Review of the paper : "Optimal spectral transportation with application to music transcription"

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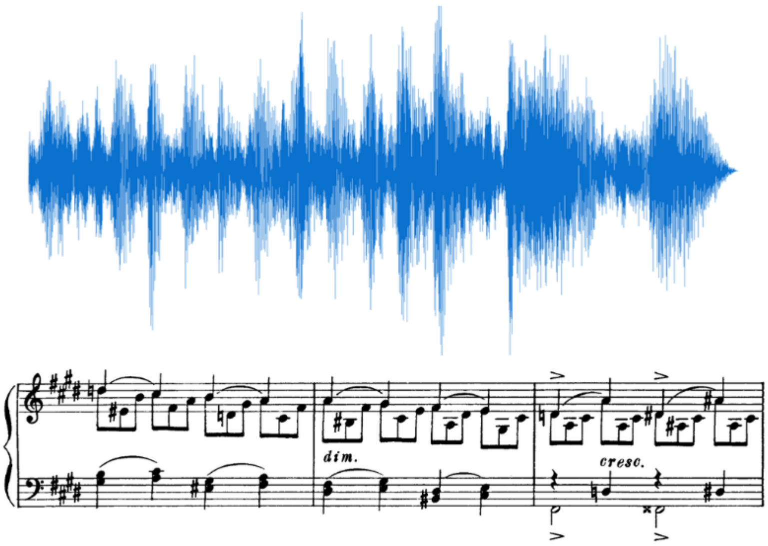

Figure 1: Music transcription on Prelude in C# minor, Op.3 No.2, S. Rachmaninoff

Abstract

The analysis of signals, especially musical tracks, covers several areas : generation of original melodies, analysis of musical style or separation of signal sources. The last point is one of the most documented, corresponding to one of the looters of the analysis of audio tracks. In this context, music transcription aims to "translate" a raw musical signal according to its composition (notes, rhythm, etc.). Thus, the challenge is to associate the spectrogram of the input signal with representations of the notes. Commonly used fit metrics for assessing the quality of decomposition typically involve comparing the data and template entries on a per-frequency basis. However, this approach is not robust against minor shifts of energy between frequency bins and changes in timbre, which leads to difficulties to perform transcription. In this context, paper [10] "Optimal spectral transportation with application to music transcription" (NIPS, 2016) leverages optimal transportation methods [1] and introduces a new fit metric that evaluates the energy frequency distributions in a holistic manner. Optimal Spectral Transportation (OST) approaches thus leads to fast, simple and efficient decomposition algorithms that perform on real-world music data. In this context, the approach of this paper is the following. Describe pros and cons of a relevant baseline for music transcription, Probabilistic Latent Component Analysis, and compare it to classical and regularized OST, in order to highlight the pros and the cons of such approaches. This paper shows that, in the context of basic synthetic data or real-world musical tracks, entropic and group-sparse optimal spectral transportation methods are powerful tools, according to various requirements [2]. Finally, some possible extensions are suggested, such as the use of quadratic regularization or unbalanced optimal transport.

[1]Source code of some implementation for the studied article [10]: https://github.com/rflamary/OST/

[2]Code : https://github.com/Lucas-Haubert/Optimal_Transport_MVA

1 Introduction

Music Transcription task (MT) aims to represent a raw input soundtrack into a relevant decomposition, by means of a dictionary of features. Several approaches have been designed in order to approximate a spectrogram $V = (v_1, ..., v_N)$ by a product $\hat{V} = WH$, where W is a dictionary of representation of notes, and H is the optimal allocation of notes, for each time step $n \in [\![1, N]\!]$. The following introduces the related problems, as well as the related work. Then, the approach of this paper is presented, as well as some justification about it.

1.1 The problem

Many MT approaches leverage non-negative matrix decomposition techniques, in order to approximate the input signal. The framework can be presented as follows : consider the spectrogram (time representation of successive frequency spectra) V, with columns v_n for $n \in [\![1, N]\!]$, of the input signal. The columns of V represent the spectrum of the signal at time n. The problem of music transcription consists in the following : Knowing a dictionary of features W and an entry spectrogram V,

$$\text{Find } H \text{ such that } \hat{V} = WH \approx V \tag{MT Problem}$$

which is equivalent to :

$$\text{Find } h_n \text{ such that } \hat{v}_n = Wh_n \approx v_n \text{ for all } n \in [\![1, N]\!]$$

This way, W is fixed, and H (resp. h_n) describes the approximation of V (resp. v_n) according to the dictionary. In particular, the matrix W should be able to represent the notes in a relevant way. This point is treated in the following of this section.

