Advanced Quantum Mechanics
Exercises Set 2′
(Fill the gaps)

Due date: June 30$^{\text{th}}$, 2023

Exercise 1: Show that a state $|\Psi\rangle$ of a bipartite system is a product state if and only if it has Schmidt number $n_s = 1$.

Exercise 2: Show that a state $|\Psi(1,2)\rangle$ of a bipartite system is a product state if and only if $\hat{\rho}^{(1)} = \mathrm{Tr}_2\, |\Psi(1,2)\rangle$ (and thus also $\hat{\rho}^{(2)} = \mathrm{Tr}_1\, |\Psi(1,2)\rangle\big)$ is a pure state.

Exercise 3: Let $\hat{\rho}^{(1)}$ be a state operator in Hilbert space H_1 written in terms of normalized state vectors $\{|\varphi_i\rangle\,, i = 1, \cdots, n\}$ as

$$\hat{\rho}^{(1)} = \sum_{i=1}^{n} p_i\, |\varphi_i\rangle\, \langle\varphi_i| \quad \text{with} \quad p_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1 \tag{1}$$

Use a set of mutually orthogonal and normalized vectors $\{|\chi_i\rangle\,, i = 1, \cdots, n\}$ from a Hilbert space H_2 to construct an entangled pure bipartite state $|\Psi(1,2)\rangle$ in $H_1 \otimes H_2$ such that

$$\hat{\rho}^{(1)} = \mathrm{Tr}_2\, \hat{\rho}^{(12)}, \quad \text{with} \quad \hat{\rho}^{(12)} = |\Psi(1,2)\rangle\langle\Psi(1,2)| \tag{2}$$

$\hat{\rho}^{(12)}$ is then an extension of $\hat{\rho}^{(1)}$.

Exercise 4: Show that two purifications $|\Psi(1,2)\rangle$ and $|\Phi(1,2)\rangle$ in $H_1 \otimes H_2$ differ by a unitary operator.

Exercise 5: Let $\hat{\rho}^{(12)}$ be a general state operator (*not necessarily pure*) in a tensor-product Hilbert space $H_1 \otimes H_2$. Show that if $\hat{\rho}^{(1)}$ is a pure state, then $\hat{\rho}^{(12)}$ is a product $\hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}$, with $\hat{\rho}^{(2)}$ in H_2 (where $\hat{\rho}^{(2)}$ can be mixed). Hint: Choose an arbitrary purification of $\hat{\rho}^{(12)}$ and make use of the result of Exercise 2.

Exercise 6 (Monogamy of entanglement): Suppose that $\hat{\rho}^{(12)} = |\Psi(1,2)\rangle\langle\Psi(1,2)|$ is an entangled state of particles 1 and 2 . Use the result of Exercise 5 to show that an extension $\hat{\rho}^{(123)}$ of $\hat{\rho}^{(12)}$ would imply that particles 1 and 2 are both uncorrelated with particle 3, that is, if particles 1 and 2 are entangled, each of then cannot be entangled with particle 3.

Monogamy of entanglement is valid even when $\hat{\rho}^{(12)}$ is not pure; but we will not deal with entanglement for general quantum states (*i.e.* not necessarily pure).