y^{n-j}}{(n-j)!}$$

We make the claim that:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^j}{j!} \frac{y^{n-j}}{(n-j)!} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{x^p}{p!} \frac{y^q}{q!}$$

Clearly, given some pair $(p,\ q)$ characterizing a unique term of the right-hand sum, there will exist a unique term in the left-hand sum with $n = p + q$ and $j = p$ that is equal to this term.

In addition, given some pair $(n,\ j)$ characterizing a unique term in the left-hand sum, there exists a unique term in the right-hand sum with $p = j$ and $q = n - j$ that is equal to this term.

Thus, there is a one-to-one correspondence between the terms of the sums, so:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^j}{j!} \frac{y^{n-j}}{(n-j)!} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{x^p}{p!} \frac{y^q}{q!} = \Big(\sum_{p=0}^{\infty} \frac{x^p}{p!}\Big)\Big(\sum_{q=0}^{\infty} \frac{x^q}{q!}\Big) = e^x e^y$$

and the proof is complete.

Note that this proof can easily be generalized to aribtrary sums, using induction. $\square$

From Problem 4, we note that A is similar to an upper triangular matrix, so $A = QBQ^{-1}$, where B is upper-triangular. It follows that:

$$\det(e^A) = \det(e^{QBQ^{-1}}) = \det(Qe^BQ^{-1})$$

Since the determinant is invariant under change of basis, it follows that:

$$\det(Qe^BQ^{-1}) = \det(e^B)$$

Since trace is invariant under change of basis, it follows that $\text{trace}(A) = \text{trace}(B) = 0$, which implies that:

$$\sum_k b_{kk} = 0$$

where b_{ij} is the $(i,\ j)$-th element of the matrix B.

The determinant of an upper-triangular matrix is simply the product of the diagonal entires. Since e^B is a sum of powers of an upper-diagonal matrix, it is also upper-diagonal, so its determinant will be the product of its diagonal.

Therefore, from Part 1 and Proposition 4:

$$\det(e^B) = \prod_k e^{b_{kk}} = \exp\Big[\sum_k b_{kk}\Big] = e^0 = 1$$

and the proof is complete, $\det(e^A) = \det(e^B) = 1$. Note that it is easy to see that $e^0 = 1$, from the definition of the exponential function.

3 Problem 3

Part 1

Clearly, the commutator is a valid bilinear map:

$$[A+B,\ Y] = (A+B)Y - Y(A+B) = AY - YA + BY - YB = [A,\ Y] + [B,\ Y]$$
$$[\lambda X,\ Y] = \lambda XY - Y(\lambda X) = \lambda[X,\ Y]$$

where we can verify that the same linearity holds true for the second entry in a similar fashion.

In addition, given $X \in GL(n,\ \mathbb{R})$, we have $[X,\ X] = XX - XX = 0$. Finally:

$$[X,\ [Y,\ Z]] + [Z,\ [X,\ Y]] + [Y,\ [Z,\ X]] = [X,\ YZ - ZY] + [Z,\ XY - YX] + [Y,\ ZX - XZ]$$
$$= XYZ - XZY - YZX + ZYX + ZXY - ZYX - XYZ + YXZ + YZX - YXZ - ZXY + XZY = 0$$

Trust me, all the terms cancel. Therefore, $GL(n,\ \mathbb{R})$ equipped with the commutator is a real Lie algebra (we already know that $M_n(\mathbb{R})$ is a vector space over $\mathbb{R}$).

The dimension of $GL(n,\ \mathbb{R})$ is n^2, as it is easy to verify that the list of matrices with a 1 in entry $(i,\ j)$ and 0s everywhere else, for all $i,\ j$ from 1 to n is a linearly independent spanning set, and has n^2 elements.

Part 2

Proposition 6. $\text{trace}(XY) = \text{trace}(YX)$

Proof.

$$\text{trace}(XY) = \sum_k (XY)_{kk} = \sum_k \sum_r X_{kr} Y_{rk} = \sum_r \sum_k Y_{rk} X_{kr} = \sum_r (YX)_{rr} = \text{trace}(YX)$$

□

First, we note that given X and Y in the vector space, we will have:

$$\text{trace}(XY - YX) = \text{trace}(XY) - \text{trace}(YX) = \text{trace}(XY) - \text{trace}(XY) = 0$$

Thus, $[X,\ Y]$ is an element of $SL(n,\ \mathbb{R})$, so the commutator is a valid bilinear map.

