2. FUNDAMENTALS OF MHD WAVES

– Part 1: Homogeneous medium –

Roberto Soler

Solar Physics Group
Universitat de les Illes Balears (Spain)

Solar Magnetohydrodynamics: Applications
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Ideal MHD equations

$$
\begin{aligned}
\text{Continuity} \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\nabla\cdot\mathbf{v} \\
\text{Momentum} \qquad \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} &= -\nabla p + \frac{1}{\mu}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} \\
\text{Energy (pressure)} \qquad \frac{\mathrm{D}p}{\mathrm{D}t} &= \frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} \\
\text{Induction} \qquad \frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\left(\mathbf{v}\times\mathbf{B}\right) \\
\text{State} \qquad p &= \rho R\frac{T}{\tilde{\mu}}
\end{aligned}
$$

along with the solenoidal condition $\nabla\cdot\vec{B} = 0$.

Linearization in a homogenous and static plasma

- We assume that each physical quantity can be written as a homogeneous equilibrium value plus a perturbation.

$$\rho = \rho_0 + \rho_1, \qquad T = T_0 + T_1,$$
$$p = p_0 + p_1, \qquad \vec{v} = \vec{v}_1,$$
$$\vec{B} = \vec{B}_0 + \vec{B}_1,$$

- After neglecting all non-linear terms in the perturbations, the ideal MHD equations become:

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\rho_0 \nabla \cdot \vec{v}_1, \\
\rho_0 \frac{\partial \vec{v}_1}{\partial t} &= -\nabla p_1 + \frac{1}{\mu}\left(\nabla \times \vec{B}_1\right) \times \vec{B}_0, \\
\frac{\partial \vec{B}_1}{\partial t} &= \nabla \times \left(\vec{v}_1 \times \vec{B}_0\right), \\
\frac{\partial p_1}{\partial t} &= \frac{\gamma p_0}{\rho_0}\frac{\partial \rho_1}{\partial t}, \qquad \frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}.
\end{aligned}$$

Governing equations for MHD waves

- We define the auxiliary variables Δ (compressibility) and Γ (field-aligned vorticity) as

$$\Delta = \nabla \cdot \vec{v}_1 \qquad \Gamma = \frac{(\nabla \times \vec{v}_1) \cdot \vec{B}_0}{|\vec{B}_0|}$$

- From the linearized MHD equations we obtain the governing equations for the MHD waves.

Alfvén waves

$$\left[\frac{\partial^2}{\partial t^2} - \frac{v_{\mathrm{A}}^2}{|\vec{B}_0|^2}\left(\vec{B}_0 \cdot \nabla\right)^2\right]\Gamma = 0$$

Magnetoacoustic waves

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial t^2} - \left(v_{\mathrm{s}}^2 + v_{\mathrm{A}}^2\right)\nabla^2\right]\Delta + \frac{v_{\mathrm{s}}^2 v_{\mathrm{A}}^2}{|\vec{B}_0|^2}\left(\vec{B}_0 \cdot \nabla\right)^2 \nabla^2 \Delta = 0$$

A new kind of waves

Unmagnetized gas

- **Acoustic waves:** Perturbations in a gas propagate isotropically at the sound velocity of the medium, v_{s}, as variations in pressure.

Magnetized plasma

- **MHD waves:** Due to the presence of the magnetic field, waves propagating in a plasma are also driven by the magnetic force in addition to the pressure force.
  - **Alfvén waves:** Incompressible and driven exclusively by the magnetic tension. They propagate non-dispersively along the magnetic field lines at the Alfvén velocity, v_{A}.
  - **Magnetoacoustic waves:** Compressible waves that arise due to the combined effect of the pressure gradient and the magnetic force. Two types of magnetoacoustic waves are possible: slow and fast.

$$\text{Sound velocity} \rightarrow v_{\mathrm{s}} = \sqrt{\frac{\gamma p_0}{\rho_0}} \qquad \text{Alfvén velocity} \rightarrow v_{\mathrm{A}} = \frac{|\vec{B}_0|}{\sqrt{\mu \rho_0}}$$

A Nobel Price winner

- Hannes Alfvén, *Nature*, 1942. **Nobel Prize 1970.**

A Nobel Price winner

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Anemotaxis in Drosophila

Cole[1] has observed that *Drosophila melanogaster* sometimes walks against an air current. Flügge[2] showed that this reaction only occurred when the air was scented, and must therefore be regarded as orientation by smell. However, reactions to air currents without smell do occur in Drosophila.

Comparing several species of the genus, a division can be made between those which show a distinct positive anemotaxis and those which do not. *Drosophila virilis* +, *D. virilis americana*, *D. subobscura* and *D. funebris* turn very sharply towards a tube from which air is flowing and start walking against it, so long as they are not blown away. *D. melanogaster* and *D. buskii*, on the other hand, show no reaction, whereas *D. pseudo-obscura* shows a slight reaction.

Removal of the wings or antennæ or both does not abolish the reaction in the species showing it. Therefore Fraenkel and Gunn's[3] contention that anemotaxis depends on the perception of body deformation caused by the wind seems plausible for Drosophila.

The fact that the two light species observed do not react to wind whereas the four dark species do, may be an indication that the former are not exposed to strong air currents in their natural environment, while the latter are. As wind is a factor which increases evaporation, this is in agreement with some recently published deductions[4], to the effect that a dark cuticle provides better protection against desiccation than a light one.

H. Kalmus.

Department of Biometry,
University College, London,
at Rothamsted Experimental Station,
Harpenden, Herts.
Sept. 3.

[1] Cole, W. H., *J. Anim. Behaviour*, **7**, 71 (1917).
[2] Flügge, C., *Z. vergl. Physiol.*, **20**, 463 (1934).
[3] Fraenkel and Gunn, "Orientation of Animals" (Oxford, 1940).
[4] Kalmus, H., Nature, **1:8**, 428 (1941).

Nomenclature of Biological Movement

Having read recently with very great interest Fräenkel and Gunn's "Orientation of Animals", I set about making for my own clarification a classification of all the cases I know of what used to be called 'tropistic' movements.

It is clear that three major groupings are possible: a *taxis*, which is a bodily movement of an animal or motile plant in a direction determined by the direction of the stimulus; a *kinesis*, which is a change of rate of movement of an animal (or perhaps motile plant) in response to a change of intensity of a stimulus, but not in a direction determined by the direction of the stimulus—and often producing an aggregating effect superficially similar to that of a taxis; and an *orientation*, which is the placing of the body (usually if not always animal) in a direction determined by the direction of the stimulus. To these three classes many of the cases can be referred.

