1. INTRODUCTION

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Solar Magnetohydrodynamics: Applications
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Magnetohydrodynamics

1. Magnetohydrodynamics (MHD) is the most extended physical theory to describe the **macroscopic behavior** of a plasma permeated by electric and magnetic fields.
2. MHD combines the equations of **fluid dynamics** together with the equations of **electromagnetism**.
3. MHD is a **single-fluid theory** where the distinction between the different species that form the plasma has been lost, and **global or average quantities** are used instead.
4. Therefore, MHD makes sense for situations in which the temporal and spatial scales involved are **much larger** than the scales associated with the interactions of the individual plasma components.
5. We distinguish between **ideal MHD** and **dissipative MHD**.

Ideal MHD equations

- The **ideal** version of MHD neglects all dissipative effects:

$$
\begin{aligned}
\text{Continuity} \qquad \frac{D\rho}{Dt} &= -\rho\nabla\cdot\mathbf{v} \\
\text{Momentum} \qquad \rho\frac{D\mathbf{v}}{Dt} &= -\nabla p + \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \rho\mathbf{g} \\
\text{Energy (pressure)} \qquad \frac{Dp}{Dt} &= \frac{\gamma p}{\rho}\frac{D\rho}{Dt} \\
\text{Induction} \qquad \frac{\partial\mathbf{B}}{\partial t} &= \nabla\times(\mathbf{v}\times\mathbf{B}) \\
\text{State} \qquad p &= \rho R\frac{T}{\tilde{\mu}}
\end{aligned}
$$

along with the solenoidal condition $\nabla\cdot\vec{B} = 0$.

- The plasma motions and the magnetic field are **coupled** because of the Lorentz force in the momentum equation and the inductive term in the induction equation.

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- The plasma motions and the magnetic field are **coupled** because of the Lorentz force in the momentum equation and the inductive term in the induction equation.

Continuity equation

- The usual form of the continuity equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{v} = 0,$$

or, alternatively,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0,$$

where we have used $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$ to write the material or total derivative for time variations following the motion.

1. The continuity equation expresses the fact that the density at a point increases ($\frac{\partial\rho}{\partial t} > 0$) if mass flows into the surrounding region ($\nabla\cdot(\rho\mathbf{v}) < 0$) and decreases if it flows out.
2. If the plasma is incompressible ($\nabla\cdot\mathbf{v} = 0$), then $\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = 0$, which means that the density is simply advected by the flow, but there is no compression or rarefaction of plasma.

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Momentum equation

- The momentum equation or equation of motion is

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} + \mathbf{F}_{visc.}$$

- It is equivalent to the Navier-Stokes equation of fluid dynamics with the addition of the Lorentz force.
- The viscous force, $\mathbf{F}_{visc.}$, is a nonideal term given by

$$\mathbf{F}_{visc.} = -\nabla \cdot \hat{\Pi},$$

where $\hat{\Pi}$ is the (very complicated) 5-component plasma viscosity tensor.

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The viscosity tensor

- First derived by Braginskii (1965): *Transport Processes in a Plasma.*
- Good summary in Fitzpatrick (2014): *Plasma Physics*

$$\hat{\Pi} = \sum_{n=0}^{4} \hat{\Pi}_n$$

$$\begin{aligned}
\hat{\Pi}_0 &= -3\nu_0 \left(\hat{b}\hat{b} - \frac{1}{3}\hat{I}\right)\left(\hat{b}\hat{b} - \frac{1}{3}\hat{I}\right) : \nabla \mathbf{v} \\
\hat{\Pi}_1 &= -\nu_1 \left(\hat{I}_\perp \cdot \hat{W} \cdot \hat{I}_\perp + \frac{1}{2}\hat{I}_\perp \left(\hat{b} \cdot \hat{W} \cdot \hat{b}\right)\right) \\
\hat{\Pi}_2 &= -\nu_2 \left(\hat{I}_\perp \cdot \hat{W} \cdot \hat{b}\hat{b} + \hat{b}\hat{b} \cdot \hat{W} \cdot \hat{I}_\perp\right) \\
\hat{\Pi}_3 &= \nu_3 \left(\hat{b} \times \hat{W} \cdot \hat{I}_\perp - \hat{I}_\perp \cdot \hat{W} \times \hat{b}\right) \\
\hat{\Pi}_4 &= \nu_4 \left(\hat{b} \times \hat{W} \cdot \hat{b}\hat{b} - \hat{b}\hat{b} \cdot \hat{W} \times \hat{b}\right)
\end{aligned}$$

- with $\hat{W} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T - \hat{I}\frac{2}{3}\nabla \cdot \mathbf{v}$ the rate-of-strain tensor, $\hat{I}$ the identity tensor, $\hat{b} = \mathbf{B}/|\mathbf{B}|$, and $\hat{I}_\perp = \hat{I} - \hat{b}\hat{b}$.

