3. TRANSVERSE OSCILLATIONS OF CORONAL LOOPS

– Part 2: The kink mode of a magnetic cylinder –

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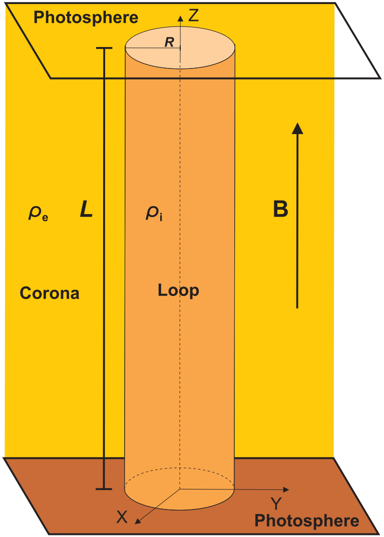
Standard coronal loop model: straight magnetic cylinder



- Curvature neglected
- Homogeneous magnetic field along the tube
- Radius: R, length: L
- Abrupt density jump at $r = R$
- Overdense loop: $\rho_\mathrm{i} > \rho_\mathrm{e}$, with both densities uniform
- Cylinder footpoints fixed at the photosphere.

This system supports MHD normal modes

Linearized MHD equations

- To study MHD normal modes of the cylinder we consider the linearized MHD equations.

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\rho_0 \nabla \cdot \vec{v}_1, \\
\rho_0 \frac{\partial \vec{v}_1}{\partial t} &= -\nabla p_1 + \frac{1}{\mu}\left(\nabla \times \vec{B}_1\right) \times \vec{B}_0, \\
\frac{\partial \vec{B}_1}{\partial t} &= \nabla \times \left(\vec{v}_1 \times \vec{B}_0\right), \\
\frac{\partial p_1}{\partial t} &= v_{\mathrm{s}}^2 \frac{\partial \rho_1}{\partial t}, \qquad \frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}.
\end{aligned}
$$

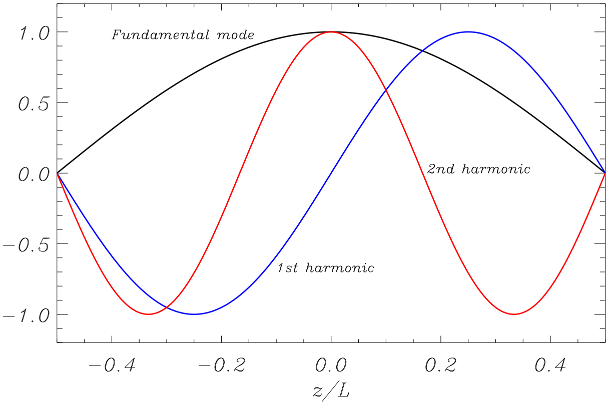
- $\beta = 0$ approximation in the solar corona $v_{\mathrm{s}} \ll v_{\mathrm{A}}$ (gas pressure gradient is neglected compared to magnetic force)
- We shall use the total pressure perturbation, P', as our main variable

Normal modes

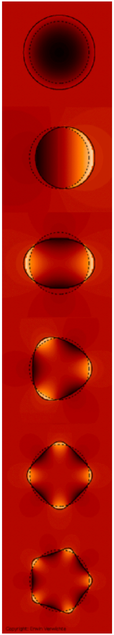
- We use cylindrical coordinates: r, φ, z
- We assume perturbations of the from:
  $P' \sim P'(r) \exp\left(im\varphi + ik_z z - i\omega t\right)$
- Radial dependence is contained in $P'(r)$
- Longitudinal dependence depends on the value of k_z:
  - $k_z = \frac{\pi}{L} \longrightarrow$ Fundamental mode
  - $k_z = \frac{2\pi}{L} \longrightarrow$ First harmonic
  - $k_z = \frac{3\pi}{L} \longrightarrow$ Second harmonic
  - and so on...
- Azimuthal dependence depends on the value of m:
  - Sausage modes ($m = 0$): expansion and contraction of tube cross-section
  - Kink mode ($m = 1$): lateral displacement of the tube
  - Fluting modes ($m \geq 2$): perturbations of the tube boundary
  - Only kink modes displace laterally the cylinder axis
- The frequency of oscillation, ω, needs to to be determined from the **dispersion relation**.

