Computer Vision

Course 5

Sharpening Spatial Filters

The principal objective of sharpening is to highlight transitions in intensity. These filters are applied in electronic printing, medical imaging, industrial inspection, autonomous guidance in military systems.

Averaging – analogous to integration

Sharpening – spatial differentiation

Image differentiation enhances edges and other discontinuities (noise, for example) and deemphasizes areas with slowly varying intensities.

For digital images, discrete approximation of derivatives are used

$$\frac{\partial f}{\partial x} \approx f(x+1, y) - f(x, y)$$

$$\frac{\partial^2 f}{\partial x^2} \approx f(x+1, y) - 2f(x, y) + f(x-1, y)$$

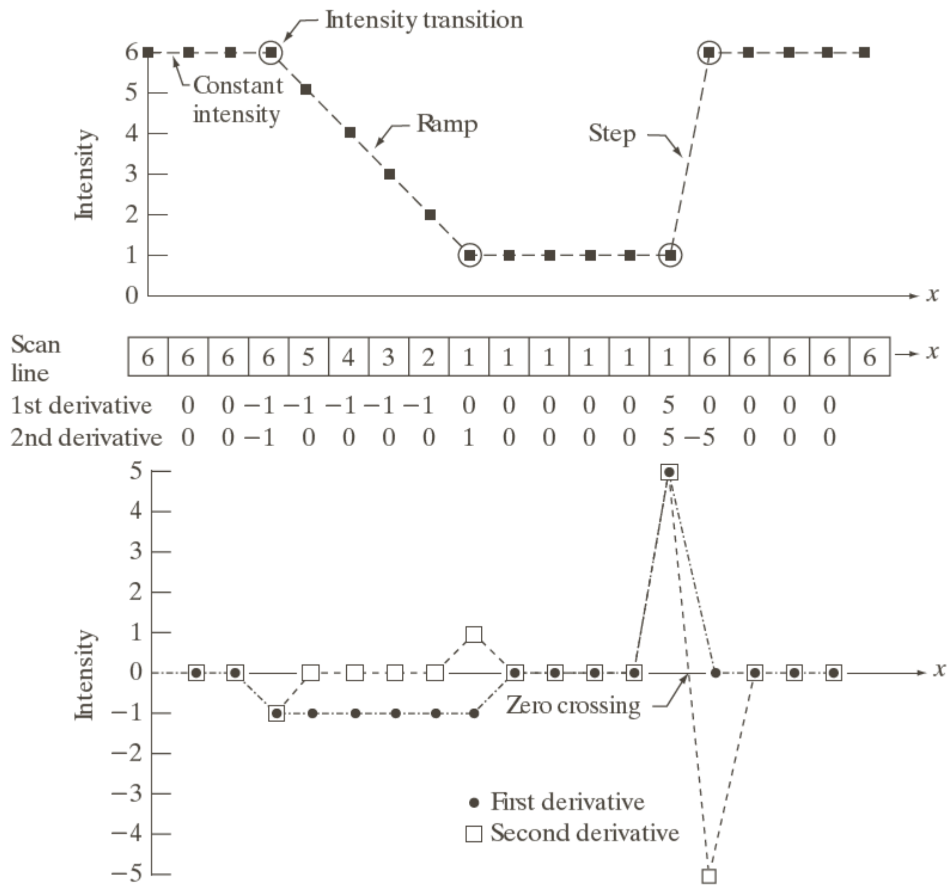

Illustration of the first and second derivatives of a 1-D digital function

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Using the Second Derivative for Image Sharpening – Laplacian operator

Isotropic filters – the response of this filter is independent of the direction of the discontinuities in the image. Isotropic filters are *rotation invariant*, in the sense that rotating the image and then applying the filter gives the same result as applying the filter to the image and then rotating the result.

The simplest isotropic derivative operator is the Laplacian:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

This operator is linear.

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$$\frac{\partial^2 f}{\partial x^2} \approx f(x+1, y) - 2f(x, y) + f(x-1, y)$$

$$\frac{\partial^2 f}{\partial y^2} \approx f(x, y+1) - 2f(x, y) + f(x, y-1)$$

$$\nabla^2 f(x, y) \approx f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1) - 4f(x, y)$$

0	1	0
1	−4	1
0	1	0

1	1	1
1	−8	1
1	1	1

0	−1	0
−1	4	−1
0	−1	0

−1	−1	−1
−1	8	−1
−1	−1	−1

Filter mask that approximate the Laplacian

The Laplacian being a derivative operator highlights gray-level discontinuities in an image and deemphasizes regions with slowly varying gray levels. This will tend to produce images that have grayish edge lines and other discontinuities, all superimposed on a dark, featureless background. Background features can be "recovered" while still preserving the sharpening effect of the Laplacian operation simply by adding the original and Laplacian images.