This setting is relevant to perform music transcription. Indeed, when W is directly related to the frequencies of the notes (see Section 1.2), the allocation matrix H refers to the notes that are played at time n. In order to bring efficient solutions to (MT Problem), several methods have been developed, involving different types of comparisons. This is the purpose of what follows.

1.2 Related work

Several works have been done to leverage efficient solutions in MT. It first concerns how V is defined. The direct definition of a spectrogram leads to a power-magnitude value of a regular short-time Fourier transform (see [11]). It may also consist in a specific audio transformation, like in [3] or [6]. The success of MT is in direct relation with such a definition, but also the way of representing W. These issues are treated in the following of this section.

Now comes the question of robustness. Indeed, the challenge is to deal with the variability of real notes. A too simple dictionary models a note characterized by its fundamental ν_0 by a vector of zero-coefficients, except on ν_0 and its multiples (harmonic frequencies). However, real-world instruments can produce displacements of energy from a frequency bin to another or variation of timbre (about the intensities of the harmonics). In such a setting, a naive dictionary of notes and fit measure can lead to a large gap between the actual input and its representation. Some works tried to fix such an issue, as [3] and [4], in which the time-invariant semi-parametric models are used to adjust the columns of W to the data. However, the variation of notes in time is not treated in such approaches, which is a capital issue, since the time component of an audio track is one of its pillars. Optimal Transport (OT) techniques may solve this problem and bring flexibility in terms of the time component, but also the robustness issue (see [10]). Before to introduce it, a relevant baseline (PLCA) is presented, in order to bring some comparison tools (see Section 1.3), but also inspire the construction of an OT-based approach.

Probabilistic Latent Component Analysis (PLCA) [8] is a relevant baseline, very used in audio processing. The idea is as follows : Approximate the spectrogram V by WH, where the columns of V are normalized and considered as probability distributions of frequency quanta. On the other side, the matrices $W \in \mathbb{R}^{M\times K}$ and $H \in \mathbb{R}^{K\times N}$ have also normalized columns, leading to a normalized approximation matrix $\hat{V} = WH$. Assuming that W is known, the PLCA approach aims to solve (MT Problem) as :

$$\min_{H\geq 0} D_{KL}(V|\hat{V}) \text{ s.t. } \forall n, \|h_n\|_1 = 1 \tag{PLCA}$$

where $D_{KL}(v_n|\hat{v}_n) = \sum_{i=1}^{M} v_i \log(v_i/\hat{v}_i)$ is the Kullback-Leibler divergence between the two distributions v_n and $\hat{v}_n$, and by extension $D_{KL}(V|\hat{V}) = \sum_{n=1}^{N} D_{KL}(v_n|\hat{v}_n)$. The mathematical aspects of KL-divergence are developed in Appendix A.1. Yet, it is important to notice that the separability of KL divergence leads to a frequency-wise analysis. This implies that little displacements in frequency of the observation v_n can change the KL value too much. Then the resolution of (MT Problem) may lack of efficiency. This gives some motivation to built some OT-based approach.

Optimal Transport theory gives relevant tools to analyse probability measures. The studied article [10] purposes an OT-based approach, called Optimal Spectral Transportation (OST). Consider the vectors v_n's as an energy distribution of intensities $v_{n1}, ..., v_{nM}$ at sampling frequencies $f_1, ..., f_M$ and a transportation matrix T as defined in (a). It is then possible to define an "OT divergence", $D_C(V|\hat{V})$ according to a cost matrix, which is the weighted sum, by T, of transportation costs. The definition of C may allow to fix the inharmonicities, for example by taking $c_{i,j} = (f_i - f_j)^2$. The above context allows to define a new approach, called Optimal Spectral Transportation :

$$\min_{H\geq 0} D_C(V|\hat{V}) \text{ s.t. } \forall n, \|h_n\|_1 = 1 \tag{OST}$$

