Mecânica Quântica Avançada
Lista 2

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IFT

List of Exercises

Exercise 1 (6.5.1 - Independence of the tensor product from the choice of basis)**.** *Verify that the definition (6.3) of the tensor product of two vectors is independent of the choice of basis in* $\mathcal{H}_1$ *and* $\mathcal{H}_2$.

Answer. Let $|n'\rangle$ and $|m'\rangle$ be two other basis of the Hilbert spaces one and two, respectively. Then, it is true that

$$\begin{aligned} |n\rangle &= \sum a_{n'} |n'\rangle \\ |m\rangle &= \sum b_{m'} |m'\rangle \end{aligned} \tag{1}$$

Thus, we can write

$$\begin{aligned} |\varphi\rangle \otimes |\chi\rangle &= \sum_{n,m} c_n d_m |n\rangle \otimes |m\rangle \\ &= \sum_{n,m} c_n d_m \left(\sum_{n'} a_{n'} |n'\rangle \right) \otimes \left(\sum_{m'} b_{m'} |m'\rangle \right) \\ &= \sum_{n',m'} a_{n'} b_{m'} \left(\sum_n c_n \right) \left(\sum_m d_m \right) |n'\rangle \otimes |m'\rangle \\ &= \sum_{n',m'} e_{n'} f_{m'} |n'\rangle \otimes |m'\rangle \end{aligned}$$

This shows that the tensor product does not depend on the choice of basis.

Exercise 2 (2 - Representação matricial de produtos tensoriais)**.** *Calculate the tensor products of two-level systems.*

Answer. We can calculate the tensor products of $|+\rangle$ and $|-\rangle$ as follows:

$$\begin{aligned}
|+\rangle \otimes |-\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
|-\rangle \otimes |+\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
|-\rangle \otimes |-\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Now for the three dimension qubits, we will write the answers directly:

$$|{+}{+}{+}\rangle = \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \qquad |{+}{+}{-}\rangle = \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$|{+}{-}{+}\rangle = \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \qquad |{+}{-}{-}\rangle = \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}$$

$$|{-}{+}{+}\rangle = \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \qquad |{-}{+}{-}\rangle = \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}$$

$$|{-}{-}{+}\rangle = \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}, \qquad |{-}{-}{-}\rangle = \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

Exercise 3 (6.5.2)**.** *Write down explicitly the 4×4 matrix $A \otimes B$, the tensor product of the 2×2 matrices A and B:*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2}$$

Answer. It is very easy do perform this calculation:

$$A \otimes B = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix}. \tag{3}$$

We just multiply each element of the first matrix by the whole second matrix.

Exercise 4 (6.5.3 - Properties of state operators). ***Part 1.*** *Show that* $\rho_{ii} \geq 0$, $\rho_{jj} \geq 0$, *and* $\det A \geq 0$, *from which* $|\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj}$. *Also deduce that if* $\rho_{ii} = 0$, *then* $\rho_{ij} = \rho^*_{ji} = 0$.
Answer. We can always write

$$\rho = \sum a_n |\phi_n\rangle \langle\phi_n| \tag{4}$$

for some states $|\phi_n\rangle$ and $a_n \geq 0$. Thus, the diagonal matrix elements are

$$\begin{aligned} \rho_{ii} &= \langle\phi_i| \left(\sum a_n |\phi_n\rangle \langle\phi_n|\right) |\phi_i\rangle \\ &= a_i \end{aligned}$$

Hence, $\rho_{ii} \geq 0$. We also note that

$$\det A = \rho_{ii}\rho_{jj} - |\rho_{ij}|^2 \tag{5}$$

where we used the fact that A is hermitian. This implies that

$$\det A \geq 0 \iff |\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj} \tag{6}$$

Using this inequality, if $\rho_{ii} = 0$, then

$$0 \leq \rho_{ij}\rho_{ji} \leq 0 \tag{7}$$

Obviously, $\rho_{ij} = 0$ or $\rho_{ji} = 0$, but is does not matter, because they are the complex conjugate of each other, so if one is zero, the other is zero as well.