The viscosity tensor

- The 5 coefficients are related and have different physical meanings:
  - $\nu_0 \longrightarrow$ Compressive or parallel viscosity
  - $\nu_1, \quad \nu_2 = 4\nu_1 \longrightarrow$ Shear or perpendicular viscosity
  - $\nu_3, \quad \nu_4 = 4\nu_3 \longrightarrow$ Gyro-viscosity (not an actual viscosity, no dissipation)
- As a first approximation, if the effect of the magnetic field is ignored, the viscosity tensor can be approximated by

$$\hat{\Pi} \approx -\nu_0 \hat{W} = -\nu_0 \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \hat{I} \frac{2}{3} \nabla \cdot \mathbf{v} \right),$$

so that the viscous force ($\mathbf{F}_{visc.} = -\nabla \cdot \hat{\Pi}$) can be cast as

$$\mathbf{F}_{visc.} \approx \rho\nu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right),$$

where $\nu = \nu_0/\rho$ is the coefficient of kinematic viscosity (assumed uniform in this formula).

The Reynolds number

- The Reynolds number, R, is a dimensionless quantity that allows us to quantify the importance of viscosity.
- We define R as the order-of-magnitude ratio of the inertia and viscosity terms of the momentum equation

$$R \equiv \frac{\left|\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\right|}{\left|\rho\nu\left(\nabla^2\mathbf{v} + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}\right)\right)\right|} \sim \frac{\rho V/T}{\rho\nu V/L^2} \sim \frac{L^2}{\nu T} \sim \frac{V\,L}{\nu},$$

  where V, L, and T are typical velocity, length, and time scales.
- In the solar corona, $V \sim 10^6$ m/s and $\nu \sim 10^{11}$ m^2/s, so that $R \sim 10^{-5}\frac{L}{1\,\mathrm{m}}$, which means that viscosity would dominate for length scales shorter than 10^5 m or 100 km.
- **In the solar corona, viscosity is only important for small scale dynamics.**

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Induction equation

- The equation that governs the evolution of the magnetic field is

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \\
&+ \nabla \times \left\{ \left[\eta_{\mathrm{A}} \frac{1}{B^2} (\nabla \times \mathbf{B}) \times \mathbf{B} \right] \times \mathbf{B} \right\} \\
&- \nabla \times \left[\eta_{\mathrm{H}} \frac{1}{B} (\nabla \times \mathbf{B}) \times \mathbf{B} \right]
\end{aligned}
$$

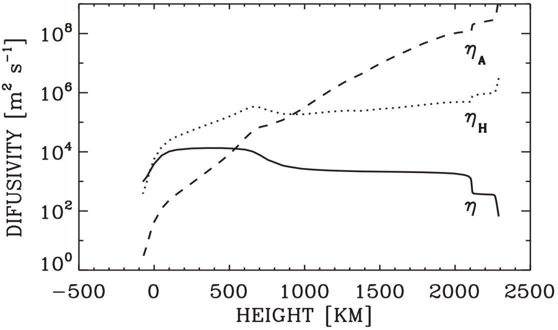
$$
\eta = \frac{m_{\mathrm{e}} (\nu_{\mathrm{ei}} + \nu_{\mathrm{en}})}{\mu e^2 n_{\mathrm{e}}}, \qquad \eta_{\mathrm{A}} = \frac{\rho_{\mathrm{n}} B^2}{\mu (\rho_{\mathrm{i}} + \rho_{\mathrm{n}})^2 \nu_{\mathrm{ni}}}, \qquad \eta_{\mathrm{H}} = \frac{B}{\mu e n_{\mathrm{e}}}
$$

- Nonideal terms
  - Ohm's diffusion: electron collisions
  - Ambipolar diffusion: ion-neutral collisions **(only if the plasma is partially ionized!)**
  - Hall's term: drifting of charged particles across the magnetic field (not an actual diffusion, no dissipation)

Induction equation: Nonideal terms in the chromosphere

$$\eta = \frac{m_e(\nu_{ei} + \nu_{en})}{\mu e^2 n_e}, \qquad \eta_A = \frac{\rho_n B^2}{\mu(\rho_i + \rho_n)^2 \nu_{ni}}, \qquad \eta_H = \frac{B}{\mu e n_e}$$

$B = 100$ G



(From Khomenko & Collados 2012)

Relative importance depends on magnetic field strength!