Longitudinal dependence (standing modes)

- $k_z = \frac{n\pi}{L}$
  - $n = 1$ Fundamental mode
  - $n = 2$ First harmonic (second mode)
  - $n = 3$ Second harmonic (third mode)



Azimuthal dependence



m = 0 (Sausage or breathing mode)

m = 1 (kink mode)

m = 2 (first fluting mode)

m = 3 (second fluting mode)

m = 4 (third fluting mode)

m=5 (fourth fluting mode)

Radial dependence

- Governing equation for $P'(r)$

$$\frac{\partial^2 P'}{\partial r^2}+\left[\frac{1}{r}-\frac{\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho(r)\left(\omega^2-\omega_{\mathrm{A}}^2(r)\right)\right)}{\rho(r)\left(\omega^2-\omega_{\mathrm{A}}^2(r)\right)}\right]\frac{\partial P'}{\partial r}+\left(\frac{\rho(r)\left(\omega^2-\omega_{\mathrm{A}}^2(r)\right)}{B^2/\mu}-\frac{m^2}{r^2}\right)P'=0$$

- If density is uniform ($\rho=$ constant) we get the Bessel Equation of order m

$$\frac{\partial^2 P'}{\partial r^2}+\frac{1}{r}\frac{\partial P'}{\partial r}+\left(\frac{\rho\left(\omega^2-\omega_{\mathrm{A}}^2\right)}{B^2/\mu}-\frac{m^2}{r^2}\right)P'=0$$

- The quantity $k_\perp$ plays the role of the wavenumber in the radial direction

$$k_\perp^2=\frac{\rho\left(\omega^2-\omega_{\mathrm{A}}^2\right)}{B^2/\mu}=\frac{\omega^2-k_z^2 v_{\mathrm{A}}^2}{v_{\mathrm{A}}^2}$$

- $k_\perp$ is different inside and outside the tube

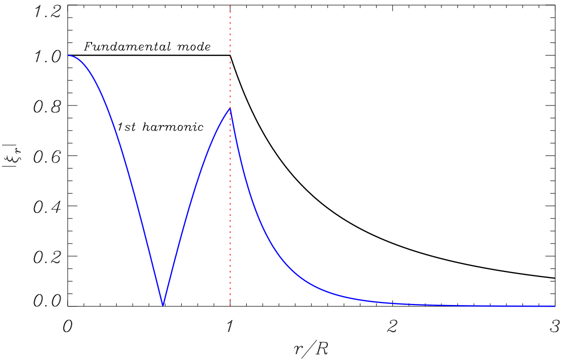
Trapped modes

- Internal "radial" wavenumber:

$$k_{\perp,\mathrm{i}}^2 = \frac{\omega^2 - k_z^2 v_{\mathrm{A},i}^2}{v_{\mathrm{A},i}^2} > 0$$

- External "radial" wavenumber:

$$\tilde{k}_{\perp,\mathrm{e}}^2 = -\frac{\omega^2 - k_z^2 v_{\mathrm{A},e}^2}{v_{\mathrm{A},e}^2} < 0$$



Trapped modes

- We seek solutions that represent trapped modes: oscillatory inside and evanescent outside
- Physical solution:

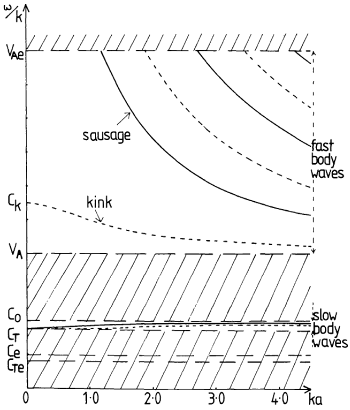
$$P'(r) = \begin{cases} A_\mathrm{i} J_m(k_{\perp,\mathrm{i}} r), & \text{if} \quad r \leq R \\ A_\mathrm{e} K_m(\tilde{k}_{\perp,\mathrm{e}} r), & \text{if} \quad r > R \end{cases}$$

