Mecânica Quântica Avançada
Prova 2

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Prova 2

Exercise 1. *The Galilean boosts, a.k.a. pure Galilean transformations, form a subgroup of a larger, 10-dimensional group named Galilei (or Galileo) group of space-time transformations:*

$$\begin{aligned} \vec{x} &\to \vec{x}' = R\vec{x} + \vec{a} + \vec{v}t \\ t &\to t' = t + s \end{aligned}$$

where in the addition to the displacement $\vec{a}$ and boost velocity $\vec{v}$ studied so far, one also has a spatial rotation R and time displacement s. Let $g = (R, \vec{a}, \vec{v}, s)$ denote such a transformation. Show that the composition law for $g_3 = g_2 g_1$, with $g_3 = (R_3, a_3, v_3, s_3)$ is:

$$\begin{aligned} R_3 &= R_2 R_1 \\ \vec{a}_3 &= \vec{a}_2 + R\vec{a}_1 + \vec{v}_2 s_1 \\ \vec{v}_3 &= \vec{v}_2 + R_2 \vec{v}_1 \\ s_3 &= s_2 + s_1 \end{aligned}$$

Answer. If we apply g_3 to the pair $\{\vec{v}, t\}$, we get:

$$\begin{aligned} \vec{x} &\to \vec{x}'' = R_3\vec{x} + \vec{a}_3 + \vec{v}_3 t \\ t &\to t'' = t + s_3 \end{aligned}$$

Now let us apply g_1 to the initial pair, so that later we may apply g_2 as well:

$$\begin{aligned} \vec{x} &\to \vec{x}' = R_1\vec{x} + \vec{a}_1 + \vec{v}_1 t \\ t &\to t' = t + s_1 \end{aligned}$$

Applying g_2 here:

$$\begin{aligned} \vec{x}' &\to \vec{x}'' = R_2(R_1\vec{x} + \vec{a}_1 + \vec{v}_1 t) + \vec{a}_2 + \vec{v}_2 t' \\ t' &\to t'' = (t + s_1) + s_2 = t + (s_2 + s_1) \end{aligned}$$

Rearrenging terms in the first of these equations

$$\begin{aligned}\vec{x}'' &= R_2R_1\vec{x} + R_2\vec{a}_1 + \vec{a}_2 + R_2\vec{v}_1t + \vec{v}_2t + \vec{v}_2s_1 \\ &= (R_2R_1)\vec{x} + (R_2\vec{a}_1 + \vec{a}_2 + \vec{v}_2s_1) + (R_2\vec{v}_1 + \vec{v}_2)t\end{aligned}$$

Equating this to the transformation of g_3, we get:

$$\begin{aligned}R_3 &= R_2R_1 \\ \vec{a}_3 &= R_2\vec{a}_1 + \vec{a}_2 + \vec{v}_2s_1 \\ \vec{v}_3 &= R_2\vec{v}_1 + \vec{v}_2 \\ s_3 &= s_2 + s_1\end{aligned}$$

Exercise 2. ***Item 1.*** *Use the relations*

$$\begin{aligned}&\langle j'm'|\vec{J}^2|jm\rangle = j(j+1)\hbar^2\delta_{jj'}\delta mm' \\ &\langle j'm'|J_0|jm\rangle = m\hbar\delta_{mm'} \\ &\langle j'm'|J_\pm|jm\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\delta_{jj'}\delta_{m',m\pm1}\end{aligned} \tag{1}$$

to find the operators S_x, S_y, S_z for spin $j = 1/2$.
Answer. First of all, when $j = 1/2$, then $m = \pm 1/2$. The easiest is S_z:

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \tag{2}$$

The other two we can find as following:

$$S_\pm = S_x \pm iS_y \tag{3}$$

Thus, inverting:

$$\begin{aligned}S_x &= \frac{S_+ + S_-}{2} \\ S_y &= \frac{S_+ - S_-}{2i}\end{aligned}$$

