Mecânica Quântica Avançada
Lista 4

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IFT

List of Exercises

Exercise 1 (6.5.2 - Rotation and $SU(2)$)**.** *The $SU(2)$ group is the group of 2×2 unitary matrices of unit determinant.*

Part 1. *Show that if $U \in SU(2)$, then U has the form*

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1 \tag{1}$$

Answer. Generically, we must have

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2}$$

First of all, being unitary means that $U^{-1} = U^\dagger$. That means

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

This leads to four equations:

$$\begin{cases} |a|^2 + |b|^2 = 1 \\ ac^* + bd^* = 0 \\ a^*c + b^*d = 0 \\ |c|^2 + |d|^2 = 1 \end{cases} \tag{4}$$

Moreover, the unit determinant yields the last condition

$$ad - bc = 1 \tag{5}$$

Using this equation in the second equation of the system multiplied by d we have:

$$\begin{aligned}
adc^* + bdd^* &= 0 \\
(1+bc)c^* + b|d|^2 &= 0 \\
c^* + b|c|^2 + b|d^2| &= 0 \\
c^* + b &= 0 \\
c^* &= -b \\
c &= -b^*
\end{aligned}$$

Using this in the second equation:

$$\begin{aligned}
ac^* + bd^* &= 0 \\
-ab + bd^* &= 0 \\
a &= d^* \\
d &= a^*
\end{aligned}$$

Thus, we can write the most general $SU(2)$ matrix as

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{6}$$

Part 2. *Show that in the neighborhood of the identity we can write*

$$U = I - i\tau, \quad \tau = \tau^\dagger \tag{7}$$

and that τ is expressed as a function of the Pauli matrices as

$$\tau = \frac{1}{2}\sum_{i=1}^{3} \theta_i \sigma_i, \quad \theta_i \to 0 \tag{8}$$

Answer. In the neighborhood of the identity, we can write our (complex) coefficients as:

$$a = x + iy \approx 1 + i\frac{\theta_3}{2} \tag{9}$$

And

$$b = w + iz \approx \frac{1}{2}\left(\theta_2 + i\theta_1\right) \tag{10}$$

with $\theta_i \to 0$. That is

$$\begin{aligned}
U &= \frac{1}{2}\begin{pmatrix} 2 + i\theta_3 & \theta_2 + i\theta_1 \\ -\theta_2 + i\theta_1 & 2 - i\theta_3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{i}{2}\left(\begin{pmatrix} \theta_3 & 0 \\ 0 & -\theta_3 \end{pmatrix} + \begin{pmatrix} 0 & \theta_1 \\ \theta_1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & \theta_2 \\ -\theta_2 & 0 \end{pmatrix}\right) \\
&= I + \frac{i}{2}\left(\sigma_3\theta_3 + \sigma_1\theta_1 + \sigma_2\theta_2\right) \\
&= I + i\tau
\end{aligned}$$

as desired.

Part 3. *We take $\theta = (\Sigma_i \theta_i^2)^{1/2}$ and $\theta_i = \theta\hat{n}$, where $\hat{n}$ is a unit vector. Assuming that the θ_i are finite, we define $U_{\hat{n}}(\theta)$ as*

$$U_{\hat{n}}(\theta) = \lim_{N\to\infty} \left(U_{\hat{n}} \left(\frac{\theta}{N} \right) \right)^N \tag{11}$$

Show that

$$U_{\hat{n}}(\theta) = e^{-i\theta\vec{\sigma}\cdot\hat{n}/2} \tag{12}$$

Conversely, any $SU(2)$ matrix of this form.
Answer. Since θ/N is small, we can taylor-expand U:

$$U_{\hat{n}} \left(\frac{\theta}{N} \right) = I - \frac{i\theta}{2N} (\vec{\sigma} \cdot \hat{n}) \tag{13}$$

Thus, we can use the well-known limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e \tag{14}$$

