Mecânica Quântica Avançada
Lista 1

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List of Exercises

Exercise 1 (4.1)**.** *Show that a necessary and sufficient condition for $|\psi\rangle$ to be an eigenvector of a Hermitian operator A is that the dispersion (4.8) $\Delta_\psi A = 0$.*

Answer. Vamos iniciar mostrando que se $|\psi\rangle = 0$, então $\Delta_\psi A = 0$. Ora, por definição:

$$\Delta_\psi A = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2 \tag{1}$$

É fácil ver que

$$\langle\psi|A^2|\psi\rangle = \langle\psi|A^\dagger|\,|A|\psi\rangle = a^2 \tag{2}$$

E também

$$\langle\psi|A|\psi\rangle^2 = (a)^2 = a^2 \tag{3}$$

Logo, é trivial que $\Delta_\psi A = 0$.

Agora vamos assumir que a dispersão é nula, ou seja, $\Delta A = 0$. Então ,por definição:

$$0 = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \tag{4}$$
$$\Rightarrow \langle A^2\rangle = \langle A\rangle^2 \tag{5}$$

Exercise 2 (4.4.2 - 1)**.** *Let $|\psi\rangle$ be a vector (not normalized) in the Hilbert space of states and H be a Hamiltonian. The expectation value $\langle H\rangle_\psi$ is*

$$\langle\psi\rangle_\psi = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \tag{6}$$

Show that if the minimum of this expectation value is obtained for $|\psi\rangle = |\psi_m\rangle$ and the maximum for $|\psi\rangle = |\psi_M\rangle$, then

$$H|\psi_m\rangle = E_m|\psi_m\rangle, \qquad H|\psi_M\rangle = E_M|\psi_M\rangle \tag{7}$$

where E_m and E_M are the smallest and largest eigenvalues.

Answer. É evidente que

$$\langle H \rangle_{\psi_m} = \frac{\langle \psi_m | H | \psi_m \rangle}{\langle \psi_m | \psi_m \rangle} = E_m \tag{8}$$

Portanto, é evidente que se $\langle H \rangle_{\psi_m}$ for mínimo, então E_m também é. Vale um raciocínio análogo para $E_M = \langle H \rangle_{\psi_M}$.

Exercise 3 (4.4.2 - 2)**.** *We assume that the vector* $|\varphi\rangle$ *depends on a parameter* $\alpha : |\varphi\rangle = |\varphi(\alpha)\rangle$. *Show that if*

$$\left. \frac{\partial \langle H \rangle_{\varphi(\alpha)}}{\partial \alpha} \right|_{\alpha=\alpha_0} = 0, \tag{9}$$

then $E_m \leq \langle H \rangle_{\varphi(\alpha_0)}$ *if* α_0 *corresponds to a minimum of* $\langle H \rangle_{\varphi(\alpha)}$, *and* $\langle H \rangle_{\varphi(\alpha_0)} \leq E_M$ *if* α_0 *corresponds to a maximum. This result forms the basis of an approximation method called the variational method (Section 14.1.4).*

Answer. Vamos abrir a derivada:

$$\frac{\partial \langle H \rangle \varphi(\alpha)}{\partial \alpha} = \frac{1}{\langle \psi | \psi \rangle} \left(\langle \psi | H | \, \partial_\alpha \, |\psi\rangle + \left(\partial_\alpha \langle \psi | \right) |H|\psi\rangle \right) - \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle^2} \left(\left(\partial_\alpha \langle \psi | \right) |\psi\rangle + \langle \psi | \, \partial_\alpha \, |\psi\rangle \right) \tag{10}$$

Então, em α_0

$$\langle \psi | H | \partial_\alpha \psi \rangle + \langle \partial_\alpha \psi | H | \psi \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \big(\langle \partial_\alpha \psi | \psi \rangle + \langle \psi | \partial_\alpha \psi \rangle \big) \tag{11}$$