Induction equation in the fully ionized corona

- In the fully ionized coronal plasma, the induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) - \nabla \times \left[\frac{1}{\mu e n_e} (\nabla \times \mathbf{B}) \times \mathbf{B} \right]$$

- We quantify the importance of Hall's term with the dimensionless parameter H:

$$H \equiv \frac{|\nabla \times (\mathbf{v} \times \mathbf{B})|}{\left| \nabla \times \left[\frac{1}{\mu e n_e} (\nabla \times \mathbf{B}) \times \mathbf{B} \right] \right|} \sim \frac{V B / L}{B^2 / (L^2 \mu e n_e)} \sim \frac{\Omega_i}{V / L},$$

where $\Omega_i = \frac{eB}{m_p} \approx 10^5$ rad/s is the ion cyclotron frequency, and $V \sim 10^6$ m/s is the typical (Alfvén) velocity, so that

$$H \sim 0.1 \frac{L}{1 \text{ m}}.$$

- **In the corona, Hall's term is only important for length scales shorter than 10 m!**
- In the solar wind, Hall's term is more important because Ω_i is small.

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The magnetic Reynolds number

- As in the case of the viscosity, we can define a dimensionless quantity that allows us to quantify the importance of Ohm's diffusion.
- We define the magnetic Reynolds number R_m as the order-of-magnitude ratio of the inductive and diffusive terms of the induction equation

$$R_m \equiv \frac{|\nabla \times (\mathbf{v} \times \mathbf{B})|}{|\nabla \times (\eta \nabla \times \mathbf{B})|} \sim \frac{V\, B/L}{\eta B/L^2} \sim \frac{V\, L}{\eta},$$

  If V is related to the Alfvén velocity, R_m is also called the Lundquist number.
- In the solar corona, $V \sim 10^6$ m/s and $\eta \sim 1$ m^2/s, so that $R_m \sim 10^6 \frac{L}{1\,\mathrm{m}}$, which means that Ohm's diffusion would dominate for length scales shorter than 10^{-6} m or 1 μm.
- **In the solar corona, Ohm's diffusion is completely negligible unless sharp discontinuities ($L \rightarrow 0$) are considered!**
- **Current sheets $\rightarrow$ Magnetic reconnection**

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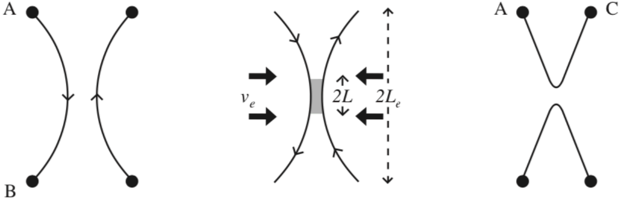
Magnetic reconnection

- In an ideal medium ($\eta = 0$), plasma elements preserve their magnetic connections.
- If non-ideal effects are important in a localized region ($L \rightarrow 0$), plasma elements can change their magnetic connectivity by the process of magnetic reconnection
- Ohm's diffusion is essential for this process to happen.



(From Priest 2014)

- Reconnection events are behind solar flares.

Ideal induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

1. **Magnetic flux is conserved**: the total magnetic flux through a certain surface remains constant as it moves with the plasma. In other words, the plasma elements that initially form a flux tube continue to do so at later times.
2. **Magnetic field lines are topology are conserved**: if two plasma elements lie on a field line initially, they will always do so.
3. **The magnetic field is frozen into the plasma**: the plasma can move freely along field lines but, in motion perpendicular to them, either the field lines are dragged with the plasma or the field lines push the plasma.

Energy equation

- The general form of the energy equation can be cast as

$$\frac{\mathrm{D}e}{\mathrm{D}t} + (\gamma - 1)e\nabla \cdot \mathbf{v} = \rho\mathcal{L},$$

  where $e = \frac{p}{(\gamma-1)\rho}$ is the internal energy, $\mathcal{L}$ is the heat-loss function (nonideal term), and $\gamma = 5/3$ is the adiabatic index.
- Usually, the energy equation is written in terms of the pressure and density as

$$\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = (\gamma - 1)\mathcal{L}$$

- If we take $\mathcal{L} = 0$, the plasma is thermally isolated from its surroundings, in the sense that there is no exchange of heat. The change of state is said to be **adiabatic** and $p/\rho^{\gamma} =$ constant for each plasma element following the motion.

