Trapped-ion Quantum Computer

: From Ion Trapping to Gate Implementation

*Note: Coursework Project of EE337 (FA24)

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***Abstract*—This document is the final project for the *EE337: Introduction to Quantum Information* course. Each student is assigned a different quantum computing theme that is not covered in class and is required to structure the content in a document using LaTeX. Students are also required to present and defend their work to the professor and classmates. The *EE337* course focuses on quantum information with the assumption that there is a qubit.**

This project provides an opportunity to explore what happens underneath the hood by understanding the bridge between the physical and informational aspects of quantum systems in a trapped ion quantum computer. This document focuses on the implementation of trapped ions from a physics perspective and how they can act as carriers of information. Additionally, it discusses how gates for trapped ions can be implemented and used for computation. Finally, this project highlights the advantages of the trapped ion quantum computer in comparison to the superconducting quantum computer.

***Index Terms*—Hamiltonian, Hyper Fine Structure, Laser Cooling, Matter Light Interaction**

I. Hamiltonian

In a classical system, the total energy of one-dimensional linear motion (x) can be written using the following equation:

$$H(x,p) = \frac{p^2}{2m} + V(x), \quad (p = mv) \tag{1}$$

In a quantum system, the total energy can be redefined as $\hat{H}$ and the momentum operator as $\hat{p}$. The momentum operator $\hat{p}$ can be written as:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{2}$$

Then the Hamiltonian operator can be defined as the sum of kinetic energy and potential energy, which only depends on the one-dimensional variable x:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{3}$$

The time-dependent Schrödinger equation with a potential term $V(x)$ is given by

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t) = E\psi(x,t)$$

where $\psi(x,t)$ can be expressed as the product $\psi(x,t) = \psi(x)\cdot\psi(t)$ if the wave function is separable and $\psi(t)$ can be removed. E is allowed energy and acts as an eigenvalue applied to $\psi(x,t)$ $\hbar$ is the reduced Planck's constant, m is the particle mass, and $V(x)$ represents the potential energy as a function of position x. Considering only x, Schrödinger equation can be written as

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E\psi(x) \tag{4}$$

By multiplying $\psi(x)$ by (3) and and comparing it with (4), the time-independent Schrödinger equation in terms of the Hamiltonian operator $\hat{H}$ is obtained:

$$\hat{H}\psi(x) = E\psi(x) \tag{5}$$

From (5), it can be seen that the Hamiltonian operator $\hat{H}$ acts on the wave function $\psi(x)$, giving back $\psi(x)$ with eigenvalue E. The Hamiltonian operator $\hat{H}$ is fundamentally related to the energy of a quantum system, encapsulating both its kinetic and potential energy.

II. Hyper Fine Structure

A. Definition

Hyperfine coupling is caused by the interaction between the magnetic moments arising from the spins of both the nucleus and electrons in atoms. When the atom is ionized and becomes an ion, this spin interaction causes the splitting of the fine structure of spectral lines into smaller components called hyperfine structure, which is caused by the shift in the energy levels of the electron.

B. Term Symbol and Representation

The system can be described by J(Total electronic angular momentum of the system) which can be chosen within the range of $|L-S|$ and $|L+S|$, where L is orbital angular momentum of electron, and S is spin quantum number of electron. The quantum state of electron in the system can be represented as ${}^{2S+1}L_J$. For example, the values of (J,S,L) for the state ${}^2S_{1/2}$ are ($\frac{1}{2}$, $\frac{1}{2}$, 0) and the values of (J,S,L) for the state ${}^2P_{1/2}$ are ($\frac{1}{2}$, $\frac{1}{2}$, 1). In the context of hyperfine