Now, it is sufficient to show that $SL(n,\ \mathbb{R})$ is a vector space, as we have already proved the sufficient propoerties of the commutator above.

Clearly, given two trace 0 matrices, their vector sum (component-wise addition) will result in a matrix that also has trace 0. The same is clearly true for component-wise scalar multiplication. Finally, it is clear that the 0 matrix is in $SL(n,\ \mathbb{R})$. Thus, it is a vector space.

It is easy to verify that trace is a linear map. Clearly, $SL(n,\ \mathbb{R})$ is the null-space of the trace operator when it maps from $M_n(\mathbb{R})$ to $\mathbb{F}$. Thus, by rank-nullity theorem:

$$\dim\ M_n(\mathbb{R}) = \dim\ SL(n,\ \mathbb{R}) + \dim\ \mathbb{F}\ \Rightarrow\ \dim\ SL(n,\ \mathbb{R}) = n^2 - 1$$

where we note that range trace $= \mathbb{F}$, as there exists a matrix with non-zero trace λ, and all other elements of $\mathbb{F}$ will simply be scalar multiples of λ.

Part 3

Clearly, $[X,\ Y] = 0$ is a valid bilinear map from V to V, as $0 \in V$. In addition, $[X,\ X] = 0$, by definition of the map.

It is easy to verify that the Jacobi identity also holds.

Part 4

Consider $X,\ Y \in SO(n,\ \mathbb{R})$. We note, from the basic properties of transposition:

$$(XY-YX)^T = (XY)^T-(YX)^T = Y^TX^T-X^TY^T = (-Y)(-X)-(-X)(-Y) = YX-XY = -(XY-YX)$$

Thus, the given bracket is a valid bilenear map. It is easy to verify that set $SO(n,\ \mathbb{R})$ is closed under scalar multiplication and vector addition, and contains the 0 matrix. Therefore, $SO(n,\ \mathbb{R})$ is a vector space.

Proposition 7. *The dimension of $SO(n,\ \mathbb{R})$ is $n(n-1)/2$.*

Proof. Consider the set B of matrices m:

$$B = \{m \mid m \in M_n(\mathbb{R}),\ m_{ij} = -1,\ m_{ji} = 1\}$$

with 0s in all other entries, and $1 \leq i \leq n$ and $j < i$. Clearly, every element of B is in $SO(n,\ \mathbb{R})$, so $\text{span}(B) \subset SO(n,\ \mathbb{R})$. In addition, consider some $M \in SO(n,\ \mathbb{R})$. We must have:

$$M^T = -M \ \Rightarrow\ M_{ij} = -M_{ji}$$

for all i and j from 1 to n. We note that the above equation implies that $M_{ii} = -M_{ii}$, so $M_{ii} = 0$. In other words, the main diagonal of M is all 0s.
Let $\mathcal{M}^{ij}$ be the matrix with a 1 in entry $(i,\ j)$ and 0s everywhere else. Using the above facts, we will have:

$$M = \sum_{i,\ j} M_{ij}\mathcal{M}^{ij} = \sum_{i=1}^{n}\sum_{j<i} M_{ij}(\mathcal{M}^{ij} - \mathcal{M}^{ji})$$

By definition, $\mathcal{M}^{ij} - \mathcal{M}^{ji}$ is an element of B, so it follows that B is a linear combination of elements of B. Therefore, $SO(n,\ \mathbb{R}) \subset \text{span}(B)$.

We have inclusion both ways, so $SO(n,\ \mathbb{R}) = \text{span}(B)$. Finally, we note that all elements of B are linearly independent, as each matrix in B contains a non-zero entry in some entry $(i,\ j)$ where no other element of B has a non-zero entry.

Thus, by definition, B is a basis for $SO(n,\ \mathbb{R})$. Clearly, there are $\binom{n}{2} = n(n-1)/2$ elements in B, so the dimension of $SO(n,\ \mathbb{R})$ is $n(n-1)/2$. This completes the proof. ☐

Part 5

Using a very similar proof to Proposition 3, it is clear that the product of two upper diagonal matrices with 0s on the main diagonal is also an upper-diagonal matrix with 0s on the main diagonal. Thus, given X and Y in the Heisenberg group, $XY - YX$ is also in the Heisenberg group as well.

It follows that $[X,\ Y]$ is a valid bilinear map into the Heisenberg group.

It isn't difficult to see that $H(3, \mathbb{R})$ is closed under vector addition and scalar multiplication, and contains the 0 vector. Therefore, $H(3,\ \mathbb{R})$ is in fact a real Lie algebra.

