The replicating portfolio of a constant product market with bounded liquidity

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Abstract

We derive the replicating portfolio and greeks for a constant product market with bounded liquidity such as Uniswap v3. The portfolio value is concave in the relative price of pool assets, short volatility, and can be effectively hedged in the same manner as a vanilla option.

1 Introduction

A constant product market (CPM) is a mechanism to trade between a pair of assets without an order book. Liquidity providers deposit some quantity of two assets into a pool, and any swap preserves the product of reserves. Arbitrage ensures that the value of each side of the pool is equal.

From a liquidity provider perspective, the situation is similar to selling an option. The value of pool is concave in price, so always under-performs a linear future. The swap fee revenue must be enough to compensate liquidity providers for this under-performance. This is similar to a delta-hedged option seller requiring contributions from theta to offset losses from gamma.

Constant product markets, and their generalization as constant function markets have been extensively characterised by Angeris and coauthors in a series of papers (see in particular (2) and (4)). The basic replicating portfolio and greeks were discussed in (6) and generalised for for constant function markets in (3).

Bounded liquidity CPMs such as Uniswap v3 (1) generalize the mechanism by allowing liquidity providers to concentrate liquidity over an interval. This improves capital efficiency

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and reduces slippage over the interval. The bounded position behaves like an ordinary CPM with a larger amount of 'virtual' reserves while the price is within the bound.

The value of the pool is a known function of the price and can be perfectly replicated with a spanning portfolio of bonds, futures, and options. We provide the replicating portfolio in section 5 and illustrate with an example.

2 Preliminaries

For the most part we follow the notation of (2) who examine constant product markets such as Uniswap.

Definition 1: Bounded liquidity position

A bounded liquidity position is a tuple $(t, R_\beta, R_\alpha, p_a, p_b)$ satisfying

$$\left(R_\alpha + \frac{L}{\sqrt{p_b}}\right)(R_\beta + L\sqrt{p_a}) = L^2 \tag{1}$$

This can be seen as an ordinary constant product market on virtual reserves

$$R_\alpha^{virtual} R_\beta^{virtual} = L^2$$

while the price $p = R_\beta^{virtual}/R_\alpha^{virtual}$ is between p_a and p_b.

A transaction depositing Δ_β at time t will receive Δ_α satisfying

$$(R_\alpha^{virtual} - \Delta_\alpha)(R_\beta^{virtual} + \Delta_\beta) = L^2 \tag{2}$$

Actual reserves are updated according to $R_\beta \mapsto R_\beta + \Delta_\beta$, $R_\alpha \mapsto R_\alpha - \Delta_\alpha$ provided the resulting actual reserves are positive.

Definition 2: Spot market

A spot market is a mechanism which exchanges Δ_α units of α for $p^t\Delta_\beta$ units of β at time t. An infinitely elastic spot market is one where p^t does is not depend on Δ_β

3 Characterising actual and virtual liquidity

From 1 and 2 virtual and actual reserves be helpfully related to virtual liquidity and price