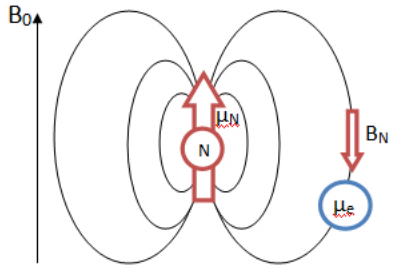

Fig. 1. Electron with its magnetic moment moving within the magnetic dipole field of the nucleus

structure, F represents the total angular momentum quantum number of the atom, which includes both the total electronic angular momentum J and the nuclear spin I. Mathematically, it is expressed as: F = J + I

The value of F determines the splitting of energy levels due to the interaction between the magnetic fields produced by the electrons and the nucleus. This interaction leads to the hyperfine structure observed in atomic spectra, which is typically much smaller than the fine structure splitting.The state $|0\rangle$ and the state $|1\rangle$ can be represented as following:

$$|0\rangle \Leftrightarrow {}^2S_{1/2}, F = 0$$

$$|1\rangle \Leftrightarrow {}^2S_{1/2}, F = 1$$

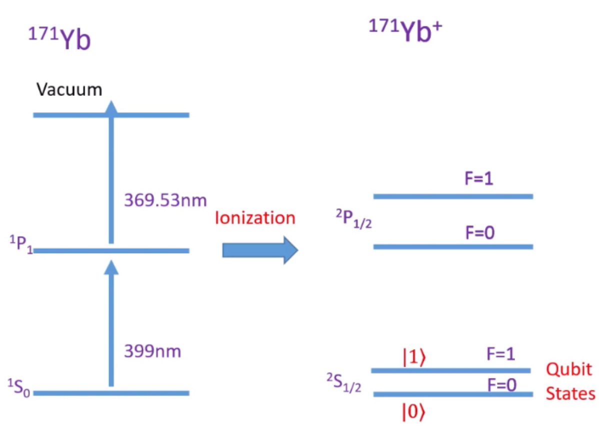

Fig. 2. Hyperfine State from Ionization of ^{171}Yb

III. Ion Trapping

A. Linear Paul Trap

A linear Paul trap, also known as a linear quadrupole ion trap, uses static and oscillating electric fields to confine ions. It consists of four parallel rods with alternating AC voltage, creating a quadrupole field for radial confinement. This static potentials push confined atom inside, which represents lattice of atom. Each ion acts as one qubit. The electric potential Φ that influences the ion's motion and confinement within the trap is given by:

$$\Phi = \Phi_0 + \frac{V_0}{2r_0^2}\cos(\Omega t)(x^2 - y^2)$$

where Ω represents the angular frequency of the oscillating electric field applied to the trap.
The solution x is:

$$x = x_0 \cos(A\tau)\left(1 + B\cos(2\tau)\right)$$

$$\tau = \frac{\Omega_0 t}{2}$$

where A is angular frequency of overall motion and B is the amplitude of the modulation in the oscillatory motion which indicates how much the secondary oscillation (with frequency 2τ) affects the primary motion.