We must only find $S\pm$:

$$\begin{aligned}S_+ &= \hbar\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix} \\ S_- &= \hbar\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\end{aligned}$$

Therefore, we see that

$$\begin{aligned}S_x &= \frac{\hbar}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \\ S_y &= \frac{\hbar}{2i}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\end{aligned}$$

Item 2. *Using the same relations, find the matrix representations of J_x, J_y, J_z for $j = 1$.*
Answer. Now with $j = 1$, we have $m = -1, 0, 1$. Again, the easiest is J_z:

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4}$$

We can again find $J_{x,y}$ by means of $J_\pm$, thus we shall find them first.

$$\begin{aligned} J_+ &= \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \\ J_- &= \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \end{aligned}$$

Summing and subtracting accordingly:

$$\begin{aligned} J_x &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ J_y &= \frac{\hbar}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

Item 3. *Show that, for $j = 1$, the cartesian components are related to the infinitesiaml generators T_x, T_y, T_z by a unitary transformation*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{5}$$

with $J_i = U^\dagger T_i U$.
Answer. All we need to do is two matrix multiplications for each coordinate in order to check that the expression works. Let us begin with the first one:

$$\begin{aligned} U^\dagger T_x U &= \frac{1}{2} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= J_x \end{aligned}$$

Doing the same for the next:

$$\begin{aligned}
U^\dagger T_y U &= \frac{1}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & i\sqrt{2} & 0 \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix} \\
&= \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&= J_y
\end{aligned}$$

Finally,

$$\begin{aligned}
U^\dagger T_z U &= \frac{1}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & -1 \\ -i & 0 & i \\ 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= J_z
\end{aligned}$$

Item 4. *Find the rotation matrix $d^{(1)}(\theta)$*

$$d^{(1)} = e^{-i\theta J_y} \tag{6}$$

Answer. Let us first show that $J_y = J_y^3$:

$$
\begin{aligned}
J_y^3 &= -\frac{1}{2i\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\\
&= \frac{1}{2i\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & -1\end{pmatrix}\\
&= \frac{1}{2i\sqrt{2}}\begin{pmatrix}0 & -2 & 0\\ 2 & 0 & -2\\ 0 & 2 & 0\end{pmatrix}\\
&= J_y
\end{aligned}
$$

Now, we can taylor expand the exponential:

$$
\begin{aligned}
d^{(1)}(\theta) &= 1 - i\theta J_y + \frac{\theta^2}{2}J_y^2 - i\frac{\theta^3}{3!}J_y^3 + \cdots\\
&= \mathbb{I} + \left(\frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right)J_y^2 - i\left(\theta + \frac{\theta^3}{3!} + \cdots\right)J_y\\
&= \mathbb{I} + (1-\cos(\theta))J_y^2 - i\sin(\theta)J_y
\end{aligned}
$$

where we used that $J_y^4 = J_y^2$. Now, we only need to plug the matrix definition of $\mathbb{I}$, J_y and J_y^2 to finish.

$$
\begin{aligned}
d^{(1)}(\theta) &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - \frac{1-\cos\theta}{2}\begin{pmatrix}-1 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & -1\end{pmatrix} + \frac{\sin\theta}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\\
&= \begin{pmatrix}(1+\cos\theta/2 & -\sin\theta/\sqrt{2} & (1-\cos\theta)/2\\ \sin\theta/\sqrt{2} & \cos\theta & -\sin\theta/\sqrt{2}\\ (1-\cos\theta)/2 & \sin\theta/\sqrt{2} & (1+\cos\theta)/2\end{pmatrix}
\end{aligned}
$$

Exercise 3. *Consider the quantum harmonic oscilator.*

Answer. The non relativistic relation is

$$T = \frac{p^2}{2M} \tag{7}$$

However, if we consider relativity, we should use

$$T = E - mc^2 \tag{8}$$

Moreover, we also have the relation

$$E^2 = (mc^2)^2 + (pc)^2 \tag{9}$$

Thus, we can write

$$
\begin{aligned}
T &= \sqrt{(mc^2)^2 + (pc)^2} - mc^2\\
&= mc^2\left(\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right)
\end{aligned}
$$