Thus,

$$U_{\hat{n}}(\theta) = e^{-i\theta\vec{\sigma}\cdot\hat{n}/2} \tag{15}$$

Part 4. *Let $\vec{V}$ be a vector in $\mathbb{R}^3$ and $\mathcal{V}$ be a Hermitian matrix of zero trace:*

$$\mathcal{V} = \begin{pmatrix} V_z & V_x - iV_y \\ V_x + iV_y & -V_z \end{pmatrix} = \vec{\sigma} \cdot \vec{V} \tag{16}$$

What is the determinant of $\mathcal{V}$? Let $\mathcal{W}$ be the matrix $U \in SU(2)$

$$\mathcal{W} = U\mathcal{V}U^{-1} \tag{17}$$

Show that $\mathcal{W}$ has the form $\sigma \cdot \vec{\mathcal{W}}$ and that $\vec{\mathcal{W}}$ is derived from $\vec{\mathcal{V}}$ by a rotation. Has this property been completely proved at this stage?
Answer. The determinant of $\mathcal{V}$ is simply

$$\det \mathcal{V} = -V_z^2 - V_x^2 - V_y^2 = -\vec{V}^2 \tag{18}$$

Now we note that

$$\det(U\mathcal{V}U^{-1}) = \det(\mathcal{V}) \tag{19}$$

Hence, $\det \mathcal{V} = \det \mathcal{W}$, which shows that $\vec{W}^2 = \vec{V}^2$. Thus, $\vec{W}$ must be a rotation of $\vec{V}$.

Part 5. *We define $\vec{V}(\theta)$ as*

$$\vec{\sigma} \cdot \vec{V}(\theta) = U_{\hat{n}}(\theta) \left(\vec{\sigma} \cdot \vec{V} \right) U_{\hat{n}}^{-1}(\theta), \qquad \vec{V}(0) = \vec{V} \tag{20}$$

Show that

$$\frac{d\vec{V}(\theta)}{d\theta} = \hat{n} \times \vec{V}(\theta) \tag{21}$$

Answer. Using the same argument of the last item, we can easily show that $\mathcal{W} = \vec{\sigma}\cdot\vec{W}$. Now, for the derivative:

$$\begin{aligned}
\frac{d\vec{\sigma}\cdot\vec{V}(\theta)}{d\theta} &= \frac{d}{d\theta}\left(e^{-i\theta\vec{\sigma}\cdot\hat{n}/2}\vec{\sigma}\cdot\vec{V}e^{i\theta\vec{\sigma}\cdot\hat{n}/2}\right)\\
&= -\left(\frac{i\hat{n}\cdot\vec{\sigma}}{2}\right)\left(e^{-i\theta\vec{\sigma}\cdot\hat{n}/2}\vec{\sigma}\cdot\vec{V}e^{i\theta\vec{\sigma}\cdot\hat{n}/2}\right)+\frac{i\vec{\sigma}\cdot\hat{n}}{2}\left(e^{-i\theta\vec{\sigma}\cdot\hat{n}/2}\vec{\sigma}\cdot\vec{V}e^{i\theta\vec{\sigma}\cdot\hat{n}/2}\right)\\
&= -\frac{i}{2}\left[\vec{\sigma}\cdot\hat{n},\vec{\sigma}\cdot\vec{V}(\theta)\right]\\
&= \sigma\cdot(\hat{n}\times\vec{V})
\end{aligned}$$

which is the desired identity. Finally, there is a two to one correspondence between $SU(2)$ and $SO(3)$, because both $U_{\hat{n}}(\theta)$ and $-U_{\hat{n}}(\theta) = U_{\hat{n}}(\theta+2\pi)$ point to the same matrix in $SO(3)$.