Isolando a quantidade de interesse:

$$\langle H \rangle_{\varphi(\alpha_0)} = \frac{\langle \psi | H | \partial_\alpha \psi \rangle + \langle \partial_\alpha \psi | H | \psi \rangle}{\langle \partial_\alpha \psi | \psi \rangle + \langle \psi | \partial_\alpha \psi \rangle} \tag{12}$$

Podemos reescrever, usando que $H = H^\dagger$:

$$\langle H \rangle_{\varphi(\alpha_0)} = \frac{\langle \partial_\alpha \psi | H | \psi \rangle^\dagger + \langle \partial_\alpha \psi | H | \psi \rangle}{\langle \partial_\alpha \psi | \psi \rangle^\dagger + \langle \partial_\alpha \psi | \psi \rangle} \tag{13}$$

Podemos expandir qualquer estado usando os autokets do hamiltoniano, que são ortonormais:

$$|\psi\rangle = \sum c_j \, |\psi_j\rangle \tag{14}$$

De modo que

$$\partial_\alpha \, |\psi\rangle = \sum \frac{\partial c_j}{\partial \alpha} \, |\psi_j\rangle \tag{15}$$

Assim, vale que

$$\langle \partial_\alpha \psi | \psi \rangle = \sum c_j \frac{\partial c_j^*}{\partial \alpha} \tag{16}$$

E também

$$\langle \partial_\alpha \psi | H | \psi \rangle = \sum E_j c_j \frac{\partial c_j^*}{\partial \alpha} \tag{17}$$

Considere o denominador:

$$
\begin{aligned}
\langle \partial_\alpha \psi | \psi \rangle^\dagger + \langle \partial_\alpha \psi | \psi \rangle &= \sum c_j \frac{\partial c_j^*}{\partial \alpha} + c_j^* \frac{\partial c_j}{\partial \alpha} \\
&= \sum \frac{\partial}{\partial \alpha}(c_j^* c_j) \\
&= \sum \partial_\alpha |c_j|^2 \\
&= \partial_\alpha \sum |c_j|^2
\end{aligned}
$$

Considerando, enfim, o numerador e fazendo os mesmos cálculos:

$$
\begin{aligned}
\langle \partial_\alpha \psi | H | \psi \rangle^\dagger + \langle \partial_\alpha \psi | H | \psi \rangle &= \sum E_j c_j \frac{\partial c_j^*}{\partial \alpha} + E_j c_* \frac{\partial c_j}{\partial \alpha} \\
&= \sum E_j \partial_\alpha |c_j|^2 \\
&= \partial_\alpha \sum E_j |c_j|^2
\end{aligned}
$$

É óbvio, então, que

$$
E_m \partial_\alpha \sum |c_j|^2 \leq \partial_\alpha \sum E_j |c_j|^2 \leq E_M \partial_\alpha \sum |c_j|^2 \tag{18}
$$

Portanto, concluímos que

$$
E_m \leq \langle H \rangle_{\varphi(\alpha_0)} \leq E_M \tag{19}
$$

Exercise 4 (4.4.2 - 3)**.** *If H acts in a two-dimensional space, its most general form is*

$$
H = \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix},
$$

where b can always be chosen to be real. Parametrizing $|\varphi(\alpha)\rangle$ as

$$
|\varphi(\alpha)\rangle = \begin{pmatrix} \cos \alpha/2 \\ \sin \alpha/2 \end{pmatrix}
$$

find the values of α_0 by seeking the extrema of $\langle \varphi(\alpha) | H | \varphi(\alpha) \rangle$. Rederive (2.35).