If we taylor expand with $p \geq mc^2$ up to first order, we get the original expression for T, which indicates this equation is correct. Now, let us consider also second order corrections:

$$\begin{aligned} T &= mc^2\left(\sqrt{1+\left(\frac{p}{mc}\right)^2}-1\right) \\ &= mc^2\left(\frac{1}{2}\left(\frac{p}{mc}\right)^2-\frac{1}{8}\left(\frac{p}{mc}\right)^4\right) \\ &= \frac{p^2}{2m}-\frac{p^4}{8m^3c^2} \end{aligned}$$

Using this, we can write our hamiltonian as:

$$H = H_0 + W \tag{10}$$

where

$$\begin{aligned} H_0 &= \frac{P^2}{2m}+\frac{1}{2}m\omega^2X^2 \\ W &= -\frac{P^4}{8m^3c^2} \end{aligned}$$

We know, then, that our $1/c^2$ correction to the ground state eigenenergy is:

$$\Delta E = \langle 0^{(0)}|W|0^{(0)}\rangle \tag{11}$$

where $|0^{(0)}\rangle$ is the non-perturbed ground state ket. We can write

$$\hat{P} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a) \tag{12}$$

Thus,

$$\begin{aligned} \hat{P}^4 &= \left(\frac{\hbar m\omega}{2}\right)^2(a^\dagger-a)^4 \\ &= \left(\frac{\hbar m\omega}{2}\right)^2(a^\dagger-a)(a^\dagger-a)(a^\dagger-a)(a^\dagger-a) \\ &= \left(\frac{\hbar m\omega}{2}\right)^2((a^\dagger)^2-2N+a^2)((a^\dagger)^2-2N+a^2) \\ &= \left(\frac{\hbar m\omega}{2}\right)^2((a^\dagger)^4-2(a^\dagger)^2N+(a^\dagger)^2a^2-2N(a^\dagger)^2+4N^2-2Na^2+a^2(a^\dagger)^2-2a^2N+a^4) \end{aligned}$$

where we used the hermiticity of N. Now, let us find the braket of interest, writing the constants in front of $\hat{P}^4$ simply as c for now.

$$\begin{aligned} \hat{P}^4 &= c\,\langle 0|(a^\dagger)^4-2(a^\dagger)^2N+(a^\dagger)^2a^2-2N(a^\dagger)^2+4N^2-2Na^2+a^2(a^\dagger)^2-2a^2N+a^4|0\rangle \\ &= c\,\langle 0|(a^\dagger)^4-2N(a^\dagger)^2+a^2(a^\dagger)^2|0\rangle \\ &= c(2\sqrt{6}\,\langle 0|4\rangle-4\sqrt{2}\,\langle 0|2\rangle+2\,\langle 0|0\rangle) \\ &= 2c \end{aligned}$$

Thus,

$$\Delta E = -2\left(\frac{\hbar m\omega}{2}\right)^2\left(\frac{1}{8m^3c^2}\right) = -\frac{\hbar^2\omega^2}{16mc^2} \tag{13}$$

Exercise 4. *Consider two spin-1, identical, non-interacting particles.*

Item 1. *Suppose the spacial part of the vector state is symmetric under pair exchange. Let $|m\rangle = \{+, 0, -\}$. If possible, build the 3-particle state in the following scenarios. Can the state be written as a eigenket of the total spin $\vec{S}$? If yes, do it and find the total spin.*

1. *All particles in the state $|+\rangle$.*
   Answer. The total state is:
$$|\psi\rangle = |+++\rangle \tag{14}$$
   The total spin is $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$, thus
$$\begin{aligned} \vec{S}\,|+++\rangle &= \left(\vec{S}_1 + \vec{S}_2 + \vec{S}_3\right)|+++\rangle \\ &= 3\hbar\,|+++\rangle \end{aligned}$$
   so the total spin is $3\hbar$ and our ket is an eigenket of $\vec{S}$.