Exercise 2 (6.5.6 - The center of mass and the reduced mass)**.** *Let us take two particles of masses m_1 and m_2 moving on a line. We use X_1 and X_2 to denote their position operators and P_1 and P_2 to denote their momentum operators. The position and momentum operators of two different particles commute. We define the operators X and P as*

$$X = \frac{m_1X_1+m_2X_2}{m_1+m_2}, \qquad P = P_1+P_2 \tag{22}$$

and $\hat{X}$ and $\hat{P}$ as

$$\hat{X} = X_1 - X_2, \qquad \hat{P} = \frac{m_2P_1 - m_1P_2}{m_1+m_2} \tag{23}$$

Part 1. *Calculate the commutators$[X,\hat{P}]$ and $[\hat{X},P]$ and show that they equal zero.*
Answer. We note that

$$[X_i,P_j] = i\hbar\delta_{ij}, \quad [X_i,X_j] = [P_i,P_j] = 0 \tag{24}$$

So, with $k = 1/(m_1+m_2)$,

$$\begin{aligned}
[X,P] &= k[m_1X_1+m_2X_2,P_1+P_2]\\
&= k\left(m_1([X_1,P_1]+[X_1,P_2])+m_2([X_2,P_1]+[X_2,P_2])\right)\\
&= i\hbar
\end{aligned}$$

And

$$\begin{aligned}
[X,\hat{P}] &= k^2\left([m_1X_1+m_2X_2,m_2P_1-m_1P_2]\right)\\
&= k^2\left(m_1m_2[X_1,P_1]-m_1^2[X_1,P_2]+m_2^2[X_2,P_1]-m_1m_2[X_2,P_2]\right)\\
&= 0
\end{aligned}$$

At last

$$\begin{aligned}
[\hat{X},P] &= [X_1-X_2,P_1+P_2]\\
&= [X_1,P_1]+[X_1,P_2]-[X_2,P_1]-[X_2,P_2]\\
&= i\hbar - i\hbar\\
&= 0
\end{aligned}$$

Part 2. *Write the hamiltonian*

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + V(X_1 - X_2) \tag{25}$$

in terms of the new operators.
Answer. We begin noting that $V(X_1 - X_2) = V(\hat{X})$. Moreover,

$$\begin{aligned}
\hat{P}^2 &= \frac{m_2^2 P_1^2 - m_1 m_2 P_1 P_2 - m_1 m_2 P_2 P_1 + m_1^2 P_2^2}{(m_1 + m_2)^2} \\
&= \frac{m_2^2 P_1^2 + m_1^2 P_2^2}{(m_1 + m_2)^2} - 2\frac{m_1 m_2 P_1 P_2}{(m_1 + m_2)^2}
\end{aligned}$$

Now let us add to this $m_1 m_2 P^2/(m_1 + m_2)^2$:

$$\begin{aligned}
\hat{P}^2 + \frac{m_1 m_2 P^2}{(m_1 + m_2)^2} &= \frac{1}{(m_1 + m_2)^2}\left(m_2^2 P_1^2 + m_1^2 P_2^2 + m_1 m_2 P_1^2 + m_1 m_2 P_2^2\right) \\
&= \frac{1}{(m_1 + m_2)^2}\left(m_2 P_1^2 (m_2 + m_1) + m_1 P_2^2 (m_1 + m_2)\right) \\
\frac{1}{2m_1 m_2}\left(\hat{P}^2 + \frac{m_1 m_2 P^2}{(m_1 + m_2)^2}\right) &= \frac{1}{(m_1 + m_2)}\left(\frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2}\right) \\
\frac{\hat{P}^2}{2\mu} + \frac{P^2}{2M} &= \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2}
\end{aligned}$$

Hence, putting everything together:

$$H = \frac{\hat{P}^2}{2\mu} + \frac{P^2}{2M} + V(\hat{X}) \tag{26}$$

As we only make use of the squares of the X's and P's, that is, of their norms, we can generalize this result to three dimensions without almost any extra effort whatsoever. The differences are that $P_1 P_2 \to \vec{P}_1 \cdot \vec{P}_2$ (still commute, as shown) and also $\hat{X} \to \vec{\hat{X}}$. Thus, we would have

$$H = \frac{\hat{P}^2}{2\mu} + \frac{P^2}{2M} + V(\vec{\hat{X}}) \tag{27}$$

Part 3. *The following example of an entangled state was used in the original article of Einstein, Podolsky, and Rosen (Section 6.2.1). The wave function of two particles is written as*

$$\psi(x_1, x_2; p_1, p_2) = \delta(x_1 - x_2 - L)\delta(p_1 + p_2) \tag{28}$$

where L is a constant length. Why is it possible to write such a wave function? What is its physical interpretation? Measurement of x_1 determines x_2, and measurement of p_1 determines p_2. Develop the analogy with the example of Section 6.3.1.
Answer. This wavefunction characterizes a pair of particles whose CM has a definite position and momentum. It is clear that because of this definiteness, measuring one position or one momentum gives away the value of the other. It is normalized, as must be. This wavefunction is the most "deterministic" allowed by the hamiltonian we derived in the last item.