Part 2. *Show that if there exists a maximal test giving 100% probability for the quantum state described by a state operator* ρ, *then this state is a pure state. Also show that if* ρ *describes a pure state, and if it can be written as*

$$\rho = \lambda\rho' + (1-\lambda)\rho'', 0 \leq \lambda \leq 1 \tag{8}$$

then $\rho = \rho' = \rho''$. *Hint: first demonstrate that if* ρ' *and* ρ'' *are generic state operators, then* ρ *is a state operator. The state operators form a convex subset of Hermitian operators.*
Answer. Let us suppose the state in which there is a probability of 1 is $|\psi\rangle$ and let $P_\psi = |\psi\rangle\langle\psi|$. Then having a probability of one means:

$$\mathrm{tr}\left(P_\psi \rho\right) = 1 \tag{9}$$

In general, we could write

$$\rho = \sum_i c_i |\psi_i\rangle \langle\psi_i| \tag{10}$$

where we define $\psi_0 \equiv \psi$. Then, if we use this definition into the expression above, we can easily see that both equations together imply

$$c_0 = 1 \tag{11}$$

However, as $\sum_i |c_i|^2 = 1$, we must have $c_i = 0 \ \forall \ i > 0$. Hence, $\rho = |\psi\rangle\langle\psi|$ is a pure state. We will suppose that $\rho \neq \rho' \neq \rho''$. If that is the case, then the state ρ can be described as a statistical mixture of ρ' (with $p = \lambda$) and ρ'' (with $p = 1 - \lambda$). As we know ρ is a pure state, it is false the assertion that $\rho \neq \rho' \neq \rho''$. Thus, we prove the result asked.

Exercise 5 (4)**.** *Ao ter provado o item 2 do exercício 6.5.3 do Le Bellac, você provou que um operador de estado ρ correspondente a um estado puro não pode ser escrito como combinação linear de dois outros operadores de estado genéricos ρ_1 e ρ_2. Esse resultado tem a ver com a seguinte observação: "A preparação de um estado puro é única enquanto que a preparação de um estado misto é sempre ambígua". Você poderia explicar o que uma coisa tem a ver com a outra, isto é, o que o resultado que você provou tem a ver com a preparação de um estado físico e possíveis resultados de medida?*

Answer. Dado um operador de estado puro, existe uma e apenas uma maneira de prepará-lo, que é coloca-lo no estado $|\psi\rangle\langle\psi|$. No entanto, para um estado misto, para qualquer $\lambda \in \mathbb{R}$, sempre existirão operadores ρ_1, ρ_2 tais que ρ seja uma combinação linear deles ponderada por λ, de modo que é impossível saber exatamente qual foi usada na preparação do estado.

Exercise 6 (5)**.** *Considere a seguinte matriz densidade de um spin 1/2:*

$$\rho = \frac{1}{4}\mathbb{I} + \frac{1}{2}|+,\hat{a}\rangle\langle +,\hat{a}| \tag{12}$$

onde $|+,\hat{a}\rangle$ é o autoestado da projeção do operador de spin ao longo de um eixo a com autovalor $+\hbar/2$. Calcule a probabilidade como função de θ de encontrar o valor $-\hbar/2$ ao se medir o spin ao longo de um eixo b, em que θ é o ângulo entre a e b, i.e. $\hat{a}\cdot\hat{b} = \cos\theta$.