This is a "naive" version of OST, since W must correspond to realistic note templates (pure note spectra) and is also computationally heavy, due to the large dimension of the inputs. Moreover, this method is not harmonic-invariant, which means that different octaves for the same note suffer from that same cost that other random notes. To fix it, paper [10] introduces a cost matrix C_h defined by :

$$c_{i,j} = \min_{q=1,\dots,q_{max}} (f_i - qf_j)^2 + \epsilon\delta_{q\neq 1} \tag{1}$$

where q_{max} is the ceiling of f_i/f_j and ϵ is a small value (which aims at discriminate the octave with some degree of freedom). This way, the cost matrix handles harmonicity and local invariance, which is in favour of the robustness of the MT method to inharmonicities and variation of timbre. It is also capital to choose the template dictionary W in a cleaver way, in order to provide correct computational cost and accuracy. In particular, the use of C_h is powerful when associated to a dictionary of Dirac vectors placed at the fundamental frequencies $\nu_1, ..., \nu_K$ of the notes to identify. This way, a spectral sample v_n can be transported to a mixture of Dirac vectors placed at the corresponding fundamental frequencies, which deletes the problem of choosing a particular dictionary W. In what follows, the method consisting in taking this W and $C = C_h$ is called Optimal Spectral Transportation "OST".

Some optimization by means of linear programming (LP) or regularization can be made, in order to provide several approaches to OST. These notions are detailed in Appendix A.3. The idea is to work on the hypothesis that $K < M$, where K is the size of the dictionary W, i.e. the number of pure representative notes, and M is the number of features in the representation vectors. Thus, some simplification can be made on (OST), providing fast and efficient solutions. In particular, the entropic and group regularizations bring relevant analysis tools.

Since 2016, other works on MT have been made, especially leveraging deep learning techniques. For instance, [1] involves LSTM neural networks to capture the key features of the inputs. Later, transformer-based approaches were designed (see [2] and [5]) to perform MT. In the context of the study article [10], this paper focuses on the non neural networks based approaches and the use of OT techniques in order to deepen the method proposed by R. Flamary and N. Courty.

1.3 Contributions

The approach of this paper is to study the original article [10]. In particular, some implementation of classic OST and regularized OST methods is available on GitHub. However, this material is not sufficient to reproduce the main corpus of experiments, as it was developed in MATLAB and the source code is no longer available. The first contribution is then a completion of such a code, in order to develop the following methods : OST, OST_e and OST_g, since each of them rely on Optimal Spectral Transportation techniques, and are alternatives of each other, according to the way of regularizing.

In a second time, one focus in given on reproducing the experiments of the study article [10] in order to analyze them under a new point of view. In particular, this paper focuses on the two following aspects.

- **Simulated data :** Generate a template of simulated harmonic templates, as shown in Figures 2 and 3 by means of Gaussian kernels and exponential damping. Then, apply OST methods on simulated inharmonicities and variations of timbre to compare their efficiency.

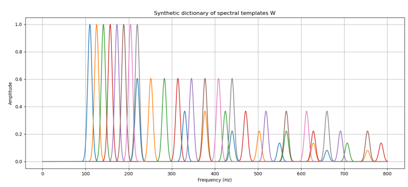

Figure 2: 8 simulated harmonic templates

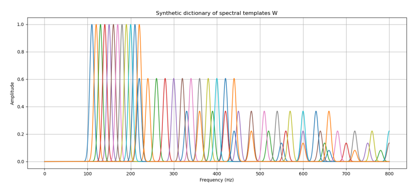

Figure 3: 12 simulated harmonic templates

- **Real-world data :** Consider piano recordings from the dataset MAPS and produce a spectrogram by means of Hann windows and overlap in order to induce as less irregularity as possible. As exposed in Figure 4 extracted from paper [10], real-world music instrument can produce time varying signals, due to their physical composition. It is then relevant to explore the pros and cons of different types of regularizations in order to fix this natural issue.

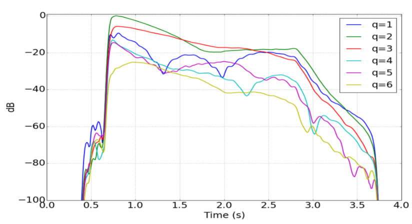

Figure 4: Intensity of 6 harmonics of one piano note during time

2 Contributions

The content of this section is the introduction of the contributions of this paper, as an extension of [10]. To this end, two main experiments are considered : music transcription on simulated sparse data, and real-world audio tracks.

2.1 Simulated spectra

The purpose of this first approach is to evaluate the efficiency of OST-based methods on MT. To do so, sparse dictionaries of harmonic spectra are build (see Figures 2 and 3). In order to model sparse representations, the idea is to produce the densities with Gaussian kernels placed at fundamental frequencies and harmonics. For instance, the first template consists in setting a fundamental at 110 Hz, then build the harmonics at 220 Hz, 330 Hz, and so on.