Exercise 3. *So far we have considered Galilean boosts by a speed v (in one spatial dimension) at $t = 0$ and obtained, by demanding that the expectation values of the position $\hat{X}$, speed $d\hat{X}/dt$ and momentum $\hat{P}$ transform like their classical counter-parts, that the unitary operator $\hat{U}(v)$ that implements such boosts in a Hilbert space is given by (employing the active point of view of the boost)*

$$\hat{U}(v) = e^{imv\hat{X}/\hbar} \tag{29}$$

where m is the mass of the particle. Show that this generalizes for an arbitrary time t to

$$\hat{U}(v) = e^{imv\hat{X}/\hbar}e^{-itv\hat{P}/\hbar}e^{-imv^2t/2\hbar} = e^{-itv\hat{P}/\hbar}e^{imv\hat{X}/\hbar}e^{imv^2t/2\hbar} \tag{30}$$

Show also that the last result generalizes in three dimensions.

Answer. If we consider the galilean transformation at a time t, we must ensure that

$$\begin{aligned} U^\dagger \hat{X}_i U &= \hat{X}_i - \hat{V}_0 t \\ U^\dagger \hat{P}_i U &= \hat{P}_i - \hat{V}_0 m_i \end{aligned}$$

Let us assume

$$U = U_X U_P \tag{31}$$

with U_X depending only on $\hat{X}$ and U_P only on $\hat{P}$. Then, we can write

$$\begin{aligned} U_P^\dagger \hat{X}_i U_P &= \hat{X}_i - \hat{V}_0 t \\ U_X^\dagger \hat{P}_i U_X &= \hat{P}_i - \hat{V}_0 m_i \end{aligned}$$

from which follows that

$$\begin{aligned} [U_P, \hat{X}_i] &= \hat{V}_0 t U_P \\ \text{and} \quad [U_P, \hat{X}_i] &= \hat{V}_0 m_i U_X \end{aligned}$$

Using Schrödinger's equation:

$$\begin{aligned} -i\hbar \frac{\partial U_p}{\partial p_{ik}} &= v_{0k} t U_P \\ i\hbar \frac{\partial U_x}{\partial x_{ik}} &= v_{0k} m_1 U_x \end{aligned}$$

We can easily integrate this equations to give:

$$\begin{aligned} U_p &= c_p e^{p\hat{P}\cdot\vec{v}_0 t/\hbar} \\ U_x &= c_x e^{-(i/\hbar)\sum m_i \hat{x}_i \cdot \vec{v}_0} \end{aligned}$$

However, $|c_p| = |c_x| = 1|$ (by the unitarity). Thus,

$$U = c(t, \vec{v}_0) e^{p\hat{P}\cdot\vec{v}_0 t/\hbar} e^{iM\hat{X}\cdot\vec{v}_0/\hbar} \tag{32}$$

where $\hat{X}$ describes the position of the center of mass. Now we use the identity

$$e^{A+B} = e^{-[A,B]/2} e^A e^B \tag{33}$$

And using the fact that $e^{A+B} = e^{B+A}$, we can write

$$e^A e^B = e^{[A,B]} e^B e^A \tag{34}$$

Using this in our case is

$$e^{p\hat{P}\cdot\vec{v}_0 t/\hbar} e^{iM\hat{X}\cdot\vec{v}_0/\hbar} = e^{iMv_0^2 t/\hbar} e^{i\hat{P}\cdot\vec{v}_0 t/\hbar} e^{-iM\hat{X}\cdot\vec{v}_0} \tag{35}$$