But the responses of sessile plant organs do not seem to be so conveniently classified. The thigmotropism of Clematis tendrils appears to warrant that name, for the response is a directional one. But the same cannot be said of the so-called 'thigmotropism' of Mimosa leaflets, Mimulus stigma or Berberis stamen, for here the response is not in a direction determined by that of the stimulus. The response in these cases appears to bear a closer resemblance to the photonasty of Oxalis leaflets and the thermonasty of Tulipa flowers. Is one then justified (disregarding—as often becomes necessary for purposes of coherence—mere etymological niceties) in putting these responses under the heading of 'thigmonasty' (or possibly in some cases—as in Stiles's "Plant Physiology"—'seismonasty')? By the same token may the response of Mimosa leaflets to a lighted match in the neighbourhood be called a 'thermonasty', and to ammonia vapour, a 'chemonasty'?

Cyril Bibby.

20 Dellcott Close,
Welwyn Garden City.
Sept. 4.

An Inland Record of *Triglochin maritimum* L.

In a previous note[1] I recorded the occurrence of the halophilic alga *Percursaria percursa* Rosenv. from a salt-spring situated at Aldersey[2], Cheshire. Continuing the ecological survey of this spring, I am now able to report the presence of the halophyte *Triglochin maritimum* L. (Naiadaceæ). It occurs in three small colonies growing in the salt-spring and appears to be well established. I should add that during the past year the spring has maintained a salinity of 1,642 parts per 100,000 with only slight variation.

Triglochin maritimum, popularly known as seaside arrow-grass, is a plant frequently found in salt marshes. Its occurrence in a non-littoral region is exceptional and would seem to be unique in so far as the county of Cheshire is concerned. Search of the literature has revealed the fact that Triglochin has occasionally been recorded from the counties of Cambridgeshire, Staffordshire and Surrey.

In Cambridgeshire, the plant appears to have been first reported from Tydd Marsh by Skrimshire[3]. A. E. Evans[4], however, points out that it was never plentiful in that county and has not been found there during recent years. In Staffordshire, J. E. Bagnall[5] mentions records of Triglochin by the independent observers, Shaw, Stokes and Brown. In Surrey, C. E. Britton[6] records its discovery by W. A. Todd from the Thames near Putney.

I am indebted to Mr. A. A. Dallman, of Doncaster, for certain of the references quoted.

Frederick Burke.

12 Queen's Road,
Chester.
Aug. 29.

[1] Burke, F., Nature, **149**, 331 (1942).
[2] Sherlock, Mem. Geol. Survey, Mineral Resources of Gt. Brit., Rock-salt and Brine, **18**, 111 (1921).
[3] Relhan, R., "Flora Cantabrigiensis", second edition, 145 (1802).
[4] Evans, A. E., "Flora of Cambridgeshire", 165 (1939).
[5] Bagnall, J. E., "Flora of Staffordshire", 57 (1901).
[6] Britton, C. E., *J. Bot.*, **48**, 186 (1910).

Existence of Electromagnetic-Hydrodynamic Waves

If a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an e.m.f. which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid.

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Thus a kind of combined electromagnetic-hydrodynamic wave is produced which, so far as I know, has as yet attracted no attention.

The phenomenon may be described by the electrodynamic equations

$$\text{rot } H = \frac{4\pi}{c} i$$

$$\text{rot } E = -\frac{1}{c}\frac{dB}{dt}$$

$$B = \mu H$$

$$i = \sigma(E + \frac{v}{c} \times B);$$

together with the hydrodynamic equation

$$\partial \frac{dv}{dt} = \frac{1}{c}(i \times B) - \text{grad } p,$$

where σ is the electric conductivity, μ the permeability, ∂ the mass density of the liquid, i the electric current, v the velocity of the liquid, and p the pressure.

Consider the simple case when $\sigma = \infty$, $\mu = 1$ and the imposed constant magnetic field H_0 is homogeneous and parallel to the z-axis. In order to study a plane wave we assume that all variables depend upon the time t and z only. If the velocity v is parallel to the x-axis, the current i is parallel to the y-axis and produces a variable magnetic field H' in the x-direction. By elementary calculation we obtain

$$\frac{d^2H'}{dz^2} = \frac{4\pi\partial}{H_0^2}\frac{d^2H'}{dt^2},$$

which means a wave in the direction of the z-axis with the velocity

$$V = \frac{H_0}{\sqrt{4\pi\partial}}.$$

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and as solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have $H_0 = 15$ gauss and $\partial = 0{\cdot}005$ gm. cm.$^{-3}$, the velocity of the waves amounts to

$$V \sim 60 \text{ cm. sec.}^{-1}.$$

This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of H_0 and ∂ refer to a distance of about 10^{10} cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance proceeding as an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in *Arkiv för matematik, astronomi och fysik*.

H. Alfvén.

Kgl. Tekniska Högskolan,
Stockholm.
Aug. 24.

Energy of Dissociation of Carbon Monoxide

The energies of dissociation of a number of diatomic molecules have been determined from spectroscopic data, apparently with high accuracy, by the observation of predissociation limits. During the last few years the following values have been proposed for CO: $D(\text{CO}) = 6{\cdot}92$ [1], $8{\cdot}41$ [2], $9{\cdot}14$ [3], $10{\cdot}45$ [4] e.v.; while values of $9{\cdot}85$ and $11{\cdot}11$ also appear possible[3]. Controlled electron experiments suggest $9{\cdot}6$ [5].

The value obtained by extrapolation of the vibrational levels of the ground state is about 11, and support for this value has been given by Kynch and Penney[6]. Herzberg[7] has recently summarized evidence favouring $9{\cdot}14$.

At first sight, the strongest argument for $9{\cdot}14$ is the observation by Faltings, Groth and Harteck[8] that CO is decomposed by the xenon line at 1295 A., but not by that at 1470 A., from which they conclude that $8{\cdot}44 < D(\text{CO}) < 9{\cdot}57$. This conclusion is not based on an examination of the initial act of absorption. The only known absorption in the 1295 A. region is that corresponding to the fourth positive bands. The origins of the (9,0) and (10,0) bands lie at 76,839 cm.$^{-1}$ and 78,010 cm.$^{-1}$. The xenon line 1295 A. = 77,172 cm.$^{-1}$ falls between these bands and, if absorbed from the lowest vibrational level of CO, would correspond approximately to the line $P(35)$ of (10,0). This gives as the upper limit of $D(\text{CO})$ (when the rotational energy is taken into account) a value of 79,722 cm.$^{-1}$ = $9{\cdot}88$ e.v. (not $9{\cdot}57$ e.v. as stated by Herzberg[7]). Actually, it is doubtful whether such a high rotational line as $P(35)$ would be observed at room temperature, and absorption, if it is due to CO, would probably occur from a higher vibrational level, corresponding perhaps to the (13,2) band, in which case the dissociation limit may be placed as high as $10{\cdot}1$.