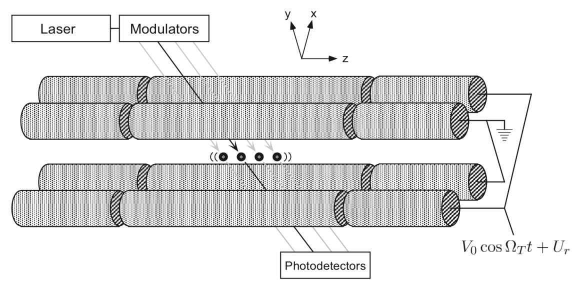

Fig. 3. Linear Paul Trap

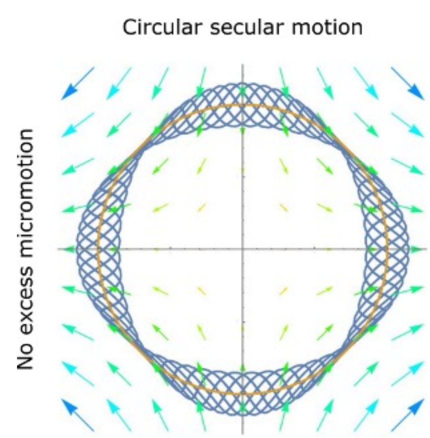

Fig. 4. Trapped Ion moving along, not stationary

B. Laser Cooling

Laser cooling excites the $^2S_{1/2}$ state to the $^2P_{1/2}$ state by shooting a photon. From Fig. 4, the ion is moving and will absorb the photon. The ion becomes excited but will later decay, emitting a photon in any direction. If the direction of the photon emitted by the decaying ion is different from the direction of the moving ion, the ion will have less momentum in the original direction due to the conservation of total momentum. On the other hand, if the two directions are the same, the ion will still have velocity in the original direction. With a 50% chance, if this process happens many times, the velocity will decrease, and the temperature (T), which is related to the ion's velocity, will also decrease.

$$kT \propto mv^2$$

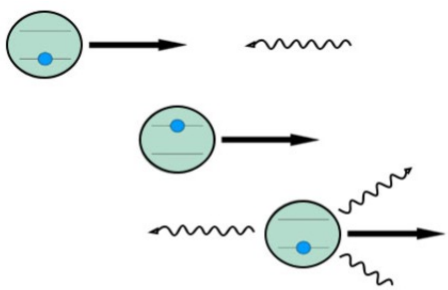

Fig. 5. Laser Cooling

C. Doppler Cooling

In order for an ion to absorb a photon, it needs a photon with a certain amount of energy. If the photon and ion are coming closer from different directions, due to the Doppler effect, the photon appears to the ion as having higher energy than it actually has. In this case, the ion needs a photon with smaller energy, which it can absorb using the redshifted photon. However, since the photon with smaller energy cannot be absorbed, the same redshifted photon won't be absorbed by an ion moving away. Therefore, with photons coming from multiple directions, any ion matched to the direction will be cooled down.

IV. Qubit

A. Initialization to $|0\rangle$

- Add Cooling Laser
- Pump: ${}^2S_{1/2},\ F=1 \rightarrow {}^2P_{1/2},\ F=0$
- Decay: ${}^2P_{1/2} \rightarrow |0\rangle$: ${}^2S_{1/2},\ F=0$
- Those decayed to $|1\rangle$ again: pump again.

B. Measurement

- Assume the amount of energy of the photon is sufficient for the ion to excite from $|1\rangle$ to ${}^2P_{1/2},\ F=0$.
- Ion in the state $|0\rangle$ cannot be excited with the same amount of energy of the photon and will stay in $|0\rangle$.
- If the state was $|0\rangle$, then nothing happens.
- If the state was $|1\rangle$, then the ion will emit a photon when it decays.
- If there is photon emission, it means the state was $|1\rangle$.

V. Gates

A. Single-qubit Gates

Implementing Single-qubit gates rely on shooting laser pulse to trap ion. Laser pulse is electromagnetically fluctuating ($E_0\cos(\omega t)$: E-field oscillating at any point) and hits the atom. Recall Schrödinger equation:

$$i\hbar\frac{d\psi}{dt} = \hat{H}\psi \tag{6}$$

$$\psi(x) = |\psi(x)\rangle \quad \text{(State of Trapped Ion)}$$

The energy levels of $|0\rangle$ and $|1\rangle$ are E_0 and E_1, respectively. Before the laser pulse is applied, these are the eigen energies for the original Hamiltonian H_0:

$$H_0|0\rangle = E_0|0\rangle, H_0|1\rangle = E_1|1\rangle$$

However, when laser pulse applied and there is an interaction, electron will interact with the electric field of laser pulse. There is interaction hamiltonian from electric dipole ($\mathbf{d} = e\mathbf{r}$) interaction

