Chapter 10

The Structure of White Dwarfs

10.1 Introduction

Up until now a few thousand white dwarfs have been discovered. They are located in the Hertzsprung-Russell diagram in a region well beneath the main sequence in the spectral classes A and F. One of the best known white dwarfs is the companion of the brightest star for terrestrial observers: Sirius. Its faint companion Sirius B, which was discovered in 1862, is sometimes visible in sufficiently large amateur telescopes. The problem is not the weak magnitude of the B-component, but the large magnitude contrast with the bright Sirius A. Sirius B is a white dwarf with a mass of about $0.9M\odot$ and an absolute magnitude of +11.5. Both components of the Sirius binary have an effective temperature of about 10,000 K.

White dwarfs are the final stage in the evolution of low mass stars with initial masses smaller than $3M\odot$. Our Sun will eventually evolve into such a state, in which all energy production has stopped. The small amount of energy radiated away in space and responsible for the faint magnitude is supplied by the thermal energy still resident in the star from earlier evolutionary stages. Since this energy supply is limited, the white dwarf will slowly become fainter and will finally fade out. This process takes billions of years but leads to an inevitable end. Estimates conclude that there should be about 10 billion inactive white dwarfs in our galaxy.

The position of white dwarfs in the HR-diagram shows that they have a very small radius compared to normal main-sequence stars. On the other hand, the masses of white dwarfs are in the order of one solar mass. These two facts show that these stars have very large densities. The average density of a white dwarf is about 10^6 g/cm^3, compared to only 1.42 for the Sun. The central densities of white dwarfs are even higher. The physical state of the matter in the interior of white dwarfs can only be explained in terms of

electron degeneracy. This state allows extremely high densities and pressures. Some "normal" terrestrial laws of physics, such as the perfect gas law, no longer hold and should be replaced by an appropriate law for degenerate matter. An excellent and detailed description of the mathematical deduction of the formulae for degenerate matter may be found in *Structure and Evolution of the Stars* by Schwarzschild or in *Principles of Nucleosynthesis* by Clayton. Important theoretical work on white dwarfs was performed by Chandrasekhar.

10.2 Electron Degeneracy

The particles of a gas, for instance the free electrons in the stellar interior, do not all have the same velocity. The majority of the velocities is clustered around some average value, but some particles may have a lower or higher velocity. Under normal terrestrial or stellar temperatures and densities, the velocity distribution, i.e., the number of particles having a certain speed v, is a Maxwell-Boltzmann distribution, a clock-shaped function, shifting towards higher velocities as the temperature increases. Since the momentum of a particle is defined as the product of its mass and its velocity, the momentum distribution has the same shape.

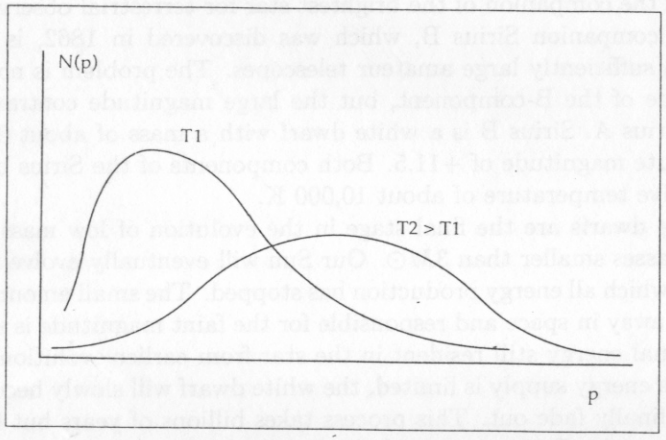


Fig. 10.1: *Maxwell-Boltzmann distributions.*

If we increase the density, keeping the temperature constant, the distribution is increased proportionally, but stays in the same momentum region. However, as the density becomes extremely high, an important physical process creates deviations from the normal Maxwell-Boltzmann distribution. This process is a consequence of the exclusion principle of Pauli: the maximum number of electrons in a volume dV and with momentum between p

and $p + dp$ is limited by

$$N_{e,\max}\, dV\, dp_x dp_y dp_z = \frac{2}{h^3} dV\, dp_x dp_y dp_z. \tag{1}$$

