A New Analytic Method to Approximate Trigonometric Functions based upon Legendre Polynomials

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*Abstract*— This paper presents a new analytic approximation method which calculates very accurate values for forward trigonometric functions. The method integrates the product of Legendre polynomials with the sine function and then derives a solution from the quotient. The resulting approximations are valid over the entire range [-∞,+∞] and thus are suited for computer subroutines. We also demonstrate a method to evaluate certain derivatives containing trigonometric functions, and likewise, to evaluate certain definite integrals for which no closed form exists.

*IndexTerms*—Elementary function approximation, evaluating definite integrals and derivatives, Legendre polynomials.

1. **INTRODUCTION**

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inding rational approximations to trigonometric functions has a long history with much effort devoted to approximating these functions by polynomials. Many of these approximations and their degree of error within an indicated range are summarised in [1]. In his survey [2], Cody describes methods such as analytic expansions and minimax approximations for practical rational and polynomial

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odd Legendre polynomials and the sine function. These analytic solutions generate a set of approximations *T*(*n*,*a*) ≈ tan(*a*), which when evaluated numerically in the narrow range

0 < *a* < π/4, permit us to derive highly accurate approximations to sine, cosine and other trigonometric functions over the entire range [-∞, +∞]. Numerous examples of trigonometric approximations in terms of the *T*(*n*,*a*) polynomial quotients are presented including evaluations of the sine integral and Bessel functions.

1. **QUOTIENT APPROXIMATION METHOD**
2. ***Theoretical Approach***

Recently, while evaluating some integrals involving the Legendre polynomials, we ran across the definite integral

*I* (*n*,*a*)  *x*1 *P* (*t*)*G*(*at*)*dt*  *N* (*n*,*a*)sin(*a*)  *M* (*n*,*a*)cos(*a*)

*n*

approximations of elementary functions. Without analytic expansions that were economical, many authors have used the Remez algorithm to minimize the maximum error of the



*x*0

*an*1

(1)

polynomial employed for the given interval [3]. A recent trend has been to shift from software to actual microcode on the CPU chip itself. In the design for a numerical coprocessor such as [4], a rational approximation was used for tangent, but more recently in the design for the Intel IA-64 [5], the cheap hardware multiply and add instructions were preferred. Despite this shift from the use of rational approximations,

polynomial quotients such as Pade approximations[6] still

where *Pn*(t) are the Legendre polynomials and G(a,t) equals

cos(at) for even n and sin(at) for odd n. Here N(n,a) and *M*(*n*,*a*) are polynomials in *a* for fixed *n*. What is interesting about this equality is ( because of the oscillatory nature of Pn(t) ) that when n gets large but a<1, the product of an+1 I(n,a) becomes very small so that one has to a good approximation

*I* (*n*, *a*)*a* (*n*1)

receive considerable attention.

Our purpose here to introduce a new rational approximation

cos(*a*)

 0  *N* (*n*, *a*) tan(*a*)  *M* (*n*, *a*)

(2)

technique [7], based on polynomial quotients, derived by solving integrals whose integrand consists of the product of

provided that cos(a) does not vanish. We thus have the estimate

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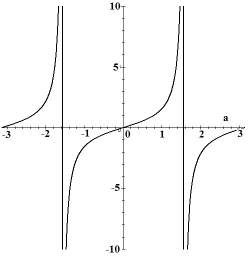
tan(*a*)  *T* (*n*, *a*)  [ *M* (*n*, *a*)]

*N* (*n*, *a*)

(3)

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with the order of the approximation determined by the value of *n*. A measure of the error in tan(a) for given values of *n* and

*a* is determined by the value of ε = *I*(*n*,*a*)*an*+1/[*N*(*n*,*a*)cos(*a*)]. Working out the first few approximations for tan(a), we find-

*T* (1, *a*)  *a*

*T* (2, *a*) 

*T* (3, *a*) 