$$H_I = e\mathbf{r}\cdot\mathbf{E_0}\cos(\omega t)$$

With H_I, the state $|\psi\rangle$ can be written as:

$$|\psi\rangle = C_0|0\rangle e^{-i\omega_0 t} + C_1|1\rangle e^{-i\omega_1 t}$$

$$\omega_0 = \frac{E_0}{\hbar}, \quad \omega_1 = \frac{E_1}{\hbar}, \quad |C_0|^2 + |C_1|^2 = 1$$

where $|0\rangle$ and $|1\rangle$ are no longer eigenvectors (basis). Substitute $|\psi\rangle$ into the Schrödinger equation and find C_0 and C_1. The equations are:

$$i\frac{dC_0}{dt} = \Omega\cos(\omega t)e^{-i\omega_L t}C_1 \tag{7}$$

$$i\frac{dC_1}{dt} = \Omega^*\cos(\omega t)e^{i\omega_L t}C_0 \tag{8}$$

$$\omega_L = \frac{E_1 - E_0}{\hbar} \quad \text{(Frequency of energy difference)}$$

$$\Omega = \frac{\langle 0|e\mathbf{r}\cdot\mathbf{E_0}|1\rangle}{\hbar}$$

.

From (7), by Euler's formula $\cos(\omega t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$, we have:

$$i\frac{dC_0}{dt} = \Omega\cos(\omega t)e^{-i\omega_L t}C_1 \tag{9}$$

$$= \Omega C_1\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)e^{-i\omega_L t} \tag{10}$$

$$= \frac{\Omega C_1}{2}\left(e^{i(\omega-\omega_L)t} + e^{-i(\omega+\omega_L)t}\right) \tag{11}$$

where ω is the light frequency of the laser on the trapped ion, and $\omega_L = \frac{E_1 - E_0}{\hbar}$.

In this equation, $e^{-i(\omega+\omega_L)t}$ is an oscillating term which rotates fast. If $\omega \approx \omega_L$, the effect on the system of $e^{-i(\omega+\omega_L)t}$ term is ignored. Therefore, by approximation, we have:

$$i\frac{dC_0}{dt} = \frac{\Omega C_1}{2}e^{i(\omega-\omega_L)t} \tag{12}$$

Similarly,

$$i\frac{dC_1}{dt} = \frac{\Omega^* C_0}{2} e^{-i(\omega-\omega_L)t} \tag{13}$$

First, substitute equation (12) into (13):

$$\frac{d^2C_1}{dt^2} + i(\omega-\omega_L)\frac{dC_1}{dt} + \left|\frac{\Omega}{2}\right|^2 C_1 = 0$$

If $\omega \approx \omega_L$, this simplifies to:

$$\frac{d^2C_1}{dt^2} + \left|\frac{\Omega}{2}\right|^2 C_1 = 0$$

Solving this equation, we get:

$$\psi = \cos\left(\frac{\theta}{2} + \omega_R' t\right) e^{-\frac{i}{2}\left(-\frac{\pi}{2}+\omega_L t\right)}|0\rangle$$

$$+ \sin\left(\frac{\theta}{2} + \omega_R' t\right) e^{\frac{i}{2}\left(-\frac{\pi}{2}+\omega_L t\right)}|1\rangle$$

where $\omega_R' = \frac{\Omega}{2}$. This means the initial angle of the state $\frac{\theta}{2}$ is changing at the rate of $\omega_R' t$. From the general state on the Bloch sphere, we have:

$$\psi = e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

For gate implementation that switches from $|0\rangle$ to $|1\rangle$, apply a pulse such that $\Omega t = \pi \Leftrightarrow \omega_R' t = \frac{\pi}{2}$. The resulting state is:

$$\psi' = \cos\left(\frac{\theta}{2} + \frac{\pi}{2}\right) e^{-\frac{i}{2}\left(-\frac{\pi}{2}+\omega_L t\right)}|0\rangle$$

$$+ \sin\left(\frac{\theta}{2} + \frac{\pi}{2}\right) e^{\frac{i}{2}\left(-\frac{\pi}{2}+\omega_L t\right)}|1\rangle$$

$$= e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right)|0\rangle + e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right)|1\rangle$$

where $\frac{\phi}{2} = -\frac{\pi}{2} + \omega_L t$.