Answer. Queremos calcular:

$$\begin{aligned} p(\hbar/2,\hat{b}) &= \operatorname{tr}\Big(\rho\,|+,\hat{b}\rangle\langle +,\hat{b}|\Big) \\ &= \frac{|\langle +,\hat{b}|+,\hat{b}\rangle|^2}{4} + \frac{|\langle +,\hat{a}|+,\hat{b}\rangle|^2}{2} \end{aligned}$$

Precisamos, então, considerar o termo $\langle +,\hat{a}|-,\hat{b}\rangle$. Pela Fig(1), vemos que o produto interno dos

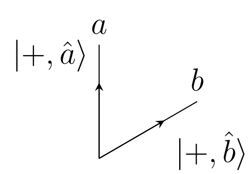

Figure 1: The vectors $\hat{a}$ and $\hat{b}$.

vetores pode ser escrito como $\cos\theta$, pois têm norma unitária. Portanto,

$$p(+\hbar/2,\hat{b}) = \frac{1}{4} + \frac{\cos^2\theta}{2} \tag{13}$$

Como precisa ser que $p(+\hbar/2,\hat{b}) + p(-\hbar/,\hat{b}) = 1$, então vale que

$$p(-\hbar/2,\hat{b}) = \frac{1}{4} + \frac{\sin^2\theta}{2} \tag{14}$$

Exercise 7 (6). *Considere a seguinte matriz densidade de dois spins 1/2:*

$$\rho = \frac{\mathbb{I}}{8} + \frac{1}{2}\left|\Psi_-\right\rangle\left\langle\Psi_-\right| \tag{15}$$

onde $|\Psi_-\rangle$ *é o estado singleto (i.e. o estado de spin total igual a zero). Suponhamos que medimos um dos spins ao longo de um eixo* a *e o outro ao longo de um eixo* b*, em que* $\hat{a}\cdot\hat{b} = \cos\theta$*. Qual é a probabilidade (como função de* θ*) de encontramos* $+\hbar/2$ *para ambos spins nestas medidas?*

Answer. O estado de singleto é

$$|\Psi_-\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \tag{16}$$

Portanto,

$$\rho = \frac{1}{8}\mathbb{I} + \frac{1}{4}\left(|+-\rangle\langle+-| - |+-\rangle\langle-+| - |-+\rangle\langle+-| + |-+\rangle\langle-+|\right) \tag{17}$$

O estado que queremos medir é

$$|\phi\rangle = |+\rangle_a |+\rangle_b \tag{18}$$

onde o índice significa o eixo em relação ao qual estamos considerando. Sejam θ_a e θ_b os ângulos que os eixos a e b fazem com o eixo z. Então,

$$\begin{aligned}
|+\rangle_a &= \cos(\theta_a/2)|+\rangle + \sin(\theta_a/2)|-\rangle \\
|+\rangle_b &= \cos(\theta_b/2)|+\rangle + \sin(\theta_b/2)|-\rangle
\end{aligned}$$

O projetor que projeta no estado desejado é $P_\phi = |\phi\rangle\langle\phi|$. Ou seja,

$$\begin{aligned}
|\phi\rangle &= \left(\cos(\theta_a/2)|+\rangle + \sin(\theta_a/2)|-\rangle\right)\left(\cos(\theta_b/2)|+\rangle + \sin(\theta_b/2)|-\rangle\right) \\
&= \cos\frac{\theta_a}{2}\cos\frac{\theta_b}{2}|+\rangle|+\rangle + \cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}|+\rangle|-\rangle + \sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}|-\rangle|+\rangle + \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}|-\rangle|-\rangle
\end{aligned}$$

Portanto, o operador é

$$\begin{aligned}
&P_\phi = |\phi\rangle\langle\phi| \\
&= \left(\cos\frac{\theta_a}{2}\cos\frac{\theta_b}{2}|+\rangle|+\rangle + \cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}|+\rangle|-\rangle + \sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}|-\rangle|+\rangle + \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}|-\rangle|-\rangle\right) \\
&\times\left(\cos\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\langle+|\langle+| + \cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\langle+|\langle-| + \sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\langle-|\langle+| + \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\langle-|\langle-|\right) \\
&= \cos^2\frac{\theta_a}{2}\cos^2\frac{\theta_b}{2}|+\rangle|+\rangle\langle+|\langle+| + \sin^2\frac{\theta_a}{2}\sin^2\frac{\theta_b}{2}|-\rangle|-\rangle\langle-|\langle-| + \\
&\quad\cos^2\frac{\theta_a}{2}\sin^2\frac{\theta_b}{2}|+\rangle|-\rangle\langle+|\langle-| + \sin^2\frac{\theta_a}{2}\cos^2\frac{\theta_b}{2}|-\rangle|+\rangle\langle-|\langle+| + \\
&\quad\cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}|+\rangle|-\rangle\langle-|\langle+| + \cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}|-\rangle|+\rangle\langle+|\langle-| + \cdots
\end{aligned}$$