Taking the first act of absorption as

$$\text{CO}(X^1\Sigma) + h\nu = \text{CO}(A^1\Pi),$$

and assuming a life not less than 10^{-8} sec. for $A^1\Pi$, then at atmospheric pressure each molecule experiences at least 100 collisions before radiating. It seems to us that this gives a reasonable chance for a reaction such as

$$\text{CO}(A^1\Pi) + \text{CO}(X^1\Sigma) = \text{CO}_2 + \text{C}$$

to proceed with quantum efficiency approaching unity. The state of the carbon atom might be either ^1D or ^3P; the former if spin is to be conserved, the latter if not. In either case the reaction is strongly exothermic. The failure of the xenon line 1470 to induce photodissociation may be due to the reaction requiring an activation energy.

Estimates of $D(\text{CO})$ less than 10 take no account of the non-crossing rule of Hund, and Neumann and Wigner[9]. This rule states that potential energy curves of molecular states of identical species cannot cross. Whether the rule is rigorous when the nuclear and electronic motions are not separated needs further examination, but at least we see no reason for anticipating a failure of the rule in the lowest energy curve of CO. If this curve has only one turning point then the non-crossing rule requires unequivocally that $D(\text{CO}) > 10{\cdot}3$, and would agree well with the predissociation limit at $11{\cdot}11$ e.v.

The dissociation energy of CO^+ is $2{\cdot}6$ e.v. less than that of CO ($D(\text{CO}^+) = D(\text{CO}) + I(\text{C}) - I(\text{CO})$). Three electronic states of CO^+ are known, namely, $X^2\Sigma^+$, $A^2\Pi$ and $B^2\Sigma^+$, extrapolating to dissociation limits of about $9{\cdot}8$ (a very long extrapolation), $9{\cdot}2$ and $9{\cdot}4$ e.v. respectively. Since the two $^2\Sigma^+$ states must give different products of dissociation, it would appear, on the evidence of the $B^2\Sigma^+$ state, that $D(\text{CO}^+)$ is $7{\cdot}4$, and $D(\text{CO})$ is about 10, and on the evidence of the $A^2\Pi$ state that $D(\text{CO}^+)$ is $9{\cdot}2$ and $D(\text{CO})$ is $11{\cdot}8$. All that may fairly be deduced from present evidence on CO^+ is that $D(\text{CO})$ is unlikely to be much less than 10.

We have also re-examined nitrogen. The accepted value $D(\text{N}_2) = 7{\cdot}38$ is based on the identification of the upper state of the Vegard–Kaplan bands with the

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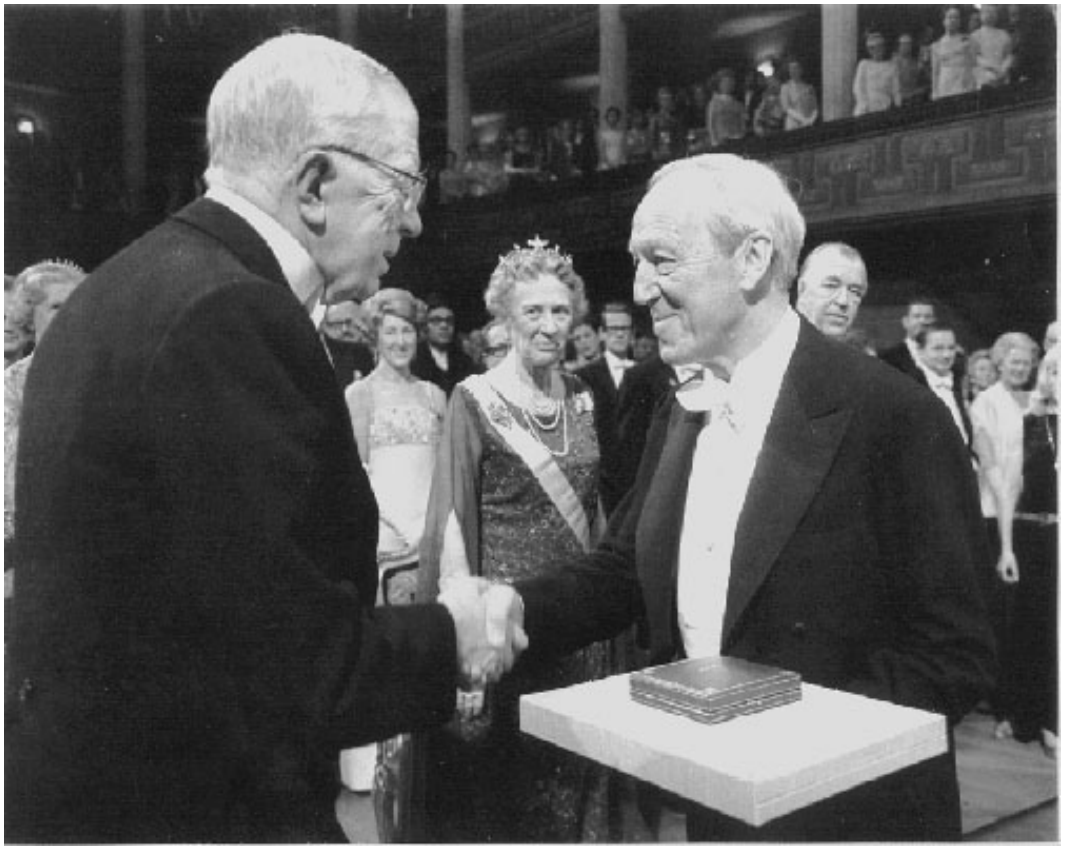
A Nobel Price winner