Clearly:

$$B = \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

is a basis for $H(3,\ \mathbb{R})$ so the Lie algebra has dimension 3.

Part 6

We know that the cross product is anti-commutative. In other words, $X \times Y = -(Y \times X)$. This implies that $X \times X = -(X \times X)$, so $X \times X = 0$. Thus, $[X,\ X] = 0$.

We also note that:

$$[X,\ [Y,\ Z]] + [Z,\ [X,\ Y]] + [Y,\ [Z,\ X]] = X \times (Y \times Z) + Z \times (X \times Y) + Y \times (Z \times X)$$

Part 7

Part 8

Part 9

I fucked up the notation a bit by not using the fancy symbols to denote GL, SL and SO, so I'll refer to SL (as it is used in the notes) as S.

In Problem 2, we showed that if A has trace equal to 0, then $\det(e^A) = 1$. It follows that given some element of $SL(n,\ \mathbb{C})$ (which has trace equal to 0), then the determinant of the exponential of this element is 1, so it is in $S(n,\ \mathbb{C})$. Thus, $\exp : SL(2,\ \mathbb{C}) \ \rightarrow\ S(n,\ \mathbb{C})$ is a valid map.

Now, let us pick some element $M \in S(n,\ \mathbb{C})$. Since this vector space is over the complex field, it follows that M is similar to a matrix in Jordan form. Thus, $M = Q^{-1}PQ$, where P is in Jordan form.

It follows that P is either diagonal (with determinant 1, as determinant is invariant under change of basis), or of the form:

$$P = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \tag{1}$$

For the purpose of this exercise, we will assume that the function $f : \mathbb{C}\ \rightarrow\ \mathbb{C}$ with $f(x) = e^x$ is surjective (this follows from Euler's formula).

If $P = \text{diag}(a,\ b)$, we note that $\det(P) = ab = 1$. We choose the matrix N to have an entry x in the top right corner such that $e^x = a$ (this follows from the surjectivity of e^x). We then choose N to have the entry $-x$ in the bottom right corner. We note that $e^{-x} = b$, as b is the unique multiplicative inverse of a, and:

$$ae^{-x} = e^x e^{-x} = e^0 = 1$$

Clearly, N has trace 0, so it is in $SL(n,\ \mathbb{C})$. We know that trace is invariant under change of basis, so $Q^{-1}NQ$ is also in $SL(n,\ \mathbb{R})$. Finally, we note that:

$$\exp(Q^{-1}NQ) = Q^{-1}\exp(N)Q = Q^{-1}\exp\left[\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}\right]Q = Q^{-1}\exp\left[\begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}\right]Q$$

$$= Q^{-1}\exp\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right]Q = Q^{-1}PQ = M$$

In addition, if $M = Q^{-1}PQ$, with P of the form of equation (1), then we note that:

$$\exp\left[Q^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}Q\right] = Q^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}Q$$

$$\exp\left[Q^{-1}\begin{pmatrix} i\pi & 1 \\ 0 & -i\pi \end{pmatrix}Q\right] = Q^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}Q$$

Part 10

Note that if A has two unique eigenvalues, then it can be diagonalized, and we can use the same proof is was presented in Question 2 to conclude that $\det(e^A) = 1$.

Otherwise, A has less than two unique eigenvalues. It follows that A is not invertible. Therefore, we must have $\det A = 0$. Using the rational canonical form, we note that:

$$A = Q^{-1} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} Q$$

It is easy to see that:

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, using the definition of the matrix exponential, it is clear that:

$$\exp\left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] = I + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

So the exponential of this matrix has determinant 1. Finally, we see that:

$$\exp(A) = Q^{-1} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} Q$$

Since determinant remains invariant under change of basis, it follows that $\det(e^A) = 1$ in this case as well.

We have shown that for any $A \in SL(n, \mathbb{R})$, the exponential e^A has determinant 1. Therefore, $\exp : SL(n, \mathbb{R}) \to S(n, \mathbb{R})$ is a valid map.

Part 11

Proposition 8. *The map* $\exp : SL(n, \mathbb{R}) \to S(n, \mathbb{R})$ *is not surjective.*

Proof. Consider the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

The characteristic polynomial of this matrix is clearly $(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$ (as the matrix is upper diagonal). ☐

4 Problem 4

Part 1

Clearly, g is a vector subspace of itself, so it is a Lie subalgebra of g. In addition, it is clear that $[g, g] \subset g$, as the bracket is a bilinear map into g. Therefore, g is an ideal, by definition.

