Mecânica Quântica Avançada
Lista 2

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IFT

List of Exercises

Exercise 1 (6.5.1 - Independence of the tensor product from the choice of basis)**.** *Verify that the definition (6.3) of the tensor product of two vectors is independent of the choice of basis in $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Answer. Let $|n'\rangle$ and $|m'\rangle$ be two other basis of the Hilbert spaces one and two, respectively. Then, it is true that

$$\begin{aligned} |n\rangle &= \sum a_{n'} |n'\rangle \\ |m\rangle &= \sum b_{m'} |m'\rangle \end{aligned} \tag{1}$$

Thus, we can write

$$\begin{aligned} |\varphi\rangle \otimes |\chi\rangle &= \sum_{n,m} c_n d_m |n\rangle \otimes |m\rangle \\ &= \sum_{n,m} c_n d_m \left(\sum_{n'} a_{n'} |n'\rangle\right) \otimes \left(\sum_{m'} b_{m'} |m'\rangle\right) \\ &= \sum_{n',m'} a_{n'} b_{m'} \left(\sum_n c_n\right)\left(\sum_m d_m\right) |n'\rangle \otimes |m'\rangle \\ &= \sum_{n',m'} e_{n'} f_{m'} |n'\rangle \otimes |m'\rangle \end{aligned}$$

This shows that the tensor product does not depend on the choice of basis.

Exercise 2 (2 - Representação matricial de produtos tensoriais)**.** *Calculate the tensor products of two-level systems.*

Answer. We can calculate the tensor products of $|+\rangle$ and $|-\rangle$ as follows:

$$
\begin{aligned}
|+\rangle \otimes |-\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|-\rangle \otimes |+\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|-\rangle \otimes |-\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Now for the three dimension qubits, we will write the answers directly:

$$
\begin{aligned}
|{+}{+}{+}\rangle &= \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \qquad |{+}{+}{-}\rangle = \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}\\
|{+}{-}{+}\rangle &= \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \qquad |{+}{-}{-}\rangle = \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}\\
|{-}{+}{+}\rangle &= \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \qquad |{-}{+}{-}\rangle = \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}\\
|{-}{-}{+}\rangle &= \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}, \qquad |{-}{-}{-}\rangle = \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}
\end{aligned}
$$

Exercise 3 (6.5.2)**.** *Write down explicitly the 4×4 matrix $A \otimes B$, the tensor product of the 2×2 matrices A and B:*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2}$$

Answer. It is very easy do perform this calculation:

$$A \otimes B = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix}. \tag{3}$$

We just multiply each element of the first matrix by the whole second matrix.

Exercise 4 (6.5.3 - Properties of state operators). ***Part 1.*** *Show that* $\rho_{ii} \geq 0$, $\rho_{jj} \geq 0$, *and* $\det A \geq 0$, *from which* $|\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj}$. *Also deduce that if* $\rho_{ii} = 0$, *then* $\rho_{ij} = \rho^*_{ji} = 0$.
Answer. We can always write

$$\rho = \sum a_n |\phi_n\rangle \langle\phi_n| \tag{4}$$

for some states $|\phi_n\rangle$ and $a_n \geq 0$. Thus, the diagonal matrix elements are

$$\begin{aligned} \rho_{ii} &= \langle\phi_i| \left(\sum a_n |\phi_n\rangle \langle\phi_n|\right) |\phi_i\rangle \\ &= a_i \end{aligned}$$

Hence, $\rho_{ii} \geq 0$. We also note that

$$\det A = \rho_{ii}\rho_{jj} - |\rho_{ij}|^2 \tag{5}$$

where we used the fact that A is hermitian. This implies that

$$\det A \geq 0 \iff |\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj} \tag{6}$$

Using this inequality, if $\rho_i i = 0$, then

$$0 \leq \rho_{ij}\rho_{ji} \leq 0 \tag{7}$$

Obviously, $\rho_{ij} = 0$ or $\rho_{ji} = 0$, but is does not matter, because they are the complex conjugate of each other, so if one is zero, the other is zero as well.

Part 2. *Show that if there exists a maximal test giving 100% probability for the quantum state described by a state operator* ρ, *then this state is a pure state. Also show that if* ρ *describes a pure state, and if it can be written as*

$$\rho = \lambda\rho' + (1 - \lambda)\rho'', 0 \leq \lambda \leq 1 \tag{8}$$

then $\rho = \rho' = \rho''$. *Hint: first demonstrate that if* ρ' *and* ρ'' *are generic state operators, then* ρ *is a state operator. The state operators form a convex subset of Hermitian operators.*
Answer.