- Hannes Alfvén, *Nature*, 1942. **Nobel Prize 1970.**

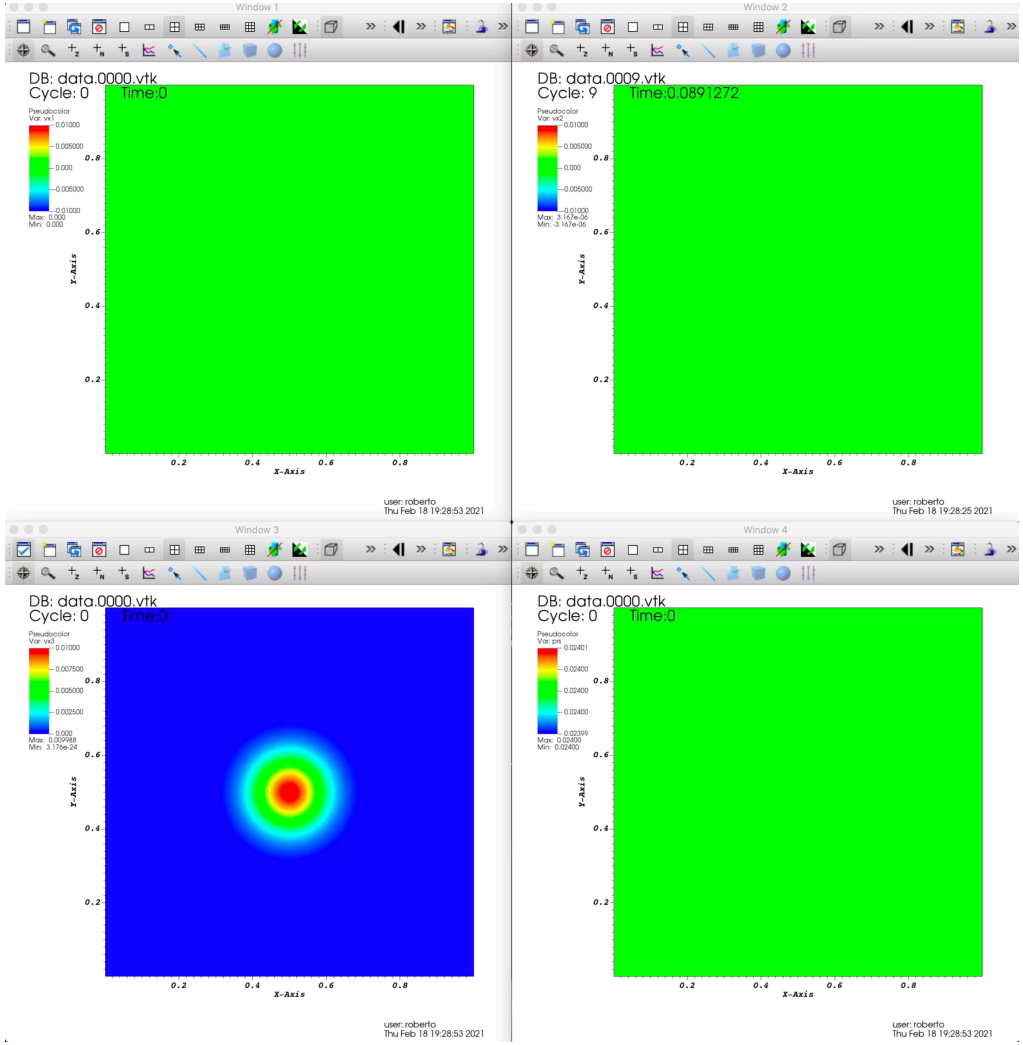


"for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics"

Visualizing MHD waves

- 2D domain, $\{x, y\}$, but $\vec{v}_1$ and $\vec{B}_1$ have the three components
- Homogeneous plasma with straight magnetic field, $\vec{B}_0 = B_0\hat{x}$

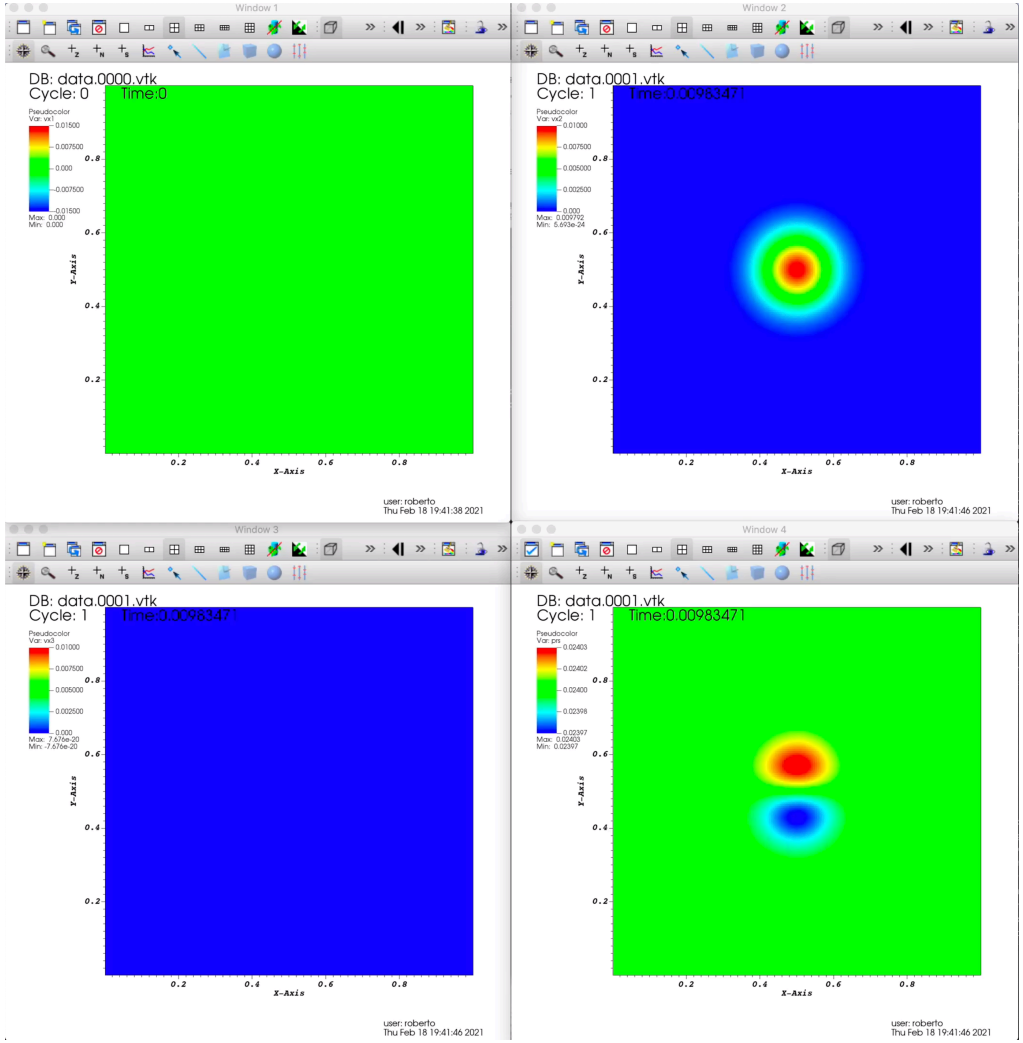
Alfvén wave



Visualizing MHD waves

- 2D domain, $\{x, y\}$, but $\vec{v}_1$ and $\vec{B}_1$ have the three components
- Homogeneous plasma with straight magnetic field, $\vec{B}_0 = B_0\hat{x}$