Answer. Começamos considerando a equação do valor médio explicitamente:

$$
\begin{aligned}
\langle \psi | H | \psi \rangle &= \begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} a+c & b \\ b & a-c \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} (a+c)\cos(\alpha/2) + b\sin(\alpha/2) \\ b\cos(\alpha/2) + (a-c)\sin(\alpha/2) \end{bmatrix} \\
&= (a+c)\cos^2(\alpha/2) + b\sin(\alpha/2)\cos(\alpha/2) + b\cos(\alpha/2)\sin(\alpha/2) + (a-c)\sin^2(\alpha/2) \\
&= a(\sin^2(\alpha/2) + \cos^2(\alpha/2)) + c(\cos^2(\alpha/2) - \sin^2(\alpha/2)) + b\sin(\alpha) \\
&= a + b\sin(\alpha) + c\cos(\alpha)
\end{aligned}
$$

Vejamos os extremos dessa função:

$$
\begin{aligned}
\partial_\alpha \langle \psi | H | \psi \rangle &= b\cos(\alpha) - c\sin(\alpha) = 0 \\
b\cos(\alpha) &= c\sin(\alpha) \\
\tan(\alpha) &= b/c \\
\alpha_0 &= \operatorname{atan}(b/c)
\end{aligned}
$$

A Eq.(2.35) se refere aos autovetores e autovalores de H. Vejamos o caso do nosso ψ:

$$\begin{aligned} H\left|\psi\right\rangle &= \left(a\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \sqrt{b^2+c^2}\begin{bmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{bmatrix}\right)\begin{bmatrix}\cos(\alpha/2)\\ \sin(\alpha/2)\end{bmatrix} \\ &= a\begin{bmatrix}\cos(\alpha/2)\\ \sin(\alpha/2)\end{bmatrix} + \sqrt{b^2+c^2}\begin{bmatrix}\cos\alpha\cos(\alpha/2)+\sin\alpha\sin(\alpha/2)\\ \sin\alpha\cos(\alpha/2)-\cos\alpha\sin(\alpha/2)\end{bmatrix} \\ &= a+\sqrt{b^2+c^2}\begin{bmatrix}\cos(\alpha/2)\\ \sin(\alpha/2)\end{bmatrix} \end{aligned}$$

onde usamos a Eq.(2.34) e algumas identidades trigonométricas.

Exercise 5 (4.4.3)**.** *Let a Hamiltonian H depend on a parameter $\lambda : H = H(\lambda)$. Let $E(\lambda)$ be a nondegenerate eigenvalue and $|\varphi(\lambda)\rangle$ be the corresponding normalized eigenvector $(\|\varphi(\lambda)\|^2 = 1)$:*

$$H(\lambda)|\varphi(\lambda)\rangle = E(\lambda)|\varphi(\lambda)\rangle$$

Demonstrate the Feynman-Hellmann theorem:

$$\frac{\partial E}{\partial \lambda} = \left\langle \varphi(\lambda) \left| \frac{\partial H}{\partial \lambda} \right| \varphi(\lambda) \right\rangle.$$

Answer. Sabemos que podemos escrever

$$E(\lambda) = \langle\psi|H|\psi\rangle \tag{20}$$

Então considere:

$$\partial_\lambda E = \langle\partial_\lambda\psi|H|\psi\rangle + \langle\psi|\partial_\lambda|\psi\rangle + \langle\psi|H|\partial_\lambda\psi\rangle \tag{21}$$

Logo, é suficiente mostrar que

$$\langle\partial_\lambda\psi|H|\psi\rangle + \langle\psi|H|\partial_\lambda\psi\rangle = 0 \tag{22}$$

Considere:

$$\begin{aligned} \langle\partial_\lambda\psi|H|\psi\rangle + \langle\psi|H|\partial_\lambda\psi\rangle &= \langle\partial_\lambda\psi|E|\psi\rangle + \langle\psi|E^\dagger|\partial_\lambda\partial_\lambda\psi\rangle \\ &= E\left(\langle\partial_\lambda\psi|\psi\rangle + \langle\psi|\partial_\lambda\psi\rangle\right) \\ &= E(\partial_\lambda\left\langle\psi|\psi\right\rangle) \\ &= E\partial_\lambda 1 \\ &= 0 \end{aligned}$$

onde usamos o fato de H ser hermitiano e ter autovalores reais e de $|\psi\rangle$ ser normalizado. Portanto, fica demonstrado o teorema.