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2. *Two particles is the $|+\rangle$ state and one in the $|0\rangle$ state.*
   Answer. We can write:
$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|++0\rangle + |+0+\rangle + |0++\rangle\right) \tag{15}$$
   The total spin is:
$$\begin{aligned} \vec{S}\,|\psi\rangle &= \frac{1}{\sqrt{3}}\left(\vec{S}_1 + \vec{S}_2 + \vec{S}_3\right)\left(|++0\rangle + |+0+\rangle + |0++\rangle\right) \\ &= \frac{\hbar}{\sqrt{3}}\left(2\,|++0\rangle + 2\,|+0+\rangle + 2\,|0++\rangle\right) \\ &= \frac{2\hbar}{\sqrt{3}}\left(|++0\rangle + |+0+\rangle + |0++\rangle\right) \end{aligned}$$
   so, again, our ket is an eigenket of the total spin with value $2\hbar$.

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3. *The three particles in different states.*
   Answer. We can write
$$|\psi\rangle = \frac{1}{\sqrt{3!}}\left(|+0-\rangle + |+-0\rangle + |-+0\rangle + |-0+\rangle + |0-+\rangle + |0+-\rangle\right) \tag{16}$$
   As for the total spin, it is quite easy to see that $\vec{S} = 0$, because $\vec{S}\,|kk'k''\rangle = (+\hbar) + (0\hbar) + (-\hbar) = 0$ if all states are different. Thus, $|\psi\rangle$ is also an eigenket of the total spin with eigenvalue 0.

Item 2. *Now do the same, but supposing an anti-symmetric state vector.*

1. *All states in the $|+\rangle$ state.*
   Answer. There is no such anti-symmetrical state like this.

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2. Two particles is the $|+\rangle$ state and one in the $|0\rangle$ state.
Answer. We can write the state using the slater determinant:

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{3}} \det \begin{bmatrix} |+\rangle & |+\rangle & |0\rangle \\ |+\rangle & |+\rangle & |0\rangle \\ |+\rangle & |+\rangle & |0\rangle \end{bmatrix} \\
&= 0
\end{aligned}$$

so the answer is again no.

3.The three particles in different states. We can write the state using the slater determinant:

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{3!}} \det \begin{bmatrix} |+\rangle & |+\rangle & |-\rangle \\ |+\rangle & |+\rangle & |-\rangle \\ |+\rangle & |+\rangle & |-\rangle \end{bmatrix} \\
&= \frac{1}{\sqrt{3!}} \left(|{+}0{-}\rangle - |{+}-0\rangle + |{-}+0\rangle - |{-}0{+}\rangle + |0-+\rangle - |0+-\rangle \right)
\end{aligned}$$

Here, the total spin is again zero and our state is an eigenket of $\vec{S}$.

Exercise 5. *Consider the hamiltonian:*

$$\hat{H} = \sum_{\alpha} T_\alpha a_\alpha^\dagger a_\alpha + \sum_{\alpha,\beta,\gamma} V_{\alpha\beta\gamma} \left(a_\alpha^\dagger a_\beta a_\gamma + a_\alpha^\dagger a_\beta^\dagger a_\gamma \right) \tag{17}$$

where $V_{\alpha,\beta,\gamma} = V^*_{\alpha,\beta,\gamma}$ and $T_\alpha = T^*_\alpha$ are symmetrical in all indices and the operators $a_\alpha, a_\alpha^\dagger$ satisfy the bosonic commutation relations.