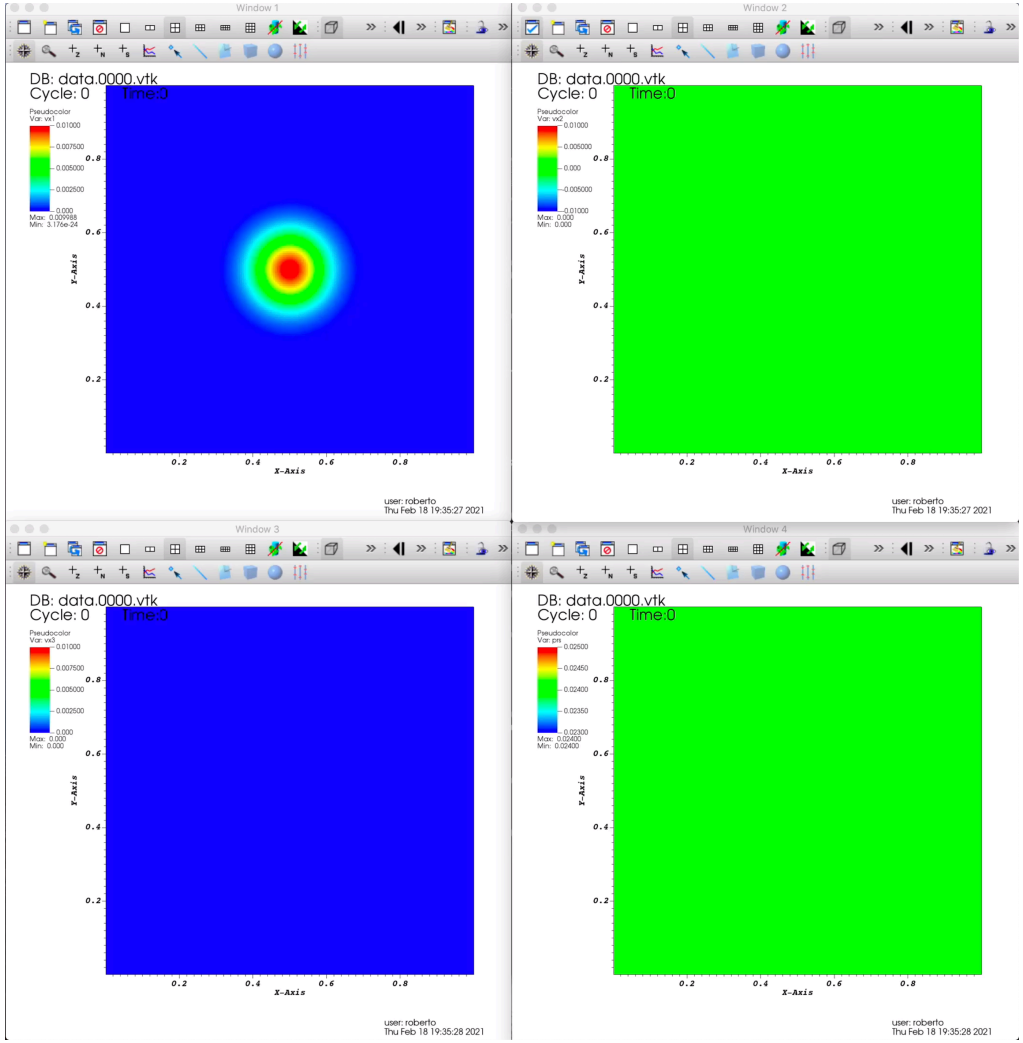
Fast wave



Visualizing MHD waves

- 2D domain, $\{x, y\}$, but $\vec{v}_1$ and $\vec{B}_1$ have the three components
- Homogeneous plasma with straight magnetic field, $\vec{B}_0 = B_0\hat{x}$

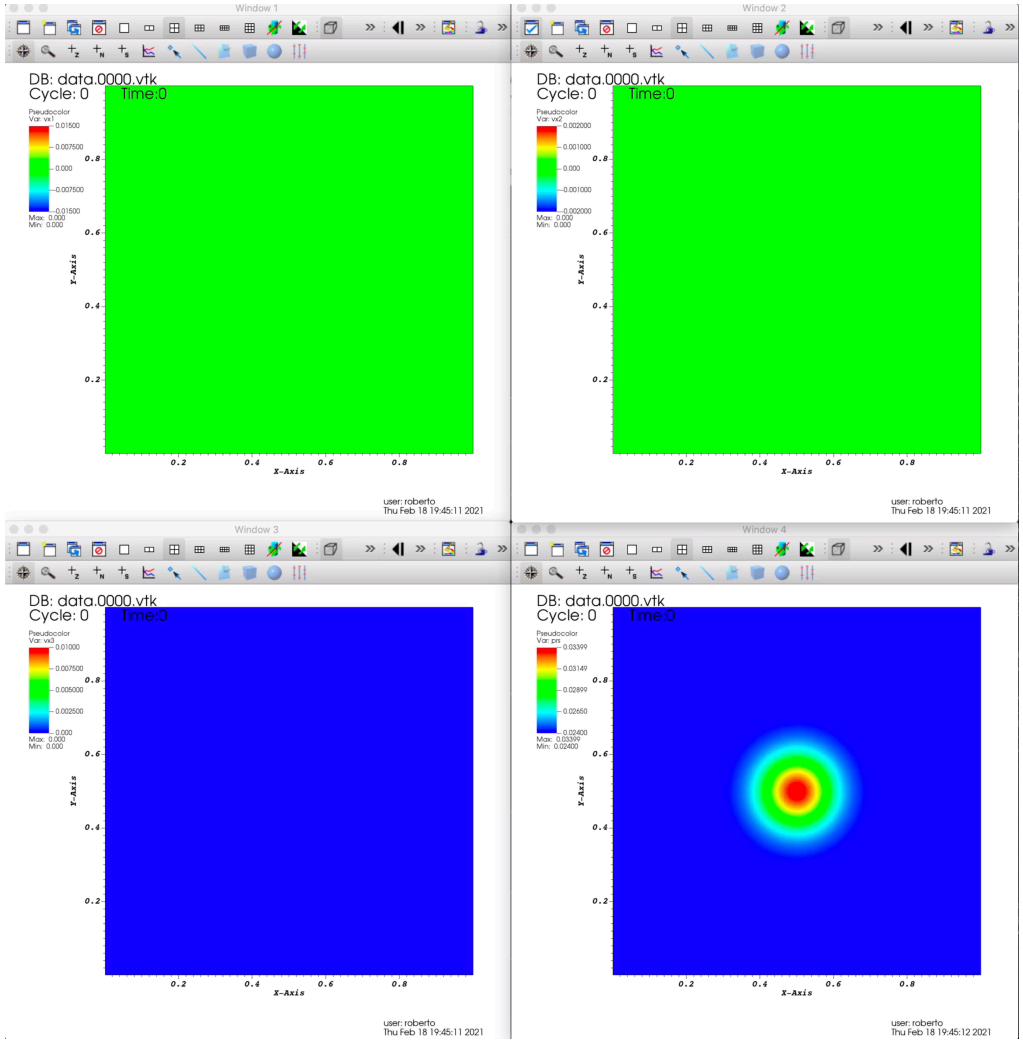
Slow wave



Visualizing MHD waves

- 2D domain, $\{x, y\}$, but $\vec{v}_1$ and $\vec{B}_1$ have the three components
- Homogeneous plasma with straight magnetic field, $\vec{B}_0 = B_0\hat{x}$

Pressure pulse = Slow wave + Fast wave



Monocromatic waves

- We consider perturbations in the form of monocromatic plane waves

$$\sim \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) = \exp\left(ik_x x + ik_y y + ik_z z - i\omega t\right)$$