Thus, in three dimensions, we can write:

$$\hat{U}(v) = e^{i\hat{P}\cdot\vec{v}_0 t/\hbar} e^{-iM\hat{X}\cdot\vec{v}_0} e^{iMv_0^2 t/\hbar} \tag{36}$$

Exercise 4. *The Galilean boosts, a.k.a. pure Galilean transformations, form a subgroup of a larger, 10-dimensional group named Galilei (or Galileo) group of space-time transformations:*

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}' = R\vec{x} + \vec{a} + \vec{v}t \\ t &\rightarrow t' = t + s \end{aligned}$$

where in the addition to the displacement $\vec{a}$ and boost velocity $\vec{v}$ studied so far, one also has a spatial rotation R and time displacement s. Let $g = (R, \vec{a}, \vec{v}, s)$ denote such a transformation. Show that the composition law for $g_3 = g_2 g_1$, with $g_3 = (R_3, a_3, v_3, s_3)$ is:

$$\begin{aligned} R_3 &= R_2 R_1 \\ \vec{a}_3 &= \vec{a}_2 + R\vec{a}_1 + \vec{v}_2 s_1 \\ \vec{v}_3 &= \vec{v}_2 + R_2\vec{v}_1 \\ s_3 &= s_2 + s_1 \end{aligned}$$

Answer. If we apply g_3 to the pair $\{\vec{v}, t\}$, we get:

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}'' = R_3\vec{x} + \vec{a}_3 + \vec{v}_3 t \\ t &\rightarrow t'' = t + s_3 \end{aligned}$$

Now let us apply g_1 to the initial pair, so that later we may apply g_2 as well:

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}' = R_1\vec{x} + \vec{a}_1 + \vec{v}_1 t \\ t &\rightarrow t' = t + s_1 \end{aligned}$$

Applying g_2 here:

$$\begin{aligned} \vec{x}' &\rightarrow \vec{x}'' = R_2(R_1\vec{x} + \vec{a}_1 + \vec{v}_1 t) + \vec{a}_2 + \vec{v}_2 t' \\ t' &\rightarrow t'' = (t + s_1) + s_2 = t + (s_2 + s_1) \end{aligned}$$

Rearrenging terms in the first of these equations

$$\begin{aligned} \vec{x}'' &= R_2R_1\vec{x} + R_2\vec{a}_1 + \vec{a}_2 + R_2\vec{v}_1 t + \vec{v}_2 t + \vec{v}_2 s_1 \\ &= (R_2R_1)\vec{x} + (R_2\vec{a}_1 + \vec{a}_2 + \vec{v}_2 s_1) + (R_2\vec{v}_1 + \vec{v}_2)t \end{aligned}$$

Equating this to the transformation of g_3, we get:

$$\begin{aligned} R_3 &= R_2 R_1 \\ \vec{a}_3 &= R_2\vec{a}_1 + \vec{a}_2 + \vec{v}_2 s_1 \\ \vec{v}_3 &= R_2\vec{v}_1 + \vec{v}_2 \\ s_3 &= s_2 + s_1 \end{aligned}$$

Exercise 5 (10.6.1 - Properties of $\vec{J}$)**.** *Show by explicit calculation that* $[J^2, J_z] = 0$. *Also verify the identities (10.5) to (10.9).*

Answer. Let us begin proving the commutation relation. We can write $J^2 = J_x^2 + J_y^2 + J_z^2$. It is easy to see that:

$$\begin{aligned} [J^2, J_z] &= [J_x^2, J_z] + [J_y^2, J_z] + [J_z^2, J_z] \\ &= J_x[J_x, J_z] + [J_x, J_z]J_x + J_y[J_y, J_z] + [J_y, J_z]J_y \\ &= J_x(-i\hbar J_y) + (-i\hbar J_y)J_x + J_y(i\hbar J_x) + (i\hbar J_x)J_y \\ &= 0 \end{aligned}$$

The next property is $[J_z, J_\pm] = \pm J_\pm$:

$$\begin{aligned} [J_z, J_\pm] &= [J_z, J_+ \pm iJ_y] \\ &= [J_z, J_x] \pm i[J_z, J_y] \\ &= iJ_y \pm (-i^2)J_x \\ &= \pm(J_x \pm iJ_y) \\ &= \pm J_\pm \end{aligned}$$