Os termos omitidos são aqueles que não aparecem em ρ, de modo que serão zero ao se tomar o traço de $P_\phi\rho$. Vamos, então, calcular a probabilidade

$$\begin{aligned}
p_{++} &= \operatorname{tr}\left(P_\phi\rho\right) \\
&= \frac{1}{8}\left(\cos^2\frac{\theta_a}{2}\cos^2\frac{\theta_b}{2}+\sin^2\frac{\theta_a}{2}\sin^2\frac{\theta_b}{2}+\cos^2\frac{\theta_a}{2}\sin^2\frac{\theta_b}{2}+\sin^2\frac{\theta_a}{2}\cos^2\frac{\theta_b}{2}\right)+ \\
&\quad \frac{1}{4}\left(\cos^2\frac{\theta_a}{2}\sin^2\frac{\theta_b}{2}-\sin\frac{\theta_a}{2}\cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\cos\frac{\theta_b}{2}-\sin\frac{\theta_a}{2}\cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\cos\frac{\theta_b}{2}+\sin^2\frac{\theta_a}{2}\cos^2\frac{\theta_b}{2}\right) \\
&= \frac{1}{8}+\frac{1}{4}\left(\cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}-\sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\right)^2 \\
&= \frac{1}{8}+\frac{1}{4}\sin^2\left(\frac{\theta_b-\theta_a}{2}\right)
\end{aligned}$$

Ora, é fácil ver que $\theta = |\theta_b - \theta_a|$. Como o seno ao quadrado é uma função par, $\sin^2\theta = \sin^2(-\theta)$. Portanto, podemos escrever

$$p_{++} = \frac{1}{8}+\frac{1}{4}\sin^2\left(\frac{\theta}{2}\right) \tag{19}$$

Exercise 8 (7)**.** *Considere um sistema bipartite descrito por um operador de estado ρ^{AB} que evolui unitariamente:*

$$i\hbar\frac{d\rho^{AB}}{dt} = [H_{AB}, \rho_{AB}] \tag{20}$$

com $H_{AB} = H_A + H_B + V_{AB}$ onde H_A depende somente das coordenadas do subsistema A, H_B depende somente das coordenadas do subsistema B e V_{AB} depende das coordenadas de ambos subsistemas. Mostre que o operador de densidade reduzido do sistema A, i.e. $\rho^A = \operatorname{tr}_B(\rho^{AB})$, obedece à seguinte equação de evolução temporal:

$$i\hbar\frac{d\rho^{A}}{dt} = \left[H_A, \rho^A\right] + \operatorname{tr}_B\left[V_{AB}, \rho^{AB}\right] \tag{21}$$

Você acabou de mostrar que enquanto o sistema bipartite evolui unitariamente, o subsistema A não evolui unitariamente em geral. No curso de Física Estatística você, muito provavelmente, vai provar esse resultado novamente.

Answer. Vamos aplicar o traço em B na Eq.(20) e deduzir a expressão que queremos. Obviamente, o lado esquerdo trivialmente dá a expressão que queremos, então focaremos no lado direito. Considere