- The linearized equations become

$$\begin{aligned}
\omega\rho_0\vec{v}_1 &= \vec{k}\left(p_1 + \frac{\vec{B}_0\cdot\vec{B}_1}{\mu}\right) - \frac{\vec{k}\cdot\vec{B}_0}{\mu}\vec{B}_1 \\
\omega p_1 &= \gamma p_0 \vec{k}\cdot\vec{v}_1 \\
\omega\vec{B}_1 &= \left(\vec{k}\cdot\vec{B}_1\right)\vec{B}_0 - \left(\vec{k}\cdot\vec{B}_0\right)\vec{v}_1
\end{aligned}$$

- Same expressions for propagating waves (ω prescribed and k given by DR) and for standing waves (k prescribed and ω given by DR)
- Relevant perturbations for Alfvén waves: $v_{\perp,1}$ and $B_{\perp,1}$
- Relevant perturbations for magnetocoustic waves: $v_{\mathrm{n},1}$, $v_{\|,1}$, $B_{\mathrm{n},1}$, $B_{\|,1}$, and p_1

Dispersion relations

- We take the equilibrium magnetic field along the z-direction: $\vec{B}_0 = B_0 \hat{e}_z$
- θ is the angle that the wavevector forms with the equilibrium magnetic field: $\theta = \arctan\left(\sqrt{k_x^2 + k_y^2}/k_z\right)$

Alfvén waves

$$\omega^2 = k_z^2 v_{\mathrm{A}}^2 = k^2 v_{\mathrm{A}}^2 \cos^2\theta$$

Magnetoacoustic waves

$$\omega^4 - \left(v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2\right) k^2 \omega^2 + v_{\mathrm{A}}^2 v_{\mathrm{s}}^2 k^4 \cos^2\theta = 0$$

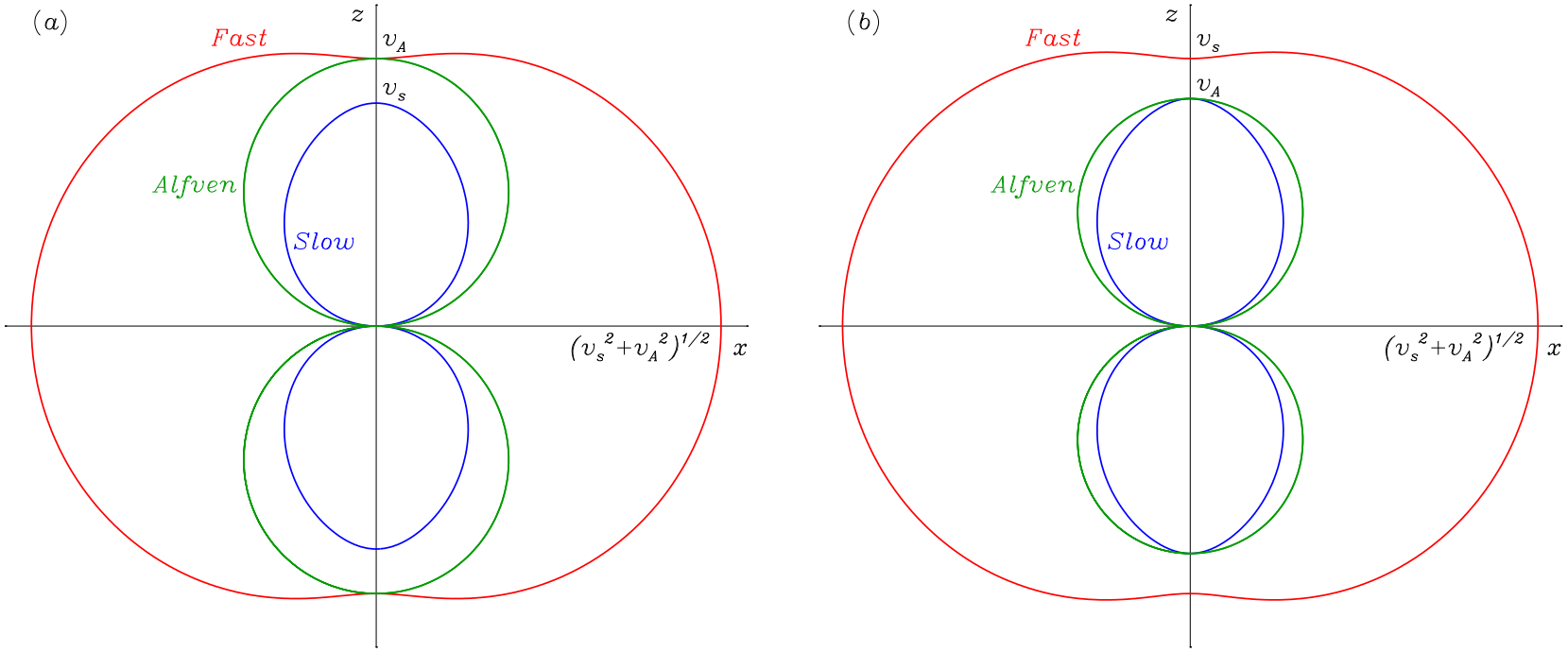
$$\omega_{\pm}^2 = \frac{k^2}{2}\left[\left(v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2\right) \pm \sqrt{\left(v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2\right)^2 - 4 v_{\mathrm{A}}^2 v_{\mathrm{s}}^2 \cos^2\theta}\right]$$

$+ \longrightarrow$ Fast wave $\qquad - \longrightarrow$ Slow wave

Friedrichs diagrams

- Polar plot of the phase speed, ω/k

$v_\mathrm{s} < v_\mathrm{A}$ $\qquad$ $v_\mathrm{s} > v_\mathrm{A}$



- Conditions in the solar corona: $v_\mathrm{s} \ll v_\mathrm{A}$

$$\text{Fast wave} \rightarrow \omega^2 \approx k^2 v_\mathrm{A}^2 \qquad \text{Slow wave} \rightarrow \omega^2 \approx k^2 v_\mathrm{s}^2 \cos^2\theta$$

MHD waves in the solar corona

Alfvén waves

- Incompressible waves
- Driven by magnetic tension (magnetic waves)
- Propagate strictly along the magnetic field
- Motions are perpendicular to the magnetic field
- Energy flows along the magnetic field at the Alfvén velocity

MHD waves in the solar corona

Fast magnetoacoustic waves

- Compressible waves
- Driven by both magnetic force and gas pressure (but gas pressure NOT needed!)
- Propagate almost isotropically (can propagate across the field)
- Motions have components normal and along to the field
- Energy flows fairly isotropically approximately at the Alfvén velocity

- If $v_{\mathrm{s}} = 0$ (no gas pressure)
  - $\omega_{+} = kv_{\mathrm{A}}$ (fast wave remains)
  - $\omega_{-} = 0$ (slow wave absent)