3*a* 3  *a*2

15*a*  *a*3

15  6*a*2

105*a*  10*a*3

*T* (4, *a*)  105  45*a*2  *a*4

945*a*  105*a*3  *a*5

*T* (5, *a*) 

*T* (6, *a*) 

*T* (7, *a*) 

*T* (8, *a*) 

*T* (9, *a*) 

945  420*a* 2  15*a*4

10395  1260*a*3  21*a*5

10395  4725*a*2  210*a* 4  *a*6

135135*a*  17325*a*3  378*a*5  *a*7

135135  62370*a*2  3150*a*4  28*a*6

2027025*a*  270270*a*3  6930*a*5  36*a*7

2027025  945945*a*2  51975*a* 4  630*a*6  *a*8

34459425*a*  4729725*a*3  135135*a*5  990*a*7  *a*9

34459425  16216200*a* 2  945945*a* 4  13860*a*6  45*a*8

(4)

**Fig.1. Plot of the Quotient Approximation *T*(9,*a*).**

The *T*(9,*a*) approximation also predicts the location of singularities of the tan(*a*) function at –π/2 and π/2 to twelve

One notices that when ‘a’ gets small each of these tangent

approximations go as tan(a)≈a. With increasing n the approximations become more and more accurate. For T(9,1) we get an error estimate of ε=1.439x10-16. Also, one has the equality-

digit accuracy. This is achieved by setting the denominator of

*T*(9,*a*) to zero.

To generate approximate values for other trigonometric functions we start with *T*(*n*,*a*) and note the following

tan(*a*)  lim *T* (*n*, *a*)

*n*  

(5)

sin(*a*)  *S* (*n*, *a*) 

1  *T* (*n*, 2*a*   )2

 4

*T* (*n*, *a*)

1  *T* (*n*, *a*)2

(6)

The above quotients are very easy to generate compared to standard Pade approximates and generally are more accurate for the same polynomial powers in the quotients. The only

and

1  *T* (*n*, 2*a*   )2

4

drawback noticed with this approximation method is that the quotients become rather lengthy as n is increased beyond n=10 or so.

To confirm the above ε estimate for n=9 and a=1, we find-

cos(*a*)  *C*(*n*, *a*) 

*a*

1  *T* (*n*, )

1

1  *T* (*n*, *a*)2

 2

*a*

1  *T* (*n*, )

2

2

. (7)

2

*T* (9,1)  1.557407724654890208...

*compared to*

tan(1)   / 2  1.55740772465490223...

Thus one indeed has a 16 digit accurate estimate for tan(1)=π/2 and for π. A series expansion of the T(9,1) quotient matches the standard Taylor expansion out to O(a17). By decreasing the value of ‘a’ or increasing n , the accuracy of our approximations will improve further. For n=9 and a=1/10,

Note that the second forms for the functions *S*(*n*,*a*) and *C*(*n*,*a*) do not involve the square root of a function and so are somewhat simpler computationally. The approximations for sine and cosine have an accuracy comparable with that of the tangent approximation. Thus, calculating approximations for tan(π/6), sin(π/6) and cos(π/6) respectively, we obtain:



*T* (9, )  0.57735026918962576453 ...

6



one finds a result accurate to 35 digits. One also obtains a very accurate representation of the standard tan(a) curve in –π<a<π as shown.

*S* (9,

*C* (9,

)  0.50000000000000000003...

6

 )  0.86602540378443864678... 6

In each case, these values are accurate to 18 places compared to the known values 1/sqrt(3), 1/2 and sqrt(3)/2, respectively. Once *T*(*n*,*a*) has been determined, approximations for other trigonometric functions (excluding the inverse functions), are

possible. As an example, we may calculate secant(1) as

2

**FURTHER RESULTS FROM THE METHOD**

From the above theoretical results and the properties of trigonometric functions, it is clear that to be able to calculate tan(*a*) and other trigonometric functions for any argument, it is sufficient to have highly accurate approximations for tan(*a*) for the limited range 0 < a < π/4. This fact follows from well

sec( 1)  1  [ *K* (9,0.5)]

