BMEN 509 - Lab #2

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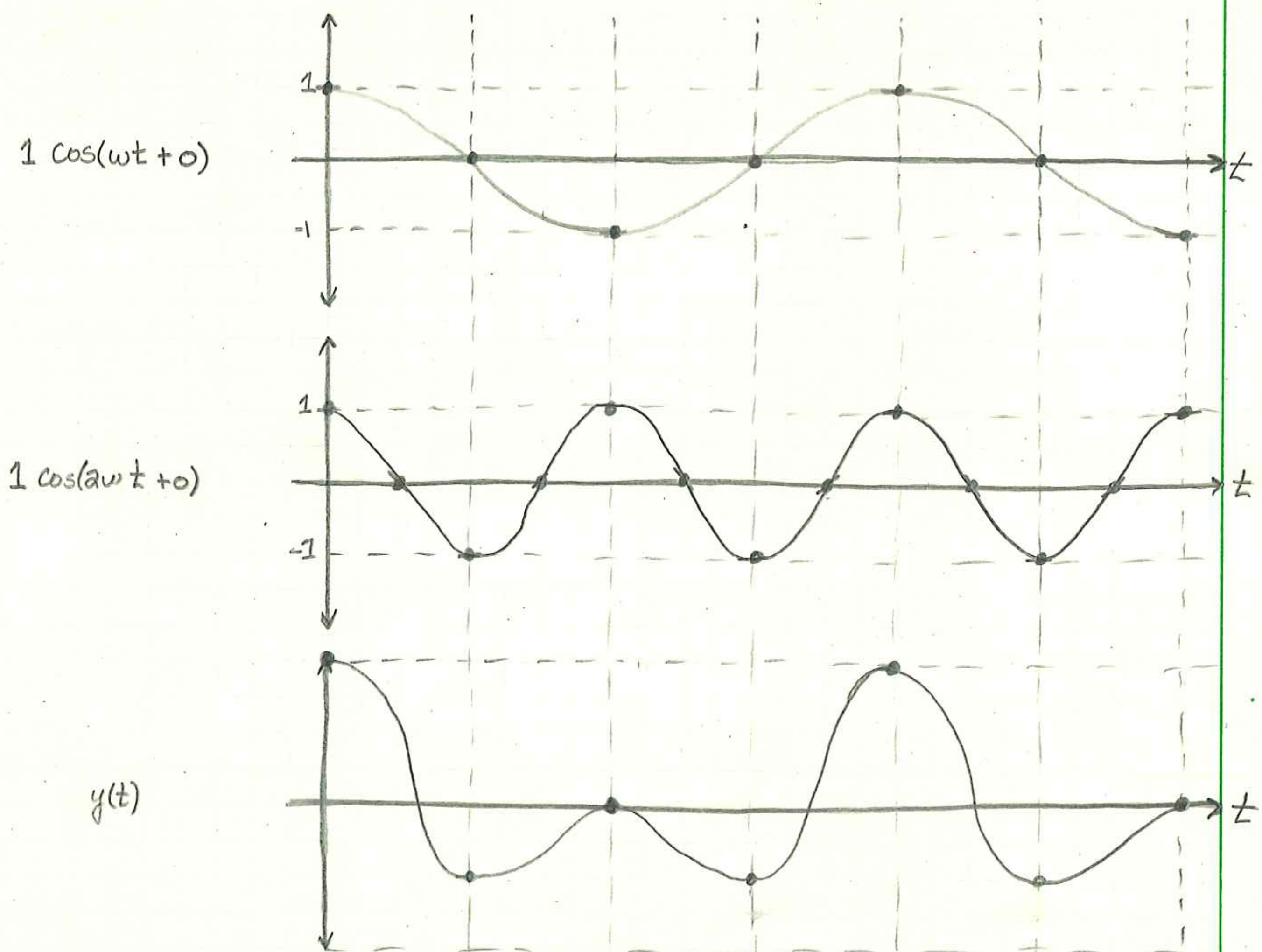
Objectives
- Understand the Fourier Transform
- Understand and implement frequency domain filtering