The next one is $[J_+, J_-] = 2J_z$:

$$\begin{aligned} [J_+, J_-] &= [J_x + iJ_y, J_x - iJ_y] \\ &= [J_x, J_x] - i[J_x, J_y] + i[J_y, J_x] + [J_y, J_y] \\ &= -i(iJ_z) + i(-iJ_z) \\ &= 2J_z \end{aligned}$$

The next one is $J^2 = \frac{1}{2}(J_-J_+ + J_+J_-) + J_z^2$:

$$\begin{aligned} \frac{1}{2}(J_-J_+ + J_+J_-) + J_z^2 &= \frac{1}{2}\Big((J_x - iJ_y)(J_x + iJ_y) + (J_x + iJ_y)(J_x - iJ_y)\Big) + J_z^2 \\ &= \frac{1}{2}\Big((J_x^2 + iJ_xJ_y - iJ_yJ_x + J_y^2) + (J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2)\Big) + J_z^2 \\ &= J_x^2 + J_y^2 + J_z^2 \\ &= J^2 \end{aligned}$$

The next one is $J_+J_- = J^2 - J_z(J_z - 1)$:

$$\begin{aligned} J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2 \\ &= J^2 - J_z^2 - i[J_x, J_y] \\ &= J^2 - J_z^2 + J_z \\ &= J^2 - J_z(J_z - 1) \end{aligned}$$

The next one is $J_+J_- = J^2 - J_z(J_z+1)$:

$$\begin{aligned}
J_+J_- &= (J_x - iJ_y)(J_x + iJ_y) \\
&= J_x^2 + iJ_xJ_y - iJ_yJ_x + J_y^2 \\
&= J^2 - J_z^2 + i[J_x, J_y] \\
&= J^2 - J_z^2 - J_z \\
&= J^2 - J_z(J_z+1)
\end{aligned}$$

Exercise 6 (10.7.2 - Rotation of angular momentum)**.** *Let $\mathcal{R}$ be a rotation (10.30) by angles (θ, ϕ). Show that the vector*

$$U(\mathcal{R})\,|jm\rangle = e^{-i\phi J_z} e^{-i\theta J_y}\,|jm\rangle \tag{37}$$

is an eigenvector of the operator

$$J_x \sin\theta\cos\phi + J_y \sin\theta\sin\phi + J_z\cos\theta = \vec{J}\cdot\hat{n} \tag{38}$$

with eigenvalue m. Here $\hat{n}$ is the unit vector in the direction (θ, ϕ). Hint: adapt (8.29).

Answer. First of all, we expect $U(\mathcal{R})\,|jm\rangle$ to be an eigenvector of $\vec{J}\cdot\hat{n}$ because the first term is the ket $|jm\rangle$ along the (θ,ϕ) direction and the second is the momentum operator projected in the $\hat{n}$ direction, however we see that $\hat{n}$ points along the (θ,ϕ) direction. As $|jm\rangle$ is an eigenket of $\vec{J}$, it is natural to expect their projections along the same axis to also be a pair of eigenket and operator as well. Let us show this mathematically.

Let α be an auxiliary, small angle. Then a rotation about $\hat{n}$ of an angle α is

$$R_{\hat{n}}(\alpha) = e^{-i\alpha\vec{J}\cdot\hat{n}} \tag{39}$$

Or, as a product of rotations (decomposing):

$$R_{\hat{n}(\theta,\phi)}(\alpha) = R_z(\phi)R_y(\theta)R_z(\alpha)R_y(-\theta)R_z(-\phi) \tag{40}$$

Using the generators (exponential forms) and letting $\alpha \to 0$, we can show that

$$\vec{J}\cdot\hat{n}(\theta,\phi) = e^{-i\phi J_z}e^{-i\theta J_y}J_z e^{i\theta J_y}e^{i\phi J_z} \tag{41}$$