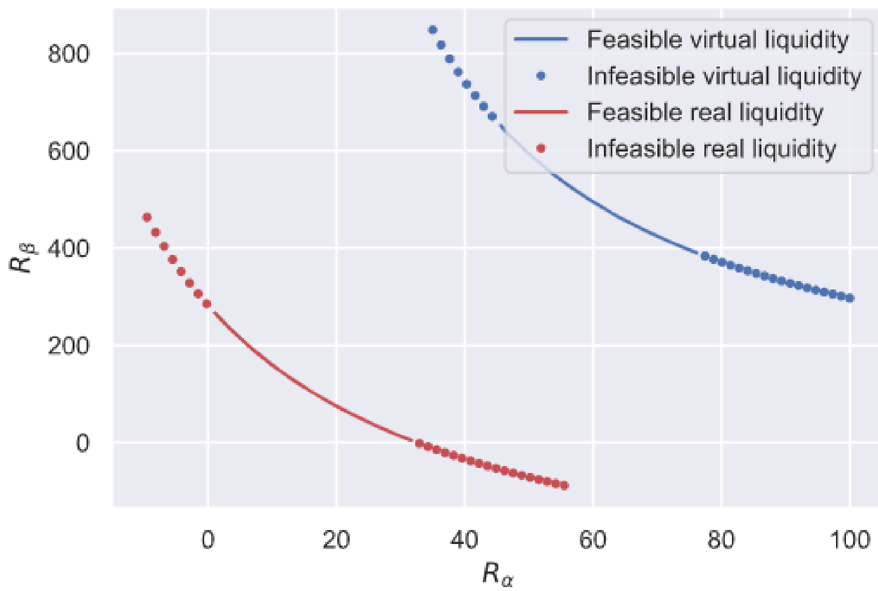


Figure 1: Virtual and real liquidity

$$R_\alpha^{virtual} = \frac{L}{\sqrt{p}} \tag{3}$$

$$R_\beta^{virtual} = L\sqrt{p} \tag{4}$$

$$R_\alpha = \frac{L}{\sqrt{p}} - \frac{L}{\sqrt{p_b}} \tag{5}$$

$$R_\beta = L\sqrt{p} - L\sqrt{pa} \tag{6}$$

The relationship between actual reserves and virtual reserves in 1 is illustrated in figure 1. Points on the virtual liquidity curve are feasible if they correspond to positive values for actual liquidity R_α and R_β.

While virtual liquidity is unbounded, actual liquidity is limited by the availability of reserves. The maximum actual liquidity supported by a some virtual liquidity L is obtained on the boundaries $p = p_a$ and $p = p_b$

$$R_\alpha^+ = \frac{L}{\sqrt{p_a}} - \frac{L}{\sqrt{p_b}} \tag{7}$$

$$R_\beta^+ = L\sqrt{p_b} - L\sqrt{p_a} \tag{8}$$

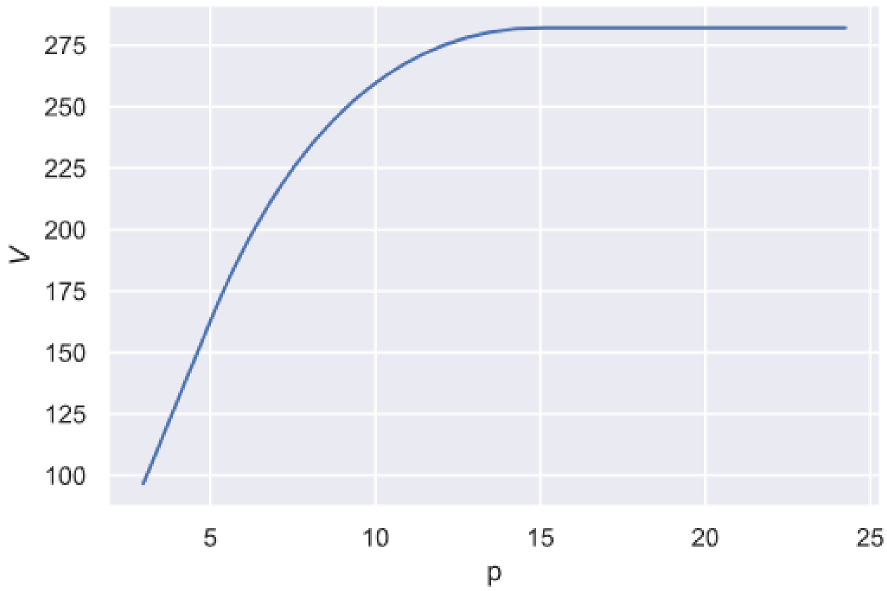


Figure 2: Value function of bounded liquidity position

4 The value of a bounded liquidity position

The total value of actual liquidity measured in units of β is

$$V = R_\alpha p + R_\beta \tag{9}$$

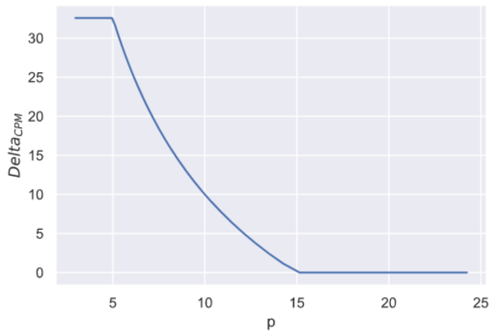
Using the conditions from 1 and 2 and the bounds for actual liquidity this is