The basic way to use the Laplacian for image sharpening is given by:

$$g(x,y) = f(x,y) + c\nabla^2 f(x,y)$$

The (discrete) Laplacian can contain both negative and positive values – it needs to be scaled.

Blurred image of the North Pole of the Moon; Laplace filtered image

 

Sharpening with $c=1$ and $c=2$

Unsharp Masking and Highboost Filtering

- process used in printing and publishing industry to sharpen images
- subtracting an unsharp (smoothed) version of an image from the original image

1. Blur the original image
2. Subtract the blurred image from the original (the resulting difference is called the mask)
3. Add the mask to the original

Let $\bar{f}(x,y)$ be the blurred image. The mask is given by:

$$g_{\text{mask}}(x,y) = f(x,y) - \bar{f}(x,y)$$

$$g(x,y) = f(x,y) + k \cdot g_{\text{mask}}(x,y)$$

k* = *1 – unsharp masking

k > ***1*** – *highboost filtering*

original image

blurred image (Gaussian filter 5×5, σ=3)

mask – difference between the above images

unsharp masking result

highboost filter result ($k=4.5$)

The Gradient for (Nonlinear) Image Sharpening

$$\nabla f \equiv grad(f) \equiv \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix}$$

- the gradient points in the direction of the greatest rate of change of f at location (x,y).

The *magnitude* (*length*) of the gradient is defined as:

$$M(x,y) = \mathrm{mag}(\nabla f) = \sqrt{g_x^2 + g_y^2}$$

M(x,y) is an image of the same size as the original called the *gradient image* (or simply as the *gradient*). ***M(x,y)*** is rotation invariant (isotropic) (the gradient vector ∇f is not isotropic). In some applications, the following formula is used:

$$M(x,y) \approx |g_x| + |g_y| \quad \text{(not isotropic)}$$

Different ways of approximating g_x and g_y produce different filter operators.

Roberts cross-gradient operator (1965)

$$g_x \approx f(x+1,y+1) - f(x,y) = \Delta_1$$

$$g_y \approx f(x,y+1) - f(x+1,y) = \Delta_2$$

$$M(x,y)=\sqrt{\Delta_1^2+\Delta_2^2}$$

$$M(x,y)\approx|\Delta_1|+|\Delta_2|$$

Sobel operators

$$g_x \approx \big(f(x-1,y+1)+2f(x,y+1)+f(x+1,y+1)\big)- \\ \big(f(x-1,y-1)+2f(x,y-1)+f(x+1,y-1)\big)$$

$$g_y \approx \big(f(x+1,y-1)+2f(x+1,y)+f(x+1,y+1)\big)- \\ \big(f(x-1,y-1)+2f(x-1,y)+f(x-1,y+1)\big)$$

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

−1	0
0	1

0	−1
1	0

Roberts cross gradient operators

−1	−2	−1
0	0	0
1	2	1

−1	0	1
−2	0	2
−1	0	1

Sobel operators

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Filtering in the Frequency Domain

Filter: a device or material for suppressing or minimizing waves or oscillations of certain frequencies

Frequency: the number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable

Fourier series and Transform

Fourier in a memoir in 1807 and published in 1822 in his book *La Théorie Analitique de la Chaleur* states that any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (called now a *Fourier series*). Even function that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighing function – the *Fourier* *transform*. Both representations share the characteristic that a function expressed in either a Fourier series or transform, can be reconstructed (recovered) completely via an inverse process, with no loss of information. It allows us to work in the "Fourier domain" and then return to the original domain of the function without losing any information.