Let us show this more explicitly:

$$\begin{aligned}
\vec{J}\cdot\hat{n} &= J_x\sin\theta\cos\phi + J_y\sin\theta\sin\phi + J_z\cos\theta \\
&= (J_x\cos\phi + J_y\sin\phi)\sin\theta + J_z\cos\theta \\
&= e^{-i\phi J_z}J_x e^{i\phi J_z}\sin\theta + J_z\cos\theta\left(e^{-i\phi J_z}e^{i\phi J_z}\right) \\
&= e^{-i\phi J_z}\left(J_x\sin\theta + J_z\cos\theta\right)e^{i\phi J_z} \\
&= e^{-i\phi J_z}e^{-i\theta J_y\theta}J_z e^{i\theta J_y}e^{i\phi J_z}
\end{aligned}$$

where we used the fact that J_z commutes with any $f(J_z)$. Now, applying this form of the angular momentum rotated operator onto our rotated vector, we get

$$\begin{aligned}
\vec{J}\cdot\hat{n}(\theta,\phi)\mathcal{R}(\theta,\phi)\,|jm\rangle &= \left(e^{-i\phi J_z}e^{-i\theta J_y}J_z e^{i\theta J_y}e^{i\phi J_z}\right)e^{-i\phi J_z}e^{-i\theta J_y}\,|jm\rangle \\
&= e^{-i\phi J_z}e^{-i\theta J_y}J_z\,|jm\rangle \\
&= me^{-i\phi J_z}e^{-i\theta J_y}\,|jm\rangle \\
&= m\mathcal{R}(\theta,\phi)\,|jm\rangle
\end{aligned}$$

as we wanted to show.

Exercise 7 (10.7.4 - The angular momenta $j = 1/2$ and $j = 1$.)***. Part 1.*** *Use (10.23) to find the operators S_x, S_y, and S_z for spin 1/2.*
Answer. Let us calculate the values. As $j = 1/2$, $m = \pm 1/2$. Then,

$$\begin{aligned}
\left\langle \frac{1}{2} \middle| J_\pm \middle| \frac{1}{2} \right\rangle &= \sqrt{\frac{3}{4} - \frac{1}{4}}\delta_{1/2,1/2\pm 1} = 0 \\
\left\langle \frac{1}{2} \middle| J_- \middle| -\frac{1}{2} \right\rangle &= 0 \\
\left\langle \frac{1}{2} \middle| J_+ \middle| -\frac{1}{2} \right\rangle &= \sqrt{\frac{3}{4} + \frac{1}{4}} = \frac{1}{2} \\
\left\langle -\frac{1}{2} \middle| J_- \middle| +\frac{1}{2} \right\rangle &= \sqrt{\frac{3}{4} + \frac{1}{4}} = \frac{1}{2} \\
\left\langle -\frac{1}{2} \middle| J_+ \middle| +\frac{1}{2} \right\rangle &= 0 \\
\left\langle -\frac{1}{2} \middle| J_\pm \middle| -\frac{1}{2} \right\rangle &= 0
\end{aligned}$$

Here we remember that

$$J_\pm = J_x \pm iJ_y \tag{42}$$

which means that

$$J_x = \frac{J_+ + J_-}{2}, \quad J_y = \frac{J_+ - J_y}{2i} \tag{43}$$

Now, using the order $\{\left|\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\right\rangle\}$ we have:

$$J_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{44}$$

and

$$J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{45}$$

Thus, we see that $J_x = \sigma_x/2$ and $J_y = \sigma_y/2$. Using (10.22), we easily see that $J_z = \sigma_z/2$.

Part 2. *Again using (10.23), calculate the 3×3 matrix representations of J_x, J_y, and J_z for angular momentum $j = 1$.*