The experimentation can be described as follows. Extract one template from W, the dictionary of generated harmocis, and apply a transformation : inharmonicity or variations of intensities of the Gaussians (variation of timbre). Figure 5 displays shift transformation, which can occur in the case or real-world instruments : The following can be observed : Since the shift is

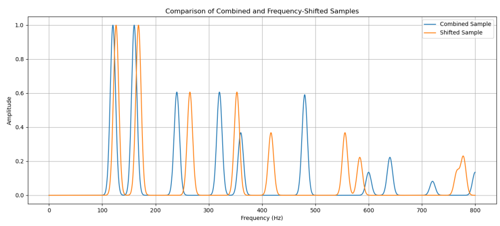

Figure 5: Shifted simulated templates from the 12-templates W

propagated, the lower (from fundamentals) frequencies do not suffer from such a transformation. However, there is an increasing gap between the harmonics when the degree augments. In such a context, when such a spectrum (even this one s very simplistic !) represents v_n, then a little shift of frequency can harm the analysis on a frequency-wise based comparison. In this context, leveraging OST methods is relevant.

The analysis show that the l_1 estimation errors between h_n and $\hat{h}_n$ is better when using sparse-group regularization. This can be explained as follows : Group regularization (see Appendix A.3) leverages the group structure of the transportation matrix T, then $\tilde{T}$, which is relevant in the case of few templates bases, with high distinction (Gaussian pics with low standard deviation) among a wide dictionary. The same analysis comes from perturbations by variation of timbre.

2.2 Real-world musical data

On the other hand, it is interesting to focus on real-world musical data, that differ from simplistic synthetic one. Indeed, as stated on Figure 4, the amplitudes of harmonics of a piano note may vary when time passes. Thus, a fixed amplitude pattern as in Figures 2 and 3, or even more complete, cannot be well suited in this situation, since the Gaussian pics are of time-independent amplitude.

In this context, entropic regularization is adapted, since a better distributed transportation (inducted by $\Omega_e(\tilde{T})$, see Appendix A.3) may capture features in a better way. The provided code gives the key to reproduce such experiments. Also note that Appendix A develops considerations about the choice of parameters and hyperparameters.

3 Conclusion

In this paper, some OT-based approaches were studied in order to perform music transcription. These methods have shown their efficiency in the case of concrete tasks.

3.1 Overview of the OT-based approach

The idea of paper [10] is to consider a global measure of fit when it comes to compare the power spectrum distributions. (PLCA) approach is a widely used baseline in music transcription tasks, which involves a frequency-wise comparison of spectra v_n's and $\hat{v}_n$'s, due to its separability. This setting can actually lead to a lack of robustness in front of two issues : small displacements of frequencies in v_n from a bin to another (inharmonicities), or variation of harmonic intensities, which is relative to the timbre. To fix this, R. Flamary and N. Courty proposed an OT-based approach, leveraging transportation and cost matrices in order to built a so-called "OT-divergence", a powerful tool in order to compare v_n and $\hat{v}_n$ in a global way. Optimal Spectral Transportation (OST) then aims at designing both cost and transportation matrices to better estimate $\hat{V}$. Several improvements have been proposed and tested, which is the purpose of the next section.

3.2 Results and analysis

First [10] obtains relevant results about the efficiency of entropic and group regularizations. Indeed, group-regularized versions of OST have been shown useful when it comes to deal with synthetic harmonic templates. Indeed, by introducing frequency shifts and variation of timbres on combined harmonic templates, that are by construction very sparse (few components over a wide dictionary), it comes that exploiting the group structure of the transportation matrix in (OST) leads to better l_1-errors. On the other hand, handling real-world musical data, like piano recordings, is better managed with entropic regularization, which tend to distribute the transportation in the computation of the "OT-divergence" between v_n and $\hat{v}_n$, leading to a better representation accuracy.