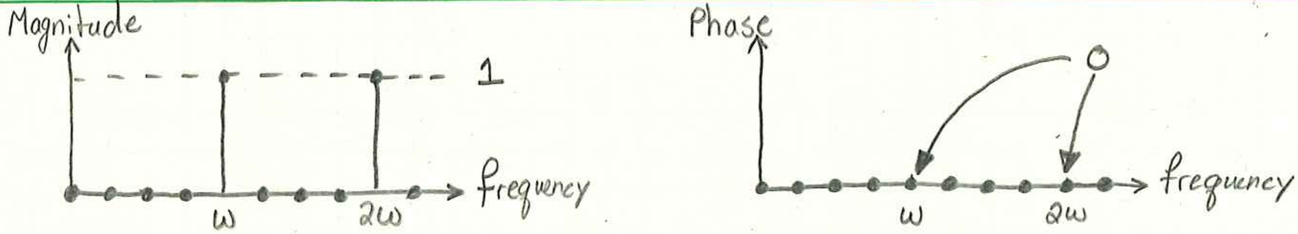
One Dimensional Example
Consider adding two cosine functions.

$$y(t) = \underset{\text{amplitude}}{1}\cos(\omega t + \underset{\text{phase}}{0}) + 1\cos(\underset{\text{frequency}}{2\omega} t + 0)$$



Properties of $y(t)$
- Periodic with freque
- Represented using only cosine functions!

We could plot the parameters of each cosine - the magnitude and phase - as a function of frequency.



We could express $y(t)$ as the sum of many cosine functions, most with a magnitude of zero.

See Example 1 for concrete examples.

Generalized Fourier Transform

In general, any function can be represented by a sum of cosine functions.

$$y(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t + \phi_n)$$

The n^{th} component   amplitude   frequency   phase

However, it is more convienient to express the formula using complex exponentials.

Boring Math

$$\cos(\alpha) = \frac{1}{2}\left[e^{j\alpha} + e^{j\alpha}\right]$$

$$y(t) = \sum_{n=0}^{\infty} \frac{A_n}{2}\left[e^{j(\omega_n t + \phi_n)} + e^{-j(\omega_n t + \phi_n)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{A_n}{2} e^{j\phi_n} e^{j\omega_n t} + \sum_{n=0}^{\infty} \frac{A_n}{2} e^{-j\phi_n} e^{-j\omega_n t}$$

let $\frac{A_n e^{j\phi_n}}{2} = M_n e^{j\phi_n}$ ← Phase

Magnitude (notice $M = \frac{A}{2}$!)

Result

$$y(t) = \sum_{n=-\infty}^{+\infty} M_n e^{j\phi_n} e^{j\omega_n t}$$

The Fourier Transform takes a signal, $y(t)$, and returns the magnitude – M_n – and phase – ϕ_n – of many cosine functions. Adding these cosine functions 'reconstructs' the signal $y(t)$. The result of the Fourier Transform is:

- A complex number, with phase and magnitude
- The magnitude and phase change with frequency.

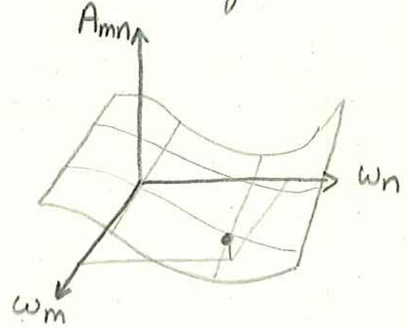
See Example 2.

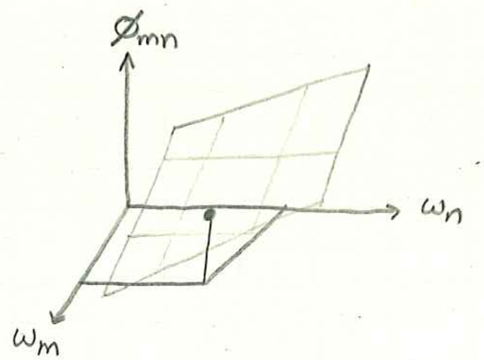
Fourier Transform for Images

Images are two dimensional signals. As before, we can represent images as a sum of 2D cosine functions.

$$I(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \underbrace{A_{mn}}_{\text{amplitude}} \cos(\underbrace{\omega_m x + \omega_n y}_{\text{frequencies}} + \underbrace{\phi_{mn}}_{\text{phase}})$$

The 2D Fourier Transform gives the components - A_{mn}, ϕ_{mn} - for various frequencies ω_m, ω_n.