Answer. In this case as $j = 1$, we have $m = -1, 0, 1$. We have the mean values:

$$\begin{aligned}
\langle 1, J_\pm, 1\rangle &= 0, & \langle -1, J_\pm, -1\rangle &= 0\\
\langle 1, J_+, 0\rangle &= \sqrt{2}, & \langle 1, J_-, 0\rangle &= 0\\
\langle 1, J_+, -1\rangle &= 0, & \langle 1, J_-, -1\rangle &= 0\\
\langle 0, J_+, 1\rangle &= 0, & \langle 0, J_-, 1\rangle &= \sqrt{2}\\
\langle 0, J_+, 0\rangle &= 0, & \langle 0, J_-, 0\rangle &= 0\\
\langle 0, J_+, -1\rangle &= \sqrt{2}, & \langle 0, J_-, -1\rangle &= 0\\
\langle -1, J_+, 1\rangle &= 0, & \langle -1, J_-, 1\rangle &= 0\\
\langle -1, J_+, 0\rangle &= 0, & \langle -1, J_-, 0\rangle &= \sqrt{2}
\end{aligned}$$

Thus, using the $\{1, 0, -1\}$ ordering

$$J_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix} \tag{46}$$

And

$$J_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix} \tag{47}$$

Again using

$$J_x = \frac{J_+ + J_-}{2}, \quad J_y = \frac{J_+ - J_y}{2i} \tag{48}$$

we have

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0 \end{pmatrix}$$

And, finally, J_z from (10.22):

$$= \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1 \end{pmatrix}$$

Part 3. *Show that for $j = 1$, J_x, J_y, and J_z are related to the infinitesimal generators (8.26) T_x, T_y, and T_z by a unitary transformation which takes the Cartesian components of $\hat{r}$ to the spherical components (10.64): $J_i = U^\dagger T_i U$ with*

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1\\ -i & 0 & -i\\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{49}$$

Answer. This exercises boils down to a simple calculation and matrix multiplication:

$$\begin{aligned}
J_x &= \frac{1}{2}\begin{pmatrix}-1 & i & 0\\ 0 & 0 & \sqrt{2}\\ 1 & i & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}-1 & 0 & 1\\ -i & 0 & -i\\ 0 & \sqrt{2} & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & \sqrt{2} & 0\\ \sqrt{2} & 0 & \sqrt{2}\\ 0 & \sqrt{2} & 0\end{pmatrix} = J_x\\
J_y &= \frac{1}{2}\begin{pmatrix}-1 & i & 0\\ 0 & 0 & \sqrt{2}\\ 1 & i & 0\end{pmatrix}\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix}\begin{pmatrix}-1 & 0 & 1\\ -i & 0 & -i\\ 0 & \sqrt{2} & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & -i\sqrt{2} & 0\\ -i\sqrt{2} & 0 & i\sqrt{2}\\ 0 & i\sqrt{2} & 0\end{pmatrix} = J_y\\
J_z &= \frac{1}{2}\begin{pmatrix}-1 & i & 0\\ 0 & 0 & \sqrt{2}\\ 1 & i & 0\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}-1 & 0 & 1\\ -i & 0 & -i\\ 0 & \sqrt{2} & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -2\end{pmatrix} = J_z
\end{aligned}$$

Part 4. *Calculate the rotation matrix $d^{(1)}(\theta)$:*

$$d^{(1)}(\theta) = e^{-i\theta J_y} \tag{50}$$

and verify (10.39).
Answer. We first begin noting that

$$J_y^3 = -\frac{i}{2\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}^3 = -\frac{i}{2\sqrt{2}}\begin{pmatrix}0 & 2 & 0\\ -2 & 0 & 2\\ 0 & -2 & 0\end{pmatrix} = J_y \tag{51}$$

Thus, when we taylor expand the exponential:

$$\begin{aligned}
e^{-i\theta J_y} &= 1 - i\theta J_y + \frac{\theta^2}{2}J_y^2 - i\frac{\theta^3}{3!}J_y^3 + \frac{\theta^4}{4!}J_y^4 + \cdots\\
&= 1 + \Big(\frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\Big)J_y^2 - i\Big(\theta + \frac{\theta^3}{3!} + \cdots\Big)J_y\\
&= 1 - (1-\cos\theta)J_y^2 - i\sin\theta J_y
\end{aligned}$$

where 1 is the identity. Plugging the matrices, we easily see that

$$d^{(1)}(\theta) = \begin{pmatrix}\frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta)\\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta\\ \frac{1}{2}(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta)\end{pmatrix} \tag{52}$$