Exercise 6 (4.4.4)**.** *We consider a two-level system with Hamiltonian H represented by the matrix*

$$H = \hbar\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

in the basis

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

According to (2.35), the eigenvalues and eigenvectors of H are

$$\begin{array}{ll} E_+ = \hbar\sqrt{A^2+B^2}, & |\chi_+\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \\ E_- = -\hbar\sqrt{A^2+B^2}, & |\chi_-\rangle = -\sin\frac{\theta}{2}|+\rangle + \cos\frac{\theta}{2}|-\rangle \end{array}$$

with

$$A = \sqrt{A^2+B^2}\cos\theta, \quad B = \sqrt{A^2+B^2}\sin\theta, \quad \tan\theta = \frac{B}{A}$$

Part 1. *The state vector $|\varphi(t)\rangle$ at time t can be decomposed on the $\{|+\rangle, |-\rangle\}$ basis:*

$$|\varphi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$$

Write down the system of coupled differential equations which the components $c_+(t)$ and $c_-(t)$ satisfy.

Answer. Começamos escrevendo:

$$|\varphi(t)\rangle = \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} \tag{23}$$

Agora montamos a equação de Schrödinger:

$$\begin{aligned} H\,|\varphi(t)\rangle &= -i\hbar\partial_t\,|\varphi(t)\rangle \\ \hbar\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} &= -i\hbar\begin{pmatrix} \partial_t c_+(t) \\ \partial_t c_-(t) \end{pmatrix} \\ \begin{pmatrix} Ac_+ + Bc_- \\ Bc_+ - Ac_- \end{pmatrix} &= -i\begin{pmatrix} \dot{c}_+(t) \\ \dot{c}_-(t) \end{pmatrix} \end{aligned}$$

Ou seja, temos as equações diferenciais:

$$i\dot{c}_+ = Ac_+ + Bc_- \tag{24}$$

$$i\dot{c}_- = Bc_+ - Ac_- \tag{25}$$

Part 2. *Let $|\varphi(t=0)\rangle$ be decomposed on the $\{|\chi_+\rangle, |\chi_-\rangle\}$basis:*

$$|\varphi(t=0)\rangle = |\varphi(0)\rangle = \lambda\,|\chi_+\rangle + \mu\,|\chi_-\rangle, \quad |\lambda|^2 + |\mu|^2 = 1$$

Show that $c_+(t) = \langle + \mid \varphi(t)\rangle$ is written as

$$c_+(t) = \lambda \mathrm{e}^{-\mathrm{i}\Omega t/2}\cos\frac{\theta}{2} - \mu\mathrm{e}^{\mathrm{i}\Omega t/2}\sin\frac{\theta}{2}$$

with $\Omega = 2\sqrt{A^2+B^2}$. Here $\hbar\Omega$ is the energy difference of the two levels. Show that $c_+(t)$ (as well as $c_-(t)$) satisfies the differential equation

$$\ddot{c}_+(t) + \left(\frac{\Omega}{2}\right)^2 c_+(t) = 0.$$

Answer. Vamos começar pela segunda parte, mostrando a validez da equação diferencial. Considere as EDOs que obtemos e vamos deriva-las mais uma vez (faremos as contas para c_+, pois são análogas para c_-.):

$$i\ddot{c}_+ = A\dot{c}_+ + B\dot{c}_- \tag{26}$$

Mas note, também, que:

$$\begin{aligned} A\dot{c}_+ &= -iA^2 c_+ - iABc_- \\ B\dot{c}_- &= -iB^2 c_+ + iABc_- \end{aligned}$$