Although this upper limit corresponds to a considerable number of electrons, the densities needed to reach this limit are present in some types of stars. Under normal circumstances the top of the Maxwell-Boltzmann distribution is far below the limit value, but in the interior of white dwarfs, this limit is attained. We therefore say that the interior of white dwarfs is *electron degenerate*. For the even higher densities of neutron stars, the distribution of atomic nuclei is affected in a similar way, which is then called *ion degeneracy*.

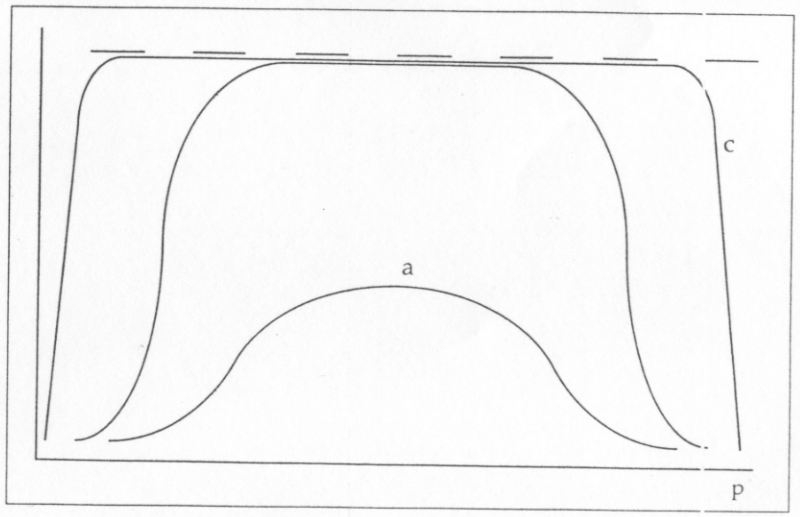


Fig. 10.2: *From a Maxwell-Boltzmann distribution to electron degeneracy.*

What does this electron degeneracy mean for the momentum distribution of the electrons? In Fig. 10.2 we have plotted three distributions for densities a, b, and c. For density a, the lowest one, the distribution is still Maxwellian since its top is far below the limit. At density b, however, the top would exceed the maximum allowed number of electrons. Therefore, the electrons above this maximum are now located at the slopes of the ideal distribution. These slopes become steeper. If we keep increasing the density to level c, we finally end up with an almost rectangular distribution in which all the volumes and all the momentum values between zero and a certain maximum momentum p are filled with the maximum number. We then call the matter *completely electron degenerate*.

Degenerate matter obeys its own physical laws. These laws have been studied profoundly by various astronomers so that it is now possible to use a generally accepted and known basic theory. In particular, the equation of state is no longer the well known ideal gas law. For electron degeneracy, the new law may be written using a dimensionless parameter:

$$x = \frac{p_o}{m_e c} \tag{2}$$

where

m_e = mass of the electron;
c = speed of light;
p_o = maximum momentum of the degenerate momentum distribution.

Deriving the equation of state is mathematically rather complicated and may be found in most books on stellar structure. We only present the result of the computations. The pressure P is given by

$$P = 6.01 \times 10^{22} f(x) \tag{3}$$

with

$$f(x) = x\left(2x^2 - 3\right)\left(1 + x^2\right)^{1/2} + 3\ln\left(x + \left(1 + x^2\right)^{1/2}\right). \tag{4}$$

where "ln" stands for the natural logarithm. The density is then given by

$$\rho = 9.82 \times 10^5 \mu_e x^3 \tag{5}$$

with the mean molecular weight $\mu_e = 2$ for the interior of a white dwarf. A definition of the mean molecular weight and some examples are given in the chapter on the stellar atmospheres.