 1.8508157176 8092561791 1...

known identities for tan(*a*) plus the less well known identity

1  [ *K* (9,0.5)]2

and this result is accurate to 22 places.



tan(

4

* *A*) tan( 

4

 *A*)  1 . (8)

One question seems pertinent at this point, have previous rational approximations every come close to the analytic solutions of equation (4)? An investigation of the rational

Thus, to find an approximation for tan(13π/9) using *n* = 4, we may use

polynomials employed by Koren and Zinaty in [4] shows this is true for their tangent formula. They reduce an argument α to the integer *N* and the fraction *x* as follows:

tan(

13

9

4

)  tan( ) 

9

1



*T* (4, )

 4 /   2 *N*  *x*

If 2*N* is odd they add 1 and then *x* becomes –(1-*x*). This enables them to calculate tan(α) as tan(𝜋*x*/4) when N is even

and –cot(𝜋*x*/4) when *N* is odd. With this done their formula

18

 5.67128181961770953099441843986

yielding a 30 digit accurate result.

With this new method to accurately approximate trigonometric functions, it also becomes possible to estimate the values of various definite integrals that include these functions. Consider first the following integral containing tan(*t*) and then apply the *n* = 4 approximation:

(page 1033) matches the form of *T*(9,*a*) with *a* replaced by *x*. To afford a direct comparison with *T*(9,*a*), we reduced their

1

*S*  

*t*0

tan(*t*)4

*t*

*dt* 

1



*a*0

*T* (4, *a*)4

*a*

*da* . (9)

coefficients on page 1037, multiplying by 4/𝜋 as necessary.

After this multiplication, their coefficients *a*0 and *b*0 corresponding to the repeated value 34459425 in *T*(9,*a*) are identical to 23 digits! We then calculated a number to

We find at once the 15 place accurate approximation *S* ≈ 0.815400592038350 for this integral, which cannot be evaluated in closed form.

For another application, consider the zeroth order Bessel Function of the First Kind. It can be defined by the integral

rationalize the average of these two coefficients to match our value. Multiplying throughout by this number (and truncating for brevity) results in:

*J*0 (*x*) 

2  / 2

Re  exp[*ix* sin(*t*)]*dt*

 *t*0

(10)

tan(*a*) 

*a*(34459425  4730898*a*2  135285*a*4  993.28*a*6 

1.0087*a*8 )

Note here that the range 0 < *t* < π/2 is sufficiently small such that sin(*t*) is well approximated by its *T*(*n*,a) form. Using *n* = 3, *x* = 1 and *t* = *a*, we find the 10 place accurate result

34459425  16217373*a*2  946486*a*4  13887*a*6  45.24*a*8

(9)

*J* 0 (1) 

2 *a* / 2 Re 

*a*

exp[*i T* (3, *a*) /

]*da*

1  *T* (3, *a*)2

which can be compared directly with *T*(9,*a*). It can be seen

that the minimax algorithm that they employed has perturbed the values in the analytic solution to achieve their desired error

 0

 0.7651976865...

(11)

goals in the interval -1 < *x* < 1. Their equation does produce

23 digit accuracy but this cannot be increased by simply

Another integral which can be well approximated by our

quotient method is the sine integral

switching to *T*(11,*a*). It seems that some of the effort expended on minimax approximations in the last 50 years might have been better spent developing analytic solutions.