- J_m: Bessel function of the first kink of order m
- K_m: Modified Bessel function of the second kink of order m
- A_i and A_e are constants
- To find the dispersion relation, we impose continuity of P' and ξ_r at $r = R$

Dispersion relation

$$\frac{\tilde{k}_{\perp,\mathrm{e}}}{\rho_e\left(\omega^2-k_z^2 v_{\mathrm{A},e}^2\right)}\frac{K_m'\left(\tilde{k}_{\perp,\mathrm{e}}R\right)}{K_m\left(\tilde{k}_{\perp,\mathrm{e}}R\right)}-\frac{k_{\perp,\mathrm{i}}}{\rho_i\left(\omega^2-k_z^2 v_{\mathrm{A},i}^2\right)}\frac{J_m'\left(k_{\perp,\mathrm{i}}R\right)}{J_m\left(k_{\perp,\mathrm{i}}R\right)}=0$$

Edwin & Roberts (1983)



- Classification according o the phase speed of the modes: ω/k_z
  - Transverse (fast) modes: $v_{\mathrm{A},i} < \omega/k_z < v_{\mathrm{A},e}$
  - Longitudinal (slow) modes $v_{\mathrm{T,i}} < \omega/k_z < v_{s,\mathrm{i}}$ (not present when $\beta = 0$ because $v_s = 0$)
  - Torsional/Rotational (Alfvén) modes: $\omega/k_z = v_{\mathrm{A},i}$

Animations of different oscillatory modes

- Solar Wave Theory Group in Sheffield

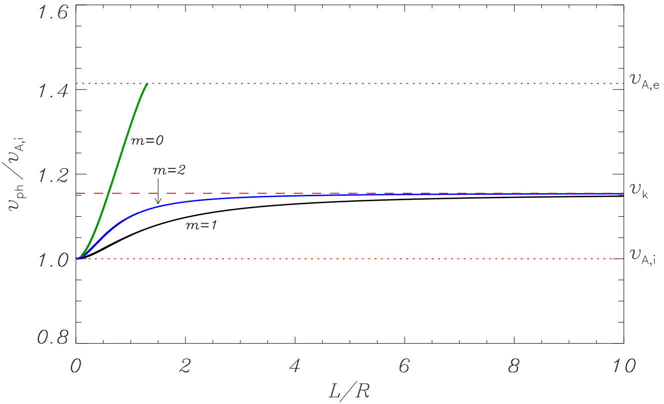
  http://swat.group.shef.ac.uk/fluxtube.html

- Centre for Fusion, Space and Astrophysics in Warwick

  http://www2.warwick.ac.uk/fac/sci/physics/research/cfsa/research/wpc/vis/

Longitudinally fundamental, radially fundamental modes

- Phase velocity: $v_{\mathrm{ph}} = \omega / k_z$ with $k_z = \frac{\pi}{L}$
- Solution of the dispersion relation for $v_{\mathrm{A},i} < v_{\mathrm{ph}} < v_{\mathrm{A},e}$
- Modes with $m \neq 0$ converge when $L/R \gg 1$.



$$v_k = \sqrt{\frac{\rho_i v_{\mathrm{A},i}^2 + \rho_e v_{\mathrm{A},e}^2}{\rho_i + \rho_e}} = v_{\mathrm{A},i}\sqrt{\frac{2\zeta}{\zeta + 1}} \qquad \zeta = \frac{\rho_i}{\rho_e}$$