The two constant factors in (3) and (5) are computed in centimeter-gram-second units. The function $f(x)$ is a strictly increasing function of x, starting at $f(0) = 0$, with each positive value of x corresponding to a unique value $f(x)$ and vice-versa. In this way the density and the pressure are linked to each other in an unambiguous way. Therefore, it is important to realize that the equation of state does not contain the temperature as the ideal gas law does. This fact makes the computation of the structure of a white dwarf much easier than the structure of a normal main-sequence star. It is possible to compute the variations of density, pressure, radius and mass throughout the star without any knowledge of the temperature. The separation of the hydrostatic variables and the thermodynamic variables (T, energy, ...) simplifies the calculations considerably.

10.3 Equations of Structure

10.3.1 Equation of State

The equation of state for a white dwarf is the one for electron degeneracy and is given by (3,4,5). Given the density, it is easy to compute the value of x by (5) and then P by (3,4). By contrast, computing the density from the pressure requires some basic numerical techniques that will be explained in section 4.

10.3.2 Hydrostatic Equilibrium and Continuity of Mass

The equation of state is a local equation applicable at local points in the star. The two other relevant equations are differential equations that describe the variation of the pressure and the mass throughout the star as functions of the distance r to the center of the star. As usual M_r stands for the amount of mass inside a sphere with radius r. The equation of hydrostatic equilibrium containing the variation of the pressure is

$$dP = -\rho G \frac{M_r}{r^2} dr, \tag{6}$$

with G being the gravitational constant. A proof of this equation is given in the chapter on polytropes. The equation of continuity, giving the variation of mass as a function of the distance r is

$$dM_r = 4\pi\rho r^2 dr. \tag{7}$$

The goal of the next section is to solve these two equations in combination with the equation of state to obtain the structure of the white dwarf.

10.4 Numerical Method

10.4.1 Initial Conditions

The set of equations for the structure of white dwarfs contains two differential equations of the first order. This means that there are two initial conditions to be chosen. The first initial condition, the one for the mass as a function of the distance to the center is trivial: $M_r = 0$ at the center where $r = 0$. The second initial condition is the central pressure P_c. It is better, however, to start from a choice of the central density and use this to compute the central pressure with (3,4,5). For instance, taking a central density of 10^8 results in (5) having a value of the parameter $x = 3.7064$, and hence

$$f(X) = 354.307 \text{ and } \log P_c = 25.3282.$$

Since the initial condition for the mass is always zero, there is only one free initial condition for our problem. This single value will determine the complete structure of the white dwarf, its total mass, and its radius. We therefore say that the models for white dwarfs computed with this method are a one-parameter family of the central density. The choice of this central density is also limited since we are dealing with electron degeneracy. A lower limit for the logarithm of the central density is $\log \rho_c = 6$, while on the other hand we may take 12 as an upper limit. At higher densities the atomic nuclei also become degenerate, and we end up with a neutron star, once again with another equation of state. Therefore, the choice of the central density is limited by the condition

$$10^6 < \rho_c < 10^{12}. \tag{8}$$

10.4.2 Computing the Density from the Pressure

Before writing down the numerical formulae of the equations of structure, we will present a way to compute the density from the pressure. It is not difficult to compute the pressure from the density since in that direction we only have to compute x with (5) and introduce x in (3,4). On the other hand, the computation of ρ from P needs the computation of x for a given value of $f(x)$. For a given P we should compute x from

$$f(x) = \frac{P}{6.01 \times 10^{22}}.$$

Let us therefore first compute

$$K = \frac{P}{6.01 \times 10^{22}}. \tag{9}$$

The problem is then to find a value for x so that $f(x) = K$, or in other words we need to search the zero point of the function $f(x) - K$. This may easily be performed by successive approximations using Newton's method. This method is discussed in Chapter 1 on numerical techniques. In this case it may be applied in ideal circumstances since it is always possible to find a good starting value, and $f(x) - K$ is strictly monotone. The successive approximations are