As time goes by, both $\frac{\theta}{2}$ in sin and cos is rotating ($\theta \to \theta + \Omega t$, where Ω is the Rabi frequency). Applying the laser pulse to trap the ion, where the state of the qubit is mapped to the Bloch sphere, it rotates from the north pole to the south pole at the rate of Ω (Rabi frequency).

ϕ is changing at the speed of ω_L: the angle ϕ is changing from $-\frac{\pi}{2}$ at the rate of ω_L ($\phi \to \phi + \omega_L t$, where ω_L is the Larmor frequency). If the correct pulse is applied to the state on the Bloch sphere, then the state keeps rotating from the north pole to the south pole.

To summarize, from the split of the two energy levels of $|0\rangle$ and $|1\rangle$, the reason it can have Rabi oscillation is because ω_L and ω_R' are derived from Ω, which is the dipole interaction between the atom's electric dipole and the electric field in the laser you applied.

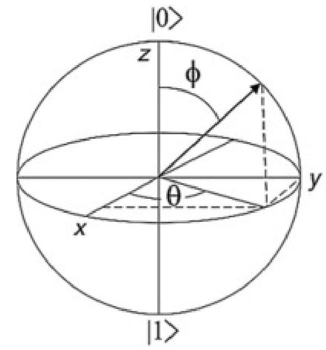

Fig. 6. Bloch Sphere Representation

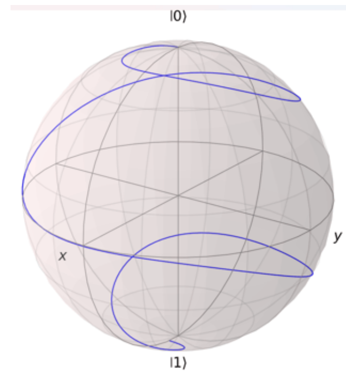

Fig. 7. Rabi Oscillation on the Bloch Sphere

B. Two-qubit gates

Array of trapped ions which is lattice can be represented with phonon state and qubit state. Phonons are quantized units of vibrational energy in a lattice, representing mechanical vibrations of atoms within a solid. A $|0\rangle$ phonon state involves no lattice vibrations for the ions to interact through, effectively isolating them and preventing the entangling operation. On the other hand, $|1\rangle$ phonon state allows the ions to interact in a way that facilitates the entangling operation required for control gate operation. This $|1\rangle$ phonon state effectively couples the motion of one ion to another, enabling controlled operations where the state of one ion (control qubit) affects the other (target qubit). Also, there are two qubit states: ground $|g\rangle$ ($^2S_{1/2}, F = 0$) and excited $|e\rangle$ ($^2S_{1/2}, F = 1$), depending on the energy level of an electron in an ion. Based on the phonon state of the lattice and the qubit state of the ion, there are 4 possible states: $|g0\rangle$, $|g1\rangle$, $|e0\rangle$, $|e1\rangle$ for a single qubit. Similarly, the states can be represented as: $|g_A g_B 0\rangle$, $|g_A g_B 1\rangle$, $|e_A g_B 0\rangle$, $|g_A e_B 0\rangle$, $|e_A e_B 0\rangle$, $|e_A g_B 1\rangle$, $|g_A e_B 1\rangle$, $|e_A e_B 1\rangle$ for two qubits.

Consider applying a CZ gate on two trapped ions A and B with 0 phonon state. (When the system is cooled down, it becomes a 0 phonon state with no vibration.)