Logo,

$$A\dot{c}_+ + B\dot{c}_- = -i(A^2 + B^2)c_+ \tag{27}$$

Portanto,

$$i\ddot{c}_+ = -i(A^2 + B^2)c_+ \tag{28}$$

que, simplificando, implica na expressão desejada

$$\ddot{c}_+ + \left(\frac{\Omega}{2}\right)^2 c_+ = 0 \tag{29}$$

Vamos considerar agora a outra parte. Ela nos dá condições iniciais:

$$|\varphi_0\rangle = \begin{pmatrix} \lambda\cos(\theta/2) - \mu\sin(\theta/2) \\ \lambda\sin(\theta/2) + \mu\cos(\theta/2) \end{pmatrix} \tag{30}$$

Ou seja,

$$\begin{aligned} c_+(0) &= \lambda\cos(\theta/2) - \mu\sin(\theta/2) \\ c_-(0) &= \lambda\sin(\theta/2) + \mu\cos(\theta/2) \end{aligned}$$

Usando a equação diferencial que deduzimos, é fácil ver que:

$$\begin{aligned} c_+(t) &= A_+ e^{i\Omega t/2} + B_+ e^{-i\Omega t/2} \\ c_-(t) &= A_- e^{i\Omega t/2} + B_- e^{-i\Omega t/2} \end{aligned}$$

Isso obviamente nos diz que:

$$\begin{aligned} A_+ + B_+ &= \lambda\cos(\theta/2) - \mu\sin(\theta/2) \\ A_- + B_- &= \lambda\sin(\theta/2) + \mu\cos(\theta/2) \end{aligned}$$

*

Part 3. *We assume that $c_+(0) = 0$. Find λ and μ up to a phase as well as $c_+(t)$. Show that the probability of finding the system in the state $|+\rangle$ at time t is*

$$\mathrm{p}_+(t) = \sin^2\theta \sin^2\left(\frac{\Omega t}{2}\right) = \frac{B^2}{A^2 + B^2}\sin^2\left(\frac{\Omega t}{2}\right).$$

Answer. Se $c_+(0) = 0$, então precisa ser que

$$\lambda \cos(\theta/2) - \mu \sin(\theta/2) = 0 \tag{31}$$

Logo, é verdade que

$$\lambda = \mu \tan(\theta/2) \tag{32}$$

Mas, pela normalização:

$$\begin{aligned} |\lambda|^2 + |\mu|^2 &= 1 \\ |\mu \tan(\theta/2)|^2 + |\mu|^2 &= 1 \\ |\mu|^2 |\tan(\theta/2)|^2 + |\mu|^2 &= \\ |\mu|^2 &= \frac{1}{1 + |\tan(\theta/2)|^2} \end{aligned}$$

Logo, a menos de uma fase e^{ia},

$$\begin{aligned} \mu = &= \frac{1}{\sqrt{1 + |\tan(\theta/2)|^2}} \\ \lambda &= \frac{\tan(\theta/2)}{\sqrt{1 + |\tan(\theta/2)|^2}} \end{aligned}$$

Vamos supor, daqui em diante, $\theta \in [0, \pi]$ para tirarmos a tangente do módulo. Usando a solução geral de $c_+(t)$, sabemos que precisa valer

$$A_+ + B_+ = 0 \tag{33}$$

Exercise 7 (4.4.5)**.** *Let $|\varphi(0)\rangle$ represent the state vector at time $t = 0$ of an unstable particle, or more generally that of an unstable quantum state such as an atom in an excited state, and let $\mathrm{p}(t)$ be the probability (survival probability) that it has not decayed at time t. The particle is assumed to be isolated from external influences (but not from quantized fields), so that the Hamiltonian H that governs the decay is time-independent. Let $|\Psi(t)\rangle$ be the state vector at time t of the full quantum system*

$$|\Psi(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right) |\varphi(0)\rangle.$$

The probability amplitude for finding the state of the quantum system at time t in $|\varphi(0)\rangle$ is

$$c(t) = \langle \varphi(0) \mid \Psi(t) \rangle = \left\langle \varphi(0) \middle| \exp\left(-\frac{\mathrm{i}Ht}{\hbar}\right) \middle| \varphi(0) \right\rangle$$

and the survival probability is

$$\mathrm{p}(t) = |c(t)|^2 = |\langle \Psi(t) \mid \varphi(0) \rangle|^2 = \langle \Psi(t) | \mathcal{P} | \Psi(t) \rangle$$

where $\mathcal{P} = |\varphi(0)\rangle\langle\varphi(0)|$ is the projector on the initial state.