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Contributions to the heat-loss function

- The heat-loss function represents the net effect of all the sinks and sources of energy has the following contributions

$$\mathcal{L} = -\nabla \cdot \mathbf{q} - L + \mathbf{J} \cdot \mathbf{E}^{*} + Q_{\nu} + H_{other}$$

  - Heat flux due to thermal conduction: $\mathbf{q} = -\hat{\kappa} \nabla T$
  - Radiative cooling: L
  - Joule heating: $\mathbf{J} \cdot \mathbf{E}^{*}$
  - Viscous heating: Q_{ν}
  - Other (unspecified) sources of heating: H_{other}

Thermal conduction

- In the fully ionized corona, the heat flux term can be spit as

$$\mathbf{q} = -\kappa_{\parallel} \nabla_{\parallel} T - \kappa_{\perp} \nabla_{\perp} T - \kappa_{\times} \hat{b} \times \nabla_{\perp} T$$

  where $\kappa_{\parallel}$ and $\kappa_{\perp}$ are the thermal conductivities parallel and perpendicular to the magnetic field, respectively, and $\kappa_{\times}$ is the cross conductivity (similar to gyro-viscosity).
- $\kappa_{\parallel}$ is mainly caused by the motion of **electrons** along the magnetic field direction
- $\kappa_{\perp}$ is predominantly caused by the orbital motion of **protons** across the magnetic field direction
- $\kappa_{\times}$ represents heat fluxes which are perpendicular to both the magnetic field and the direction of the temperature gradient
- For solar coronal conditions, $\kappa_{\perp}/\kappa_{\parallel} \approx 10^{-13}$ and $\kappa_{\times} \ll \kappa_{\perp}$, so that both $\kappa_{\perp}$ and $\kappa_{\times}$ can be neglected, so that

$$-\nabla \cdot \mathbf{q} \approx \nabla_{\parallel} \cdot \left(\kappa_{\parallel} \nabla_{\parallel} T\right) = (\mathbf{B} \cdot \nabla) \left[\frac{\kappa_{\parallel}}{|\mathbf{B}|^2} (\mathbf{B} \cdot \nabla T) \right]$$

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Thermal conduction time scale

- Let us consider an isobaric and static plasma: $\frac{\mathrm{D}p}{\mathrm{D}t} = 0$
- If only thermal conduction is considered, the energy equation is

$$\frac{\gamma p}{\rho}\frac{\partial \rho}{\partial t} = (\gamma - 1)\nabla_{\parallel} \cdot \left(\kappa_{\parallel} \nabla_{\parallel} T\right)$$

- We introduce $L \equiv$ length scale and $\tau_{\mathrm{cond.}} \equiv$ **thermal conduction time scale**. Then,

$$\frac{1}{\tau_{\mathrm{cond.}}}\gamma p_0 \sim (\gamma - 1)\frac{1}{L^2}\kappa_{\parallel} T_0 \quad \rightarrow \quad \tau_{\mathrm{cond.}} = \frac{\gamma p_0 L^2}{(\gamma - 1)\kappa_{\parallel} T_0}$$

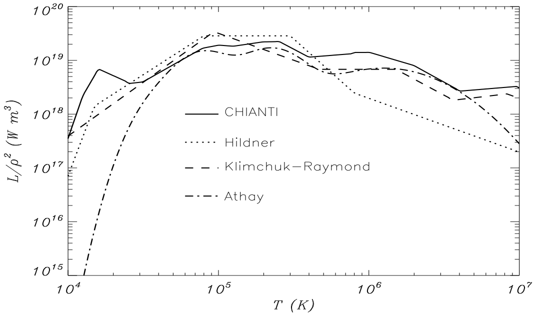
- In coronal conditions: $\tau_{\mathrm{cond.}} \sim 10^{-12} \left(\frac{L}{1\,\mathrm{m}}\right)^2$ seconds.
- Typical coronal loop length: $L \sim 10^8$ m, then $\tau_{\mathrm{cond.}} \approx 5$ hours.
- **Processes with $\tau > \tau_{\mathrm{cond.}}$ would be heavily affected by thermal conduction!**