$$\begin{aligned}
\operatorname{tr}_B\left[H_{AB}, \rho_{AB}\right] &= \operatorname{tr}_B(H_{AB}\rho^{AB}) - \operatorname{tr}_B(\rho^{AB}H_{AB}) \\
&= \operatorname{tr}_B\left(H_A\rho^{AB}+H_B\rho^{AB}+V_{AB}\rho^{AB}\right) - \operatorname{tr}_B\left(\rho^{AB}H_A+\rho^{AB}H_B\rho^{AB}V_{AB}\right) \\
&= \left(H_A\operatorname{tr}_B(\rho^{AB}) - \operatorname{tr}_B(\rho^{AB})H_A\right) + \operatorname{tr}_B\left(V_{AB}\rho^{AB}-\rho^{AB}V_{AB}\right) + \left(\operatorname{tr}_B(H_B\rho^{AB}) - \operatorname{tr}_B(\rho^{AB}H_B)\right) \\
&= [H_A, \rho_A] + \operatorname{tr}_B[V_{AB}, \rho^{AB}]
\end{aligned}$$

Portanto, juntando as duas pontas:

$$i\hbar\frac{d\rho^{A}}{dt} = \left[H_A, \rho^A\right] + \operatorname{tr}_B\left[V_{AB}, \rho^{AB}\right] \tag{22}$$

Exercise 9 (8)**.** *Mostre que sob evolução unitária (ou hamiltoniana, i.e. quando o operador densidade evolui de acordo com a Eq. (6.37) do Le Bellac) a entropia de emaranhamento é conservada no tempo.*

Answer. Podemos escrever a evolução unitária como

$$\rho(t) = U(t)\rho(0)U^*(t) \tag{23}$$

Primeiramente, notamos que

$$\begin{aligned} \ln(\rho(t)) &= \ln(U(t)\rho(0)U^*(t)) \\ &= \ln U(t) + \ln \rho_0 + \ln U^*(t) \\ &= \ln U(t) + \ln \rho_0 + -\ln U(t) \\ &= \ln \rho_0 \end{aligned}$$

Em seguida,

$$\begin{aligned} S &= -k_b \operatorname{tr}\Big(U(t)\rho_0 U^*(t) \ln \rho_0\Big) \\ &= -k_b \operatorname{tr}\big(\rho(0) \ln \rho(0)\big) \end{aligned}$$

que depende apenas do estado inicial.

Exercise 10 (9)**.** *Considere os estados de Bell.*

$$\begin{aligned} |\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle) \\ |\Phi_-\rangle &= \frac{1}{\sqrt{2}}(|{+}{+}\rangle - |{-}{-}\rangle) \\ |\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle) \\ |\Psi_-\rangle &= \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle) \end{aligned}$$

Part 1. *Escreva os estados de Bell e as correspondentes matrizes densidade na forma matricial na representação definida pela Eq. (1).*

Answer. Pelo Ex.(1), já sabemos as formas matriciais de cada um dos elementos da base. Basta realizar as somas/subtrações correspondentes.

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \qquad |\Phi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \qquad |\Psi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

Já as matrizes de densidade são apenas os produtos $\rho = |\psi\rangle \langle\psi|$.

$$\begin{aligned}
\rho(\Phi_+) &= |\Phi_+\rangle \langle\Phi_+| = \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{pmatrix}\\
\rho(\Phi_-) &= |\Phi_-\rangle \langle\Phi_-| = \frac{1}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & -1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\-1 & 0 & 0 & 1\end{pmatrix}\\
\rho(\Psi_+) &= |\Psi_+\rangle \langle\Psi_+| = \frac{1}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix}\\
\rho(\Psi_-) &= |\Psi_-\rangle \langle\Psi_-| = \frac{1}{2}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix}
\end{aligned}$$

É fácil ver que todas possuem traço 1, como deveriam.

Part 2. *Mostre que estes estados são maximamente emaranhados (ou desordenados), isto é, as entropias de emaranhamento correspondentes aos spins individuais assumem o valor máximo* $\ln 2$.
Answer. Vamos calcular a entropia do traço parcial. Primeiramente, para o B

$$\begin{aligned}
\rho_B(\Phi_+) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\rho_B(\Phi_-) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\rho_B(\Psi_+) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\rho_B(\Psi_-) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}
\end{aligned}$$

E para o A

$$\begin{aligned}
\rho_A(\Phi_+) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\rho_A(\Phi_-) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\rho_A(\Psi_+) &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\rho_A(\Psi_-) &= \frac{1}{2}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}
\end{aligned}$$