Item 1. *Is this hamiltonian hermitian? Prove your answer.*
Answer. Let us calculate it, element by element. The first term is

$$\begin{aligned}
T^\dagger &= \left(\sum_\alpha T_\alpha a_\alpha^\dagger a_\alpha \right)^\dagger \\
&= \sum_\alpha T^*_\alpha (a_\alpha^\dagger)(a_\alpha^\dagger)^\dagger \\
&= \sum_\alpha T_\alpha a_\alpha^\dagger a_\alpha \\
&= T
\end{aligned}$$

The second term is:

$$\begin{aligned}
V^\dagger &= \left(\sum_{\alpha,\beta,\gamma} V_{\alpha\beta\gamma} \left(a_\alpha^\dagger a_\beta a_\gamma + a_\alpha^\dagger a_\beta^\dagger a_\gamma \right) \right)^\dagger \\
&= \sum_{\alpha,\beta,\gamma} V^*_{\alpha\beta\gamma} \left(a_\gamma^\dagger a_\beta^\dagger a_\alpha + a_\gamma^\dagger a_\beta a_\alpha \right) \\
&= \sum_{\alpha,\beta,\gamma} V_{\gamma\beta\alpha} \left(a_\alpha^\dagger a_\beta^\dagger a_\gamma + a_\alpha^\dagger a_\beta a_\gamma \right) \\
&= \sum_{\alpha,\beta,\gamma} V_{\alpha\beta\gamma} \left(a_\alpha^\dagger a_\beta a_\gamma + a_\alpha^\dagger a_\beta^\dagger a_\gamma \right) \\
&= V
\end{aligned}$$

where we exchanged α and γ because they are just symbols we can assign any name to and then we used the fact that $V_{\alpha\beta\gamma}$ is symmetrical. Thus,

$$H = H^\dagger \tag{18}$$

Item 2. *Does this hamiltonian conserve the number of particles* $N = \sum_\alpha a_\alpha^\dagger a_\alpha$*?*
Answer. We know N is a conserved quantity if it commutes with the hamiltonian H. So, let us calculate it. We can separate our in two steps by considering:

$$\begin{aligned}
[H, N] &= HN - NH \\
&= (T+V)N - N(T+V) \\
&= (TN - NT) + (VN - NV) \\
&= [T, N] + [V, N]
\end{aligned}$$

So, let's to it in the appropriate order.

$$\begin{aligned}
[T, N] &= TN - NT \\
&= \left(\sum_\alpha T_\alpha a_\alpha^\dagger a_\alpha\right)\left(\sum_\beta a_\beta^\dagger a_\beta\right) - \left(\sum_\beta a_\beta^\dagger a_\beta\right)\left(\sum_\alpha T_\alpha a_\alpha^\dagger a_\alpha\right) \\
&= \sum_{\alpha,\beta} T_\alpha a_\alpha^\dagger a_\alpha a_\beta^\dagger a_\beta - a_\beta^\dagger a_\beta T_\alpha a_\alpha^\dagger a_\alpha \\
&= \sum_{\alpha,\beta} T_\alpha (a_\beta^\dagger a_\beta - a_\beta^\dagger a_\beta) a_\alpha^\dagger a_\alpha \\
&= 0
\end{aligned}$$

where we used the fact that $[N_\alpha, N_\beta] = 0$ and that T_α is real. Now for the second term, we must do something similar:

$$\begin{aligned}
[V, N] &= \left(\sum_{\alpha,\beta,\gamma} V_{\alpha\beta\gamma}\left(a_\alpha^\dagger a_\beta a_\gamma + a_\alpha^\dagger a_\beta^\dagger a_\gamma\right)\right)\left(\sum_\sigma a_\sigma^\dagger a_\sigma\right) - \left(\sum_\sigma a_\sigma^\dagger a_\sigma\right)\left(\sum_{\alpha,\beta,\gamma} V_{\alpha\beta\gamma}\left(a_\alpha^\dagger a_\beta a_\gamma + a_\alpha^\dagger a_\beta^\dagger a_\gamma\right)\right) \\
&=
\end{aligned}$$