Complex Numbers

$$C = R + i\, I \quad , \quad R, I \in \mathbb{R}, \; i = \sqrt{-1} \;, \quad R \text{ - real part }, \; C - \text{imaginary part}$$

$$C^* = R - i\, I \; - \; \text{ the conjugate of the complex number } C$$

$$C = |C|\,(\cos\theta + i\sin\theta) \;, \; |C| = \sqrt{R^2 + I^2} \; - \text{ complex number in polar coordinates}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \; - \text{ Euler's formula}$$

$$C = |C|\, e^{i\theta}$$

Fourier Transforms

- analyzing an image as a set of spatial sinusoids in various directions, each sinusoid having a precise frequency

One-Dimensional Fourier Transform

Continuous Fourier transform (CFT) of a continuous function ***f***:

$$F(z) = \int_{-\infty}^{+\infty} f(x) e^{-i 2\pi z x} dx = R(z) + i\, I(z)$$

The corresponding inverse Fourier transform is:

$$f(x) = \int_{-\infty}^{+\infty} F(z) e^{-i 2\pi z x} dz$$

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The magnitude function **|*F(z)*|** is called the *Fourier spectrum* of the function ***f(x)***:

$$|\,F(z)\,| = \sqrt{\left[R(z)\right]^2 + \left[I(z)\right]^2}$$

The phase angle ***Φ(z)*** of the function ***f*** is denoted by:

$$\Phi(z) = \mathrm{atan}\left[\frac{I(z)}{R(z)}\right].$$

Example:

$$f(x) = \Pi\left(\frac{x}{a}\right),\ \Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

$$F(z) = \int_{-\infty}^{+\infty} \Pi\left(\frac{x}{a}\right) e^{-i2\pi zx} dx = a \frac{\sin(za)}{za} = a\,\mathrm{sinc}(az)$$

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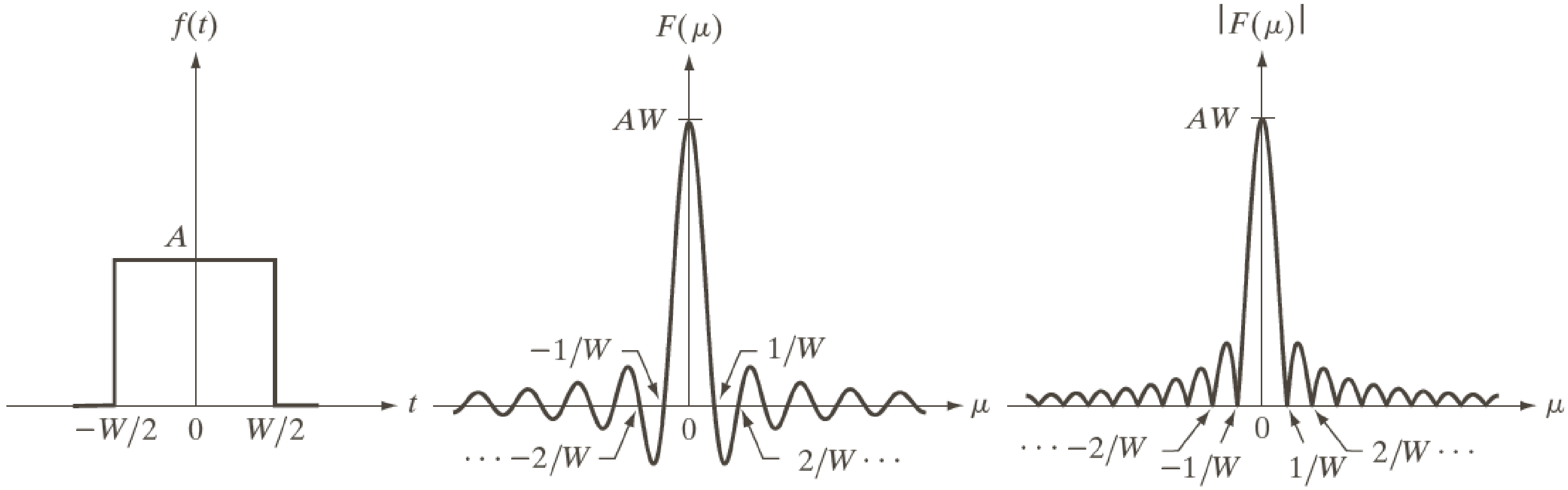

a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Two-Dimensional Fourier Transform

Continuous Fourier transform (CFT) of a continuous function ***f(x,y)***:

$$F(z,w)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-i2\pi(zx+wy)}dy\,dx = R(z,w)+i\,I(z,w)$$