MHD waves in the solar corona

Slow magnetoacoustic waves

- Compressible waves
- Driven by both magnetic force and gas pressure (but gas pressure IS needed!)
- Propagate anisotropicaly (cannot propagate across the field)
- Motions are mostly along to the magnetic field
- Energy flux is highly anisotropic along the magnetic field at the sound velocity

Propagation of energy

- To find an energy conservation equation for ideal MHD waves we consider the linearized equations and
  1. take the scalar product of the momentum equation with $\vec{v}_1$,
  2. take the scalar product of the induction equation with $\vec{B}_1/\mu$,
  3. multiply the energy equation by $p_1/\gamma p_0$.
- The result, in suffix notation, is

KE
$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 v_1^2\right) = -v_{1,i}\frac{\partial p_1}{\partial x_i} - v_{1,i}\frac{\partial}{\partial x_i}\left(\frac{B_{0,j}B_{1,j}}{\mu}\right) + \frac{v_{1,j}B_{0,i}}{\mu}\frac{\partial B_{1,j}}{\partial x_i}$$

ME
$$\frac{\partial}{\partial t}\left(\frac{B_1^2}{2\mu}\right) = -\frac{B_{0,j}B_{1,j}}{\mu}\frac{\partial v_{1,i}}{\partial x_i} + \frac{B_{0,i}B_{1,j}}{\mu}\frac{\partial v_{1,j}}{\partial x_i}$$

IE
$$\frac{\partial}{\partial t}\left(\frac{p_1^2}{2\gamma p_0}\right) = -p_0\frac{\partial v_{1,i}}{\partial x_i}$$

Propagation of energy

- We now add the equations, getting

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 v_1^2 + \frac{B_1^2}{2\mu} + \frac{p_1^2}{2\gamma p_0}\right) + \frac{\partial}{\partial x_i}\left[\left(p_1 + \frac{B_{0,j}B_{1,j}}{\mu}\right)v_{1,i} - \frac{B_{0,i}B_{1,j}}{\mu}v_{1,j}\right] = 0$$

- This has the form of a conservation equation

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{\Pi} = 0$$

- Energy density, U
- Energy flux, $\vec{\Pi}$

Alternative form of the energy flux

- The energy flux, $\vec{\Pi}$, is conveniently written in an alternative form by realizing that $\vec{\Pi}$ is the sum of the work done by the pressure perturbation and the Poynting flux (associated to electromagnetic perturbations).
- Hence,

$$\vec{\Pi} = p_1 \vec{v}_1 + \frac{\vec{E} \times \vec{B}_1}{\mu}$$

- $\vec{E}$ is the electric field. In ideal MHD:

$$\vec{E} = -\vec{v}_1 \times \vec{B}_0$$

- In the case of Alfvén waves, $p_1 = 0$ so that $\vec{\Pi} = \frac{\vec{E} \times \vec{B}_1}{\mu}$

Net energy flux for harmonic waves

- For harmonic waves, the assumed temporal dependence $\exp(-i\omega t)$ implies the presence of complex numbers and that the terms in the energy equation are oscillatory in time.
- To retain the contribution with physical meaning, we must only keep the real part and must perform a temporal average over a complete cycle (period) of the wave.
- Let us assume the pair of variables $V + v\, e^{-i\omega t}$ and $W + w\, e^{-i\omega t}$, where V and W are the background (equilibrium) values and v and w are the amplitudes of the perturbations (waves).
- The (real) energy associated to the product of the two variables is

$$\mathcal{E} = \left[V + \frac{1}{2}\left(v\, e^{-i\omega t} + v^*\, e^{i\omega t}\right)\right]\left[W + \frac{1}{2}\left(w\, e^{-i\omega t} + w^*\, e^{i\omega t}\right)\right]$$

Net energy flux for harmonic waves

$$\begin{aligned}\mathcal{E} &= V\,W + \frac{1}{2}V\left(w\,e^{-i\omega t} + w^*\,e^{i\omega t}\right) + \frac{1}{2}W\left(v\,e^{-i\omega t} + v^*\,e^{i\omega t}\right) \\ &\quad + \frac{1}{4}\left(v^* w + v w^* + v w e^{-2i\omega t} + v^* w^* e^{2i\omega t}\right)\end{aligned}$$

- Now, we average $\mathcal{E}$ over a complete period of the wave, $<\mathcal{E}>= \frac{1}{P}\int_0^P \mathcal{E}\mathrm{d}t$. All terms carrying complex exponentials average to zero. Then,
$$<\mathcal{E}>= V\,W + \frac{1}{2}\mathrm{Re}\left(vw^*\right)$$
- $V\,W$ is the energy of the background
- $\frac{1}{2}\mathrm{Re}\left(vw^*\right)$ is the actual energy associated to the waves!

Net energy flux for harmonic waves

- Therefore, the time-averaged energy density and energy flux of the ideal MHD waves is

$$\langle U \rangle = \frac{1}{2}\left(\frac{1}{2}\rho_0 v_1\, v_1^* + \frac{B_1\, B_1^*}{2\mu} + \frac{p_1\, p_1^*}{2\gamma p_0}\right)$$

$$\left\langle \vec{\Pi} \right\rangle = \frac{1}{2}\mathrm{Re}\left(p_1 \vec{v}_1^* + \frac{\vec{E} \times \vec{B}_1^*}{\mu}\right)$$

- In the case of monochromatic waves, the energy conservation equation becomes

$$\omega \langle U \rangle = \vec{k} \cdot \left\langle \vec{\Pi} \right\rangle$$

Alfvén wave energy flux

- For an Alfvén wave

$$\begin{aligned}\left\langle \vec{\Pi} \right\rangle &= \frac{1}{2}\mathrm{Re}\left(\frac{\vec{E}\times\vec{B}_1^*}{\mu}\right) = -\frac{1}{2\mu}\mathrm{Re}\left[\left(\vec{v}_1\times\vec{B}_0\right)\times\vec{B}_1^*\right] \\ &= -\frac{1}{2\mu}\mathrm{Re}\left(v_{1,\perp}B_{1,\perp}^*\right)\vec{B}_0\end{aligned}$$

- From the linearized ideal equations (with $\vec{B}_0 = B_0\hat{e}_z$) we know

$$\begin{aligned}\omega\rho_0 v_{1,\perp} &= -\frac{k_z B_0}{\mu}B_{1,\perp} \\ \omega B_{,1\perp} &= -k_z B_0 v_{1,\perp}\end{aligned}$$

  with $\omega = k_z v_{\mathrm{A}}$, so that

$$\left\langle \vec{\Pi} \right\rangle = \frac{B_0^2}{2\mu}\frac{k_z}{\omega}V^2\hat{e}_z = \frac{1}{2}\rho V^2 v_{\mathrm{A}}\hat{e}_z,$$

  with $V^2 = v_{1,\perp}v_{1,\perp}^*$ the velocity amplitude of the Alfvén wave.
- **The kinetic/magnetic energy travels along the magnetic field at the Alfvén velocity.**