$$V = \begin{cases} pR_\alpha^+, & \text{for } p \leq p_a \\ 2L\sqrt{p} - L\sqrt{p_a} - p\frac{L}{\sqrt{p_b}}, & \text{for } p_a \leq p \leq p_b \\ R_\beta^+, & \text{for } p \geq p_b \end{cases} \tag{10}$$

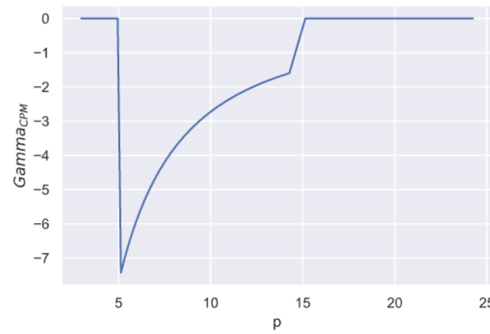
In the limiting case with $p_a = 0$, $p_b = \infty$ corresponds to an ordinary constant product market. A typical value function is figure 2.

4.1 Leverage

The bounded liquidity position supports a larger value of virtual liquidity than a standard pool. This efficiency can be characterised by the ratio of value between the virtual and actual liquidity using 9 and 3-6



(a)



(b)

Figure 3: (a) Delta (b) Gamma

$$\text{Leverage} = \frac{2}{2 - \sqrt{\frac{p}{p_b}} - \sqrt{\frac{p_a}{p}}} \tag{11}$$

4.2 Greeks

The first and second order sensitivities are

$$\begin{aligned}
\text{CPM delta:} \qquad \text{Delta}_{CPM} = \frac{\partial V}{\partial p} &= \begin{cases} R_\alpha^+, & \text{for } p \leq p_a \\ \frac{L}{\sqrt{p}} - \frac{L}{\sqrt{p_b}}, & \text{for } p_a \leq p \leq p_b \\ 0, & \text{for } p \geq p_b \end{cases} \\
\text{CPM gamma:} \qquad \text{Gamma}_{CPM} = \frac{\partial^2 V}{(\partial p)^2} &= \begin{cases} 0, & \text{for } p \leq p_a \\ -\frac{1}{2}\frac{L}{p^{3/2}} & \text{for } p_a \leq p \leq p_b \\ 0, & \text{for } p \geq p_b \end{cases}
\end{aligned} \tag{12}$$

Again the limiting case corresponds to an unbounded liquidity position for delta, and is identical for gamma.

5 The replicating portfolio

To replicate V^T we use the spanning result of Green and Jarrow (1987) which provides that any function W of a final price m^T can be replicated as

$$W(S^T) = W(A)e^{-rT} + W'(A)(p^T - Ae^{-rT}) + \int_0^A W''(K)P(K)dK + \int_A^\infty W''(K)C(K)dK \tag{13}$$

Where r is the interest rate (from here we'll assume r=0). We choose $A = p^0$ (suppressing the subscript) and replicate with a bond, futures, and strips of options on the underlying all expiring at T and for $p \in [p_a, p_b]$:

$$\begin{aligned}
\text{Face value of bond:} \quad W(p^0) = &\quad 2L\sqrt{p^0} - L\sqrt{p_a} - p\frac{L}{\sqrt{p_b}} \\
\text{Notional value of futures:} \quad W'(p^0) = &\quad \frac{L}{\sqrt{p^0}} - \frac{L}{\sqrt{p_b}} \\
\text{Notional value of options at strike } K\text{:} \quad W''(K) = &\quad -\frac{1}{2}\frac{L}{K^{3/2}}dK \qquad , \ \text{K} \in [p_a, p_b]
\end{aligned}$$

The appendix has an example of replication with limited strikes.