The magnitude function |***F(z,w)***| and the phase angle ***Φ(z,w)*** of the function ***f*** are defined by:

$$|F(z,w)|=\sqrt{\left[R(z,w)\right]^2+\left[I(z,w)\right]^2}$$

$$\Phi(z,w)=\operatorname{atan}\left[\frac{I(z,w)}{R(z,w)}\right].$$

The *power spectrum* of ***f(x,y)*** is given by the relation:

$$P(z,w) = |F(z,w)|^2 = \left[R(z,w)\right]^2 + \left[I(z,w)\right]^2$$

The corresponding inverse Fourier transform is:

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(z,w)e^{-i2\pi(zx+wy)}dw\,dz.$$

Discrete Fourier Transform (DFT)

f(x)={f(0), f(1), …, f(N-1)}

One-dimensional DFT

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) e^{-i\frac{2\pi u x}{N}}, \quad u = 0, 1, \ldots, N-1$$

The inverse transformation

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{i\frac{2\pi u x}{N}}, \quad x = 0, 1, \ldots, N-1$$

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$f = \{f(x,y);\ x=0,1,\ldots,M\text{-}1\ ;\ y=0,1,\ldots,N\text{-}1\}$

Two-dimensional DFT

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-i\,2\pi\left(\frac{u\,x}{M}+\frac{v\,y}{N}\right)},\ \ u=\overline{0,M-1}, v=\overline{0,N-1}$$

The inverse transformation

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{i\,2\pi\left(\frac{u\,x}{M}+\frac{v\,y}{N}\right)},\ \ x=\overline{0,M-1}, y=\overline{0,N-1}$$

Transformation Kernels

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)g(x,y,u,v),$$

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)h(x,y,u,v),$$

g(x,y,u,v) – forward transformation kernel

h(x,y,u,v) – inverse transformation kernel

$$g(x,y,u,v)=g_1(x,u)g_2(y,v)\rightarrow \text{the kernel is separable}$$

$$g_1\equiv g_2\rightarrow \text{the kernel is symmetric}$$

DFT – the kernel is separable and symmetric

Properties

- **Translation** – the translation of a Fourier transform pair is:

$$f(x,y)e^{i2\pi\left(\frac{u_0 x}{M}+\frac{v_0 y}{N}\right)} \quad \leftrightarrow \quad F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \quad \leftrightarrow \quad F(u,v)e^{-i2\pi\left(\frac{u_0 x}{M}+\frac{v_0 y}{N}\right)}$$

- **Rotation** – assume that ***f(x,y)*** undergoes a rotation of angle θ_0. Using polar coordinates:

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\ ,\ \ f(x,y) = \bar{f}(r,\theta)$$

$$F(u,v) = \bar{F}(s,\omega)\ ,\ \ u = s\cos\omega,\ \ v = s\sin\omega$$

$$f \leftrightarrow F,\quad \bar{f}(r,\theta+\theta_0) \leftrightarrow \bar{F}(s,\omega+\theta_0)$$

If ***f(x,y)*** is rotated by θ_0 then the Fourier transform ***F*** will be rotated by the same angle.

- **Separability**: this property ensures that the computations can be performed by decomposing the two-dimensional transform into two one-dimensional transforms

$$F(u,v) = \left[\frac{1}{M}\sum_{x=0}^{M-1} f(x,y) e^{-i\frac{2\pi u x}{M}}\right]\left[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y) e^{-i\frac{2\pi v y}{N}}\right]$$

$$f(x,y) = \left[\sum_{u=0}^{M-1} F(u,v) e^{i\frac{2\pi u x}{M}}\right]\left[\sum_{v=0}^{N-1} F(u,v) e^{i\frac{2\pi v y}{N}}\right]$$

Hence the two-dimensional DFT (and the inverse) can be computed by taking the one-dimensional DFT row-wise in the two-dimensional image and the result is again transformed column-wise by the same one-dimensional DFT.

- **Distributivity:** the DFT of the sum of two images f_1 and f_2 is identical with the sum of the DFT of these two function

$$F\left\{f_1+f_2\right\}=F\{f_1\}+F\{f_2\}.$$

The property does not hold for the product of two functions:

$$F\left\{f_1\cdot f_2\right\}\neq F\{f_1\}\cdot F\{f_2\}$$

- **Scaling property**

$$F\left\{k\,f\right\}=kF\{f\}\;,\;k\in\mathbb{R}$$

- **Convolution**

$$F\left\{ f_1 \otimes f_2 \right\} = F\{f_1\} \cdot F\{f_2\}$$

$$\left(f_1 \otimes f_2 \right)(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_1(u,v)\, f_2(x-u, v-u)\, du\, dv$$

$$\left(f_1 \otimes f_2 \right)(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f_1(u,v)\, f_2(x-u, y-v)$$