$$x_{j+1} = x_j - \frac{f(x_j) - K}{f'(x_j)} \tag{10}$$

with $f(x)$ given by (4) and its derivative $f'(x)$ by

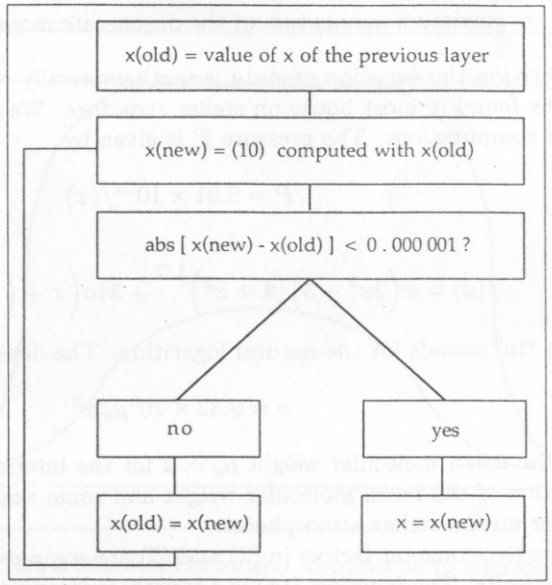
$$f'(x) = \frac{8x^4}{\sqrt{1+x^2}}. \tag{11}$$

As a starting value we may use the x-value of the previous layer. Usually, less than five iterations are needed to obtain x with sufficient accuracy. In the author's program the iteration (10,11) is stopped when two subsequent values of x are found that differ by less than 0.000001. The criterion to stop the iteration is therefore

$$\text{abs}(x_{j+1} - x_j) < 10^{-6}. \tag{12}$$

The last approximation is then used to compute the density with (5).

It is obvious that this procedure of finding x from $f(x)$ should be placed in a subprogram since we will need it two times during each step. A scheme for such a subprogram is presented in the following flowchart.



- *Example:* Compute the density from a pressure P, with $P = 2.5 \times 10^{25}$

  We find $f(x) = 415.9734$.

  Starting from $x = 5$, for instance, the approximations are as follows:

$$x(1) = 4.1951366$$
$$x(2) = 3.8940479$$
$$x(3) = 3.8545814$$
$$x(4) = 3.8539539$$
$$x(5) = 3.8539537$$

With this value of x we find $\rho = 1.124245 \times 10^8$.

10.4.3 The Iteration Procedure

We will use the well-known midpoint method to calculate the white dwarf structure. The distance between two layers is given by dr which will be taken as constant throughout the star. The numerical formulae are

$$r_{i+1/2} = r_i + 0.5dr \tag{13}$$

$$P_{i+1/2} = P_i - 0.5dr\rho_i G\frac{M_{r,i}}{r_i^2} \tag{14}$$

$$M_{r,i+1/2} = M_{r,i} + 0.5dr4\pi\rho_i r_i^2 \tag{15}$$

$$x_{i+1/2} \text{ using } (10,11) \text{ with } K = \frac{P_{i+1/2}}{6.01 \times 10^{22}} \text{ starting at } x_i$$

$$\rho_{i+1/2} = 1.964 \times 10^6 x_{i+1/2}^3 \tag{16}$$

and then finally

$$r_{i+1} = r_i + dr \tag{17}$$

$$P_{i+1} = P_i - dr\rho_{i+1/2}G\frac{M_{r,i+1/2}}{r_{i+1/2}^2} \tag{18}$$

$$M_{r,i+1} = M_{r,i} + dr4\pi\rho_{i+1/2}\, r_{i+1/2}^2 \tag{19}$$

$$x_{i+1} \text{ using } (10,11) \text{ with } K = \frac{P_{i+1}}{6.01 \times 10^{22}} \text{ starting at } x_{i+1/2}$$

$$\rho_{i+1} = 1.964 \times 10^6 x_{i+1}^3. \tag{20}$$

Because of the presence of the radius in the denominator of equation (14), this equation may not be applied to the very first step when we proceed from the center ($r = 0$) to layer one. In that case we would divide by zero. We therefore need an alternative approach to the first step.