1. Apply a π-pulse to ion A, which is the control qubit, with red-shifted energy that is smaller than the energy required to excite the ion from the ground state to the excited state. The ion in the ground state with 0 phonon state cannot absorb the

photon and excite to the second level:

$$|g_A 0\rangle \rightarrow |g_A 0\rangle$$

If the ion is already in the excited state with 0 phonon, it creates the vibration of a phonon and decays with a phase change:

$$|e_A 0\rangle \rightarrow -i|g_A 1\rangle$$

To implement the control gate, the π pulse with red-shifted energy doesn't add a phase when the control qubit is in the ground state and gets a phase change of $-i$ when the control qubit is in the excited state.

2. Apply an exact (not red-shifted) 2π-pulse to ion B, which is the target qubit. This only affects the 1 phonon state:

$$|g_B 1\rangle \rightarrow -|g_B 1\rangle$$
$$(\because \quad \omega_R' t = \frac{\Omega t}{2}, \quad \Omega t = 2\pi,$$
$$\cos(\pi+\theta) = -\cos(\theta),$$
$$\sin(\pi+\theta) = -\sin(\theta))$$

However, ions are cooled and the whole system starts with a 0 phonon state. Then how could $|g_B 1\rangle$ exist? For the case where A was in the excited state (control qubit = 1: $|e\rangle$) and a π pulse is applied, it generates 1 phonon which is applied to the whole state (both A and B). If A was in the ground state (control qubit = 0: $|g\rangle$), then the 1 phonon cannot be generated.

3. Repeat step 1: apply a red-shifted π-pulse to ion A:

$$|g_A 1\rangle \rightarrow -i|e_A 0\rangle$$

Following are all the cases for $|AB$ Phonon$\rangle$ applied to the CZ gate:

$$\begin{aligned}
|gg0\rangle &\rightarrow |gg0\rangle \rightarrow |gg0\rangle \rightarrow |gg0\rangle \\
|ge0\rangle &\rightarrow |ge0\rangle \rightarrow |ge0\rangle \rightarrow |ge0\rangle \\
|eg0\rangle &\rightarrow -i|gg1\rangle \rightarrow i|gg1\rangle \rightarrow |eg0\rangle \\
|ee0\rangle &\rightarrow -i|ge1\rangle \rightarrow i|ge1\rangle \rightarrow |ee0\rangle
\end{aligned}$$

If we only look at the qubits:

$$\begin{aligned}
|gg\rangle &\rightarrow |gg\rangle \\
|ge\rangle &\rightarrow |ge\rangle \\
|eg\rangle &\rightarrow |eg\rangle \\
|ee\rangle &\rightarrow -|ee\rangle
\end{aligned}$$

The CZ gate in matrix form is:

$$\text{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

VI. Advantage of Trapped Ion Quantum Computer

In a single ion trap, each ion can interact with every other ion directly. This is known as all-to-all connectivity, which simplifies the implementation of quantum gates between any pair of qubits without needing to move(swap) them around. Because of this full connectivity, trapped ion systems can perform multi-qubit operations more efficiently. This reduces the complexity of quantum algorithms and can lead to faster execution times. When gate operations are needed between two qubits from different traps, it is sufficient to move one qubit to the trap of the other qubit by swapping it with an unused qubit from that trap. In superconducting qubit systems, qubits are typically arranged in a fixed grid, and interactions are limited to neighboring qubits. To perform operations between distant qubits, a series of swap operations is required, which can introduce errors and increase the overall gate count.

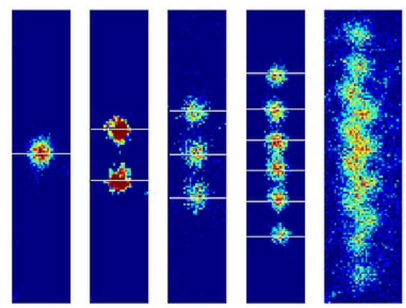

Fig. 8. Ions in a single trap

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