- **Periodicity**

$$f(x,y) = f(x+k_1M, y) = f(x, y+k_2N) = f(x+k_1M, y+k_2N)\ ,$$

$$k_1, k_2 - \text{integers}$$

$$F(u,v) = F(u+k_1M,v) = F(u,v+k_2N) = F(u+k_1M,v+k_2N)$$

$$f(x,y)(-1)^{x+y} \qquad \leftrightarrow \qquad F(u-\frac{M}{2},v-\frac{N}{2})$$

This last relation shifts the data so that ***F(0,0)*** is at the center of the frequency rectangle defined by the intervals ***[0,M-1]*** and ***[0,N-1].***

Symmetry Properties

Odd and *even* part of a function:

$$w(x,y) = w_e(x,y) + w_o(x,y)$$

$$w_e(x,y) = \frac{w(x,y)+w(-x,-y)}{2}$$

$$w_o(x,y) = \frac{w(x,y)-w(-x,-y)}{2}$$

$$w_e(x,y) = w_e(-x,-y) \ - \ \ symmetric$$

$$w_o(x,y) = -w_o(-x,-y) \ - \ \ antisymmetric$$

For digital images, evenness and oddness become:

$$w_e(x,y) = w_e(M-x, N-y)$$

$$w_o(x,y) = -w_o(M-x, N-y)$$

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} w_e(x,y)w_o(x,y) = 0$$

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M-1]$, and y, and v in the range $[0, N-1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

Fourier Spectrum and Phase Angle

Express the Fourier transform in polar coordinates:

$$F(u,v) = |F(u,v)| e^{i\phi(u,v)} \quad ,$$

$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$ is called *Fourier* or *frequency spectrum*

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$ is the *phase angle*

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$ - *the power spectrum*

$$|F(u,v)| = |F(-u,-v)|$$

$$\phi(u,v) = -\phi(-u,-v)$$

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0) = MN\,\bar{f}\ \ ,\ \ \bar{f} = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) -$$ the average value of the image f

$$|F(0,0)| = MN\,|\bar{f}|$$

Because ***MN*** usually is large, |***F(0,0)***| is the largest component of the spectrum by a factor that can be several orders of magnitude larger than other terms.

F(0,0) sometimes is called the ***dc*** *component* of the transform. (dc='direct current' – current of zero frequency)

The 2-D Convolution Theorem

2-D circular convolution:

$$f(x,y)\otimes h(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n),\ x=\overline{0,M-1},\ y=\overline{0,N-1}$$

The 2-D convolution theorem

$$f(x,y)\otimes h(x,y) \quad \leftrightarrow \quad F(u,v)H(u,v)$$

$$f(x,y)h(x,y) \quad \leftrightarrow \quad F(u,v)\otimes H(u,v)$$

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v)\, e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = \lvert F(u, v)\rvert e^{j\phi(u,v)}$
4) Spectrum	$\lvert F(u, v)\rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \lvert F(u, v)\rvert^2$
7) Average value	$\overline{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

TABLE 4.2
Summary of DFT definitions and corresponding expressions.

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Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u,v) = F(u+k_1M, v) = F(u, v+k_2N)$ $= F(u+k_1M, v+k_2N)$ $f(x,y) = f(x+k_1M, y) = f(x, y+k_2N)$ $= f(x+k_1M, y+k_2N)$
9) Convolution	$f(x,y)\bigstar h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$
10) Correlation	$f(x,y)\star h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m, y+n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

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Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

(*Continued*)

TABLE 4.3
Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the closed-form, continuous expressions.

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Name	DFT Pairs
7) Correlation theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F^*(u, v)\, H(u, v)$ $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$
8) Discrete unit impulse	$\delta(x, y) \Leftrightarrow 1$
9) Rectangle	$\text{rect}[a, b] \Leftrightarrow ab \dfrac{\sin(\pi ua)}{(\pi ua)} \dfrac{\sin(\pi vb)}{(\pi vb)} e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\dfrac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)\right]$
11) Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\dfrac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0)\right]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12) *Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\dfrac{\partial}{\partial t}\right)^m \left(\dfrac{\partial}{\partial z}\right)^n f(t, z) \Leftrightarrow (j2\pi\mu)^m (j2\pi\nu)^n F(\mu, \nu)$ $\dfrac{\partial^m f(t, z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu, \nu); \dfrac{\partial^n f(t, z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu, \nu)$
13) *Gaussian*	$A 2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow A e^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

†Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.