10.4.4 The First Step

At the first layer $r = dr$. The pressure and the mass may then be obtained by means of the Taylor series developed around zero:

$$P_1 = P_c - \frac{2}{3}G\pi\rho_c^2 dr^2 \tag{21}$$

$$M_{r,1} = \frac{4}{3}\pi\rho_c dr^3. \tag{22}$$

The density at layer one is then computed by (10,11) starting from x_0.

From this level on, we proceed with the iteration formulae given in section 4.3. The values at level zero (the center) are as follows:

$$\begin{aligned} r_0 &= 0 \\ M_{r,0} &= 0 \\ \rho_0 &= \text{free initial condition between } 10^6 \text{ and } 10^{12} \\ x_0 &= \text{from (5) with the central density} \\ P_0 &= \text{from } (3,4) \text{ with } x_0 \end{aligned}$$

We have used here the subscript 0 for the central values. They are the same as the values with subscript c.

The iteration procedure is stopped when a pressure turns out to be negative. This may occur at a certain level $i+1$ but also at a half-step, at $i+1/2$. Therefore, it *must* also be checked at half the step. The values of the radius and the mass at that moment are considered as the total radius and the total mass of the white dwarf. The small error we make by neglecting the fraction of the final unfinished step $i+1$ is never larger than about 2% of the total radius (if a normal step-size is used) and quasi zero for the mass. Professional programs sometimes contain instructions to compute the atmosphere of a white dwarf. This computation, however, would complicate our own program enormously. We have therefore limited the input physics to the model described above.

10.5 Some Remarks

The step-size dr should be given in centimeters and should be "small" compared to the solution area, in our case the radius of the star. We may hence select a step of a few times 10 km to 100 km. The Table 10-1 gives some possible step-sizes for various values of the central density. When you use these steps, the surface will be reached in about fifty steps. The values in the table are given in kilometers, don't forget to multiply by 100,000.

Table 10-1
The step-size (in km)
as a function of the central density

$\log \rho_c$:	5	6	7	8	9	10	11
dr(km) :	300	200	150	80	50	25	12

The equation of state used in the interior of white dwarfs is not valid in the outer shell of the star. In that region the effect of the temperature should be included. However this outer shell contains less than one percent of the radius and has no effect on the radius and the mass computed with our

model. More information on the atmosphere of a white dwarf may be found in advanced textbooks on stellar structure.

The output data of the model are given in centimeter-gram-second units. As usual in astrophysics, we will list the logarithms of the density and the pressure, and express the masses in solar mass ($1M\odot = 2 \times 10^{33}$ g) and the distance and radius in solar radii ($1R\odot = 6.96 \times 10^{10}$ cm). The momentum parameter x is dimensionless.

10.6 Examples and Applications

10.6.1 Detailed Model of a White Dwarf

As a first test for your program, we present in Table 10-2 the complete results for a white dwarf characterized by a central density of 10^8 g/cm^3. The step-size used is 80 km or, in other words, 8,000,000 cm.

After layer 54 the pressure becomes negative, meaning that we have arrived at the surface of the white dwarf. The values of r and M_r are therefore considered as the total radius and mass of the white dwarf. We find for this model a radius of 0.0062069 solar radii, which is 4322.2 km (less than the Earth's radius), and a mass of 1.16 solar mass (375,000 times the mass of the Earth). This clearly demonstrates that white dwarfs are very dense objects. Their mass is comparable to the mass of a small star, but the radius is only that of a small planet. The average density is more than 100,000 times the average density of the Earth and more than 400,000 times the average density of the Sun. Don't worry if the results differ in the last few significant digits. This discrepancy is a consequence of rounding effects.

