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10) a) $X \sim N(\mu, \sigma^2)$ $\quad Pr[a \leq X \leq b] = \Phi\left(\frac{b-\mu}{\sqrt{\sigma^2}}\right) - \Phi\left(\frac{a-\mu}{\sqrt{\sigma^2}}\right)$

$Pr[2 \leq X_2 \leq 3]$

$X_2 \sim N(0,2)$

$$Pr[2 \leq X_2 \leq 3] = \Phi\left(\frac{3-0}{\sqrt{2}}\right) - \Phi\left(\frac{2-0}{\sqrt{2}}\right) = 0{,}9831 - 0{,}9214$$

$Pr[2 \leq X_2 \leq 3] = 0{,}0617$ //

b) $Pr[2 \leq X_2 \leq 3 / X_3 = 2]$ (Enunciado modificado)

$$C\vec{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & \boxed{0} & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$cov[X_2, X_3] = 0$, logo X_2 e X_3 são independentes.

$Pr[2 \leq X_2 \leq 3 / X_3 = 2] = Pr[2 \leq X_2 \leq 3]$

$Pr[2 \leq X_2 \leq 3 / X_3 = 2] = 0{,}0617$ //

c) $Pr[2 \leq X_2 \leq 3 / X_3 = 2$ e $X_1 = 3]$ (Enunciado modificado)

$cov[X_2, X_3] = 0$

$Pr[2 \leq X_2 \leq 3 / X_3 = 2$ e $X_1 = 3] = Pr[2 \leq X_2 \leq 3 / X_1 = 3]$

$$X_2/X_1=3 \sim N(\mu,\sigma^2)$$

$$Pr[2\leq X_2\leq 3/X_1=3]=\Phi\left(\frac{3-\mu}{\sqrt{\sigma^2}}\right)-\Phi\left(\frac{2-\mu}{\sqrt{\sigma^2}}\right)$$

• PDF condicional de X_2 dado que $X_1=3$:

$$f_{\vec{X}}(\vec{x})=\frac{1}{\sqrt{(2\pi)^n\cdot \det C}}\cdot \exp\left(-\frac{1}{2}\cdot(\vec{x}-\vec{\mu})^T\cdot C^{-1}\cdot(\vec{x}-\vec{\mu})\right)$$

$$f_{X_2}(x_2/X_1=3)=\frac{f_{X_2,X_1}(x_2,3)}{f_{X_1}(3)}$$

• Numerador:

$$\begin{bmatrix}X_2\\X_1\end{bmatrix}\sim\vec{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}2&1\\1&1\end{bmatrix}\right)$$

$$f_{X_2,X_1}(x_2,3)=\frac{1}{\sqrt{(2\pi)^2\cdot 1}}\cdot\exp\left(-\frac{1}{2}\begin{bmatrix}x_2-0 & 3-0\end{bmatrix}\cdot\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}x_2-0\\3-0\end{bmatrix}\right)$$

$$f_{X_2,X_1}(x_2,3)=\frac{1}{\sqrt{(2\pi^2)}}\cdot\exp\left(-\frac{1}{2}\cdot(x_2^2-6x_2+18)\right)$$

• Denominador:

$$X_1\sim N(0,1)$$

$$f_{X_1}(3)=\frac{1}{\sqrt{(2\pi)^1\cdot 1}}\cdot\exp\left(-\frac{1}{2}\cdot(3-0)\cdot\frac{1}{1}\cdot(3-0)\right)$$

$$f_{X_1}(3) = \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot 9\right)$$

· Portanto:

$$f_{X_2}(x_2/X_1=3) = \frac{\frac{1}{\sqrt{(2\pi)^2}} \cdot \exp\left(\frac{-1}{2} \cdot (x_2^2 - 6x_2 + 18)\right)}{\frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot 9\right)}$$

$$f_{X_2}(x_2/X_1=3) = \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot (x_2^2 - 6x_2 + 9)\right)$$

$$f_{X_2}(x_2/X_1=3) = \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot (x_2 - 3)^2\right)$$

· Conclusão:

$$X_2/X_1=3 \sim N(3,1)$$

$$Pr[2 \leq X_2 \leq 3/X_1=3] = \Phi\left(\frac{3-3}{\sqrt{1}}\right) - \Phi\left(\frac{2-3}{\sqrt{1}}\right)$$

$$Pr[2 \leq X_2 \leq 3/X_1=3] = 0{,}5 - 0{,}1587$$

$$Pr[2 \leq X_2 \leq 3/X_1=3] = 0{,}3413$$

d) $Pr[X_2 - X_4 > 4]$

$Y = X_2 - X_4$ $\qquad [Y] = A \cdot \vec{X}$ $\qquad [Y] = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$

• Vetor média de $[Y]$:

$\mu_{\vec{Y}} = A \cdot \mu_{\vec{X}}$ $\qquad \vec{\mu}_{\vec{Y}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \mu_Y = 0$

• Matriz covariância de $[Y]$:

$C_{\vec{Y}} = A \cdot C_{\vec{X}} \cdot A^T$ $\qquad C_{\vec{Y}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \sigma_Y^2 = 6$

• Conclusão:

$$Y \sim N(0,6)$$

$$Pr[X_2 - X_4 > 4] = \Phi\left(\frac{\infty - \mu_Y}{\sqrt{\sigma_Y^2}}\right) - \Phi\left(\frac{4 - \mu_Y}{\sqrt{\sigma_Y^2}}\right)$$

$$Pr[X_2 - X_4 > 4] = \Phi\left(\frac{\infty - 0}{\sqrt{6}}\right) - \Phi\left(\frac{4 - 0}{\sqrt{6}}\right)$$

$$Pr[X_2 - X_4 > 4] = 1 - 0{,}9488$$

$$Pr[X_2 - X_4 > 4] = 0{,}0512_{//}$$