Part 1. *Let us first restrict ourselves to very short times. Show that for $t \to 0$*

$$\mathrm{p}(t) \simeq 1 - \frac{(\Delta H)^2}{\hbar^2} t^2$$

so that, for very short times, the decay law is certainly not exponential. The expectation values of H and H^2 are computed in the state $|\varphi(0)\rangle$. Note that ΔH must be finite, otherwise $|\varphi(0)\rangle$ would not belong to the domain of H^2, which would be difficult to imagine physically (see Chapter 7 for the definition of the domain of an operator).

Answer. Começamos lembrando que

$$(\Delta H)^2 = \langle H^2 \rangle - \langle H \rangle^2 \tag{34}$$

Para $t \to 0$, podemos aproximar

$$e^{-iHt/\hbar} \approx 1 - \frac{iHt}{\hbar} - \frac{H^2 t^2}{2\hbar^2} \tag{35}$$

Logo,

$$\begin{aligned} c(t) &= \langle \phi_0 | 1 - \frac{iHt}{\hbar} - \frac{H^2 t^2}{2\hbar^2} | \phi_0 \rangle \\ &= \langle \phi_0 | \phi_0 \rangle - \frac{it}{\hbar} \langle H \rangle - \frac{t^2 \langle H^2 \rangle}{2\hbar^2} \\ &= 1 - \frac{it \langle H \rangle}{\hbar} - \frac{t^2 \langle H^2 \rangle}{2\hbar^2} \end{aligned}$$

Portanto,

$$\begin{aligned} p(t) &= |c(t)|^2 \\ &= 1 - \frac{t^2 \langle H^2 \rangle}{\hbar^2} + \frac{t^2 \langle H \rangle^2}{\hbar^2} + \frac{t^4 \langle H^2 \rangle^2}{\hbar^4} \\ &= 1 - \frac{t^2}{\hbar^2} \left(\langle H^2 \rangle - \langle H \rangle^2 \right) + \mathcal{O}(t^4) \\ &= 1 - \frac{t^2 (\Delta H)^2}{\hbar^2} \end{aligned}$$

como queríamos mostrar.

Part 2. *A more general result is obtained as follows. Show first that*

$$\Delta \mathcal{P}^2 = \langle \mathcal{P} \rangle - \langle \mathcal{P} \rangle^2$$

and use (4.27) to deduce the inequality $\left(\Delta H = (\langle H^2 \rangle - \langle H \rangle^2)^{1/2} \right)$

$$\left| \frac{\mathrm{dp}(t)}{\mathrm{d}t} \right| \leq \frac{2\Delta H}{\hbar} \sqrt{\mathrm{p}(1-\mathrm{p})}$$

Integrating this differential equation, derive

$$\mathrm{p}(t) \geq \cos^2 \left(\frac{t \Delta H}{\hbar} \right) \quad 0 \leq t \leq \frac{\pi \hbar}{2\Delta H}$$

/
Answer. Sabemos que $\mathcal{P} = |\psi_0\rangle\langle\psi_0|$. Assim, como $\mathcal{P}$ é um projetor, é trivial que seja idempotente:

$$\mathcal{P}^2 = (|\psi_0\rangle\langle\psi_0|)(|\psi_0\rangle\langle\psi_0|) = |\psi_0\rangle\langle\psi_0|\psi_0\rangle\langle\psi_0| = |\psi_0\rangle\langle\psi_0| = \mathcal{P} \tag{36}$$

À partir disso, decorre diretamente a propriedade de:

$$(\Delta\mathcal{P})^2 = \langle\mathcal{P}\rangle - \langle\mathcal{P}\rangle^2 \tag{37}$$