Part 2

Clearly, $\{0\}$ is a Lie subalgebra of g, as it is a vector subspace and $[0, 0] = 0$. We then note that given $0 \in \{0\}$ and $G \in g$, we will have:

$$[G, 0] = [G, G - G] = [G, G] - [G, G] = 0$$

so $[g, \{0\}] \subset \{0\}$. Therefore, $\{0\}$ is an ideal.

Part 3

Recall that an Abelian Lie algebra is, by definition, an n-dimensional vector space V equipped with the bracket $[X,\ Y] = 0$.

Consider some arbitrary vector subspace $U \subset V$. Clearly, given $u,\ v \in U$, we will have $[u,\ v] = 0$, which is in U, so U is a Lie subalgebra. Furthermore, given $u \in U$ and $v \in V$, we have $[v,\ u] = 0 \in V$, so U is an ideal.

Therefore, all vector subspaces of V are ideal under the Abelian Lie algebra.

Part 4

Proposition 9. *The only one-dimensional Lie algebra is the Abelian one.*

Proof. Let V be a one-dimensional vector space, so $V = \text{span}(v)$ over some field $\mathbb{F}$, where v is some element of V.

Let $[\cdot,\ \cdot]$ be a bilinear map from $V \times V$ to V. Assume that this bracket along with V form a Lie algebra. Clearly, given two elements av and bv of V, we must have:

$$[av,\ bv] = a[v,\ bv] = ab[v,\ v] = ab \cdot 0 = 0$$

Therefore, the bracket must take all pairs of elements in V to 0. It follows that if V, along with some bracket form a Lie algebra, then it must be the Abelian Lie algebra. □

Part 5a

5 Problem 5

Part 1

Proposition 10. *For some natural k (or $k = 0$), $g^{(k)} \subset g_{(k)}$.*

Proof. We proceed by induction. Clearly, this is true for the case of $k = 0$ and $k = 1$. We note that for $k \geq 1$:

$$g_{(k)} = [g_{(k-1)},\ g]$$

and:

$$g^{(k)} = [g^{(k-1)},\ g^{(k-1)}]$$

Assume that for the case of k, the proposition holds true. In the case of $k+1$ we have $g_{(k+1)} = [g_{(k)},\ g]$ and $g^{(k+1)} = [g^{(k)},\ g^{(k)}]$. From the inductive hypothesis, $g_{(k)} \subset g_{(k)}$. Finally, we note that $g^{(k)}$ is a subalgebra of g, so $g^{(k)} \subset g$.

It follows from the definition of the bracket that:

$$g^{(k+1)} = [g^{(k)},\ g^{(k)}] \subset [g_{(k)},\ g^{(k)}] \subset [g_{(k)},\ g] = g_{(k+1)}$$

and the proof by induction is complete. □

Corollary 1. *Every nilpotent Lie algebra is solvable*

Proof. Let g be a nilpotent Lie algebra. It follows that there exists some k such that $g_{(k)} = 0$. In the case that $k = 0$, this proposition clearly holds. Otherwise, there is some k such that $[g_{(k-1)},\ g] = \{0\}$.

From above, we know that $[g^{(k-1)},\ g^{(k-1)}] \subset [g_{(k-1)},\ g] = \{0\}$, so it follows that $g^{(k)} = 0$. By definition, the Lie algebra is solvable and the proof is complete. □

Part 2

Consider an Abelian Lie algebra g. Given two elements $x,\ y \in G$, we note that $[x,\ y] = 0$, so it follows that $g_{(1)} = [g,\ g] = 0$. Thus, every Abelian Lie algebra is nilpotent (and solvable).

Part 3

Proposition 11. *Every 2-dimensional Lie algebra is solvable.*

Proof. Let g be a two-dimensional Lie algebra. It follows that $g = \text{span}(v,\ w)$, for two linearly independent vectors v and w. Given two elements in g, of the form $av + bw$ and $cv + dw$, we note that:

$$[av + bw,\ cv + dw] = [av + bw,\ cv] + [av + bw,\ dw] = bc[w,\ v] + ad[v,\ w] = (ad - bc)[v,\ w] = z$$

where $z \in g$. Therefore, $g^{(1)}$ has dimension 1. It follows (from a previous question), that $g^{(1)}$ must be the Abelian Lie algebra. Therefore, $g^{(2)} = [g^{(1)},\ g^{(1)}] = \{0\}$, so g is solvable. $\square$

Part 4

Part 5

Part 6