Appendix A. Example: Replicating a bounded liquidity position with options

A liquidity provider provides 10 units of α to a liquidity position bounded on $p \in [p_a, p_b] = (5, 15)$ with a current price $p^1 = 10$.

To satisfy 1 the liquidity provider must also provide 159.611 units of β corresponding to $L \approx 172.34$

The corresponding virtual liquidity can be calculated directly

$$R_\alpha^{virtual} = \frac{L}{\sqrt{p}} \approx 54.49 \tag{14}$$

$$R_\beta^{virtual} = L\sqrt{p} \approx 544.95 \tag{15}$$

$$\tag{16}$$

The initial value of the pool (in units of β) is

$$V^1 = (p^1 R_\alpha^1 + R_\beta^1) = 10 * 10 + 159.61 = 259.61$$

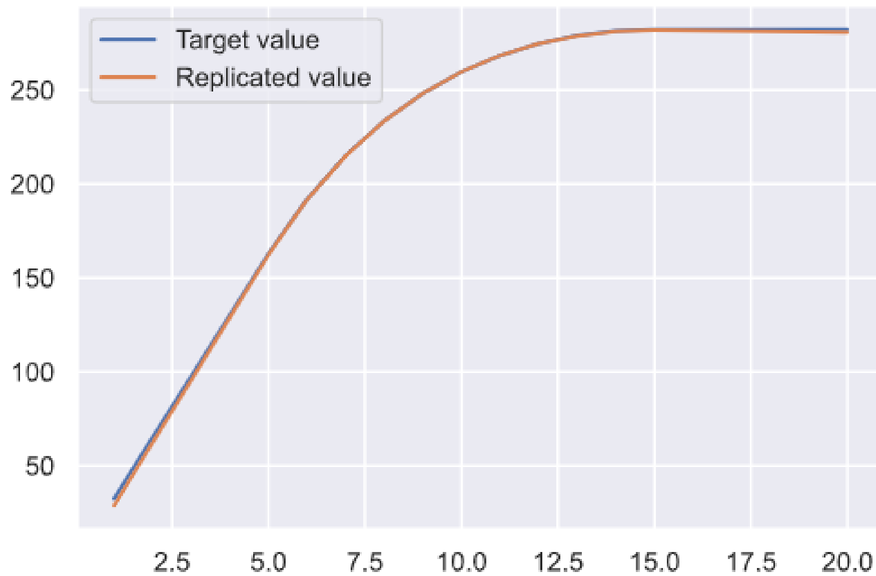


Figure 4: Constant product market value vs replicated value

The value of the virtual liquidity supported between the price range is

$$V^{1,Virtual} = (p^1 R_\alpha^{1,virtual} + R_\beta^{1,virtual}) \approx 10 * 54.49 + 544.95 \approx 1089.90$$

If we hold a discrete number of strikes $K = (5, 6, \ldots, 15)$ payoff at expiry is a discretized version of 13 (again with zero rates)

$$W(p^T) = W(p^0) + W'(p^0)(p^T - p^0) + \sum_{K \leq p^0} W''(K)P(K)\Delta K + \sum_{K > p^0} W''(K)C(K)\Delta K$$

Table 1 has the notional value of four strikes each for the calls and puts. Figure 4 compares the total value of the pool and the replicating portfolio for values of the exchange rate at T.

References

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Table 1: Option notional by strike

Strike K	Call notional	Put notional
5	0	-7.71
6	0	-5.86
7	0	-4.65
8	0	-3.80
9	0	-3.19
10	0	-2.72
11	-2.36	0
12	-2.07	0
13	-1.83	0
14	-1.64	0
15	-1.48	0

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