and here we find that the remainder of the system poles are controlled by the feedback matrix $\mathbf{F}$, where the control law is of the form

$$\mathbf{u}_k = -\mathbf{F}\begin{bmatrix} \mathbf{x}_{1k} \\ \hat{\mathbf{x}}_{2k} \end{bmatrix}$$

8.8 ALGORITHM FOR GAIN CALCULATIONS FOR SINGLE-INPUT SYSTEMS

In Section 8.2 we considered the calculation of state feedback gains to place all of the n closed-loop system poles at arbitrary locations in the z-plane. We found that expanding the closed-loop characteristic determinant with the symbolic feedback gains present was a tedious task subject to potential algebraic errors. Similar problems were encountered in Section 8.5 when placing observer poles. Ackermann (1972) and Bass and Gura (1965) have developed algorithms suitable for automating the process of pole placement in the state feedback context for problems with a single control effort or in the state estimation context involving a single system output.

We shall consider the Ackermann algorithm here, as it is slightly more efficient computationally. We are seeking a state feedback gain vector $\mathbf{f}^T$ in the control law

$$u_k = -\mathbf{f}^T\mathbf{x}_k \tag{8.8.1}$$

for the system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{b}u_k \tag{8.8.2}$$

which will pace the closed-loop characteristic roots at arbitrary z-domain locations as long as complex roots occur in conjugate pairs. The closed-loop characteristic equation is

$$\alpha_c(z) = z^n + d_{n-1}z^{n-1} + \cdots + d_1 z + d_0 \tag{8.8.3}$$

This characteristic equation is also given by

$$\alpha_c(z) = \det[z\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{f}^T] \tag{8.8.4}$$

The problem here is that if the d_i are given, then find the vector $\mathbf{f}^T$ to cause the characteristic polynomials of (8.8.3) (known) and (8.8.4) to be the same without a manual expansion of (8.8.4).

We shall assume that there exists a transformation $\mathbf{T}$ that will transform

the system into controllable canonical form with state vector $\mathbf{w}_k$ or

$$\mathbf{x}_k = \mathbf{T}\mathbf{w}_k \tag{8.8.5}$$

Substitute (8.8.5) into (8.8.2) to give the controllable canonical form, which is

$$\mathbf{w}_{k+1} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{w}_k + \mathbf{T}^{-1}\mathbf{b}u_k \tag{8.8.6}$$

and the control law in these new coordinates is

$$u_k = -\mathbf{f}^T\mathbf{T}\mathbf{w}_k \tag{8.8.7}$$

where we shall refer to vector $\mathbf{f}^T\mathbf{T}$ as $\mathbf{f}_W^T$. In the $\mathbf{x}_k$ coordinate system the controllability matrix is

$$\mathbf{C}_x = [\mathbf{A}^{n-1}\mathbf{b} \quad \mathbf{A}^{n-2}\mathbf{b} \quad \cdots \quad \mathbf{A}\mathbf{b} \quad \mathbf{b}] \tag{8.8.8}$$

and in the $\mathbf{w}_k$ coordinates it is

$$\mathbf{C}_w = [\mathbf{A}_1^{n-1}\mathbf{b}_1 \quad \mathbf{A}_1^{n-2}\mathbf{b}_1 \quad \cdots \quad \mathbf{A}_1\mathbf{b}_1 \quad \mathbf{b}_1] \tag{8.8.9}$$

where, for the controllable canonical form, the matrix $\mathbf{A}_1$ and vector $\mathbf{b}_1$ are

$$\mathbf{A}_1 = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \mathbf{b}_1 = \mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{8.8.10}$$

If we note the definitions of $\mathbf{A}_1$ and $\mathbf{b}_1$ given above, the $\mathbf{w}_k$ controllability matrix is

$$\mathbf{C}_w = \mathbf{T}^{-1}[\mathbf{A}^{n-1}\mathbf{b} \quad \mathbf{A}^{n-2}\mathbf{b} \quad \cdots \quad \mathbf{A}\mathbf{b} \quad \mathbf{b}] \tag{8.8.11}$$

and upon recognizing the controllability matrix $\mathbf{C}_x$ the relation between the controllability matrices is

$$= \mathbf{T}^{-1}\mathbf{C}_x \tag{8.8.12}$$

Thus the desired inverse transformation is

$$\mathbf{T}^{-1} = \mathbf{C}_w\mathbf{C}_x^{-1} \tag{8.8.13}$$

but at this point we do not know $\mathbf{A}_1$ and $\mathbf{b}_1$, and hence $\mathbf{C}_w$ is unknown. The open-loop characteristic equation for the controllable canonical form is from $\mathbf{A}_1$ in (8.8.10):

$$\begin{aligned} \alpha_0(z) &= \det[z\mathbf{I} - \mathbf{A}_1] \\ &= z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0 \end{aligned} \tag{8.8.14}$$