Então todos são proporcionais a $\pm I/2$. Ora, a entropia é

$$S_j = -k_b \operatorname{tr} \begin{pmatrix} (1/2)\ln(1/2) & 0 \\ 0 & (1/2)\ln(1/2) \end{pmatrix} = k_b \ln 2 \tag{24}$$

Exercise 11 (10)**.** *Considere o seguinte vetor de estado de dois spins 1/2 :*

$$|\Psi(1,2)\rangle = \cos\theta\,|+-\rangle - \sin\theta\,|-+\rangle \tag{25}$$

onde $0 \le \theta \le \pi/2$*. Obtenha a matriz densidade reduzida de um dos spins e calcule a correspondente entropia de emaranhamento. Para que valor de* θ *essa entropia é máxima?*

Answer. A matriz de densidade é:

$$\begin{aligned} \rho &= |\Psi(1,2)\rangle\langle\Psi(1,2)| \\ &= \cos^2\theta\,|+-\rangle\langle+-| - \sin\theta\cos\theta\,|+-\rangle\langle-+| - \sin\theta\cos\theta\,|-+\rangle\langle+-| + \sin^2\theta\,|-+\rangle\langle-+| \end{aligned}$$

Isso gera uma matriz 4×4. Vamos considerar a seguinte ordenação dos vetores de base $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta & 0 \\ 0 & -\sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

Obviamente, o emaranhamento é máximo para $\theta = \pi/4$, pois essa escolha não favorece nenhum estado e gera uma matriz de Bell ($\rho(\Psi_-)$, em particular). Tomando o traço parcial:

$$\rho_B = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{27}$$

Thus,

$$S_B = -k_b \operatorname{tr} \begin{pmatrix} \sin^2\theta\ln\sin^2\theta & 0 \\ 0 & \cos^2\theta\ln\cos^2\theta \end{pmatrix} = -2k_b(\sin^2\theta\ln\sin\theta + \cos^2\theta\ln\cos\theta) \tag{28}$$

Exercise 12 (6.5.4 - Fine structure and the Zeeman effect in positronium)**.** *Positronium is an electron–positron bound state very similar to the electron–proton bound state of the hydrogen atom.*

Part 1. *Calculate the energy of the ground state of positronium as a function of that of the hydrogen atom. We recall that the positron mass is equal to the electron mass.*
Answer. We recall that the groundstate energy of the hydrogen atom is

$$E_0^H = -\frac{\mu_H e^4}{8h^2\varepsilon_0^2} \tag{29}$$

However the reduced mass is

$$\mu_H = \frac{mM}{m+M} \approx m \tag{30}$$

where M is the proton's mass and m the electron's mass. Now, however, in the positronium:

$$\mu_P = \frac{m^2}{2m} = \frac{m}{2} \tag{31}$$

Thus,

$$E_0^P = -\frac{\mu_P e^4}{8h^2\varepsilon_0^2} \approx \frac{E_0^H}{2} \tag{32}$$

Part 2. *In this exercise we are interested solely in the spin structure of the ground state of positronium. The space of states to be taken into account is then a four-dimensional space $\mathcal{H}$, the tensor product of the spaces of spin-1/2 states of the electron and the positron. Following the notation of Section 6.1.2, we use $|\varepsilon_1\varepsilon_2\rangle$ to denote a state in which the z component of the electron spin is $\hbar\varepsilon_1/2$ and that of the positron spin is $\hbar\varepsilon_2/2$, with $\varepsilon = \pm 1$. Determine the action of the operators $\sigma_{1x}\sigma_{2x}$, $\sigma_{1y}\sigma_{2y}$, and $\sigma_{1z}\sigma_{2z}$ on the four basis states $|{+}{+}\rangle$, $|{+}{-}\rangle$, $|{-}{+}\rangle$, and $|{-}{-}\rangle$ of $\mathcal{H}$. Deduce the action of the operator*

$$\vec{\sigma_1} \cdot \vec{\sigma_2} \tag{33}$$

on these states.
Answer. In order to do this, we will calculate the matrix elements and build the matrix representation. Using the $\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$ order to do so. Then,

$$\sigma_{1x}\sigma_{2x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{34}$$