See Example 3.

We can visualize the magnitude - A_{mn} - and phase - ϕ_{mn} - as another image! The intensity of the pixels correspond to the value of the magnitude or phase, whichever you are visualizing.

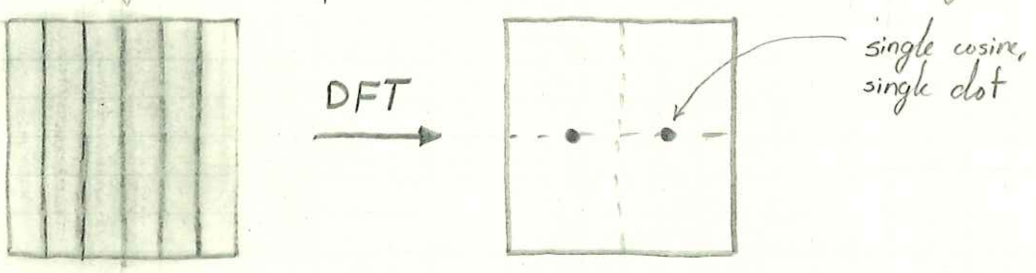


Image (Cosine) | Magnitude of DFT

See Example 4 for real images.

Frequency Domain Filtering

One important feature of the Fourier Transform is that convolution in the spatial domain is multiplication in the frequency domain

$$j(x,y) = i(x,y) * l(x,y) \underset{IDFT}{\overset{DFT}{\rightleftarrows}} J(K_x,K_y) = I(K_x,K_y) \bullet L(K_x,K_y)$$

spatial index   convolution   frequency index   multiplication

It is standard to denote indexing in the frequency domain with (K_x, K_y), similar to how (x,y) denote spatial indexing.

See Example 5 for visualizing the frequency domain after filtering.

Many imaging systems introduce a blur similar to a Gaussian. We may want to uncover this blur with the Frequency domain. We can model a blur as:

$$\text{Image} = \text{Actual} * \text{Blur}$$

Take the Fourier Transform:

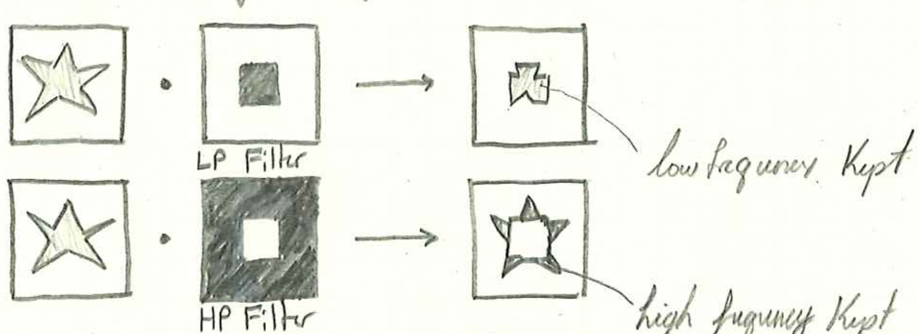
$$\text{Image}_{\mathcal{F}} = \text{Actual}_{\mathcal{F}} \bullet \text{Blur}_{\mathcal{F}}$$

Now divide!

$$\text{Blur}_{\mathcal{F}} = \frac{\text{Image}_{\mathcal{F}}}{\text{Actual}_{\mathcal{F}}}$$

See Example 6 for an example of 'uncovering' a gaussian blur.

Finally, we can filter our image data in the frequency domain by multiplication. Consider the following examples:



See Example 7 for an example of applying a filter in the frequency domain.