Advanced Quantum Mechanics
Exercises Set 4

Due date: May 15^{th}, 2023

Exercises 1 and 2: exercises from Le Bellac 8.5.2 and 8.5.6.

Exercise 3. So far we have considered Galilean boosts by a speed v (in one spatial dimension) at $t = 0$ and obtained, by demanding that the expectation values of the position $\hat{X}$, speed $d\hat{X}/dt$ and momentum $\hat{P}$ transform like their classical counterparts, that the unitary operator $\hat{U}(v)$ that implements such boosts in a Hilbert space is given by (employing the active point of view of the boost)

$$\hat{U}(v) = e^{imv\hat{X}/\hbar}$$

where m is the mass of the particle. Show that this generalizes for an arbitrary time t to

$$\hat{U}(v) = e^{imv\hat{X}/\hbar} e^{-itv\hat{P}/\hbar} e^{-imv^2 t/2\hbar} = e^{-itv\hat{P}h} e^{imv\hat{X}/\hbar} e^{imv^2 t/2\hbar}$$

Show also that the last result generalizes in three dimensions to

$$\hat{U}(v) = e^{im\vec{v}\cdot\hat{\vec{X}}/h} e^{-it\vec{v}\cdot\hat{\vec{P}}/h} e^{-im\vec{v}^2 t/2h} = e^{-it\vec{v}\cdot\hat{\vec{P}}/h} e^{im\vec{v}\cdot\hat{\vec{X}}/h} e^{im\vec{v}^2 t/2h}$$

Exercise 4. The Galilean boosts, *a.k.a.* pure Galilean transformations, form a subgroup of a larger, 10-dimensional group named Galilei (or Galileo) group of space-time transformations:

$$\begin{aligned} \vec{x} \to \vec{x}\,' &= R\vec{x} + \vec{a} + \vec{v}t \\ t \to t' &= t + s \end{aligned}$$

where in the addition to the displacement $\vec{a}$ and boost velocity $\vec{v}$ studied so far, one also has a spatial rotation R and time displacement s. Let $g = (R, \vec{a}, \vec{v}, s)$ denote such a transformation.

Show that the composition law for $g_3 = g_2 g_1$, with $g_3 = (R_3, a_3, v_3, s_3)$ is:

$$\begin{aligned} R_3 &= R_2 R_1 \\ \vec{a}_3 &= \vec{a}_2 + R_2\vec{a}_1 + \vec{v}_2 s_1 \\ \vec{v}_3 &= \vec{v}_2 + R_2\vec{v}_1 \\ s_3 &= s_2 + s_1 \end{aligned}$$

Exercises 5–7: exercises from Le Bellac 10.7.1, 10.7.2, and 10.7.4.