3.3 Possible improvements and extensions

Knowing the conclusions of this paper, it is then possible to purpose some improvements. First, a wide analysis on the impact of regularization have been made, especially entropic and group, or sparse, regularization. It may be relevant to test another form of regularization, such as quadratic regularization, introduced in [7]. Entropic regularization have been shown efficient in the case of real musical data, like piano recording, handling consistency in the note's interpretations. However, it maintains a dense transportation plan, by opposition of group methods for example. By relaxing primal and dual OT music transcription problems by means of quadratic regularization, it is then possible to maintain a sparse transportation plan, which can be relevant when analysing music data. Still concerning OT-based approach, incorporating an unbalanced optimal transportation framework may be also interesting. In classical OT, only probability distributions, subject to a hard assignment, are compared. It is assumed that the entire mass is transported from a distribution to another. Yet, this setting is not robust in the case of outliers or noise. In the situation of music transcription, where the goal is to analyse spectrograms V, i.e. their time entries v_n, it may be relevant to prevent from outliers, that can occur due to the real-world nature of musical data. Finally, as mentioned in Section 1.2, more recent music transcription approaches can be studied, involving deep learning methods. Since audio tracks consist in sequential data, RNN, LSTM or even Transformer architectures can be used to perform music transcription on real-world data.

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A Connection with the course

Several notions of the course "Computational Optimal Transport" of Prof. G. Peyré [9] have been used in this paper. This section then introduces / recalls the main results about those notions, and explains their applications in the study of music transcription.

A.1 K-L divergence

The Kullback-Leibler divergence is an important tool in measure theory. Consider two distributions P and Q, then $D_{KL}(P|Q)$ is a measure of how the distribution P is far from the reference Q. Note that KL divergence is not a metric, since it is not symmetric (due to log) and does not satisfy the triangle inequality. However, it is a powerful tool with important properties, in that $D_{KL}(P|Q) = 0$ means that $P = Q$ almost surely, then have identical quantities of information.

In the setting of (MT problem), $D_{KL}(V|\hat{V}) = \sum_{n=1}^{N} D_{KL}(v_n|\hat{v}_n)$. This implies separability of the KL divergence according to the input arguments. In this way, the coefficients of the v_ns are compared to the one of the $\hat{v}_n$s, and a frequency-wise comparison is made. Thus, considering the KL divergence as the fit measure between the observation and the model prediction may lead to a lack of robustness in the presence of local and small displacements of frequencies.

A.2 Kantorovitch relaxation

Kantorovitch relaxation is an extension of Monge discrete matching problem. In the case of two discrete probability distributions with the same number of points M, the latter is about bijections σ in the set of bijections $Perm(M)$ and can be expressed as follows :

$$\min_{\sigma \in Perm(M)} \frac{1}{M} \sum_{i=1}^{M} C_{i,\sigma(i)} \tag{Monge}$$

where C is a cost matrix that quantifies the loss of assigning i to $\sigma(i)$. This model is easy to understand, but very limiting, in that it allows only a deterministic assignment (one element to the other), and not a probabilistic one (distributed assignments).

In the situation of MT, where capturing complex features between v_n and $\hat{v_n}$ is capital, it is relevant to introduce a transportation matrix that belongs to the set :

$$\Theta = \{T \in \mathbb{R}^{M \times M} | \forall i, j \in [\![0, M]\!], \sum_{j=1}^{M} t_{i,j} = v_{ni}, \sum_{j=1}^{M} t_{i,j} = \hat{v}_{nj}\} \tag{a}$$

This way, an amount of every coefficient of v_n is transported to an entry of $\hat{v_n}$. This amount is described by the coefficients $t_{i,j}$. On the other hand, a cost matrix C is involved to quantify the loss of transportation of mass $t_{i,j}$ from v_i to $\hat{v}_j$, for all i and j. This way, the Kantorovitch relaxation problem is defined as :

$$\text{Find } D_C(v, \hat{v}) = \min_{T \in \Theta} \sum_{i,j} c_{i,j} t_{t,j} \tag{Kantorovitch}$$

$D_C(v, \hat{v})$, and by extension $D_C(V, \hat{V})$ defines an interesting fit measure, as developed in Section 1.2. It is refereed as "OT divergence" in the following of the paper.

A.3 Optimization and types of regularization

The purpose of this section is to introduce the various range of solving approaches for OST. In particular, linear programming and regularization are used.

Linear Programming (LP) : Consider problem (OST) in its naive shape. The objective function being separable, the focus is made on the allocation columns h_n. Using that $D_C(v_n|\hat{v}_n)$ has the shape $\sum_{i,j} c_{i,j}t_{i,j}$ ($=< T, C >$, Frobenius inner product), the MT problem can be expressed as a LP problem :

$$\min_{h_n \geq 0, T \geq 0} < T, C > \text{ s.t. } T1_M = v_n \text{ and } T^T 1_M = Wh_n = \hat{v}_n \tag{LP}$$

Despite of a simple to implement resolution, this method involves a heavy complexity, due the the dimensions of the inputs. Typically, the simplex algorithm is used to solve such a problem.