10.6.2 The Chandrasekhar Limit

The only free parameter in our model is the central density. It determines the whole structure, the mass and the radius. The dependency of these quantities as a function of the central density has been studied by the Indian astronomer S. Chandrasekhar, one of the greatest pioneers in the study of stellar structure. His conclusions are twofold:

- The radius decreases and the mass increases as a function of the central density, indicating that the smallest white dwarfs also have the largest mass.
- There is an upper limit for the mass of white dwarfs.

This upper limit, the "Chandrasekhar limit," is the largest mass a white dwarf can have: $1.44M\odot$. It is reached when the central density gets larger

Table 10-2

i	r	M_r	$\log P$	$\log \rho$	x
0	0.0000000	0.0000000	25.3282545	8.0000000	3.7064047
1	0.0001149	0.0001072	25.3264264	7.9986669	3.7026143
2	0.0002299	0.0008266	25.3223687	7.9957082	3.6942157
3	0.0003448	0.0028057	25.3143447	7.9898585	3.6776664
4	0.0004598	0.0066209	25.3024675	7.9812019	3.6533124
5	0.0005747	0.0127843	25.2868627	7.9698326	3.6215712
6	0.0006897	0.0217263	25.2676438	7.9558366	3.5828753
7	0.0008046	0.0337837	25.2449163	7.9392949	3.5376739
8	0.0009195	0.0491918	25.2187876	7.9202904	3.4864463
9	0.0010345	0.0680817	25.1893685	7.8989093	3.4296985
10	0.0011494	0.0904827	25.1567734	7.8752410	3.3679567
11	0.0012644	0.1163279	25.1211198	7.8493776	3.3017589
12	0.0013793	0.1454634	25.0825269	7.8214131	3.2316467
13	0.0014943	0.1776604	25.0411147	7.7914425	3.1581567
14	0.0016092	0.2126283	24.9970016	7.7595604	3.0818132
15	0.0017241	0.2500288	24.9503038	7.7258600	3.0031211
16	0.0018391	0.2894899	24.9011332	7.6904319	2.9225603
17	0.0019540	0.3306195	24.8495963	7.6533634	2.8405817
18	0.0020690	0.3730174	24.7957930	7.6147370	2.7576036
19	0.0021839	0.4162861	24.7398150	7.5746303	2.6740095
20	0.0022989	0.4600402	24.6817453	7.5331146	2.5901468
21	0.0024138	0.5039135	24.6216567	7.4902546	2.5063268
22	0.0025287	0.5475650	24.5596112	7.4461074	2.4228246
23	0.0026437	0.5906827	24.4956590	7.4007223	2.3398802
24	0.0027586	0.6329864	24.4298373	7.3541401	2.2576998
25	0.0028736	0.6742292	24.3621702	7.3063922	2.1764577
26	0.0029885	0.7141974	24.2926666	7.2575005	2.0962979
27	0.0031034	0.7527106	24.2213202	7.2074765	2.0173364
28	0.0032184	0.7896198	24.1481075	7.1563204	1.9396631
29	0.0033333	0.8248060	24.0729866	7.1040206	1.8633440
30	0.0034483	0.8581783	23.9958955	7.0505524	1.7884232
31	0.0035632	0.8896711	23.9167498	6.9958766	1.7149247
32	0.0036782	0.9192421	23.8354397	6.9399383	1.6428539
33	0.0037931	0.9468698	23.7518273	6.8826646	1.5721996
34	0.0039080	0.9725511	23.6657418	6.8239619	1.5029348
35	0.0040230	0.9962988	23.5769743	6.7637134	1.4350179
36	0.0041379	1.0181399	23.4852705	6.7017741	1.3683931
37	0.0042529	1.0381133	23.3903223	6.6379660	1.3029912
38	0.0043678	1.0562680	23.2917557	6.5720709	1.2387294
39	0.0044828	1.0726616	23.1891149	6.5038211	1.1755105
40	0.0045977	1.0873590	23.0818417	6.4328868	1.1132220
41	0.0047126	1.1004310	22.9692469	6.3588589	1.0517338
42	0.0048276	1.1119531	22.8504695	6.2812254	0.9908958
43	0.0049425	1.1220052	22.7244207	6.1993366	0.9305329
44	0.0050575	1.1306702	22.5897003	6.1123562	0.8704388
45	0.0051724	1.1380341	22.4444720	6.0191861	0.8103666
46	0.0052874	1.1441854	22.2862700	5.9183509	0.7500147
47	0.0054023	1.1492150	22.1116878	5.8078115	0.6890065
48	0.0055172	1.1532166	21.9158552	5.6846527	0.6268598
49	0.0056322	1.1562869	21.6915228	5.5445363	0.5629437
50	0.0057471	1.1585265	21.4274073	5.3807141	0.4964294
51	0.0058621	1.1600418	21.1053432	5.1823359	0.4263159
52	0.0059770	1.1609479	20.6982796	4.9333119	0.3521466
53	0.0060920	1.1613732	20.2095516	4.6361666	0.2803335
54	0.0062069	1.1614592	19.9948103	4.5060701	0.2636936