*x*

*Si*(*x*)  

*t*0

sin(*t*)

*t*

*dt* 

  / 2

 

2 *t*0

exp[*x*(cos(*t*)][cos(*x* sin(*t*))]*dt*

(12)

The second integral form is found on page 232 of the reference by Abramowitz and Stegun, ”Handbook of Mathematical Functions” [1]. This integral is well suited for approximations using the *T*(*n*,*a*) function. After a little manipulation we find

the tangent function is known. These exact values can be obtained by starting with the known values of tan(π/4) = 1 and tan(π/6) = 1/sqrt(3), and then applying the half angle formula

*b* 1  1 

 tan(2 )  tan(*b*) cos(*b*)  1 . (14)

 / 2

1  *iT* (*n*, *t*)  

*Si*(*x*)   Re 

2 *t*0

exp *x*[

1  *T* (*n*, *t*)

]*dt*

2

. (13

We can thus determine two additional exact values, tan(π/8) = sqrt(2)-1 and tan(π/12) = 2-sqrt(3). Let us now compute

Plotting this approximation using the less accurate n=3 approximation we get the following result over the wide range 0 < *x* < 15.

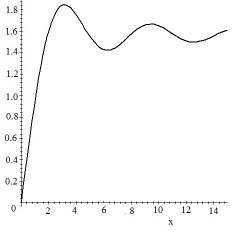
tan(π/7) which must lie between the values of sqrt(2)-1 and 1/sqrt(3). Using the approximation

tan(*a*)  *T* (*n*, *a*)

tan(*a*  *a*) 

1  tan(*a*)*T* (*n*, *a*)

. (15)

we find that the tangent of π/7 is bracketed as follows:

(  1)  *T* (*n*,  )

2

56



2

 1 

tan( ) 



7



3*T* (*n*, )

42



3

(16)

1  (

 1)*T* (*n*, )

56

 *T* (*n*, )

42

A numerical evaluation of this inequality for *n* = 9, determines tan(π/7) falls in a very limited range as follows:

0.4815746188075286443321623530569705752**193**…

< tan(π/7) < 0.4815746188075286443321623530569705752**410**…

**Fig.2. Approximation of the Si(x) Function using T(3,t)**

One can also obtain excellent approximations to the derivatives of functions at particular points. Take the following derivative evaluated at π/2:

Thus tan(π/7) has been approximated to 37 digit accuracy. This accuracy is consistent with what the error term ε given above predicts. The reason for the high accuracy follows directly from the small values of *a* appearing in these *T*(*n*,*a*) approximations (see definition of ε). The value of tan(π/10) can be similarly bracketed as

*d*exp[ tan(*x*)]  

2

 [  .

(2 

3)  *T* (*n*,  )



 (  1)  *T* (*n*, )

2

*dx* ]*x*

[1  *T* (*n*, )

2

]exp[*T* (*n*, )]

2

60  tan(

)  40

2

With *n* =9, we get the 34 digit accurate approximation

(14)

1  (2 



3)*T* (*n*, )

60

10 1  (

 1)*T* (*n*,  )

40

2

(17)

2  0.7357588823 4288464319 1047540322 9217...

*e*

1. **IMPROVING ACCURACY BY A TWO-SIDED BOUNDING METHOD**

A way to still further improve the numerical accuracy of estimates for tan(*a*) , for *a* within [0,π/4], is to apply the *T*(*n*,*a*) approximation at a nearby point for which an exact value of

These inequalities yield a value for tan(π /10) accurate to 17 digits when *n* = 5 and 39 digits when *n* = 9.

We have automated this two sided approximation approach for determining tan(*a*) for any argument within 0 < *a* < π/4.

# CONCLUSION

We have derived and applied a new analytic approximation method using Legendre Polynomials to approximate values for the forward trigonometric functions to a high order of accuracy. After obtaining approximation formulas for tan(*a*), we apply these to obtain very accurate approximations for sine and cosine and also show how these approximations may be used to find numerical values for definite integrals and derivatives of functions. A two sided bracketing approach is developed to allow an accurate measure of the digit accuracy of a given approximation to a trigonometric function at any point in the range 0 < a < π/4.

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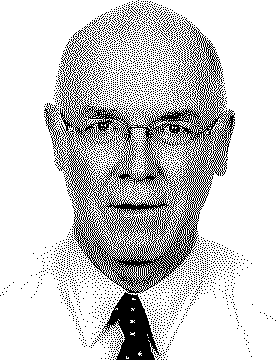
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