Página 277 de ADAPTIVE CONTROL, Astrom and Wittenmark, 1989.

Theorem 6.4—Limiting sampled-data zeros

Let G be a rational function

$$G(s) = K\frac{(s-z_1)(s-z_2),\ldots,(s-z_m)}{(s-p_1)(s-p_2),\ldots,(s-p_n)} \tag{6.64}$$

and H the corresponding pulse transfer function. Assume that $m < n$. As the sampling period $h \to 0$, m zeros of H go to 1 as $\exp(z_i h)$, and the remaining $n-m-1$ zeros of H go to the zeros of $B_{n-m}(z)$, where $B_k(z)$ is the polynomial

$$B_k(z) = b_1^k z^{k-1} + b_2^k z^{k-2} + \cdots + b_k^k \tag{6.65}$$

and

$$b_i^k = \sum_{l=1}^{i}(-1)^{i-l} l^k \begin{pmatrix} k+1 \\ i-l \end{pmatrix}, \quad i = 1,\ldots,k \tag{6.66}$$

The first five polynomials B_k are

$$B_1(z) = 1$$
$$B_2(z) = z+1$$
$$B_3(z) = z^2+4z+1$$
$$B_4(z) = z^3+11z^2+11z+1$$
$$B_5(z) = z^4+26z^3+66z^2+26z+1$$

□

Dan Rivera, Introduction to Digital Control.pdf

Zeros of Sampled Systems

References: Åström, Hagander & Sternby, Automatica, 20, 38, 1984.
Zafiriou & Morari, Int. J. Control, 42, 855, 1985.

Motivating example: Consider a system like

$$\frac{1}{(s+1)^4} \xrightarrow[\text{ZOH}]{\text{Sample with}} \frac{K^*(z^3+b_1z^2+b_2z+b_3)}{z^4+a_1z^3+a_2z^2+a_3z+a_4}$$

The poles of the discrete system correspond to

(Discrete Poles) $p_i^* \longrightarrow e^{p_iT}$ (Continuous poles)

What about the zeros? These depend on sampling, type of hold device used;

Theorem 1 (quoted in Zaf & Morari)

Let $A(s)$ be strictly proper $(m<n)$ rational function in s.

$$A(s) = \frac{K(s-v_1)\cdots(s-v_m)}{(s-w_1)\cdots(s-w_n)}$$

As $T \to 0$, m zeros of $G(z)$ go to 1 as $\exp(v_iT)$

$n-m-1$ zeros go to the zeros of $B_{n-m}(z)$

$$B_k(z) = b_1^k z^{k-1} + b_2^k z^{k-2} + \ldots + b_k^k$$ } See papers for details

(Pole Excess) $n-m$	B_{n-m}	Unstable zeros (Zeros outside unit circle)
1	1	-
2	$z+1$	-1
3	z^2+4z+1	-3.732
4	$z^3+11z^2+11z+1$	$-1, -9.899$

Theorem 2 Let $A(s)$ be a strictly proper system (rational transfer function) with $A(0) \neq 0$ and $\mathrm{Re}(w_i) < 0$. Then all zeros of $G(z)$ (ZOH-equivalent pulse transfer function) go to zero as $T \to \infty$.

As sampling time increases, system zeros become time delays...

Ejemplo:

$$G(s) = \frac{1}{(s+1)^4} \qquad \Leftrightarrow \qquad G(z) = \frac{K(z^3 + b_1 z^2 + b_2 z + b_3)}{z^4 + a_1 z^3 + a_2 z^2 + +a_3 z + a_4}$$

para $ts \to 0$

$$G(z) = \frac{K(z^3 + 11z^2 + 11z + 1)}{(z-1)^4} = \frac{K(z+9.899)\,(z+1)\,(z+0.101)}{(z-1)^4}$$

para $ts \to \infty$

$$G(z) = \frac{1}{z}$$

Lo verificamos con Matlab

```
>> s=zpk('s');G=1/(s+1)^4

Zero/pole/gain:
   1
-------
(s+1)^4

>> ts=1; Gd=c2d(G,ts)

Zero/pole/gain:
0.018988 (z+4.561) (z+0.4479) (z+0.04434)
------------------------------------------
            (z-0.3679)^4
Sampling time: 1

>> ts=.1; Gd=c2d(G,ts)

Zero/pole/gain:
3.8468e-006 (z+9.14) (z+0.9231) (z+0.09323)
--------------------------------------------
             (z-0.9048)^4
Sampling time: 0.1

>> ts=1e-6; Gd=c2d(G,ts)

Zero/pole/gain:
4.1667e-026 (z+9.899) (z+1) (z+0.101)
--------------------------------------
             (z-1)^4

Sampling time: 1e-006
>>
>> B4=[1 11 11 1]; roots(B4)
ans =
   -9.8990
   -1.0000
   -0.1010
```

Para período de muestreo grande

```
>> ts=1e2; Gd=c2d(G,ts)
Zero/pole/gain:
z^3          1
---    ⇒    ---
z^4          z

Sampling time: 100
```