And

$$\sigma_{1y}\sigma_{2y} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{35}$$

And, at last,

$$\sigma_{1z}\sigma_{2z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

Hence,

$$\vec{\sigma_1} \cdot \vec{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

Part 3. *Show that the four vectors form an orthonormal basis of $\mathcal{H}$ and that these vectors are eigenvectors of $\vec{\sigma_1} \cdot \vec{\sigma_2}$ with eigenvalues 1 or -3.*
Answer. It is easy to see that the vectors are normal and orthogonal by simple calculations. A set of orthonormal vectors with the size of the dimension of the space form a basis, thus this is a

orthonormal basis of $\mathcal{H}$. Let us see that they are eigenvectors of the operator $\vec{\sigma_1} \cdot \vec{\sigma_2}$. In respective order:

$$\vec{\sigma_1} \cdot \vec{\sigma_2} \left| I \right\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left| I \right\rangle \tag{38}$$

As for $\left| II \right\rangle$:

$$\vec{\sigma_1} \cdot \vec{\sigma_2} \left| II \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \left| II \right\rangle \tag{39}$$

And the next:

$$\vec{\sigma_1} \cdot \vec{\sigma_2} \left| III \right\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \left| III \right\rangle \tag{40}$$

At last

$$\vec{\sigma_1} \cdot \vec{\sigma_2} \left| IV \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{-3}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = -3 \left| IV \right\rangle \tag{41}$$

Part 4. *Find the projectors $\mathcal{P}_1$ and $\mathcal{P}_{-3}$ onto the subspaces of the eigenvalues 1 and -3, writing these projectors in the form*

$$\lambda I + \mu \vec{\sigma_1} \cdot \vec{\sigma_2} \tag{42}$$

Answer. In the new basis, it is easy to write the projection operators and the $\vec{\sigma_1} \cdot \vec{\sigma_2}$ operator.

$$\mathcal{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{43}$$

And also

$$\vec{\sigma_1} \cdot \vec{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{44}$$

Using the matrix elements and comparing, we can write

$$\lambda_1 + \mu_1 = 1, \quad \lambda_1 - 3\mu_1 = 0 \tag{45}$$

And

$$\lambda_2 + \mu_2 = 0, \quad \lambda_2 - 3\mu_2 = -3 \tag{46}$$

Hence, solving the equations, this is equivalent to

$$\mathcal{P}_1 = \frac{1}{4}\Big(3I + \vec{\sigma_1} \cdot \vec{\sigma_2}\Big), \qquad \mathcal{P}_{-3} = \frac{1}{4}\Big(I - \vec{\sigma_1} \cdot \vec{\sigma_2}\Big) \tag{47}$$

Part 5. *Show that the operator*

$$\mathcal{P}_{12} = \frac{1}{2}\Big(I + \vec{\sigma_1}\cdot\vec{\sigma_2}\Big) \tag{48}$$

exchanges the values of ε_1 and ε_2.
Answer. Using the matrix representation in the old basis:

$$\mathcal{P}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

where we can easily see that all vectors get exchanged.

Part 6. *The Hamiltonian H0 of the spin system in the absence of an external field is given by*

$$H_0 = E_0 I + A\vec{\sigma_1}\cdot\vec{\sigma_2} \tag{50}$$

where E_0 and $A > 0$ are constants. Find the eigenvectors and eigenvalues of H_0.
Answer. The matrix representation of this hamiltonian in the new basis is

$$H_0 = \begin{pmatrix} E_0 + A & 0 & 0 & 0 \\ 0 & E_0 + A & 0 & 0 \\ 0 & 0 & E_0 + A & 0 \\ 0 & 0 & 0 & E_0 - 3A \end{pmatrix} \tag{51}$$

where we trivially see that $\{|I\rangle, |II\rangle, |III\rangle\}$ are eigenvectors with eigenvector $E_0 + A$ and $\{|IV\rangle\}$ is the eigenvector with eigenvalue $E_0 - 3A$.