The 2-D Discrete Fourier Transform and Its Inverse

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\left(\cos\left(2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)-i\sin\left(2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)\right)$$

$$=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cos\left(2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)-i\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\sin\left(2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)$$

$$=A(u,v)+iB(u,v)=S(u,v)e^{i\Phi(u,v)}$$

f(x, y) is a digital image of size $M\times N$.

Given the transform ***F(u, v)*** we can obtain ***f(x, y)*** by using the *inverse discrete Fourier transform (IDFT)*:

Course 5

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}\quad ,\ x=0,1,\dots,M-1\ ,\ y=0,1,\dots,N-1$$

$$f(x,y)=\begin{pmatrix} 0.5000 & 0.5000 & 0.7500 & 0.7500 & 0.7500 \\ 0.5000 & 0.5000 & 0.5000 & 0.7500 & 0.7500 \\ 0.2500 & 0.5000 & 0.5000 & 0.5000 & 0.7500 \\ 0.2500 & 0.2500 & 0.5000 & 0.5000 & 0.5000 \\ 0.2500 & 0.2500 & 0.2500 & 0.5000 & 0.5000 \end{pmatrix}$$

$$F(u,v)=\begin{pmatrix} 12.50+i\,0.00 & -1.08+i\,1.48 & -0.79+i\,0.26 & -0.79-i\,0.26 & -1.08-i\,1.48 \\ 1.08-i\,1.48 & 0.00+i\,0.00 & 0.28+i\,0.09 & 0.28+i\,0.38 & 0.00-i\,0.18 \\ 0.79-i\,0.26 & -0.28-i\,0.09 & 0.00+i\,0.00 & 0.00-i\,0.77 & -0.28+i\,0.38 \\ 0.79-i\,0.26 & -0.28-i\,0.38 & 0.00+i\,0.77 & 0.00+i\,0.00 & -0.28+i\,0.09 \\ 1.08+i\,1.48 & 0.00+i\,0.18 & 0.28-i\,0.38 & 0.28-i\,0.09 & 0.00-i\,0.00 \end{pmatrix}=$$

$$\begin{pmatrix} 12.50 & -1.08 & -0.79 & -0.79 & -1.078 \\ 1.08 & 0 & 0.28 & 0.28 & 0 \\ 0.79 & -0.28 & 0 & 0 & -0.28 \\ 0.79 & -0.28 & 0 & 0 & -0.28 \\ 1.08 & 0 & 0.28 & 0.28 & 0 \end{pmatrix}+i\begin{pmatrix} 0 & 1.48 & 0.26 & -0.26 & -1.48 \\ -1.48 & 0 & 0.09 & 0.38 & -0.18 \\ -0.26 & -0.09 & 0 & -0.77 & 0.38 \\ 0.26 & -0.38 & 0.77 & 0 & 0.09 \\ 1.48 & 0.18 & -0.38 & -0.09 & 0 \end{pmatrix}$$

$$|F(u,v)|=\begin{pmatrix} 12.50 & 1.83 & 0.84 & 0.84 & 1.83 \\ 1.83 & 0 & 0.29 & 0.48 & 0.18 \\ 0.84 & 0.29 & 0 & 0.77 & 0.48 \\ 0.84 & 0.48 & 0.77 & 0 & 0.29 \\ 1.83 & 0.18 & 0.48 & 0.29 & 0 \end{pmatrix}\qquad \Phi(u,v)=\begin{pmatrix} 0 & 7 & 9 & -9 & -7 \\ -3 & 5 & 1 & 3 & -5 \\ -1 & -9 & 0 & -5 & 7 \\ 1 & -7 & 5 & 0 & 9 \\ 3 & 5 & -3 & -1 & -5 \end{pmatrix}$$

1NN, cityblock distance, "leave-one-out" cross validation

	Image features	Fourier features
Face recognition	Image features	Fourier features
Faces95	95.49%	91.53%
AT&T	98.75%	97.50%
Gender		
Feret+AR (446f+672m)	83.27%	68.52%
CBIR		
Corel 1000	53.20%	63.8%
Texture		
T	32.20%	57.60%
Brodatz	72.582%	75.79%