Resistive damping of Alfvén waves

- Let us consider the effect of Ohm's diffusion in the induction equation, so that now

$$\begin{aligned} \omega \rho_0 v_{1,\perp} &= -\frac{k_z B_0}{\mu} B_{1,\perp} \\ \omega B_{,1\perp} &= -k_z B_0 v_{1,\perp} - i k^2 \eta B_{,1\perp} \end{aligned}$$

- The dispersion relation is now

$$\omega^2 + i\omega k^2 \eta - k_z^2 v_{\mathrm{A}}^2 = 0$$

whose solutions are

$$\omega = \pm k_z v_{\mathrm{A}} \left[1 - \frac{k^4 \eta^2}{4 k_z^2 v_{\mathrm{A}}^2} \right]^{1/2} - i \frac{k^2 \eta}{2}$$

- ω is complex: **there is an imaginary part!**

Resistive damping of Alfvén waves

- Consequences:
  1. Alfvén waves are exponentially damped, with ω_{I} the exponential damping rate

     $$\exp\left(-i\omega t\right) = \exp\left[-i\left(\omega_{\mathrm{R}} + i\omega_{\mathrm{I}}\right)t\right] = \exp\left(-i\omega_{\mathrm{R}} t\right)\exp\left(-\omega_{\mathrm{I}} t\right)$$

  2. Alfvén waves are cut off ($\omega_{\mathrm{R}} = 0$) for sufficiently large k_z

     $$\omega_{\mathrm{R}} = 0 \qquad \rightarrow \qquad k_z \geq \frac{2v_{\mathrm{A}}}{\eta\cos^2\theta}$$

  3. The energy flux is no longer strictly along the magnetic field

     $$E = -\vec{v}_1 \times \vec{B}_0 + \eta\nabla\times\vec{B}_1$$

     so that

     $$\left\langle\vec{\Pi}\right\rangle = -\frac{B_0}{2\mu}\mathrm{Re}\left(v_{1,\perp}B_{1,\perp}^*\right)\hat{z} - \frac{\eta}{2\mu}\mathrm{Re}\left(ik_z B_{1,\perp}B_{1,\perp}^*\hat{z} + ik_x B_{1,\perp}B_{1,\perp}^*\hat{x}\right)$$

- However, the correction of η to the expression of the energy flux is small for coronal conditions and we can still approximate

  $$\left\langle\vec{\Pi}\right\rangle \approx -\frac{B_0}{2\mu}\mathrm{Re}\left(v_{1,\perp}B_{1,\perp}^*\right)\hat{z} = \frac{B_0^2}{2\mu}V^2\mathrm{Re}\left(\frac{k_z}{\omega^* - i\eta k^2}\right)\exp\left(-2\omega_{\mathrm{I}} t\right)\hat{z}$$

- The energy flux is no longer constant. It decreases at a rate equal to $2\omega_{\mathrm{I}}$

Resistive damping of Alfvén waves

- Wave energy equation in the presence of resistivity is no longer a conservation equation:

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{\Pi} = -H,$$

where H represents the loss of wave energy owing to magnetic diffusion:

$$H = \frac{\eta}{\mu} |\nabla \times \mathbf{B_1}|^2$$

- However, the total energy must be conserved in the system, so that the wave energy dissipated by Ohmic diffusion becomes internal energy of the plasma:

$$\frac{\mathrm{D}e}{\mathrm{D}t} + (\gamma - 1) e \nabla \cdot \mathbf{v} = \rho H$$

- If $\rho \approx$ constant, $e \uparrow$ so that $p \uparrow$ and $T \uparrow$
- **Plasma Heating!**

Non-adiabatic effects on magnetoacoustic waves

- From the linearized adiabatic energy equation, we get a relation between the perturbations of density and gas pressure, namely

$$\frac{\partial p_1}{\partial t} - \frac{\gamma p_0}{\rho_0}\frac{\partial \rho_1}{\partial t} = 0 \qquad \rightarrow \qquad p_1 = v_{\mathrm{s}}^2 \rho_1$$

- **Can we arrive at an equivalent relation in the non-adiabatic case?**
- If thermal conduction and radiative cooling are added, the linearized energy equation becomes

$$\frac{\partial p_1}{\partial t} - \frac{\gamma p_0}{\rho_0}\frac{\partial \rho_1}{\partial t} + (\gamma - 1)\left(\rho_0 L_\rho \rho_1 + \rho_0 L_T T_1 - \kappa_\parallel \frac{\partial^2 T_1}{\partial z^2}\right) = 0$$

where $L_\rho = \left(\frac{\partial L}{\partial \rho}\right)_T$ and $L_T = \left(\frac{\partial L}{\partial T}\right)_\rho$ are the partial derivatives of the cooling function with respect to ρ and T, and the background magnetic field is assumed in the z-direction.

Non-adiabatic effects on magnetoacoustic waves

- We consider perturbations in the form of monocromatic plane waves

$$\sim \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) = \exp\left(ik_x x + ik_y y + ik_z z - i\omega t\right)$$

and use the linearized equation of state to obtain the following relation between the pressure and density perturbations

$$p_1 = \Lambda^2 \rho_1$$

where $\Lambda^2 = \frac{\tilde{\gamma} p_0}{\rho_0}$ is the non-adiabatic version of the sound velocity, with $\tilde{\gamma}$ an effective adiabatic index given by

$$\tilde{\gamma} = \frac{(\gamma-1)\left(\frac{T_0}{p_0}\kappa_{\parallel}k_z^2 + \frac{\rho_0}{p_0}T_0 L_T - \frac{\rho_0}{p_0}\rho_0 L_\rho\right) - i\gamma\omega}{(\gamma-1)\left(\frac{T_0}{p_0}\kappa_{\parallel}k_z^2 + \frac{\rho_0}{p_0}T_0 L_T\right) - i\omega}$$

- **The non-adiabatic sound velocity depends on ω!**

Non-adiabatic effects on magnetoacoustic waves

- Since non-adiabatic effects only modify the sound velocity (not the Alfvén velocity), Alfvén waves are not affected.
- The last result is logical: Alfvén waves are incompressible (no p_1, no ρ_1, no T_1)
- The non-adiabatic version of the magnetoacoustic waves dispersion relation can be straight-forwardly obtained by replacing $v_{\rm s}^2$ by Λ^2, namely

$$\omega^4 - \left(v_{\rm A}^2 + \Lambda^2\right) k^2\omega^2 + v_{\rm A}^2\Lambda^2 k^4 \cos^2\theta = 0$$

- Since Λ^2 depends on ω, the dispersion relation is actually a 5th-order polynomial in ω
- **A new (mysterious) solution has been added! $\rightarrow$ THERMAL MODE**
- In addition, Λ^2 is a complex quantity, so that ω will be complex too: magnetoacoustic waves will be damped!
- SEE EXERCISE 2