Part 7. *The positronium atom is placed in a uniform, constant magnetic field $\vec{B}$ parallel to O_z. Show that the Hamiltonian becomes*

$$H = H_0 - \frac{q_e\hbar}{2m}B(\sigma_{1z} - \sigma_{2z}) \tag{52}$$

where m is the electron mass and q_e is its charge. Find the matrix representation of H in the basis $\{|I\rangle, |II\rangle, |III\rangle, |IV\rangle\}$. The parameter x is defined by

$$\frac{q_e\hbar}{2m} = -Ax \tag{53}$$

Find the eigenvalues of H and graph their behavior as a function of x.
Answer. Using the fact that the electron and positron have opposite magnetic moments,

$$H = H_0 - (\vec{\mu}_e + \vec{\mu}_p)\cdot\vec{B} = H_0 - \frac{q_e\hbar}{2m}B(\sigma_{1z} - \sigma_{2z}) \tag{54}$$

where we remember that $\sigma_{1z} = \sigma_{1z}\otimes\mathbb{I}_2$ and vice-versa. Let $\hat{O} = \sigma_{1z} - \sigma_{2z}$. Then, we can easily see that

$$\begin{aligned} \hat{O}\,|I\rangle &= 0 \\ \hat{O}\,|II\rangle &= 2\,|IV\rangle \\ \hat{O}\,|III\rangle &= 0 \\ \hat{O}\,|IV\rangle &= 2\,|II\rangle \end{aligned}$$

Using this information, we can build the matrix representation of H as

$$H = \begin{pmatrix} E_0 + A & 0 & 0 & 0 \\ 0 & E_0 + A & 0 & 2Ax \\ 0 & 0 & E_0 + A & 0 \\ 0 & 2Ax & 0 & E_0 - 3A \end{pmatrix} \tag{55}$$

The eigenvectors $\{|I\rangle, |III\rangle\}$ are trivially seen, as well as their eigenvalues $E_0 + A$ in both cases. The other two eigenvectors can be found by considering the reduced, 2×2 matrix

$$H' = E_0\mathbb{I} + A \begin{pmatrix} 1 & 2x \\ 2x & -3 \end{pmatrix} \equiv E_0\mathbb{I} + AM \tag{56}$$

where it is the matrix M that we must diagonalize. The eigenvalue equation is

$$-(1-\lambda)()\lambda + 3) - 4x^2 = 0 \tag{57}$$

Or,

$$\lambda^2 + 2\lambda - (3 + 4x^2) = 0 \tag{58}$$

which gives

$$\lambda_\pm = -1 \pm 2\sqrt{1 + x^2} \tag{59}$$

This yields the energy

$$E_\pm = E_0 - A \pm 2A\sqrt{1 + x^2} \tag{60}$$

When $x = 0$ we get our usual older eigenvectors $E_0 + A$ and $E_0 - 3A$. As $x \to \infty$, the eigenvalues tend to $\pm 2Ax$.

Exercise 13 (12)**.** *A dinâmica de um sistema de dois spins 1/2 é descrita pelo hamiltoniano:*

$$\hat{H} = -\frac{\hbar\omega}{2}\sigma_1 \cdot \sigma_2 \tag{61}$$

onde ω é uma constante. Supondo que em $t = 0$ o vetor de estado dos dois spins era $|+-\rangle$, obtenha a entropia de emaranhamento de um dos elétrons nos instantes $t = 0$ e $t = \pi/2\omega$.

Answer. No instante $t = 0$, a matriz densidade é

$$\rho = |+-\rangle\langle+-| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{62}$$

Logo, $\rho_A = |+\rangle\langle+|$. Isso implica que a entropia de emaranhamento é zero, pois $\ln \rho_A = 0$.