than 10^{12} g/cm^3. Higher masses are not possible for white dwarfs. In that case the star becomes a neutron star, to be described by physical laws other than those we have used here. With your program you are able to check the value of the Chandrasekhar limit by computing a series of models with increasing central density. Table 10-3 lists our results. Some values of the central density may look strange, but they were selected to compare our results with professional data. The latter are listed between parentheses. The professional results are based on Chandrasekhar's hydrogen degenerate configurations[1] which we have converted to pure helium with (23,24).

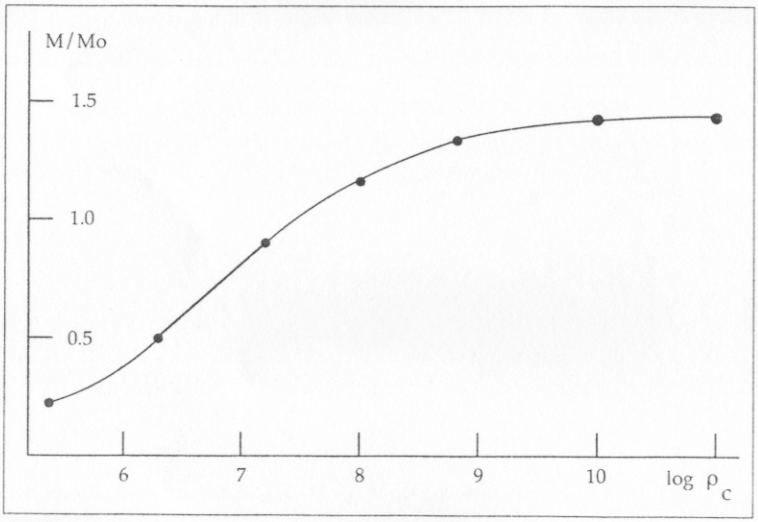


Fig. 10.3: *The mass versus the central density computed with our program.*

Table 10-3
Masses and radii computed with our program compared with the converted professional results (in parentheses).

$\log \rho_c$	M/M_0	$M/M_\odot$ $\log R/R_\odot$
5.39	0.22 (0.22)	−1.70 (−1.70)
6.29	0.50 (0.50)	−1.86 (−1.86)
7.20	0.89 (0.88)	−2.03 (−2.03)
8.00	1.16	−2.21
8.83	1.33 (1.33)	−2.41 (−2.41)
10.00	1.42	−2.72
11.00	1.435	−3.02

[1] *An Introduction to the Study of Stellar Structure*, 1939, University of Chicago Press.