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10) a) $X \sim N(\mu, \sigma^2)$ $Pr[a \le X \le b] = \Phi\left(\dfrac{b-\mu}{\sqrt{\sigma^2}}\right) - \Phi\left(\dfrac{a-\mu}{\sqrt{\sigma^2}}\right)$

$Pr[2 \le X_2 \le 3]$

$$X_2 \sim N(0,2)$$

$Pr[2 \le X_2 \le 3] = \Phi\left(\dfrac{3-0}{\sqrt{2}}\right) - \Phi\left(\dfrac{2-0}{\sqrt{2}}\right) = 0,9831 - 0,9214$

$Pr[2 \le X_2 \le 3] = 0,0617 //$

b) $Pr[2 \le X_2 \le 3 / X_3 = 2]$ (Enunciado modificado)

$C\vec{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$ $cov[X_2, X_3] = 0$, logo X_2 e X_3 são independentes.

$Pr[2 \le X_2 \le 3 / X_3 = 2] = Pr[2 \le X_2 \le 3]$

$Pr[2 \le X_2 \le 3 / X_3 = 2] = 0,0617 //$

c) $Pr[2 \le X_2 \le 3 / X_3 = 2 \text{ e } X_1 = 3]$ (Enunciado modificado)

$cov[X_2, X_3] = 0$

$Pr[2 \le X_2 \le 3 / X_3 = 2 \text{ e } X_1 = 3] = Pr[2 \le X_2 \le 3 / X_1 = 3]$

$$X_2/X_1 = 3 \sim N(\mu, \sigma^2)$$

$$Pr\left[2 \leq X_2 \leq 3 / X_1 = 3\right] = \Phi\left(\frac{3-\mu}{\sqrt{\sigma^2}}\right) - \Phi\left(\frac{2-\mu}{\sqrt{\sigma^2}}\right)$$

- PDF condicional de X_2 dado que $X_1 = 3$:

$$f\vec{X}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n \cdot \det C}} \cdot \exp\left(-\frac{1}{2} \cdot (\vec{x}-\vec{\mu})^T \cdot C^{-1} \cdot (\vec{x}-\vec{\mu})\right)$$

$$f_{X_2}(X_2/X_1=3) = \frac{f_{X_2,X_1}(X_2,3)}{f_{X_1}(3)}$$

- Numerador:

$$\begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \sim \vec{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$f_{X_2,X_1}(X_2,3) = \frac{1}{\sqrt{(2\pi)^2 \cdot 1}} \cdot \exp\left(-\frac{1}{2}\begin{bmatrix} X_2-0 & 3-0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} X_2-0 \\ 3-0 \end{bmatrix}\right)$$

$$f_{X_2,X_1}(X_2,3) = \frac{1}{\sqrt{(2\pi^2)}} \cdot \exp\left(-\frac{1}{2} \cdot (X_2^2 - 6X_2 + 18)\right)$$

- Denominador:

$$X_1 \sim N(0,1)$$

$$f_{X_1}(3) = \frac{1}{\sqrt{(2\pi)^1 \cdot 1}} \cdot \exp\left(-\frac{1}{2} \cdot (3-0) \cdot \frac{1}{1} \cdot (3-0)\right)$$

$$f_{X_1}(3) = \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot 9\right)$$

· Portanto :

$$f_{X_2}(X_2/X_1=3) = \frac{\frac{1}{\sqrt{(2\pi)^2}} \cdot \exp\left(\frac{-1}{2} \cdot (X_2^2-6X_2+18)\right)}{\frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot 9\right)}$$

$$f_{X_2}(X_2/X_1=3) = \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot (X_2^2-6X_2+9)\right)$$

$$f_{X_2}(X_2/X_1=3) = \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left(\frac{-1}{2} \cdot (X_2-3)^2\right)$$

· Conclusão :

$$X_2/X_1=3 \sim N(3,1)$$

$$Pr\left[2 \leqslant X_2 \leqslant 3/X_1=3\right] = \Phi\left(\frac{3-3}{\sqrt{1}}\right) - \Phi\left(\frac{2-3}{\sqrt{1}}\right)$$

$$Pr\left[2 \leqslant X_2 \leqslant 3/X_1=3\right] = 0,5 - 0,1587$$

$$Pr\left[2 \leqslant X_2 \leqslant 3/X_1=3\right] = 0,3413\,//$$

d) $\Pr[X_2 - X_4 > 4]$

$Y = X_2 - X_4$ $\qquad$ $[y] = A \cdot \vec{x}$ $\qquad$ $[Y] = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$

· Vetor média de $[Y]$:

$\mu_{\vec{y}} = A \cdot \mu_{\vec{x}}$ $\qquad$ $\vec{\mu}_{\vec{y}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad$ $\mu_Y = 0$

· Matriz covariância de $[Y]$:

$C_{\vec{y}} = A \cdot C_{\vec{x}} \cdot A^T$ $\qquad$ $C_{\vec{y}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad$ $\sigma_Y^2 = 6$

· Conclusão:

$$Y \sim N(0,6)$$

$\Pr[X_2 - X_4 > 4] = \Phi(x_2) + \Phi(x_4) - \Phi\left(\dfrac{4 - \mu_Y}{\sqrt{\sigma_Y^2}}\right)$

$\Pr[X_2 - X_4 > 4] = \Phi\left(\dfrac{0}{\sqrt{2}}\right) + \Phi\left(\dfrac{0}{\sqrt{4}}\right) - \Phi\left(\dfrac{4 - 0}{\sqrt{6}}\right)$

$\Pr[X_2 - X_4 > 4] = 0,5 + 0,5 - 0,9488$

$\Pr[X_2 - X_4 > 4] = 0,0512 \text{//}$