Thin tube approximation

- We focus on the longitudinally fundamental, radially fundamental modes
- We consider the limit $L/R \gg 1$ or, equivalently, $k_z R \ll 1$
- First order, asymptotic expansion for **small arguments** of the Bessel functions in the dispersion relation
- Expansions valid for $m \neq 0$

$$\frac{K_m'\left(\tilde{k}_{\perp,\mathrm{e}} R\right)}{K_m\left(\tilde{k}_{\perp,\mathrm{e}} R\right)} \approx -\frac{m}{\tilde{k}_{\perp,\mathrm{e}} R}, \qquad \frac{J_m'\left(k_{\perp,\mathrm{i}} R\right)}{J_m\left(k_{\perp,\mathrm{i}} R\right)} \approx \frac{m}{k_{\perp,\mathrm{i}} R}$$

- Approximate dispersion relation

$$\rho_i \left(\omega^2 - k_z^2 v_{\mathrm{A},i}^2\right) + \rho_e \left(\omega^2 - k_z^2 v_{\mathrm{A},e}^2\right) = 0$$

Thin tube approximation

- Analytic solution: **kink frequency**
- The same frequency of a surface MHD wave!!!

$$\omega^2 = \frac{\rho_i v_{\mathrm{A},i}^2 + \rho_e v_{\mathrm{A},e}^2}{\rho_i + \rho_e} k_z^2 = \frac{2B_0^2}{\mu\left(\rho_i + \rho_e\right)} \frac{\pi^2}{L^2} \equiv \omega_k^2$$

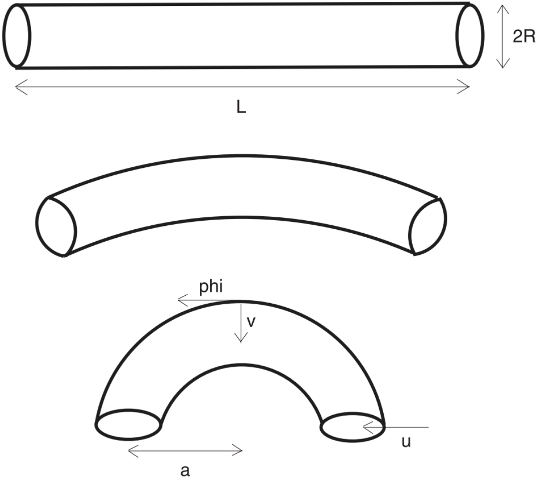
- The kink mode is a surface MHD wave in cylindrical geometry!!!
- Period of the kink mode: $P = 2\pi/\omega_k$

$$P = L\sqrt{\frac{2\mu\left(\rho_i + \rho_e\right)}{B_0^2}} = \frac{L}{v_{\mathrm{A},i}}\sqrt{\frac{2\left(\zeta + 1\right)}{\zeta}}$$

- Period of the Alfvén (string) mode, $\frac{L}{v_{\mathrm{A},i}}$, modified by the density contrast, ζ
- Only P and L are quantities that can be directly measured from observations! $\rightarrow$ problem for seismology (see part 4 of this Chapter)

Additional effect: curvature

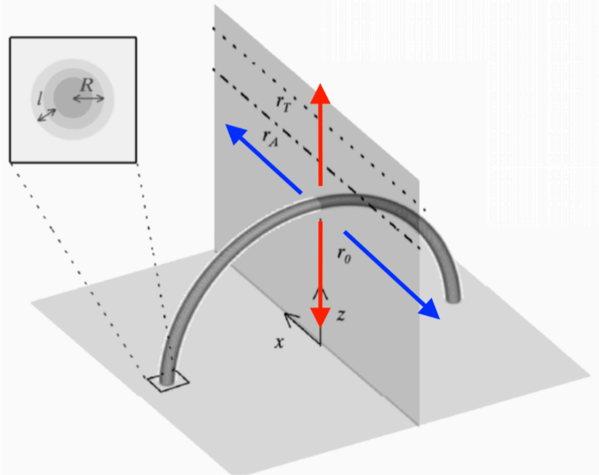
- Realistic coronal loop are not straight but curved!