A Eq.(4.27) diz que

$$\Delta_\varphi H \Delta_\varphi A \geq \frac{1}{2}\left|\langle[A,H]\rangle_\varphi\right| = \frac{1}{2}\hbar\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle_\varphi(t)\right| \tag{38}$$

onde, no nosso caso, teremos $A = \mathcal{P}$. Usando $p(t) = \langle\mathcal{P}\rangle$ e aplicando para o nosso caso:

$$\left|\frac{dp}{dt}\right| \leq \frac{2\Delta H}{\hbar}\Delta\mathcal{P} \tag{39}$$

Então, basta escrever $\Delta\mathcal{P}$ na forma desejada. Ora, isso é fácil, pois

$$\begin{aligned}\Delta\mathcal{P} &= \sqrt{\langle\mathcal{P}\rangle - \langle\mathcal{P}\rangle^2}\\ &= \sqrt{p - p^2}\\ &= \sqrt{p(1-p)}\end{aligned}$$

Portanto,

$$\left|\frac{dp}{dt}\right| \leq \frac{2\Delta H}{\hbar}\sqrt{p(1-p)} \tag{40}$$

Vamos, agora, integrar essa inequação:

$$\begin{aligned}\frac{dp}{\sqrt{p(1-p)}} &\leq \frac{2\Delta H}{\hbar}dt\\ \int\frac{dp}{\sqrt{p(1-p)}} &\leq \int\frac{2\Delta H}{\hbar}dt = \frac{2t\Delta H}{\hbar}\end{aligned}$$

Vejamos o lado esquerdo com mais cuidado. Vamos usar a substituição $u = \sqrt{p}$, onde $2du = dp/\sqrt{p}$. Assim,

$$\int\frac{dp}{\sqrt{p(1-p)}} = \int\frac{2du}{\sqrt{1-u^2}} = 2\,\mathrm{acos}(u) \tag{41}$$

Agora, notamos que isso implica

$$0 \leq \frac{2t\Delta H}{\hbar} \leq \pi \tag{42}$$

Ou,

$$0 \leq t \leq \frac{\pi\hbar}{2\Delta H} \tag{43}$$

Voltando à inequação das integrais e usando que $u = \sqrt{p}$.

$$\begin{aligned}\mathrm{acos}(u) &\leq \frac{t\Delta H}{\hbar}\\ \sqrt{p} &\geq \cos\left(\frac{t\Delta H}{\hbar}\right)\\ p(t) &\geq \cos^2\left(\frac{t\Delta H}{\hbar}\right)\end{aligned}$$

o sinal da inegualdade muda, pois $\cos(x)$ decresce no intervalo $x \in [0, \pi]$.

Part 3. *Let $|n\rangle$ be a complete set of eigenstates of the Hamiltonian*

$$H|n\rangle = E_n|n\rangle$$

Show that $c(t)$ is given by the Fourier transform of a spectral function $w(E)$

$$w(E) = \sum_n |\langle n \mid \varphi(0)\rangle|^2 \delta\left(E - E_n\right)$$

Set $E_0 = \langle H\rangle$ and give the expression of $(\Delta H)^2$ in terms of $w(E)$ and E_0.
Answer. Por definição,

$$c(t) = \langle\varphi(0)|\Psi(t)\rangle \tag{44}$$

com

$$|\Psi(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|\varphi(0)\rangle \tag{45}$$

Part 4. *If $w(E)$ has a Lorentzian shape*

$$w(E) = \frac{\Gamma\hbar}{2\pi}\frac{1}{\left(E - E_0\right)^2 + \hbar^2\Gamma^2/4}$$

show that

$$c(t) = \mathrm{e}^{-\mathrm{i}E_0 t/\hbar}\mathrm{e}^{-\Gamma t/2}$$

and that the decay law is an exponential. The width of $w(E)$ is $\hbar\Gamma$, but ΔH is infinite, Thus ΔH is a rather poor measure of energy spread, and the width $\hbar\Gamma = \Delta E$ is the physically relevant quantity.
Answer.