Cooling by radiation

- The computation of radiative losses in a plasma is very complicated.
- In general, the solution of the non-local radiative transfer problem is required.
- In the corona, the optically-thin approximation can be used, and **semi-empirical parameterizations** of the radiation function in terms of the plasma local properties can be obtained:

$$L(\rho, T) = \rho^2 \chi^* T^\alpha$$

where χ^* and α are piecewise functions of T



Radiation time scale

- As before, let us consider an isobaric and static plasma: $\frac{\mathrm{D}p}{\mathrm{D}t} = 0$
- If only radiation is considered, the energy equation is

$$\frac{\gamma p}{\rho}\frac{\partial \rho}{\partial t} = -(\gamma - 1)\rho^2 \chi^* T^\alpha$$

- We introduce $L \equiv$ length scale and $\tau_{\mathrm{rad.}} \equiv$ **radiation time scale**. Then,

$$\frac{1}{\tau_{\mathrm{rad.}}}\gamma p_0 \sim (\gamma - 1)\rho_0^2 \chi^* T_0^\alpha \quad \rightarrow \quad \tau_{\mathrm{rad.}} = \frac{\gamma p_0}{(\gamma - 1)\rho_0^2 \chi^* T_0^\alpha}$$

- $\tau_{\mathrm{rad.}}$ **is independent of L**
- In coronal conditions: $\tau_{\mathrm{rad.}} \approx 10$ hours.
- Since $\tau_{\mathrm{rad.}} \gtrsim \tau_{\mathrm{cond.}}$, radiation and thermal conduction may both play a role, although thermal conduction is typically more important.

Joule heating

- The Joule heating term takes into account plasma heating because of dissipation of electric currents, $\mathbf{J} = (\nabla \times \mathbf{B})/\mu$.
- We can split $\mathbf{J}$ into its components parallel and perpendicular to the magnetic field direction:

$$\mathbf{J}_\parallel = \frac{1}{\mu}\left[(\nabla \times \mathbf{B}) \cdot \frac{\mathbf{B}}{|\mathbf{B}|}\right]\frac{\mathbf{B}}{|\mathbf{B}|}, \qquad \mathbf{J}_\perp = \frac{1}{\mu}\frac{\mathbf{B}}{|\mathbf{B}|} \times \left[(\nabla \times \mathbf{B}) \times \frac{\mathbf{B}}{|\mathbf{B}|}\right]$$

  so that the Joule heating term can be cast as

$$\mathbf{J} \cdot \mathbf{E}^* = \mu\eta \left|\mathbf{J}_\parallel\right|^2 + \mu\left(\eta + \eta_A\right)\left|\mathbf{J}_\perp\right|^2$$

- Ohm's diffusion dissipates both parallel and perpendicular currents
- Ambipolar diffusion (in a partially ionized plasma) dissipates perpendicular currents
- The corona is fully ionized, so that $\eta_A = 0$ and $\mathbf{J} \cdot \mathbf{E}^* = \mu\eta\,|\mathbf{J}|^2$
- **Conservation of energy:** the magnetic energy dissipated by Ohm's diffusion term (in the induction equation) is converted into internal energy (in the energy equation).

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Viscous heating

- The viscous heating term takes into account plasma heating because of viscous dissipation of velocity shears.
- The viscous heating term can be cast as

$$Q_\nu = \hat{\Pi} : \nabla \mathbf{v} = \sum_{m,n} \Pi_{mn} \frac{\partial v_m}{\partial x_n}$$

- As a first approximation, we can take the non-magnetic version of $\hat{\Pi}$, so that

$$Q_\nu \approx \rho\nu \left[\frac{1}{2} \sum_{m,n} e_{mn} e_{mn} - \frac{2}{3} \left(\nabla \cdot \mathbf{v} \right)^2 \right],$$

where $e_{mn} = \frac{\partial v_m}{\partial x_n} + \frac{\partial v_n}{\partial x_m}$

- **Conservation of energy:** the kinetic energy dissipated by viscosity (in the momentum equation) is converted into internal energy (in the energy equation).

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Some conclusions

- MHD can be used to study both ideal and non-ideal processes
- A rich variety of physical mechanisms can be incorporated into non-ideal MHD

- In the solar coronal plasma, we need some kind of heating mechanism to compensate for radiative cooling
- However, diffusion processes are negligible unless very small spatial scales are considered
- For Joule and viscous heating to be efficient, the energy needs to be transferred from large scales to small scales
- MHD waves may be involved in this process. . .

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