Van Doorsselaere et al. (2004)

Additional effect: curvature

- New ingredients introduced by curvature:
  - **B** cannot be homogeneous: $\nabla \cdot \mathbf{B} = 0 \rightarrow$ **Not an easy equation to solve!**
  - There are two kink modes: horizontal and vertical polarizations



Terradas et al. (2006)

Additional effect: curvature

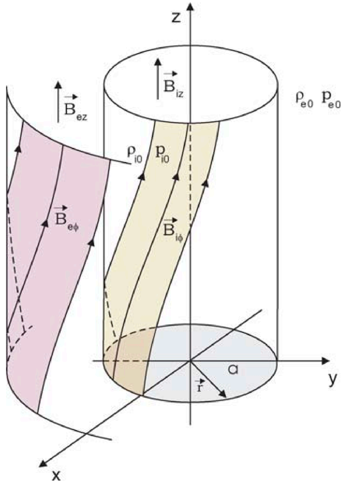
- In general, normal modes (eigenvalues) need to be solved numerically.
- Under certain conditions, it is possible to find an analytic dispersion relation using a toroidal coordinate system.
- The results of the long mathematical analysis shows:
  (Van Doorsselaere et al. 2004)

$$\omega_{\mathrm{vertical}} \approx \omega_{\mathrm{horizontal}} \approx \omega_k + \mathcal{O}\left(\frac{R}{a}\right)^2$$

  where a is the radius of curvature and R the loop inner radius
- Luckily, realistic loops have $R/a \sim R/L \ll 1$
- **Curvature can be safely neglected!**

Additional effect: magnetic twist

- Background magnetic field may have an azimuthal component: $\mathbf{B} = B_\varphi \hat{e}_\varphi + B_z \hat{e}_z$



Erdélyi & Fedun (2006)

Additional effect: magnetic twist

- Force balance condition:

$$\frac{\partial}{\partial r}\left(p+\frac{\left(B_\varphi^2+B_z^2\right)^2}{2\mu}\right)=-\frac{B_\varphi^2}{\mu\, r}$$

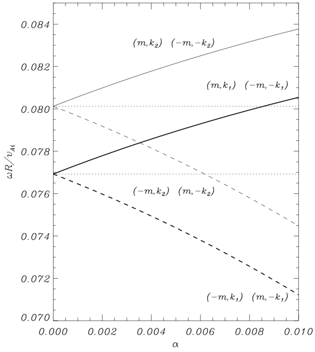
- Simplest case: uniform pressure, $p=$ constant
- Possible solution: Gold-Hoyle model of force-free uniform twist:

$$B_\varphi=B_0\frac{\Phi\, r/L}{1+\Phi^2\left(r/L\right)^2},\qquad B_z=B_0\frac{1}{1+\Phi^2\left(r/L\right)^2}$$

- Amount of twist: $\Phi=LB_\varphi/rB_z=$ constant in this model.
- Number of turns (winding) of the field over length L: $N=\Phi/2\pi$
- Maximum twist from stability analysis: $\Phi_{\max}\approx 3.3\pi$, $N_{\max}\approx 1.65$
- **Only coronal loops with weak twist are stable in the corona!**

Additional effect: magnetic twist

- In general, in the presence of twist normal modes (eigenvalues) need to be solved numerically.
- Twist breaks the degeneracy of positive and negative vales of m and k_z

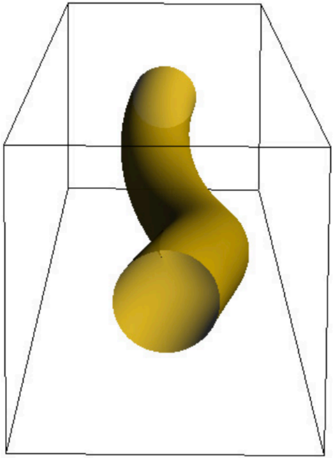


Terradas & Goossens (2006)

- However, for a standing wave the effect cancels out! $\longrightarrow$ No effect on the frequencies

Additional effect: magnetic twist

- A net effect of twist remains in the polarization of the oscillations.



Terradas & Goossens (2006)