Set up for regularization : Now consider (OST) in a clever way. Take W as a set of Dirac as explained in Section 1.2 and assume $K < M$, which is natural in the situation of MT. Thus, for all n, $\hat{v}_n = Wh_n$ contains at most K non zero coefficients, at entries i such that $f_i = \nu_k$ (denote S the set of such frequencies). Also, $T \geq 0$, hence it has only K non zero columns, indexed according to S. If $\tilde{T}$ and $\tilde{C}$ are the corresponding matrices, (OST) reduces to :

$$\min_{h_n \geq 0, T \geq 0} < \tilde{T}, \tilde{C} > \text{ s.t. } \tilde{T}1_K = v_n \text{ and } \tilde{T}^T 1_M = Wh_n = \hat{v}_n \tag{b}$$

Also h_n is affiliated to a free constraint, meaning that (b) can be reduced to :

$$\min_{\underline{\tilde{t}}_i \geq 0} \sum_k \tilde{t}_{ik}\tilde{c}_{ik} \text{ s.t. } \sum_k \tilde{t}_{ik} = v_i \text{ (then compute } h_n \text{ directly)} \tag{c}$$

where the $\underline{\tilde{t}}_i$'s are the rows of $\tilde{T}$. The solution is given by $\underline{\tilde{t}}_{ik} = v_i$ if $k = k_i^* = \arg\min_k\{\tilde{c}_{ik}\}$, 0 else. Then by constructing a labeling matrix L which is zero everywhere, except for $(i, k) = (i, k_i^*)$, the solution for (b) is $\hat{h}_n = L^T v_n$. This method is computationally efficient, since it is $\mathcal{O}(M)$ in time, bu comparison with $\mathcal{O}(KM)$ for PLCA per iteration. Remark from a computational point of view : use the cost matrix $\tilde{C}$ with $\tilde{c}_{i,k} = \min_q (f_i - q\nu_k)^2 + \epsilon\delta_{q\neq 1}$ in order to work with $\tilde{W}$. Then, $\tilde{T}$ indicates the transportation to the Diracs at positions ν_k, without the need of the vectors themselves, which brings simplicity. At this point, this method is refereed as OT_h.

OST with entropic regularization : It can be interesting to relax the hard assignment introduced by L and distribute energies a bit more in order to get a smoother estimate of transport $\tilde{T}$. Now the problem becomes :

$$\min_{h_n \geq 0, T \geq 0} < \tilde{T}, \tilde{C} > + \lambda_e \Omega_e(\tilde{T}) \text{ s.t. } \tilde{T}1_K = v_n \text{ and } \tilde{T}^T 1_M = Wh_n = \hat{v}_n \tag{OST$_e$}$$

where $\Omega_e(\tilde{T}) = \sum_{i,k} \tilde{t}_{ik} \log(\tilde{t}_{ik})$. This problem has a closed-form solution $\hat{h}_n = L_e^T v_n$ where L_e is defined with $l_{i,k} = \exp(-\tilde{c}_{ik}/\lambda_e)/\sum_p \exp(-\tilde{c}_{ip}/\lambda_e)$ (full matrix, then complexity in $\mathcal{O}(KM)$).

OST with group regularization : The group structure of T ($M - K$ non columns leading to $\tilde{T}$) is inspiring for defining OST_g. The idea is to leverage group sparse regularization when estimating $\tilde{T}$, since a small number of the K possible notes are played at each time. Some sparsity term $\lambda_g \Omega_g(\tilde{T}) = \lambda_g \sum_k \sqrt{[\![\tilde{t}_k]\!]_1}$ is then added to the objective function. Here, OST_g does not offer a closed-form solution, and it is obtained by means of a majoration-minimization procedure based on the local linearization of $\Omega_g(\tilde{T})$ (apply at each iteration classical OST with cost matrix $\tilde{C}^{(iter)} = \tilde{C} + \tilde{R}^{(iter)}$ where $\tilde{R}^{(iter)}$ has coefficients $\tilde{r}_{i,k}^{(iter)} = 1/2[\![\tilde{t}_k^{(iter)}]\!]_1^{-1/2}$).